6　加法定理

6.1　計算練習

「75° の三角比を求めたい．」(← 目標)

加法定理

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

(複号同順)

加法定理を使って 75° の三角比を求めてみよう．

$75°=30°+45°$ なので

$$\sin 75°=\sin(30°+45°)$$
$$=\sin 30°\cdot\cos 45°+\cos 30°\cdot\sin 45°$$
$$=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$$
$$=\frac{1+\sqrt{3}}{2\sqrt{2}}=\underline{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

$$\cos 75°=\cos(30°+45°)$$
$$=\cos 30°\cdot\cos 45°-\sin 30°\cdot\sin 45°$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$
$$=\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

$$\tan 75°=\tan(30°+45°)$$
$$=\frac{\tan 30°+\tan 45°}{1-\tan 30°\cdot\tan 45°}$$
$$=\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}\cdot 1}$$
$$=\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}=\frac{1+\sqrt{3}}{\sqrt{3}-1}=\underline{2+\sqrt{3}}$$

計算練習

(1) 15° の三角比を求めよ．

$15°=45°-30°$ なので

$$\sin 15°=\sin(45°-30°)$$
$$=\sin 45°\cdot\cos 30°-\cos 45°\cdot\sin 30°$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$
$$=\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

$$\cos 15°=\cos(45°-30°)$$
$$=\cos 45°\cdot\cos 30°+\sin 45°\cdot\sin 30°$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$
$$=\underline{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\tan 15°=\frac{\sin 15°}{\cos 15°}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}=\underline{2-\sqrt{3}}$$

(2) $\frac{11}{12}\pi$ の三角比を求めよ．

$\frac{11}{12}\pi=\frac{3}{4}\pi+\frac{1}{6}\pi$ なので

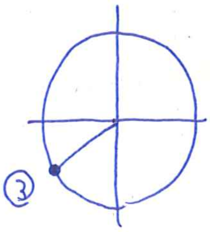
$$\sin\frac{11}{12}\pi=\sin\left(\frac{3}{4}\pi+\frac{1}{6}\pi\right)$$
$$=\sin\frac{3}{4}\pi\cdot\cos\frac{1}{6}\pi+\cos\frac{3}{4}\pi\sin\frac{1}{6}\pi$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\left(-\frac{1}{\sqrt{2}}\right)\cdot\frac{1}{2}$$
$$=\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

$$\cos\frac{11}{12}\pi=\cos\left(\frac{3}{4}\pi+\frac{1}{6}\pi\right)$$
$$=\cos\frac{3}{4}\pi\cdot\cos\frac{1}{6}\pi-\sin\frac{3}{4}\pi\cdot\sin\frac{1}{6}\pi$$
$$=-\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$
$$=\underline{-\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\tan\frac{11}{12}\pi=\frac{\sin\frac{11}{12}\pi}{\cos\frac{11}{12}\pi}$$
$$=-\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}=\underline{-(2+\sqrt{3})}$$

(3) α の動径が第 3 象限, β の動径が第 4 象限にあり,
$\sin\alpha = -\frac{3}{5}, \cos\cancel{\theta}\,\beta = \frac{4}{5}$ のとき, 以下の問いに答えよ.

(a) $\cos\alpha$ の値を求めよ.



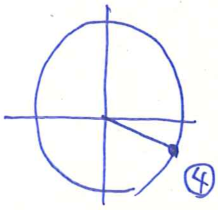
$\sin^2\alpha + \cos^2\alpha = 1$ より

$\cos^2\alpha = \frac{16}{25}$

左図から. $\cos\alpha$ は負

$\therefore \cos\alpha = -\frac{4}{5}$ //

(b) $\sin\beta$ の値を求めよ.



$\sin^2\beta + \cos^2\beta = 1$ より

$\sin^2\beta = \frac{9}{25}$

左図から. $\sin\beta$ は負

$\therefore \sin\beta = -\frac{3}{5}$ //

(c) $\sin(\alpha+\beta)$ の値を求めよ.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= -\frac{3}{5}\cdot\frac{4}{5} + \left(-\frac{4}{5}\right)\cdot\left(-\frac{3}{5}\right)$

$= 0$ //

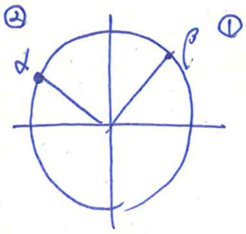
(d) $\cos(\alpha-\beta)$ の値を求めよ.

