

# Lazy evaluation illustrated

## for Haskell divers

*exploring some mental models and implementations*

Takenobu T.

Lazy,... zzz

..., It's fun!

## NOTE

- Meaning of terms are different by communities.
- There are a lot of good documents. Please see also references.
- This is written for GHC's Haskell.

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## 1. Introduction

## 1. Introduction

Basic mental models

# How to evaluate a program in your brain ?

a program

```
code  
code  
code  
:  
?
```

How to evaluate (execute, reduce) the program in your brain?

What "mental model" do you have?

# One of the mental models for C program

C program

A sequence of statements

```
main (...) {  
    code..  
    code.. } ?  
    code..  
    code..  
    code.. }
```

A nested structure

```
x = func1( func2( a ) );  
_____ ?
```

A sequence of arguments

```
y = func1( a(x), b(x), c(x) );  
_____ ? _____
```

A function and arguments

```
z = func1( m + n );  
_____ ? _____
```

How to evaluate (execute, reduce) the program in your brain?

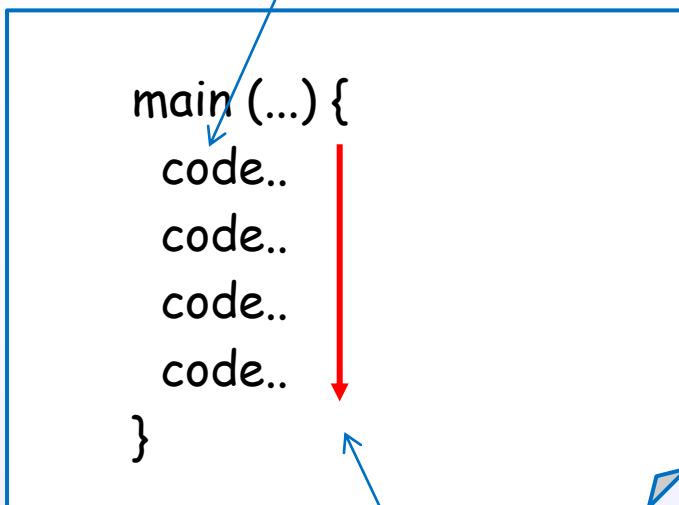
What step, what order, ... ?

# One of the mental models for C program

C program

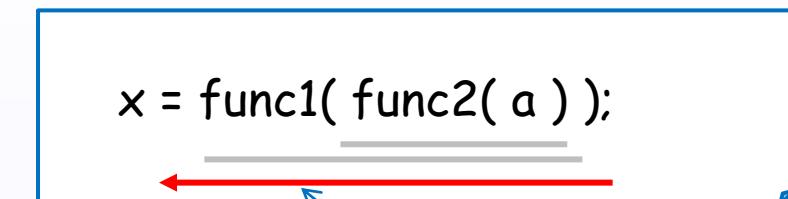
A sequence of statements

A program is a collection of statements.



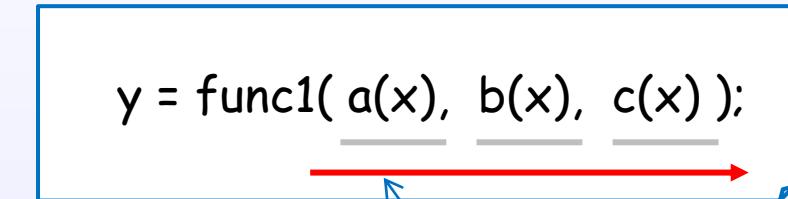
Statements are  
executed downward.

A nested structure



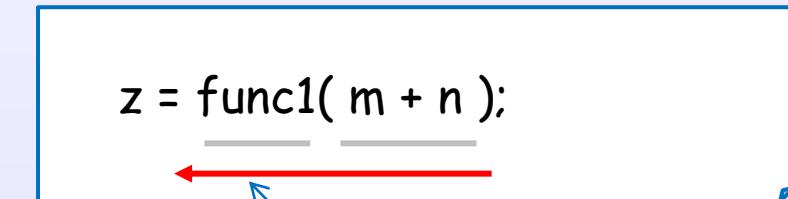
from inner to outer

A sequence of arguments



from left to right

A function and arguments



arguments first  
apply second

Each programmer has some mental models in their brain.

