2　指数関数

2.1　グラフ

指数関数

$$y = a^x$$

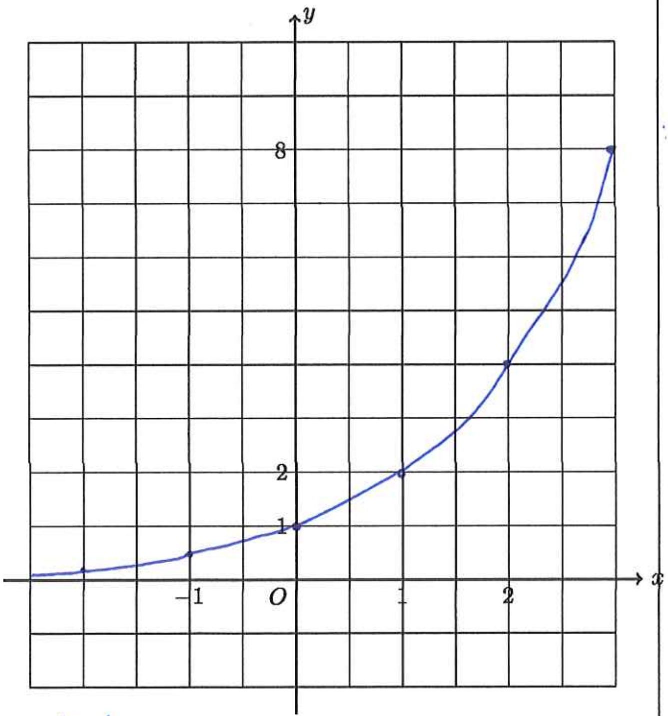
について考える．($a > 0, a \neq 1$ とする．)

指数関数 $y = a^x$ について，a を，<u>底</u>という．

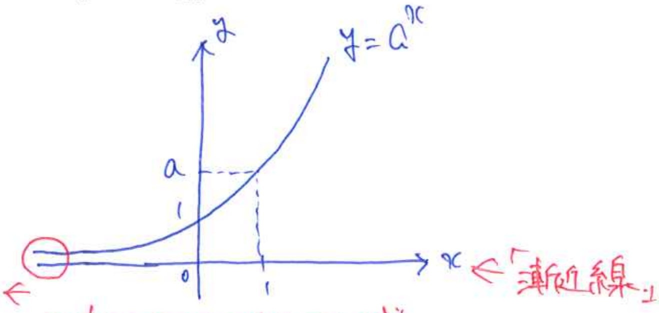
これまで，新しい関数のグラフを描くとき，まず表を描いていた．
今回も同じ手順を踏む．

