Problem 126

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1 Question

https://projecteuler.net/problem=126.

2 Number of blocks of a cuboid being covered by n layers

Assume we have a cuboid of size axbxc. Suppose, for example, $a = 3$, $b = 1$, $c = 3$. It is challenging to visualize filling the cuboid in a 3D coordinate system, especially for $n \geq 2$. To address this, we take the cross sections of the cuboid along one coordinate. In the figure below, we split the cuboid to 3 pieces of size 3x1.

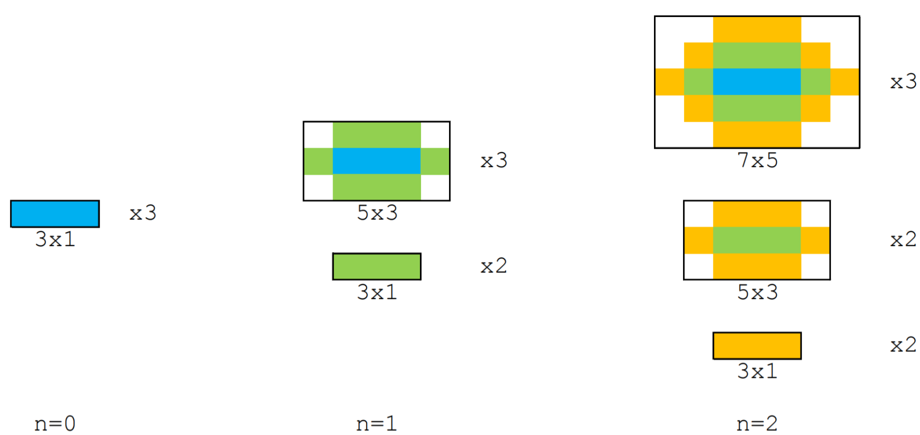

Figure 1: At $n = 0$, we have the original 3x1x3 cuboid being split to 3 pieces of size 3x1. At $n = 1$, for each 3x1 of $n = 0$, we need to cover all of their 4 sides. Other than that, we also need to cover the front of the most outer 3x1 of $n = 0$ that facing out of the paper, and to cover the back of the most inner 3x1 of $n = 0$ that facing towards the paper. The newly add pieces are illustrated by the green blocks, while the original pieces are the blue blocks. At $n = 2$, we need to cover the green blocks exposing the white blocks. The strategy of covering $n = 1$ is the same as what we do to cover $n = 0$. The newly add pieces are illustrated by the orange blocks.

From the observations above, we have that, a block of size axb grows to $(a+2)$x$(b+2)$ minus 4×1 white blocks, which turns to $(a+4)$x$(b+4)$ minus $4 \times (1+2)$ white blocks in the next iteration. In general, in the k^{th} iteration, the original tiles of size axb will turn into $(a+2k)(b+2k) - 4\sum_{i=0}^{k} i$, or $(a+2k)(b+2k) - 2k(k+1)$. Another observation is that, each iteration will add two new axb blocks, while other blocks grow.

Let $f(a, b, c, n) \to \mathbb{N}$ be a function that calculates the number of blocks of a cuboid of size axbxc being covered by n layers.

$$f(a,b,c,n) = c[(a+2n)(b+2n) - 2n(n+1)] + 2\sum_{i=0}^{n-1}[(a+2i)(b+2i) - 2i(i+1)]$$

as at iteration n, we have c cross-sections grow from the original cuboid, as other 2 per iteration.

3 Number of blocks needed to transform a cuboid of $n-1$ layers into n layers

Let $g(a,b,c,n) \to \mathbb{N}$ be a function that calculates the number of blocks needed to transform a cuboid of size axbxc being covered by $n-1$ layers into the one covered by n layers.

$$\begin{aligned}
g(a,b,c,n) &= f(a,b,c,n) - f(a,b,c,n-1) \\
&= c(a+2n)(b+2n) - c(a+2n-2)(b+2n-2) \\
&\quad - 2cn(n+1) + 2c(n-1)n + 2(a+2n-2)(b+2n-2) - 4(n-1)n \\
&= 2ab + 2ac + 4an - 4a + 2bc + 4bn - 4b + 4cn - 4c + 4n^2 - 12n + 8 \\
&= 2(ab+ac+bc) + 4(n-1)(a+b+c) + 4(n-1)(n-2)
\end{aligned}$$