6 加法定理

6.1 計算練習

「75° の三角比を求めたい．」(← 目標)

加法定理

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

(複号同順)

加法定理を使って 75° の三角比を求めてみよう．

$75° = 30° + 45°$ なので

$$\sin 75° = \sin(30° + 45°)$$
$$= \sin 30° \cdot \cos 45° + \cos 30° \cdot \sin 45°$$
$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4} \text{ //}$$

$$\cos 75° = \cos(30° + 45°)$$
$$= \cos 30° \cdot \cos 45° - \sin 30° \cdot \sin 45°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4} \text{ //}$$

$$\tan 75° = \tan(30° + 45°)$$
$$= \frac{\tan 30° + \tan 45°}{1 - \tan 30° \cdot \tan 45°}$$
$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}} \cdot 1}$$
$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \frac{1+\sqrt{3}}{\sqrt{3}-1} = 2+\sqrt{3} \text{ //}$$

計算練習

(1) 15° の三角比を求めよ．

$15° = 45° - 30°$ なので

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45° \cdot \cos 30° - \cos 45° \cdot \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4} \text{ //}$$

$$\cos 15° = \cos(45° - 30°)$$
$$= \cos 45° \cdot \cos 30° + \sin 45° \cdot \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4} \text{ //}$$

$$\tan 15° = \frac{\sin 15°}{\cos 15°}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = 2-\sqrt{3} \text{ //}$$

(2) $\frac{11}{12}\pi$ の三角比を求めよ．

$\frac{11}{12}\pi = \frac{3}{4}\pi + \frac{1}{6}\pi$ なので

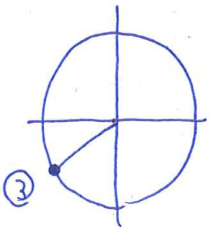
$$\sin\frac{11}{12}\pi = \sin\left(\frac{3}{4}\pi + \frac{1}{6}\pi\right)$$
$$= \sin\frac{3}{4}\pi \cdot \cos\frac{1}{6}\pi + \cos\frac{3}{4}\pi \sin\frac{1}{6}\pi$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{2}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4} \text{ //}$$

$$\cos\frac{11}{12}\pi = \cos\left(\frac{3}{4}\pi + \frac{1}{6}\pi\right)$$
$$= \cos\frac{3}{4}\pi \cdot \cos\frac{1}{6}\pi - \sin\frac{3}{4}\pi \cdot \sin\frac{1}{6}\pi$$
$$= -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= -\frac{\sqrt{6}+\sqrt{2}}{4} \text{ //}$$

$$\tan\frac{11}{12}\pi = \frac{\sin\frac{11}{12}\pi}{\cos\frac{11}{12}\pi}$$
$$= -\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = -(2+\sqrt{3}) \text{ //}$$

(3) α の動径が第 3 象限, β の動径が第 4 象限にあり,
$\sin\alpha = -\frac{3}{5}, \cos\cancel{\theta}\,\beta = \frac{4}{5}$ のとき, 以下の問いに答えよ.

(a) $\cos\alpha$ の値を求めよ.



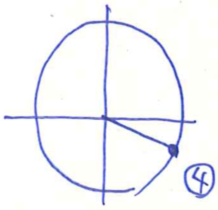
$\sin^2\alpha + \cos^2\alpha = 1$ より

$\cos^2\alpha = \frac{16}{25}$

左図から. $\cos\alpha$ は負

$\therefore \cos\alpha = -\frac{4}{5}$

(b) $\sin\beta$ の値を求めよ.



$\sin^2\beta + \cos^2\beta = 1$ より

$\sin^2\beta = \frac{9}{25}$

左図から. $\sin\beta$ は負

$\therefore \sin\beta = -\frac{3}{5}$

(c) $\sin(\alpha+\beta)$ の値を求めよ.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= -\frac{3}{5}\cdot\frac{4}{5} + \left(-\frac{4}{5}\right)\cdot\left(-\frac{3}{5}\right)$

$= 0$

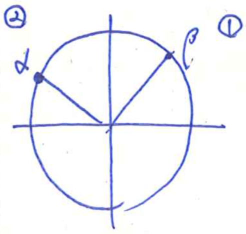
(d) $\cos(\alpha-\beta)$ の値を求めよ.