# One of the mental models for C program

Maybe, You have some implicit mental model in your brain for C program.

(1) A program is **a collection of statements**.

(2) There is the **order** between evaluations of elements.



(3) There is the **order** between termination and start of evaluations.



This is a **syntactically straightforward** model for programming languages.  
(an implicit sequential order model)

# One of the mental models for Haskell program

Haskell program

```
main = exp11 (exp12 exp13 exp14 )  
  
exp13 = exp131 exp132  
  
exp14 = exp141 exp142 exp143  
:
```

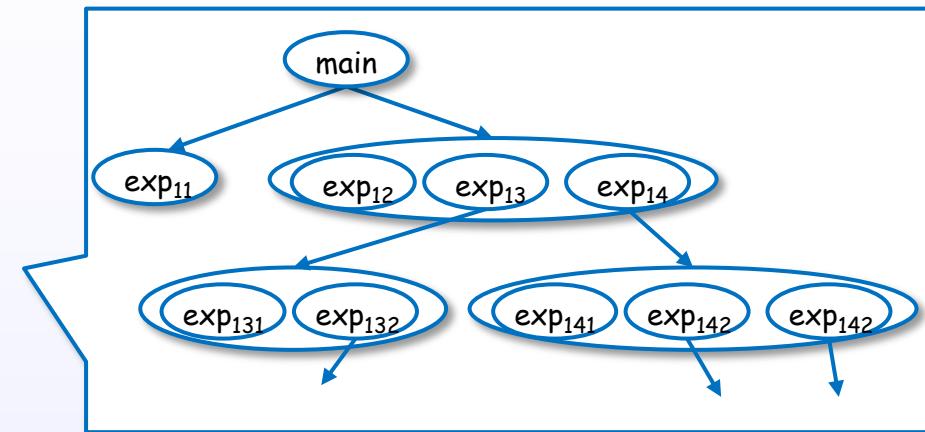
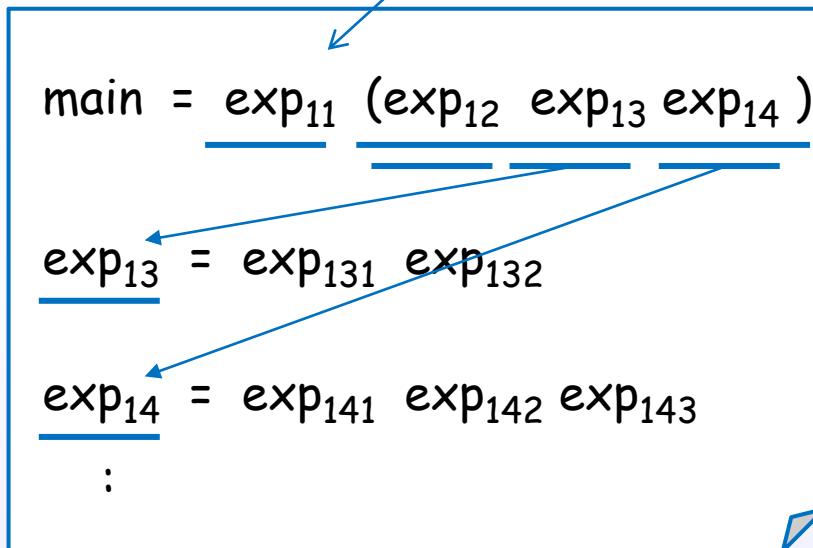


How to evaluate (execute, reduce) the program in your brain?  
What step, what order, ... ?

# One of the mental models for Haskell program

Haskell program

A program is a collection of expressions.



```
main = exp11 (exp12 (exp131 exp132) (exp141 exp142 exp143))
```

A entire program is regarded as a single expression.

The subexpression is evaluated (reduced) in some order.