(1) $y = 2^x$ について．

x	-2	-1	-0.5	0	0.5	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	4



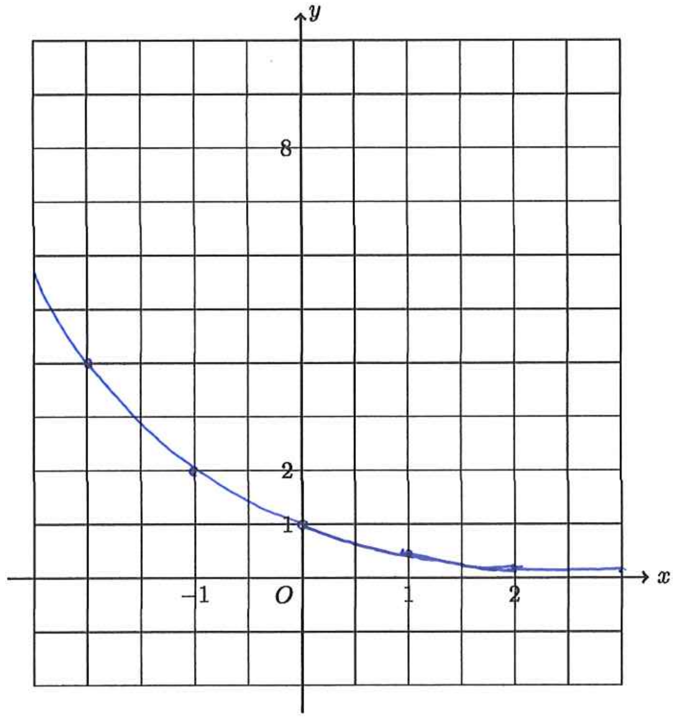
【$a>1$ のとき】



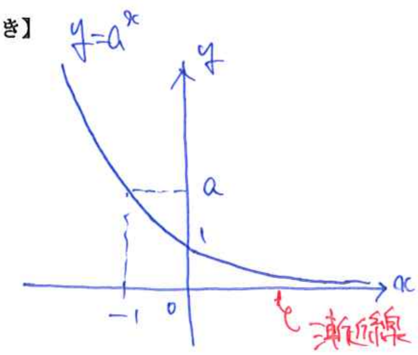
x軸にどんどん近づいていくが，
交わりはしない．

(2) $y = \left(\frac{1}{2}\right)^x$ について．

x	-2	-1	-0.5	0	0.5	1	2
y	4	2	$\sqrt{2}$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{4}$



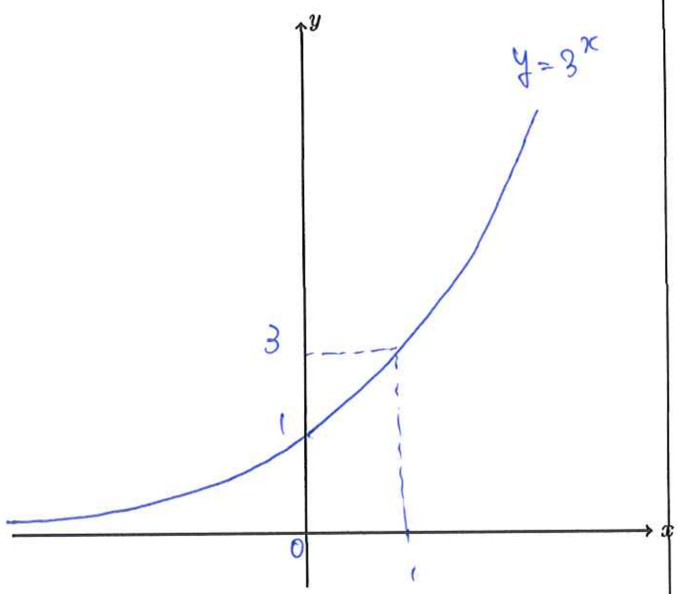
【$0<a<1$ のとき】



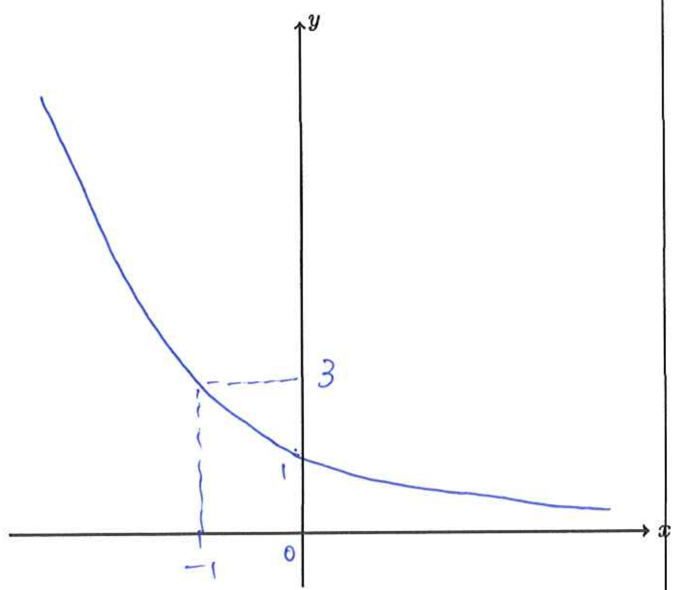
2.2 問題

以下のグラフを描け.

