6 等式の証明

6.1 問題 1

以下の等式を示せ.

(1) $a^3 - b^3 = (a-b)^3 - 3ab(-a+b)$

〈証明〉.

$$\begin{aligned}(\text{右辺}) &= (a-b)^3 - 3ab(-a+b) \\ &= a^3 - 3a^2b + 3ab^2 - b^3 \\ &\quad + 3a^2b - 3ab^2 \\ &= a^3 - b^3 = (\text{左辺}).\end{aligned}$$

$\therefore\ a^3 - b^3 = (a-b)^3 - 3ab(-a+b)$

は成立 証

(2) $(ab+1)^2 + (a-b)^2 = (a^2+1)(b^2+1)$

〈証明〉.

$$\begin{aligned}(\text{左辺}) &= (ab+1)^2 + (a-b)^2 \\ &= a^2b^2 + 2ab + 1 + a^2 - 2ab + b^2 \\ &= a^2b^2 + 1 + a^2 + b^2\end{aligned}$$

$$\begin{aligned}(\text{右辺}) &= (a^2+1)(b^2+1) \\ &= a^2b^2 + a^2 + b^2 + 1.\end{aligned}$$

よって (右辺) = (左辺)

i.e.

$(ab+1)^2 + (a-b)^2 = (a^2+1)(b^2+1)$

は成立 証

6.2 問題 2

$a+b+c=0$ のとき, 以下の等式を示せ.

$$a^3+b^3+c^3 = 3abc$$

〈証明〉.

$a+b+c=0$ より. $c = -(a+b)$.

$$\begin{aligned}(\text{左辺}) &= a^3+b^3+c^3 \\ &= a^3+b^3+(-(a+b))^3 \\ &= a^3+b^3-(a+b)^3 \\ &= a^3+b^3-(a^3+3a^2b+3ab^2+b^3) \\ &= -3a^2b - 3ab^2\end{aligned}$$

$$\begin{aligned}(\text{右辺}) &= 3abc \\ &= 3ab \times (-(a+b)) \\ &= -3a^2b - 3ab^2\end{aligned}$$

よって (左辺) = (右辺).

i.e. $a+b+c=0$ のとき,

$a^3+b^3+c^3 = 3abc$ が成立. 証

6.2.1 問題 3

$\dfrac{a}{b} = \dfrac{c}{d}$ のとき, 以下の等式を示せ.

$$\frac{a+c}{b+d} = \frac{a-c}{b-d}$$

〈証明〉.

$\dfrac{a}{b} = \dfrac{c}{d}$ より. $c = ak,\ d = bk$ (k:実数) とおくと.

$$\begin{aligned}(\text{左辺}) &= \frac{a+c}{b+d} \\ &= \frac{a+ak}{b+bk} \\ &= \frac{a\cancel{(1+k)}}{b\cancel{(1+k)}} \\ &= \frac{a}{b}\end{aligned}$$

$$\begin{aligned}(\text{右辺}) &= \frac{a-c}{b-d} \\ &= \frac{a-ak}{b-bk} \\ &= \frac{a\cancel{(1-k)}}{b\cancel{(1-k)}} \\ &= \frac{a}{b}\end{aligned}$$

よって (左辺) = (右辺)

i.e. $\dfrac{a}{b} = \dfrac{c}{d}$ のとき

$\dfrac{a+c}{b+d} = \dfrac{a-c}{b-d}$ が成立 証