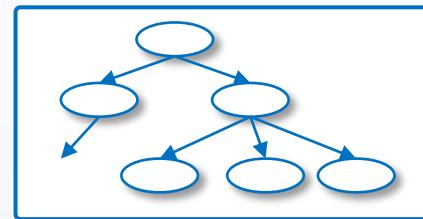
The evaluation is performed by replacement.

# One of the mental models for Haskell program

(1) A program is a collection of expressions.

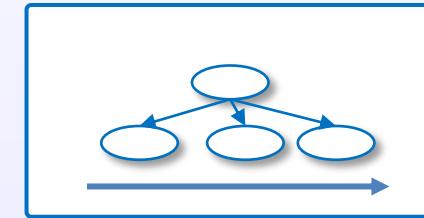
(2) A entire program is regarded as a single expression.

```
main = e (e (e (e e) e (e e e)) )
```

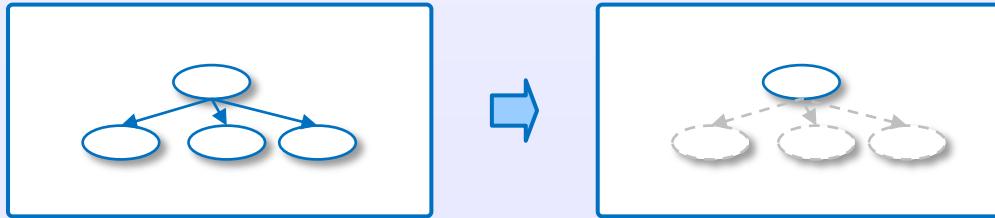


(3) The subexpressions are evaluated (reduced) in some order.

```
f = e (e (e (e e) e (e e e)) )
```



(4) The evaluation is performed by replacement.



This is an example of an expression reduction model for Haskell.

## 1. Introduction

Lazy evaluation

# Why lazy evaluation?

To avoid unnecessary computation

To manipulate infinite data structures

modularity

To manipulate streams

abstraction

pure is order free

amortizing

To manipulate huge data structures

potentially parallelism

2nd Church-Rosser theorem

out-of-order optimization

To implement non-strict semantics

asynchronization

fun

reactive

...