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(-\frac{4}{5}\right)\cdot\frac{4}{5} + \left(-\frac{3}{5}\right)\cdot\left(-\frac{3}{5}\right)$

$= \frac{-16}{5} + \frac{9}{5} = -\frac{7}{5}$

(4) α の動径が第 2 象限, β の動径が第 1 象限にあり,
$\sin\alpha = \frac{2}{3}, \cos\cancel{\theta}\,\beta = \frac{3}{5}$ のとき, $\sin(\alpha-\beta), \cos(\alpha+\beta)$ の値を求めよ.



$\sin^2\theta + \cos^2\theta = 1$ より

$\cos^2\alpha = \frac{5}{9}$

$\sin^2\beta = \frac{16}{25}$

よって、左上図から、$\cos\alpha$ は負. $\sin\beta$ は正

$\therefore \cos\alpha = -\frac{\sqrt{5}}{3}, \sin\beta = \frac{4}{5}$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{2}{3}\cdot\frac{3}{5} - \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{4}{5}$

$= \frac{6}{15} + \frac{4\sqrt{3}}{15}$

$= \frac{6+4\sqrt{3}}{15}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{3}{5} - \frac{2}{3}\cdot\frac{4}{5}$

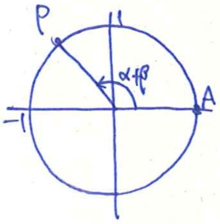
$= \frac{-3\sqrt{5}-8}{15}$

6.2 **証明**

加法定理

＜証明＞

左図において.

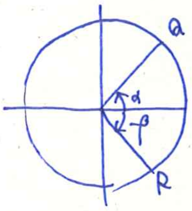


$A(1,0)$

$P(\cos(\alpha+\beta), \sin(\alpha+\beta))$

なので

$$AP^2 = \sin^2(\alpha+\beta) + (1-\cos(\alpha+\beta))^2$$
$$= \sin^2(\alpha+\beta) + 1 - 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta)$$
$$= 2 - 2\cos(\alpha+\beta)$$



同様に $Q(\cos\alpha, \sin\alpha)$

$R(\cos(-\beta), \sin(-\beta))$

また, $\cos(-\beta) = \cos\beta$.

$\sin(-\beta) = -\sin\beta$

なので

$$QR^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - (-\sin\beta))^2$$
$$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + (\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta)$$
$$= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta.$$

中心角が同じなので $AP = QR$.

$\therefore\ AP^2 = QR^2$

$$2 - 2\cos(\alpha+\beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

よって $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

この式の β を $-\beta$ へおきかえて,

$$\cos(\alpha-\beta) = \cos\alpha\cdot\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad (2)$$

また, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ より.

(2)の式の α を $\frac{\pi}{2}-\alpha$ へおきかえる.

$$(左辺) = \cos\left(\frac{\pi}{2}-\alpha-\beta\right)$$
$$= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)$$
$$= \sin(\alpha+\beta)$$

$$(右辺) = \cos\left(\frac{\pi}{2}-\alpha\right)\cdot\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right)\cdot\sin\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$\therefore\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\beta\sin\beta$

この式の β を $-\beta$ へおきかえて,

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad (1)$$

また. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ より

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

分母·分子を $\cos\alpha\cos\beta$ でわると.

$$(右辺) = \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

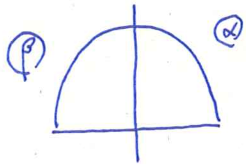
$$\therefore\ \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

この式の β を $-\beta$ へおきかえて.

$$\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \quad (3)$$

6.3 演習

(1) $\sin\alpha = \frac{3}{5}\ \left(0 < \alpha < \frac{\pi}{2}\right), \cos\beta = -\frac{4}{5}\ \left(\frac{\pi}{2} < \beta < \pi\right)$ のとき, $\sin(\alpha+\beta), \cos(\alpha-\beta), \tan(\alpha-\beta)$ の値を求めよ.



$\sin^2\theta + \cos^2\theta = 1$ より

$\cos^2\alpha = \frac{16}{25}$

$\sin^2\beta = \frac{9}{25}$

上図より $\cos\alpha > 0$, $\sin\beta > 0$

∴ $\cos\alpha = \frac{4}{5}$, $\sin\beta = \frac{3}{5}$.

