12 $\tan\theta+\dfrac{1}{\tan\theta}=3$ のとき，以下の式の値を求めよ．ただし，$0\leqq\theta\leqq\dfrac{1}{2}\pi$ とする．

(1) $\sin\theta\cos\theta$

(2) $\sin\theta+\cos\theta$

(3) $\sin^3\theta+\cos^3\theta$

(4) $\sin^4\theta+\cos^4\theta$

(5) $\dfrac{1}{\sin^4\theta}+\dfrac{1}{\cos^4\theta}$

$\tan\theta+\dfrac{1}{\tan\theta}=3$

(左辺) $=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$

$=\dfrac{1}{\sin\theta\cos\theta}$ //

$\dfrac{1}{\sin\theta\cos\theta}=3$

$\therefore \sin\theta\cos\theta=\dfrac{1}{3}$ //

(2)

$(\sin\theta+\cos\theta)^2=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$

$=1+2\sin\theta\cos\theta$

$=1+\dfrac{2}{3}$

$=\dfrac{5}{3}$

$0\leqq\theta\leqq\dfrac{\pi}{2}$ //

$\sin\theta\geqq0,\ \cos\theta\geqq0$

$\therefore \sin\theta+\cos\theta>0$

$\therefore \sin\theta+\cos\theta=\sqrt{\dfrac{5}{3}}$

$=\dfrac{\sqrt{15}}{3}$ //

(3) $\sin^3\theta+\cos^3\theta$

$=(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)$

$=\dfrac{\sqrt{15}}{3}\cdot\left(1-\dfrac{1}{3}\right)$

$=\dfrac{\sqrt{15}}{3}\cdot\dfrac{2}{3}=\dfrac{2\sqrt{15}}{9}$ //

(4) $\sin^4\theta+\cos^4\theta$

$=(\sin^2\theta+\cos^2\theta)^2-(\sin\theta+\cos\theta)$

$-\sin\theta\cdot\cos^2\theta-\cos\theta\cdot\sin^2\theta$

$=(\sin^2\theta+\cos^2\theta)^2-(\sin\theta+\cos\theta)$

$-\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta)$

$=\dfrac{2\sqrt{15}}{9}\cdot\dfrac{\sqrt{15}}{3}-\dfrac{1}{3}\cdot1$

$=\dfrac{10}{9}-\dfrac{1}{3}=\dfrac{7}{9}$ //

(5) $\dfrac{1}{\sin^4\theta}+\dfrac{1}{\cos^4\theta}$

$=\dfrac{\sin^4\theta+\cos^4\theta}{\sin^4\theta\cos^4\theta}=\dfrac{\frac{7}{9}}{\left(\frac{1}{3}\right)^4}$

$=\dfrac{\frac{7}{3^2}}{\frac{1}{3^4}}$

$=7\times9=63$ //

誘導に従って1つ1つ丁寧に解いていく!!