There are various reasons ☺

# Haskell(GHC) 's lazy evaluation

## Lazy evaluation

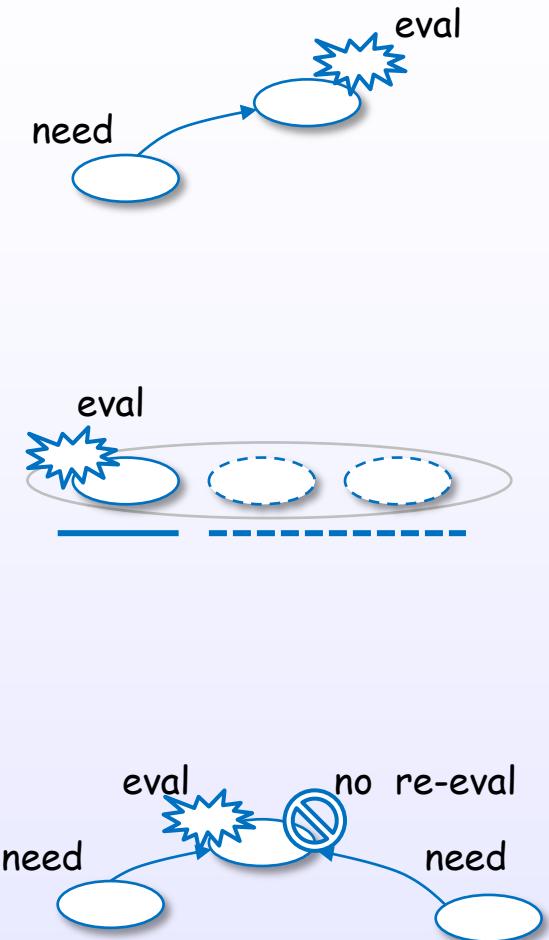
evaluate only when needed

+

evaluate only enough

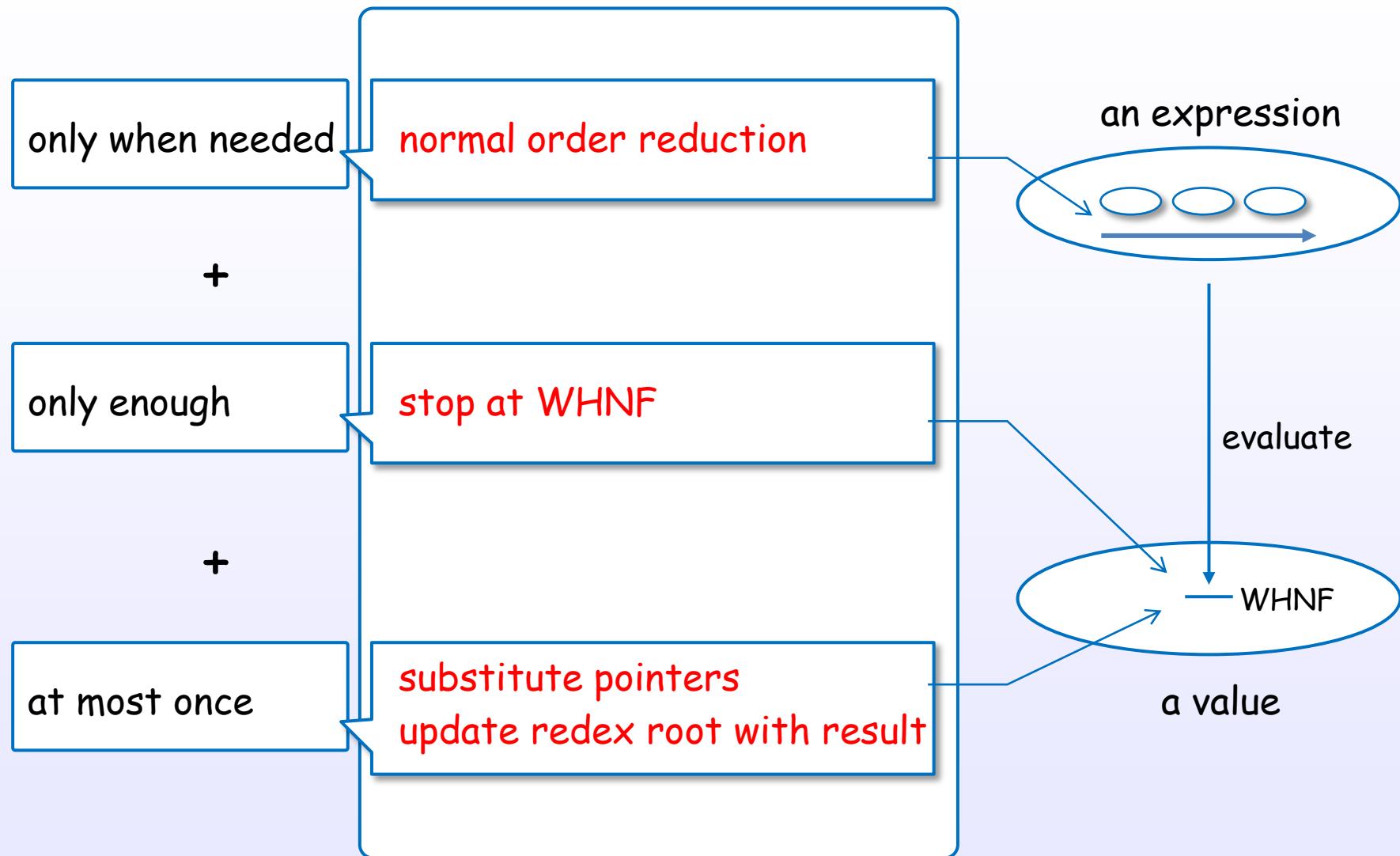
+