(1) $y = 3^x$



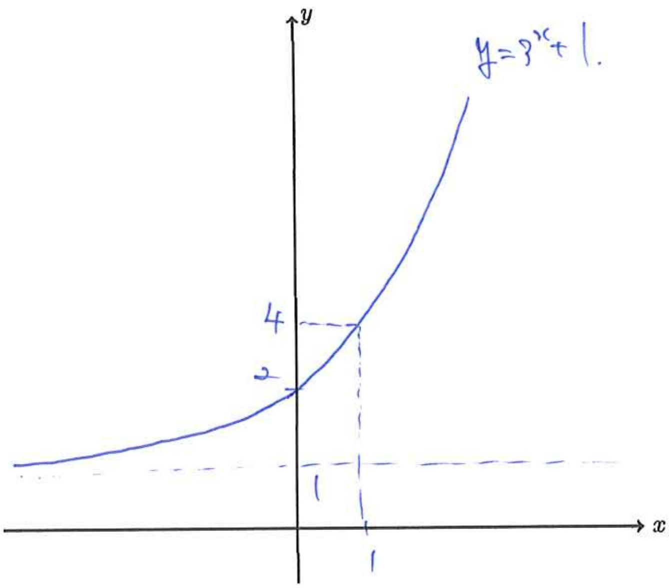
(2) $y = \left(\frac{1}{3}\right)^x$



2.3 問題 (平行移動) y軸方向に+1.

以下のグラフを描け.

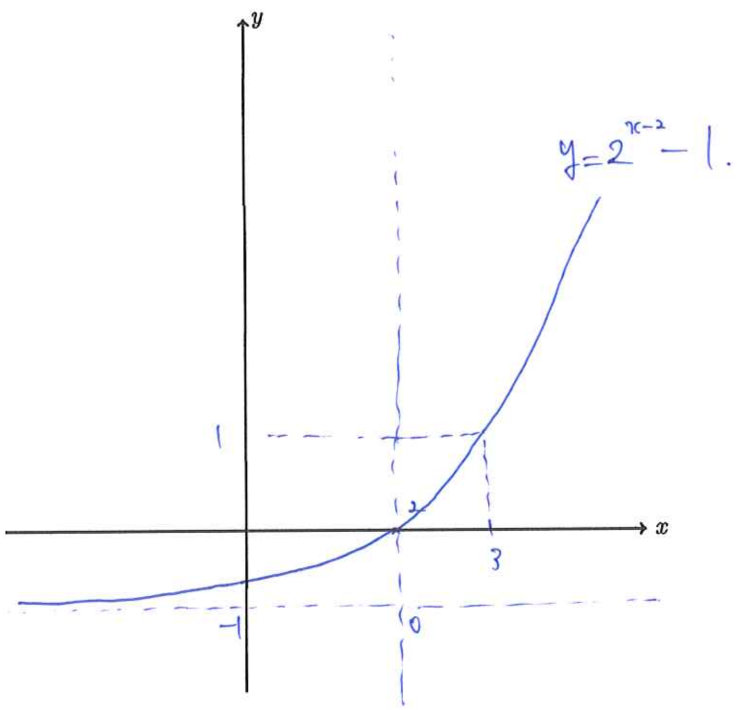
(1) $y = 3^x + 1$



x軸方向 +2

y軸方向 −1.

(2) $y = 2^{(x-2)} - 1$



2.4　大小関係比較

<u>例題</u>

以下の 3 つの数の大小関係を不等号を用いて表せ.

$$\sqrt{3},\ \ \sqrt[3]{9},\ \ \sqrt[5]{27}$$

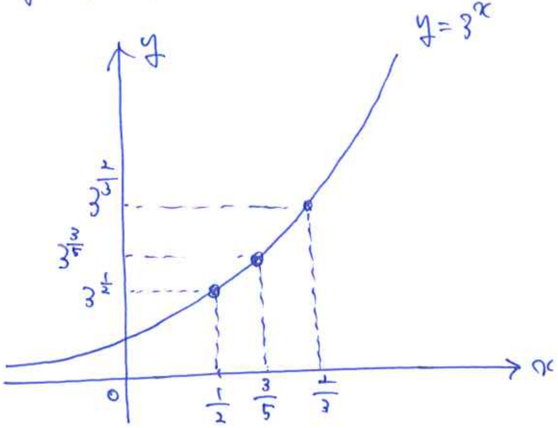
o グラフ上で比較!!