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(-\frac{4}{5}\right)\cdot\frac{4}{5} + \left(-\frac{3}{5}\right)\cdot\left(-\frac{3}{5}\right)$

$= \frac{-16}{5} + \frac{9}{5} = -\frac{7}{5}$ //

(4) α の動径が第 2 象限, β の動径が第 1 象限にあり,
$\sin\alpha = \frac{2}{3}, \cos\cancel{\theta}\,\beta = \frac{3}{5}$ のとき, $\sin(\alpha-\beta), \cos(\alpha+\beta)$ の値を求めよ.



$\sin^2\theta + \cos^2\theta = 1$ より

$\cos^2\alpha = \frac{5}{9}$

$\sin^2\beta = \frac{16}{25}$

よって、左上図から、$\cos\alpha$ は負、$\sin\beta$ は正

$\therefore \cos\alpha = -\frac{\sqrt{5}}{3}, \sin\beta = \frac{4}{5}$ 」

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{2}{3}\cdot\frac{3}{5} - \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{4}{5}$

$= \frac{6}{15} + \frac{4\sqrt{3}}{15}$

$= \frac{6+4\sqrt{3}}{15}$ //

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{3}{5} - \frac{2}{3}\cdot\frac{4}{5}$

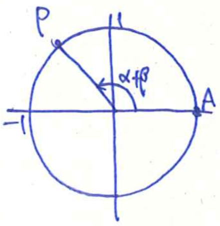
$= \frac{-3\sqrt{5}-8}{15}$ //

6.2 証明

加法定理

＜証明＞

左図において

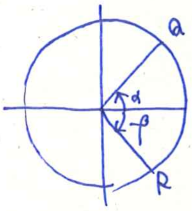


$A(1,0)$

$P(\cos(\alpha+\beta), \sin(\alpha+\beta))$

なので

$$AP^2 = \sin^2(\alpha+\beta) + (1-\cos(\alpha+\beta))^2$$
$$= \sin^2(\alpha+\beta) + 1 - 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta)$$
$$= 2 - 2\cos(\alpha+\beta)$$



同様に $Q(\cos\alpha, \sin\alpha)$

$R(\cos(-\beta), \sin(-\beta))$

また、$\cos(-\beta) = \cos\beta$.

$\sin(-\beta) = -\sin\beta$

なので

$$QR^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - (-\sin\beta))^2$$
$$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos\beta^2 + (\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta)$$
$$= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta.$$

中心角が同じなので $AP = QR$.

$\therefore AP^2 = QR^2$

$$2 - 2\cos(\alpha+\beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

よって $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

この式のβを$-\beta$へおきかえて、

$$\cos(\alpha-\beta) = \cos\alpha\cdot\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad (2)$$

よって、$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ より.

(2)の式のαを$\frac{\pi}{2}-\alpha$へおきかえる.

$$(\text{左辺}) = \cos\left(\frac{\pi}{2}-\alpha-\beta\right)$$
$$= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)$$
$$= \sin(\alpha+\beta)$$

$$(\text{右辺}) = \cos\left(\frac{\pi}{2}-\alpha\right)\cdot\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right)\cdot\sin\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\beta\sin\beta$

この式のβを$-\beta$へおきかえて、

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad (1)$$

また、$\tan\theta = \frac{\sin\theta}{\cos\theta}$ より

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

分母・分子を $\cos\alpha\cos\beta$ でわると.

$$(\text{右辺}) = \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

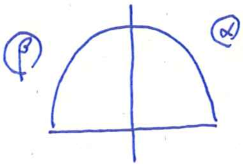
$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

この式のβを$-\beta$へおきかえて.

$$\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \quad (3)$$

6.3 演習

(1) $\sin\alpha = \frac{3}{5}\ \left(0<\alpha<\frac{\pi}{2}\right), \cos\beta = -\frac{4}{5}\ \left(\frac{\pi}{2}<\beta<\pi\right)$ のとき, $\sin(\alpha+\beta), \cos(\alpha-\beta), \tan(\alpha-\beta)$ の値を求めよ.



$\sin^2\theta + \cos^2\theta = 1$ より

$\cos^2\alpha = \frac{16}{25}$

$\sin^2\beta = \frac{9}{25}$

上図より $\cos\alpha > 0$, $\sin\beta > 0$

∴ $\cos\alpha = \frac{4}{5}$, $\sin\beta = \frac{3}{5}$.

・ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{3}{5}\cdot\left(-\frac{4}{5}\right) + \frac{4}{5}\cdot\frac{3}{5}$

$= 0$ //

・ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{4}{5}\cdot\left(-\frac{4}{5}\right) + \frac{3}{5}\cdot\frac{3}{5}$

$= -\frac{7}{25}$ //

・ $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

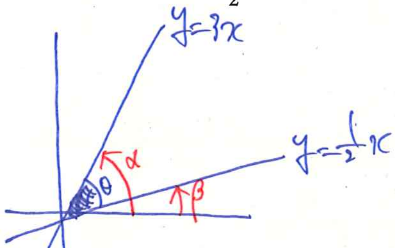
ここで, $\tan\alpha = \frac{3}{4}$, $\tan\beta = -\frac{3}{4}$ より

$\tan(\alpha-\beta) = \dfrac{\frac{3}{4}-\left(-\frac{3}{4}\right)}{1+\frac{3}{4}\cdot\left(-\frac{3}{4}\right)}$

$= \dfrac{\frac{6}{4}}{1-\frac{9}{16}} = \dfrac{\frac{6}{4}}{\frac{7}{16}}$

$= \dfrac{6\cdot 16}{7\cdot 4} = \dfrac{24}{7}$ //

(2) 2直線 $y=3x, y=\frac{1}{2}x$ のなす鋭角を求めよ.



左図のように α, β をおく.

求める角θは

$\theta = \alpha - \beta$.

ここで, $\tan\alpha = 3$, $\tan\beta = \frac{1}{2}$ より

$\tan\theta = \tan(\alpha-\beta)$

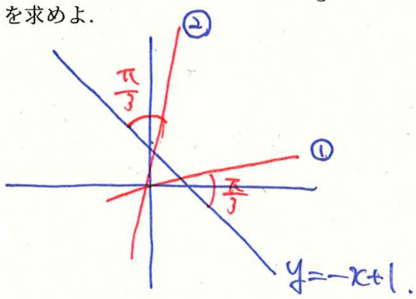
$= \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \dfrac{3-\frac{1}{2}}{1+3\cdot\frac{1}{2}} = 1$.

$\tan\theta = 1$ より, θは鋭角なので

$\theta = \frac{\pi}{4}$ //

(3) 原点を通り, 直線 $y=-x+1$ と $\frac{1}{3}\pi$ の角をなす直線の方程式を求めよ.



① は x軸とのなす角は $\frac{\pi}{3} - \frac{\pi}{4} = \frac{1}{6}\pi$.

② は, $\frac{3}{4}\pi - \frac{1}{3}\pi = \frac{5}{12}\pi$.

∴ 求める直線の傾きは

① $\tan\frac{1}{12}\pi = \dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\cdot\tan\frac{\pi}{4}}$

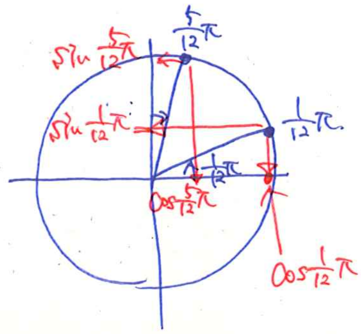
$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = 2-\sqrt{3}$

② $\tan\frac{5}{12}\pi = \dfrac{\tan\frac{3}{4}\pi - \tan\frac{\pi}{3}}{1+\tan\frac{3}{4}\pi\cdot\tan\frac{\pi}{3}}$

$= \dfrac{-1-\sqrt{3}}{1-\sqrt{3}} = 2+\sqrt{3}$

よって求める直線は

$y = (2-\sqrt{3})x$, $y = (2+\sqrt{3})x$ //



7　加法定理の応用

7.1　復習

加法定理を思い出す．

(1) $\sin(\alpha+\beta)$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

(2) $\sin(\alpha-\beta)$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

(3) $\cos(\alpha+\beta)$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

(4) $\cos(\alpha-\beta)$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

(5) $\tan(\alpha+\beta)$

$$= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

(6) $\tan(\alpha-\beta)$

$$= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

計算練習

(1) $\sin\left(\frac{1}{3}\pi+\frac{1}{4}\pi\right)$

$$= \sin\frac{1}{3}\pi\cdot\cos\frac{1}{4}\pi + \cos\frac{1}{3}\pi\cdot\sin\frac{1}{4}\pi$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(2) $\cos\left(\frac{1}{12}\pi\right)$　　$\frac{1}{12}\pi = \frac{1}{3}\pi - \frac{1}{4}\pi$