evaluate at most once



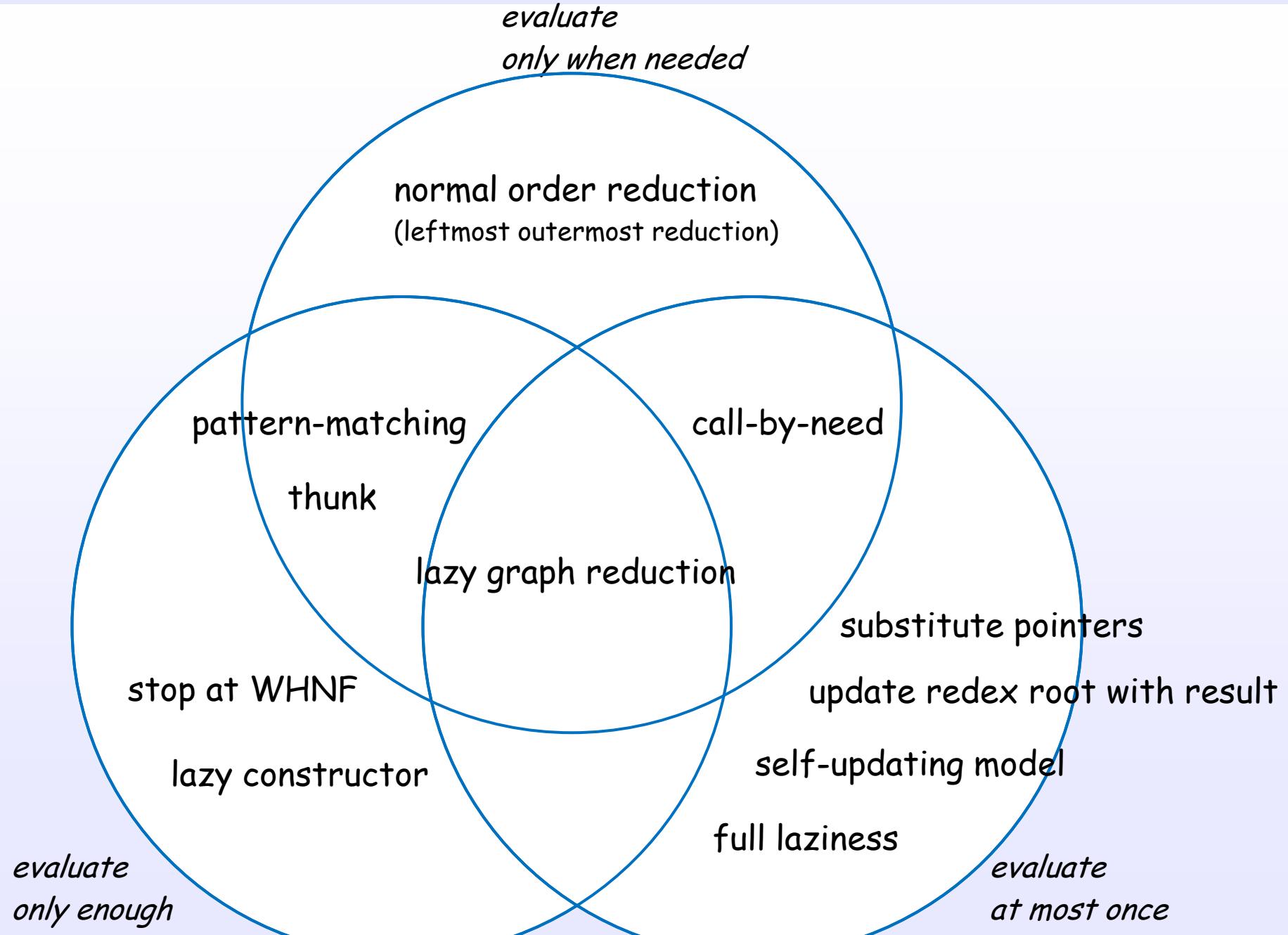
"Lazy" is "delay and avoidance" rather than "delay".

# Ingredient of Haskell(GHC) 's lazy evaluation



This strategy is implemented by lazy graph reduction.

