

# Optimization Algorithms on Riemannian Manifold

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# Riemannian Manifold

## Definition (Riemannian Manifold)

$M$  を可微分多様体とする. 任意の接空間  $T_x M$  に内積  $g : T_x M \times T_x M \rightarrow \mathbb{R}$  が定まっているとき, 組  $(M, g)$  をリーマン多様体 (Riemannian Manifold) といい,  $g$  をリーマン計量と呼ぶ.

## Definition (Gradient)

$(M, g)$  をリーマン多様体とし,  $f : M \rightarrow \mathbb{R}$  を可微分写像とする.  
 $x \in M$  について

$$\forall \xi \in T_x M, g(\operatorname{grad} f(x), \xi) = Df(x)[\xi]$$

を満たす一意な  $\operatorname{grad} f(x) \in T_x M$  を  $f$  の  $x$  での勾配 (Gradient) と呼ぶ.

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# Sphere

## Example (Sphere)

自然数  $n \geq 2$  に対して, 球面  $S^{n-1} := \{x \in \mathbb{R}^n \mid x^\top x = 1\}$  は可微分多様体となる. また,  $x \in S^{n-1}$  での接空間  $T_x S^{n-1}$  は

$$T_x S^{n-1} = \{z \in \mathbb{R}^n \mid x^\top z = 0\}$$

となる. ここで,  $g_x : T_x S^{n-1} \times T_x S^{n-1} \rightarrow \mathbb{R}$  を

$$g_x(\xi, \eta) = \xi^\top \eta$$

と定めれば,  $g_x$  は  $T_x S^{n-1}$  の内積となるので,  $S^{n-1}$  はリーマン多様体である.

# Stiefel Manifold

## Example (Stiefel Manifold)

$X = [x_1 x_2 \cdots x_p] \in \mathbb{R}^{n \times p} (n \geq p)$  で,  $\{x_i\}_{i=1}^p$  が正規直交系であるような  $n \times p$  行列全体は可微分多様体となる. この多様体をシュティーフェル多様体 (*Stiefel Manifold*) といい,  $\text{St}(p, n)$  と表す. すなわち

$$\text{St}(p, n) = \{X \in \mathbb{R}^{n \times p} \mid X^\top X = \mathbb{I}_p\}.$$

である.

$X \in \text{St}(p, n)$  での接空間  $T_X \text{St}(p, n)$  は,

$$\begin{aligned} T_X \text{St}(p, n) &= \left\{ Z \in \mathbb{R}^{n \times p} \mid X^T Z + Z^T X = 0 \right\} \\ &= \left\{ X\Omega + X_\perp K \mid \Omega \in \text{Skew}(p), K \in \mathbb{R}^{(n-p) \times p} \right\} \end{aligned}$$

となる. ここで  $\text{Skew}(p) = \{\Omega \in \mathbb{R}^{p \times p} \mid \Omega^T = -\Omega\}$  であり,  $X_\perp$  は  $\text{span}(X)^\perp = \text{span}(X_\perp)$  を満たす  $n \times (n-p)$  行列である.

$g_X : T_X \text{St}(p, n) \times T_X \text{St}(p, n) \rightarrow \mathbb{R}$  を,

$$g_X(\xi, \eta) = \text{tr}(\xi^T \eta)$$

と定めれば,  $g_X$  は内積となるので,  $\text{St}(p, n)$  はリーマン多様体である.

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# References

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