$$= \cos\left(\frac{1}{3}\pi-\frac{1}{4}\pi\right)$$

$$= \cos\frac{1}{3}\pi\cdot\cos\frac{1}{4}\pi + \sin\frac{1}{3}\pi\cdot\sin\frac{1}{4}\pi$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(3) $\sin\left(\frac{1}{12}\pi\right)$

$$= \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(4) $\sin\left(\frac{5}{12}\pi\right)$

$$= \cos\frac{1}{12}\pi$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(5) $\cos\left(\frac{5}{12}\pi\right)$

$$= \sin\frac{1}{12}\pi$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(6) $\tan\left(\frac{5}{12}\pi\right)$

$$= \frac{\sin\frac{5}{12}\pi}{\cos\frac{5}{12}\pi}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{\cancel{\sqrt{2}}(\sqrt{3}+1)}{\cancel{\sqrt{2}}(\sqrt{3}-1)}$$

$$= 2+\sqrt{3}$$

加

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

7.2　2 倍角

考える

$2\alpha = \alpha + \alpha$ と考えることで, 2α の三角比を考える.

(1) $\sin(\alpha+\alpha)$ を α の三角比で表そう.

$$= \sin\alpha\cdot\cos\alpha + \cos\alpha\cdot\sin\alpha$$
$$= 2\sin\alpha\cos\alpha$$

(2) $\cos(\alpha+\alpha)$ を α の三角比で表そう.

$$= \cos\alpha\cdot\cos\alpha - \sin\alpha\cdot\sin\alpha$$
$$= \cos^2\alpha - \sin^2\alpha$$

(3) $\cos(\alpha+\alpha)$ を $\sin\alpha$ で表そう.

$$\boxed{\begin{aligned}\sin^2\alpha+\cos^2\alpha&=1\\ \cos^2\alpha&=1-\sin^2\alpha\end{aligned}}$$

$$= \cos^2\alpha - \sin^2\alpha$$
$$= (1-\sin^2\alpha) - \sin^2\alpha$$
$$= 1 - 2\sin^2\alpha$$

(4) $\cos(\alpha+\alpha)$ を $\cos\alpha$ で表そう.

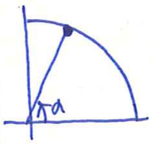
$$= \cos^2\alpha - \sin^2\alpha$$
$$= \cos^2\alpha - (1-\cos^2\alpha)$$
$$= 2\cos^2\alpha - 1$$

(5) $\tan(\alpha+\alpha)$ を $\tan\alpha$ で表そう.

$$= \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha}$$
$$= \frac{2\tan\alpha}{1-\tan^2\alpha}$$

練習問題

(1) $0 < \alpha < \frac{\pi}{2}$ で, $\sin\alpha = \frac{4}{5}$ のとき, $\sin 2\alpha$ の値を求めよ.

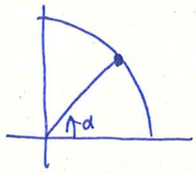


$\sin^2\alpha+\cos^2\alpha=1$ より, 左図から,

$\cos\alpha = \frac{3}{5}$.

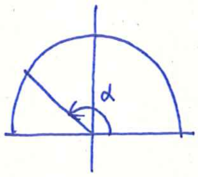
$$\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$$
$$= 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

(2) $0 < \alpha < \frac{\pi}{2}$ で, $\sin\alpha = \frac{3}{5}$ のとき, $\cos 2\alpha$ の値を求めよ.



$$\cos 2\alpha = 1 - 2\sin^2\alpha$$
$$= 1 - 2\cdot\left(\frac{3}{5}\right)^2$$
$$= 1 - \frac{18}{25} = \frac{7}{25}$$

(3) $\frac{\pi}{2} < \alpha < \pi$ で, $\cos\alpha = -\frac{\sqrt{5}}{3}$ のとき, $\sin 2\alpha, \cos 2\alpha, \tan 2\alpha$ の値を求めよ.