$\sqrt{3} = 3^{\frac{1}{2}}$

$\sqrt[3]{9} = (3^2)^{\frac{1}{3}} = 3^{\frac{2}{3}}$

$\sqrt[5]{27} = (3^3)^{\frac{1}{5}} = 3^{\frac{3}{5}}$

$y = 3^x$ について.



上グラフより.

$$3^{\frac{1}{2}} < 3^{\frac{3}{5}} < 3^{\frac{2}{3}}$$

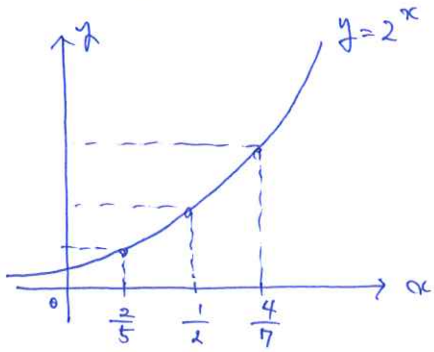
$$\therefore\ \sqrt{3} < \sqrt[5]{27} < \sqrt[3]{9}$$ //

2.5　問題

以下の数の大小関係を不等号を用いて表せ.

(1) $\sqrt{2},\ \ \sqrt[5]{4},\ \ \sqrt[7]{16}$

$\sqrt{2} = 2^{\frac{1}{2}},\quad \sqrt[5]{4} = 2^{\frac{2}{5}},\quad \sqrt[7]{16} = 2^{\frac{4}{7}}$



上図より.

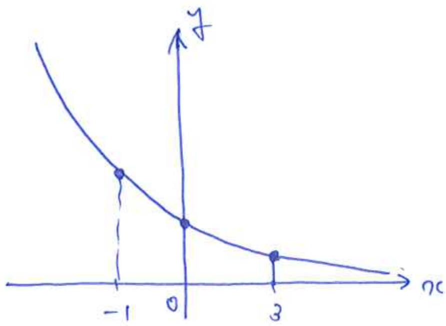
$$2^{\frac{2}{5}} < 2^{\frac{1}{2}} < 2^{\frac{4}{7}}$$

$$\therefore\ \sqrt[5]{4} < \sqrt{2} < \sqrt[7]{16}$$ //

$0.5 = \frac{1}{2}$

(2) $1,\ \ 0.5^3,\ \ 0.5^{-1}$

$1 = \left(\frac{1}{2}\right)^0,\quad 0.5^3 = \left(\frac{1}{2}\right)^3,\quad 0.5^{-1} = \left(\frac{1}{2}\right)^{-1}$



上図より.

$$\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^0 < \left(\frac{1}{2}\right)^{-1}$$

$$\therefore\ 0.5^3 < 1 < 0.5^{-1}$$ //

2.6　方程式

例題

以下の方程式を解け．

$\bigcirc^{\triangle} = \bigcirc^{\square}$

合わせて指数部の比較!!

