Harvard University
Computer Science 121

Problem Set 1

Tuesday September 22, 2015 at 11:59pm

Problem set by **Takehiro Matsuzawa**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE (See the syllabus for information)**

Note: Finite automata (FA) drawings may be done by hand or using an online drawing tool.

PART A (Graded by Sam and Serena)

PROBLEM 1 (2+2+1 points, suggested length of 3 lines)

(A) Describe informally L_1, the language accepted by the NFA on the left.

(B) Describe informally L_2, the language accepted by the NFA on the right.

(C) Write down $L_1 \cap L_2$.

$\frac{100*48}{82} = 58.54$

Solution.
(A) $\varepsilon \cup (01)^* \cup (10)^*$
(B) $\varepsilon \cup (010)^* \cup (101)^*$
$\{$ w|w is a empty string or $\}$ (C) Any string whose length is divisible by 6 and it has 1 and 0 next to each other or empty string ε because no

PROBLEM 2 (3+6+(2) points, suggested length of 1 page)

Let $S_n = \{1, 2, 3, ..., n\}$. We can represent subsets of S_n using strings of length n by setting the i^{th} character to 1 if element i is in the subset, and 0 if it is not. For example, for $n = 3$, the subset $\{3\}$ would be represented by the string 001, and the subset $\{2, 3\}$ would be represented by the string 011. Let $f_n : \{0, 1\}^n \to P(S_n)$ be a function that produces the subset represented by a string, e.g. $f_3(001) = \{3\}$ and $f_3(011) = \{2, 3\}$.

(A) Is f_n injective for all n? Surjective? Bijective? Informally explain each of your answers.

(B) In this question, we will use the machinery of DFAs to "compute" a function. We will label accept states with values, and the state that the DFA halts at after processing an input string will correspond to the output of the function it computes. Your task is to draw a DFA that computes the function f_3. *Notes:*

- If the DFA is given a string of length greater than 3, it should enter a labeled "error" non-accept state.
- If the DFA is given a string of length 3, it should halt in an accept state that corresponds to the correct subset of S_3. Thus you should have $2^3 = 8$ accept states.
- No additional explanation or description of the DFA is required beyond the drawing.
- Label the accept states in your drawing. An example of a labelled accept state that should appear in your drawing is below:

(C) (Challenge!! Not required; worth up to 2 extra credit points.) How many states are required for a DFA that computes f_n? Prove that your answer is a lower bound.

Note: On every problem set we will provide a challenge problem, generally significantly more difficult than the other problems in the set, but worth only a few points. It is recommended that you attempt these problems, but only after completing the rest of the assignment.

Solution.
(A)
It is injective. Each digit is 0 or 1. If the i^{th} character is 1, you add the i to the set. So different n-digit numbers are transformed to different elements of $P(S_n)$ by f_n because each digit is 0 or 1.

It is surjective. Each digit is 0 or 1. Elements of $P(S_n)$ represent the i^{th} digits that are 1 in n-digit number. Power set of S_n is all the digits that are 1 in all the possible n-digit numbers. Since different numbers have different locations of 0 and 1, $P(S_n)$ means the all the different possible n-digit numbers that made of 0 and 1. Elements of $P(S_n)$ are hit at least by once. Thus, this is surjective.

Since it is both surjective and injective, this is bijective.

(C) We need 2^n elements of $P(S_n)$ to explain things. Since $P(S_n)$ has 2^n elements and the formula is bijective, we need at least 2^n states of DFA to compute f_n.

PART B (Graded by Juan and Varun)

PROBLEM 3 (5+5 points, suggested length of 1/2 page)

Two FAs are "equivalent" if the languages that they accept are the same. If two FAs are not equivalent, they are "distinct". For this question, you may assume that the alphabet is $\{0, 1\}$.

(A) How many distinct DFAs are there with 1 state? Draw it/them. Describe informally the languages recognized by each.

(B) How many distinct NFAs are there with 1 state? Draw it/them. Describe informally the languages recognized by each.

Solution.
Since DFAs with one state can only have a transition to itself, we only have two distinct DFAs.

Since NFAs with one state can have a transition to nowhere, we have five distinct NFAs.

PROBLEM 4 (3+3+3 points, suggested length of 1/3 page)

Are the following statements true or false? Justify your answers with a proof or counterexample.

(A) $(L_1 \cap L_2)^* = L_1^* \cap L_2^*$

(B) $(L_1 \cup L_2) \cdot L_3 = (L_1 \cdot L_3) \cup (L_2 \cdot L_3)$, where $\cdot$ is concatenation.

(C) $\{\varepsilon\} \cdot L_1 = \emptyset \cdot L_1$

Solution.
(A)
if L_1 has $\{\ aa\}$ and L_2 has $\{\ a\}$, $L_1 \cap L_2 = \emptyset$. Thus, $(L_1 \cap L_2)^* = \{\varepsilon\}$. However, $L_1^* \cap L_2^* = \{\ aa\}^*$
(B) $(L_1 \cup L_2) = \{z : z \in L_1, z \in L_2\}$
Thus, $(L_1 \cup L_2) \cdot L_3 = \{zx : z \in L_1, z \in L_2, x \in L_3\}$
$(L_1 \cdot L_3) = \{zx : z \in L_1, x \in L_3\}$
$(L_2 \cdot L_3) = \{yx : y \in L_2, x \in L_3\}$
$(L_1 \cdot L_3) \cup (L_2 \cdot L_3) = \{yx : y \in L_2, x \in L_3\} \cup \{zx : z \in L_1, x \in L_3\} = \{ax : a \in L_1, a \in L_2, x \in L_3\}$
This is the same as $\{zx : z \in L_1, x \in L_3\}$. Thus $(L_1 \cup L_2) \cdot L_3 = (L_1 \cdot L_3) \cup (L_2 \cdot L_3)$ is true.
(C)
$\{\varepsilon\} \cdot L_1 = \{yx : y \in \{\varepsilon\}, x \in L_1\} = L_1$
Concatinating an empty string to y becomes y.
$\emptyset \cdot L_1 = \{yx : y \in \emptyset, x \in L_1\} = \emptyset$
Since there is no y, this makes $\emptyset$
Thus they are not the same.