$\sin^2\alpha+\cos^2\alpha=1$, 左図から.

$\sin\alpha = \frac{2}{3}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\cdot\frac{2}{3}\cdot\frac{-\sqrt{5}}{3} = \frac{-4\sqrt{5}}{9}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}} = -4\sqrt{5}$$

加法定理
$$\cos(\theta+\theta) = \cos\theta\cdot\cos\theta - \sin\theta\cdot\sin\theta = \cos^2\theta - \sin^2\theta$$

$\sin^2\theta + \cos^2\theta = 1$ で, $\cos^2\theta = 1-\sin^2\theta$. $\sin^2\theta = 1-\cos^2\theta$.

7.3 半角

$$\cos 2\theta = 1 - 2\sin^2\theta, \quad \cos 2\theta = 2\cos^2\theta - 1$$

を式変形して,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

また,

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

θ を $\frac{\theta}{2}$ に置き換えて,

半角

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

練習

(1) $\cos\frac{\pi}{8}$ の値を求めよ.

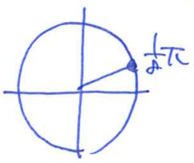
$\cos 2\theta = 2\cos^2\theta - 1$ より.

$$\cos\frac{1}{4}\pi = 2\cos^2\frac{1}{8}\pi - 1.$$

$$\frac{1}{\sqrt{2}} = 2\cos^2\frac{1}{8}\pi - 1$$

$$2\cos^2\frac{1}{8}\pi = \frac{1}{\sqrt{2}} + 1. = \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+2}{2}$$

$$\therefore \cos^2\frac{1}{8}\pi = \frac{2+\sqrt{2}}{4}$$



左図より $\cos\frac{1}{8}\pi > 0$ なので,

$$\cos\frac{1}{8}\pi = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

(2) $\sin\frac{\pi}{8}$ の値を求めよ.

$\cos 2\theta = 1 - 2\sin^2\theta$ より

$$\cos\frac{1}{4}\pi = 1 - 2\sin^2\frac{1}{8}\pi.$$

$$\frac{1}{\sqrt{2}} = 1 - 2\sin^2\frac{1}{8}\pi$$

$$2\sin^2\frac{1}{8}\pi = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\therefore \sin^2\frac{1}{8}\pi = \frac{2-\sqrt{2}}{4}$$

左下図から, $\sin\frac{1}{8}\pi > 0$ より

$$\sin\frac{1}{8}\pi = \frac{\sqrt{2-\sqrt{2}}}{2}$$

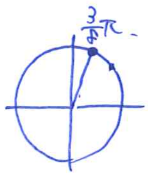
(3) $\cos\frac{3\pi}{8}$ の値を求めよ.

(1)と同様に.

$$\cos\frac{3}{4}\pi = 2\cos^2\frac{3}{8}\pi - 1.$$

$$2\cos^2\frac{3}{8}\pi = 1 - \frac{1}{\sqrt{2}}$$

$$\cos^2\frac{3}{8}\pi = \frac{2-\sqrt{2}}{4}$$



左図より $\cos\frac{3}{8}\pi > 0$.

$$\therefore \cos\frac{3}{8}\pi = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

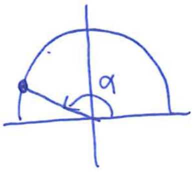
(4) $\tan\frac{3\pi}{8}$ の値を求めよ.

同様にして $\sin\frac{3}{8}\pi = \frac{\sqrt{2+\sqrt{2}}}{2}$

$$\tan\frac{3}{8}\pi = \frac{\sin\frac{3}{8}\pi}{\cos\frac{3}{8}\pi} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} \times \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \frac{2+\sqrt{2}}{\sqrt{4-2}} = \frac{2+\sqrt{2}}{\sqrt{2}} = \sqrt{2}+1$$

練習

(1) $\dfrac{\pi}{2} < \alpha < \pi$ で, $\cos\alpha = -\dfrac{4}{5}$ のとき, $\cos\dfrac{\alpha}{2}$ の値を求めよ.



$\cos 2\theta = 2\cos^2\theta - 1$. より

$\cos\alpha = 2\cos^2\dfrac{\alpha}{2} - 1$.

$2\cos^2\dfrac{\alpha}{2} = \cos\alpha + 1$.