$$16^x = 8$$

$16^x = (2^4)^x = 2^{4x}$

$8 = 2^3$

$\therefore\ 2^{4x} = 2^3$

$4x = 3$

$x = \frac{3}{4}$

(1) $9^x = 27$

$9^x = (3^2)^x = 3^{2x}$

$27 = 3^3$

$\therefore\ 3^{2x} = 3^3$

$2x = 3$　　$x = \frac{3}{2}$

(2) $2^{x+1} = \frac{1}{8}$

$\frac{1}{8} = 2^{-3}$

$\therefore\ 2^{x+1} = 2^{-3}$

$x+1 = -3$

$x = -4$

(3) $2^{x+1} = 8^x$

$8^x = (2^3)^x$

$= 2^{3x}$

$\therefore\ 2^{x+1} = 2^{3x}$

$x+1 = 3x$

$x = \frac{1}{2}$

2.7　不等式

例題

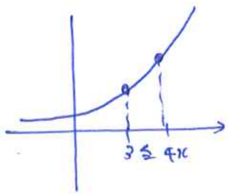
以下の方程式を解け．

$$16^x \geqq 8$$

$16^x = 2^{4x}$

$8 = 2^3$

$2^{4x} \geqq 2^3$



左図より　$3 \leqq 4x$

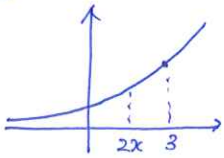
$\therefore\ \frac{3}{4} \leqq x$

(1) $9^x \leqq 27$

$9^x = 3^{2x}$

$27 = 3^3$

$3^{2x} \leqq 3^3$



左図より

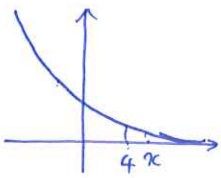
$2x \leqq 3$

$\therefore\ x \leqq \frac{3}{2}$

(2) $\left(\frac{1}{2}\right)^x \leqq \frac{1}{16}$

$\frac{1}{16} = \left(\frac{1}{2}\right)^4$

$\left(\frac{1}{2}\right)^x \leqq \left(\frac{1}{2}\right)^4$



左図より

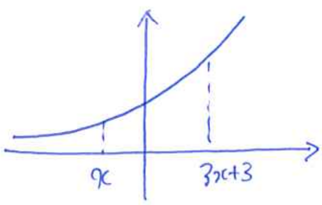
$4 \leqq x$

(3) $2^x \leqq 8^{x+1}$

$8^{x+1} = (2^3)^{x+1}$

$= 2^{3x+3}$

$2^x \leqq 2^{3x+3}$



左図より

$x \leqq 3x+3$

$-3 \leqq 2x$

$-\frac{3}{2} \leqq x$

2.8 二次関数への帰着問題

問題

以下の方程式・不等式を解け. ($0 \leqq x < 2\pi$ とする.)

(1) $2\sin^2 x - 3\sin x + 1 = 0$

$\sin x = t$ とおく.

$0 \le x < 2\pi$ より

$-1 \le \sin x \le 1$. ∴ $-1 \le t \le 1$

$2t^2 - 3t + 1 = 0$

$(2t-1)(t-1) = 0$

$t = \frac{1}{2}, 1$.

∴ $\sin x = \frac{1}{2}$ i.e. $x = \frac{\pi}{6}, \frac{5}{6}\pi$

$\sin x = 1$ i.e. $x = \frac{\pi}{2}$

以上より $x = \frac{\pi}{6}, \frac{5}{6}\pi, \frac{\pi}{2}$ //

(2) $2\sin^2 x - 3\sin x + 1 > 0$

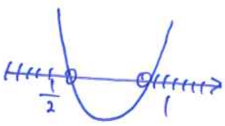
$\sin x = t$ とおく.

$0 \le x < 2\pi$ より

$-1 \le \sin x \le 1$. ∴ $-1 \le t \le 1$. —①

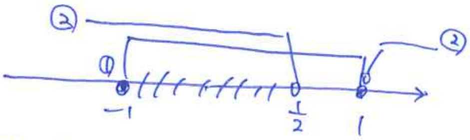
$2t^2 - 3t + 1 > 0$

$(2t-1)(t-1) > 0$



$t < \frac{1}{2}, 1 < t$ —②

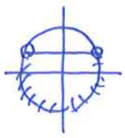
①. ②より



図より

$-1 \le t < \frac{1}{2}$

$-1 \le \sin x < \frac{1}{2}$



よって. $0 \le x < \frac{\pi}{6}, \frac{5}{6}\pi < x < 2\pi$ //

例題

以下の方程式・不等式を解け.

(1) $4^x - 5 \cdot 2^x - 24 = 0$

⇔ $(2^x)^2 - 5 \cdot 2^x - 24 = 0$

$2^x = t$ とおくと. $t > 0$.