・$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{3}{5}\cdot\left(-\frac{4}{5}\right) + \frac{4}{5}\cdot\frac{3}{5}$

$= 0$ //

・$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{4}{5}\left(-\frac{4}{5}\right) + \frac{3}{5}\cdot\frac{3}{5}$

$= -\frac{7}{25}$ //

・$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

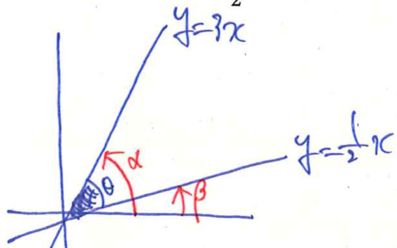
よって, $\tan\alpha = \frac{3}{4}$, $\tan\beta = -\frac{3}{4}$ より

$\tan(\alpha-\beta) = \dfrac{\frac{3}{4} - \left(-\frac{3}{4}\right)}{1 + \frac{3}{4}\left(-\frac{3}{4}\right)}$

$= \dfrac{\frac{6}{4}}{1 - \frac{9}{16}} = \dfrac{\frac{6}{4}}{\frac{7}{16}}$

$= \dfrac{6\cdot 16 4}{7\cdot 4} = \dfrac{24}{7}$ //

(2) 2 直線 $y = 3x, y = \frac{1}{2}x$ のなす鋭角を求めよ.



左図のように α, β をおく.

求める角θは

$\theta = \alpha - \beta$.

よって, $\tan\alpha = 3$, $\tan\beta = \frac{1}{2}$ より

$\tan\theta = \tan(\alpha-\beta)$

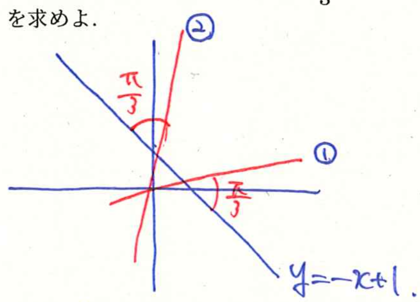
$= \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \dfrac{3 - \frac{1}{2}}{1 + 3\cdot\frac{1}{2}} = 1$.

$\tan\theta = 1$ より, θは鋭角なので

$\theta = \frac{\pi}{4}$ //

(3) 原点を通り, 直線 $y = -x+1$ と $\frac{1}{3}\pi$ の角をなす直線の方程式を求めよ.



①はx軸とのなす角は $\frac{\pi}{3} - \frac{\pi}{4} = \frac{1}{6}\pi$.

②は, $\frac{3}{4}\pi - \frac{1}{3}\pi = \frac{5}{12}\pi$.

∴, 求める直線の傾きは.

① $\tan\frac{1}{12}\pi$, $= \dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\cdot\tan\frac{\pi}{4}}$

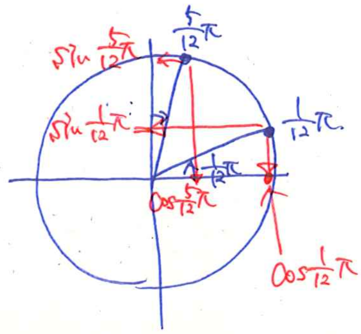
$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = 2-\sqrt{3}$

② $\tan\frac{5}{12}\pi = \dfrac{\tan\frac{3}{4}\pi - \tan\frac{\pi}{3}}{1+\tan\frac{3}{4}\pi\cdot\tan\frac{\pi}{3}}$

$= \dfrac{-1-\sqrt{3}}{1-\sqrt{3}} = 2+\sqrt{3}$

よって求める直線は

$y = (2-\sqrt{3})x$, $y = (2+\sqrt{3})x$ //



7 加法定理の応用

7.1 復習

加法定理を思い出す.

(1) $\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(2) $\sin(\alpha-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(3) $\cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(4) $\cos(\alpha-\beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(5) $\tan(\alpha+\beta)$

$= \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

(6) $\tan(\alpha-\beta)$

$= \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

計算練習

(1) $\sin\left(\frac{1}{3}\pi+\frac{1}{4}\pi\right)$

$= \sin\frac{1}{3}\pi\cdot\cos\frac{1}{4}\pi + \cos\frac{1}{3}\pi\cdot\sin\frac{1}{4}\pi$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ //

(2) $\cos\left(\frac{1}{12}\pi\right)$ $\quad \frac{1}{12}\pi = \frac{1}{3}\pi - \frac{1}{4}\pi$