$\cos^2\dfrac{\alpha}{2} = \dfrac{\cos\alpha + 1}{2}$

$= \dfrac{-\frac{4}{5} + 1}{2}$

$= \dfrac{\frac{1}{5}}{2}$

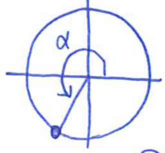
$= \dfrac{1}{10}$

左上図より,

$\cos\dfrac{\alpha}{2} < 0$.

$\therefore \cos\dfrac{\alpha}{2} = -\dfrac{1}{\sqrt{10}}$ //

(2) $\pi < \alpha < \dfrac{3\pi}{2}$ で, $\cos\alpha = -\dfrac{1}{4}$ のとき, $\sin\dfrac{\alpha}{2}, \cos\dfrac{\alpha}{2}, \tan\dfrac{\alpha}{2}$ の値を求めよ.



$\cos 2\theta = 1 - 2\sin^2\theta$ — ①

$\cos 2\theta = 2\cos^2\theta - 1$.

①より

$\cos\alpha = 1 - 2\sin^2\dfrac{\alpha}{2}$

$-\dfrac{1}{4} = 1 - 2\sin^2\dfrac{\alpha}{2}$

$-\dfrac{5}{4} = -2\sin^2\dfrac{\alpha}{2}$

$\sin^2\dfrac{\alpha}{2} = \dfrac{5}{8}$

左上図より $\sin\dfrac{\alpha}{2} < 0$

$\therefore \sin\dfrac{\alpha}{2} = -\dfrac{\sqrt{5}}{2\sqrt{2}}$

$= -\dfrac{\sqrt{10}}{4}$ //

$\sin^2\dfrac{\alpha}{2} + \cos^2\dfrac{\alpha}{2} = 1$ より

$\dfrac{10}{16} + \cos^2\dfrac{\alpha}{2} = 1$

$\cos^2\dfrac{\alpha}{2} = \dfrac{6}{16}$

左上図より $\cos\dfrac{\alpha}{2} < 0$

$\therefore \cos\dfrac{\alpha}{2} = -\dfrac{\sqrt{6}}{4}$ //

$\tan\dfrac{\alpha}{2} = \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$ より

$\tan\dfrac{\alpha}{2} = \dfrac{\sqrt{10}}{-\sqrt{6}} = -\dfrac{\sqrt{5}}{\sqrt{3}}$

$= -\dfrac{\sqrt{15}}{3}$ //

加法定理

$$\boxed{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta}$$

8　三角関数の合成

問題

以下の方程式を解け．　$(0 \leqq \theta < 2\pi)$

$$\sqrt{3}\sin x + \cos x = 1$$

$$2\cdot\left(\frac{\sqrt{3}}{2}\cdot\sin x + \frac{1}{2}\cos x\right) = 1.$$

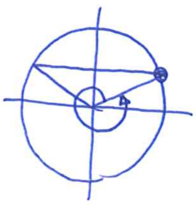
$$\frac{\sqrt{3}}{2}\cdot\sin x + \frac{1}{2}\cdot\cos x = \frac{1}{2}$$

$$\cos\frac{1}{6}\pi\cdot\sin x + \sin\frac{1}{6}\pi\cdot\cos x = \frac{1}{2}$$

$$\sin x\cdot\cos\frac{1}{6}\pi + \cos x\cdot\sin\frac{1}{6}\pi = \frac{1}{2}$$

加法定理

$$\sin\left(x + \frac{1}{6}\pi\right) = \frac{1}{2}$$



左図より

$\theta = 0,\ \frac{2}{3}\pi$.

最後の「加法定理」を用いるために、はじめに $\frac{\sqrt{3}}{2}$，$\frac{1}{2}$ の形を作った。

問題

以下の不等式を解け．　$(0 \leqq \theta < 2\pi)$

$$\sin x - \sqrt{3}\cos x \leqq 1$$

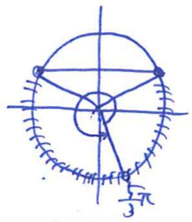
$$2\left(\frac{1}{2}\cdot\sin x - \frac{\sqrt{3}}{2}\cos x\right) \leqq 1$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x \leqq \frac{1}{2}$$

$$\sin x\cdot\left(\frac{1}{2}\right) + \cos x\left(-\frac{\sqrt{3}}{2}\right) \leqq \frac{1}{2}$$

$$\sin x\cdot\cos\frac{5}{3}\pi + \cos x\cdot\sin\frac{5}{3}\pi \leqq \frac{1}{2}$$

$$\sin\left(x + \frac{5}{3}\pi\right) \leqq \frac{1}{2}$$



左図より

$0 \leqq \theta \leqq \frac{1}{2}\pi$,

$\frac{7}{6}\pi \leqq \theta < 2\pi$

<別解>.