$t^2 - 5t - 24 = 0$

$(t+3)(t-8) = 0$

$t > 0$ より $t = 8$

∴ $2^x = 8$

$x = 3$ //

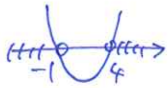
(2) $4^x - 3 \cdot 2^x - 4 > 0$

⇔ $(2^x)^2 - 3 \cdot 2^x - 4 > 0$

$2^x = t$ とおくと. $t > 0$. —①

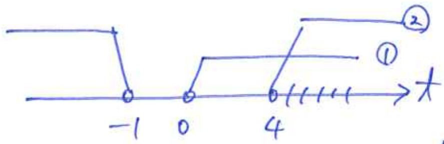
$t^2 - 3t - 4 > 0$

$(t-4)(t+1) > 0$



$t < -1, 4 < t$ —②

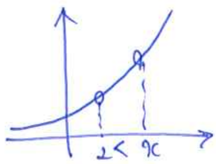
①. ②より



$4 < t$

∴ $4 < 2^x$

$2^2 < 2^x$



よって.

$2 < x$ //

2.9　練習問題

以下の方程式・不等式を解け．

(1) $9^x - 7 \cdot 3^x - 18 = 0$

$(3^x)^2 - 7 \cdot 3^x - 18 = 0$

$3^x = t$ とおくと．$t > 0$．

$t^2 - 7t - 18 = 0$

$(t-9)(t+2) = 0$

$t > 0$ より $t = 9$.

$\therefore 3^x = 9$

$\underline{x = 2}$ //

(2) $2 \cdot 4^x - 9 \cdot 2^x + 4 = 0$

$2 \cdot (2^x)^2 - 9 \cdot 2^x + 4 = 0$

$2^x = t$ とおくと $t > 0$.

$2t^2 - 9t + 4 = 0$

$(2t-1)(t-4) = 0$

$t > 0$ より

$t = \frac{1}{2}, 4$

$\therefore 2^x = \frac{1}{2}, 4$

$\underline{x = -1, 2}$ //

(3) $4^x - 6 \cdot 2^x - 16 \leqq 0$

$(2^x)^2 - 6 \cdot 2^x - 16 \leqq 0$.

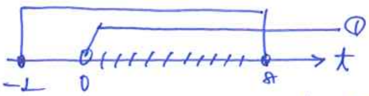
$2^x = t$ とおくと $t > 0$. — ①

$t^2 - 6t - 16 \leqq 0$

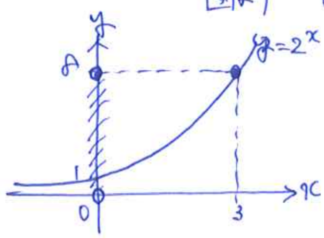
$(t-8)(t+2) \leqq 0$

$-2 \leqq t \leqq 8$ — ②

①, ②より.



図より　$0 < t \leqq 8$



$0 < 2^x \leqq 8$

左図より．

$\underline{x \leqq 3}$ //

(4) $9^x - 8 \cdot 3^x - 9 < 0$

$(3^x)^2 - 8 \cdot 3^x - 9 < 0$

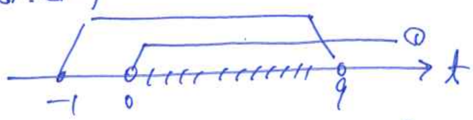
$3^x = t$ とおくと $t > 0$ — ①

$t^2 - 8t - 9 < 0$

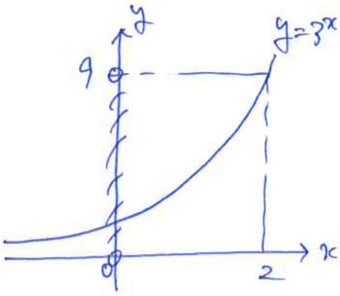
$(t-9)(t+1) < 0$

$-1 < t < 9$ — ②

①, ②より



図より　$0 < t < 9$.



$0 < 3^x < 9$.

左図より

$\underline{x < 2}$ //