$= \cos\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right)$

$= \cos\frac{1}{3}\pi\cdot\cos\frac{1}{4}\pi + \sin\frac{1}{3}\pi\cdot\sin\frac{1}{4}\pi$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ //

(3) $\sin\left(\frac{1}{12}\pi\right)$

$= \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ //

(4) $\sin\left(\frac{5}{12}\pi\right)$

$= \cos\frac{1}{12}\pi$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

(5) $\cos\left(\frac{5}{12}\pi\right)$

$= \sin\frac{1}{12}\pi$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

(6) $\tan\left(\frac{5}{12}\pi\right)$

$= \dfrac{\sin\frac{5}{12}\pi}{\cos\frac{5}{12}\pi}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \dfrac{\cancel{\sqrt{2}}(\sqrt{3}+1)}{\cancel{\sqrt{2}}(\sqrt{3}-1)}$

$= 2+\sqrt{3}$ //

加B
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

7.2　2 倍角

考える

$2\alpha = \alpha + \alpha$ と考えることで, 2α の三角比を考える.

(1) $\sin(\alpha+\alpha)$ を α の三角比で表そう.

$= \sin\alpha \cdot \cos\alpha + \cos\alpha \cdot \sin\alpha$

$= 2\sin\alpha\cos\alpha$ //

(2) $\cos(\alpha+\alpha)$ を α の三角比で表そう.

$= \cos\alpha \cdot \cos\alpha - \sin\alpha \cdot \sin\alpha$

$= \cos^2\alpha - \sin^2\alpha$ //

(3) $\cos(\alpha+\alpha)$ を $\sin\alpha$ で表そう.

$\boxed{\sin^2\alpha + \cos^2\alpha = 1 \quad \cos^2\alpha = 1 - \sin^2\alpha}$

$= \cos^2\alpha - \sin^2\alpha$

$= (1 - \sin^2\alpha) - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$ //

(4) $\cos(\alpha+\alpha)$ を $\cos\alpha$ で表そう.

$= \cos^2\alpha - \sin^2\alpha$

$= \cos^2\alpha - (1 - \cos^2\alpha)$

$= 2\cos^2\alpha - 1$ //

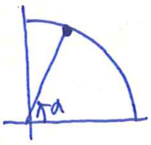
(5) $\tan(\alpha+\alpha)$ を $\tan\alpha$ で表そう.

$= \dfrac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$

$= \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$ //

練習問題

(1) $0 < \alpha < \dfrac{\pi}{2}$ で, $\sin\alpha = \dfrac{4}{5}$ のとき, $\sin 2\alpha$ の値を求めよ.



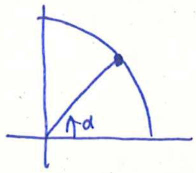
$\sin^2\alpha + \cos^2\alpha = 1$ より, 左図から.

$\cos\alpha = \dfrac{3}{5}$.

$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha$

$= 2 \cdot \dfrac{3}{5} \cdot \dfrac{4}{5} = \dfrac{24}{25}$ //

(2) $0 < \alpha < \dfrac{\pi}{2}$ で, $\sin\alpha = \dfrac{3}{5}$ のとき, $\cos 2\alpha$ の値を求めよ.

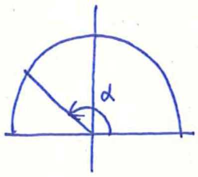


$\cos 2\alpha = 1 - 2\sin^2\alpha$

$= 1 - 2 \cdot \left(\dfrac{3}{5}\right)^2$

$= 1 - \dfrac{18}{25} = \dfrac{7}{25}$ //

(3) $\dfrac{\pi}{2} < \alpha < \pi$ で, $\cos\alpha = -\dfrac{\sqrt{5}}{3}$ のとき, $\sin 2\alpha, \cos 2\alpha, \tan 2\alpha$ の値を求めよ.



$\sin^2\alpha + \cos^2\alpha = 1$, 左図から.

$\sin\alpha = \dfrac{2}{3}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \cdot \dfrac{2}{3} \cdot \dfrac{-\sqrt{5}}{3} = \dfrac{-4\sqrt{5}}{9}$ //

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \dfrac{5}{9} - \dfrac{4}{9} = \dfrac{1}{9}$ //

$\tan 2\alpha = \dfrac{\sin 2\alpha}{\cos 2\alpha}$

$= \dfrac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}} = -4\sqrt{5}$ //