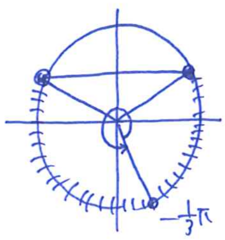
$$2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right) \leqq 1.$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x \leqq \frac{1}{2}$$

$$\sin x\cdot\left(\frac{1}{2}\right) - \cos x\cdot\left(\frac{\sqrt{3}}{2}\right) \leqq \frac{1}{2}$$

$$\sin x\cdot\cos\frac{1}{3}\pi - \cos x\cdot\sin\frac{1}{3}\pi \leqq \frac{1}{2}$$

$$\sin\left(x - \frac{1}{3}\pi\right) \leqq \frac{1}{2}$$



左図より

$0 \leqq \theta \leqq \frac{1}{2}\pi$,

$\frac{7}{6}\pi \leqq \theta < 2\pi$

スタート地点からのズレが少ない方が計算ミス減る?

$0 \leqq \theta < 2\pi$ のとき, 以下の方程式を解け.

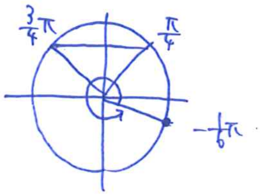
(1) $\sqrt{3}\sin x - \cos x = \sqrt{2}$

$$2\cdot\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right) = \sqrt{2}$$

$$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{\sqrt{2}}$$

$$\sin x\cdot\cos\frac{1}{6}\pi - \cos x\cdot\sin\frac{1}{6}\pi = \frac{1}{\sqrt{2}}$$

$$\sin\left(x - \frac{1}{6}\pi\right) = \frac{1}{\sqrt{2}}$$



左図より

$$\theta = \frac{\pi}{6} + \frac{\pi}{4},\ \frac{\pi}{6} + \frac{3}{4}\pi$$

$$= \frac{5}{12}\pi,\ \frac{11}{12}\pi$$

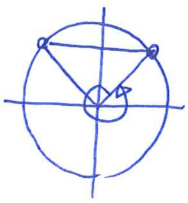
(2) $\sin x + \cos x = 1$

$$\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$

$$\sin x\cdot\cos\frac{1}{4}\pi + \cos x\cdot\sin\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$$

$$\sin\left(x + \frac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}$$



左図より

$$\theta = 0,\ \frac{1}{2}\pi$$

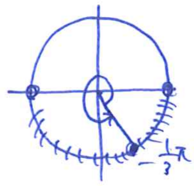
(3) $\sin x - \sqrt{3}\cos x \leqq 0$

$$2\left(\frac{1}{2}\cdot\sin x - \frac{\sqrt{3}}{2}\cos x\right) \leqq 0$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x \leqq 0$$

$$\sin x\cdot\cos\frac{\pi}{3} - \cos x\sin\frac{1}{3}\pi \leqq 0$$

$$\sin\left(x - \frac{1}{3}\pi\right) \leqq 0$$



左図より

$$0 \leqq \theta \leqq \frac{1}{3}\pi,\ \frac{4}{3}\pi \leqq \theta < 2\pi$$

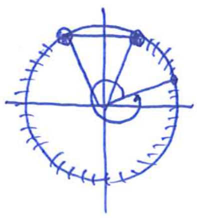
(4) $\sqrt{3}\sin x + \cos x \leqq \sqrt{3}$

$$2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) \leqq \sqrt{3}.$$

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x \leqq \frac{\sqrt{3}}{2}$$

$$\sin x\cdot\cos\frac{1}{6}\pi + \cos x\cdot\sin\frac{1}{6}\pi \leqq \frac{\sqrt{3}}{2}$$

$$\sin\left(x + \frac{1}{6}\pi\right) \leqq \frac{\sqrt{3}}{2}.$$



左図より

$$0 \leqq \theta \leqq \frac{1}{6}\pi,\ \frac{\pi}{2} \leqq \theta < 2\pi$$