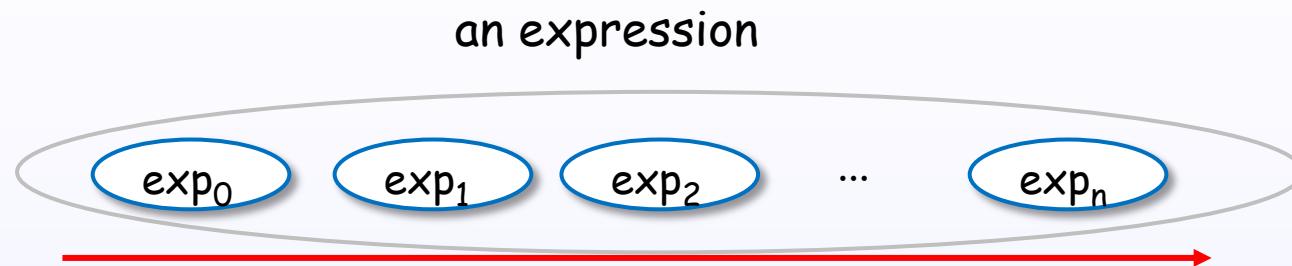
# Techniques of Haskell(GHC) 's lazy evaluation



## 1. Introduction

Simple questions

# What order?



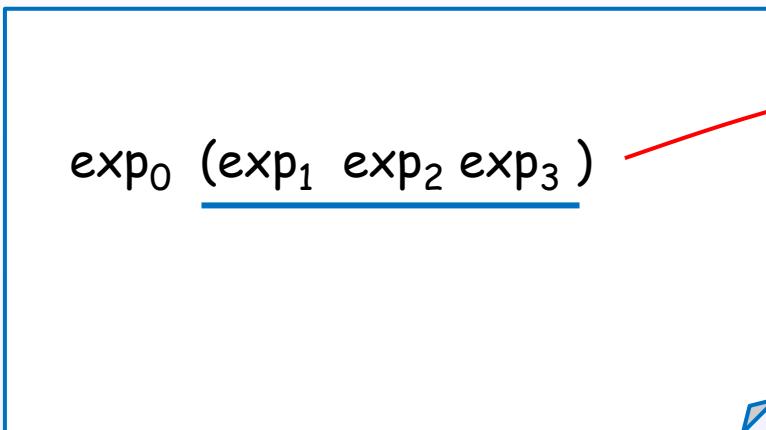
An expression is evaluated by normal order (leftmost outermost redex first).

Normal order reduction guarantees to find a normal form (if one exists).

To avoid unnecessary computation, normal order reduction chooses to apply the function rather than first evaluating the argument.

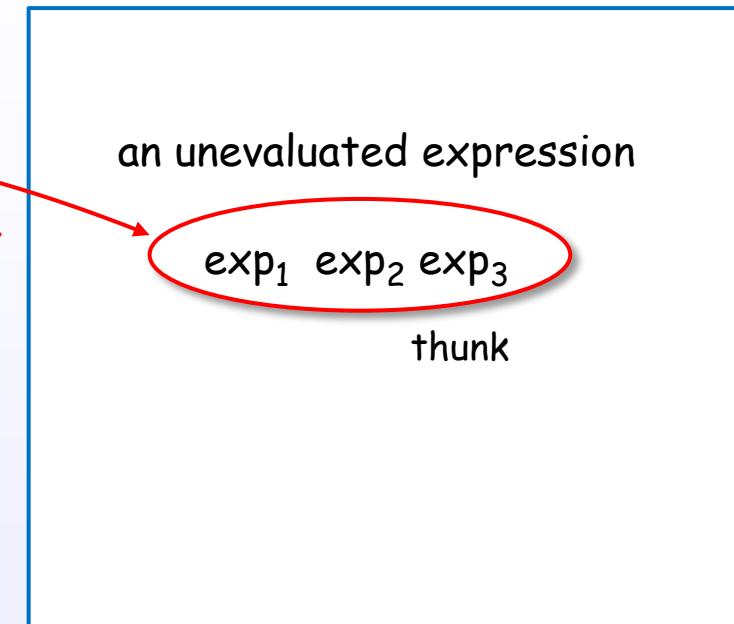
# How to postpone?

Haskell code



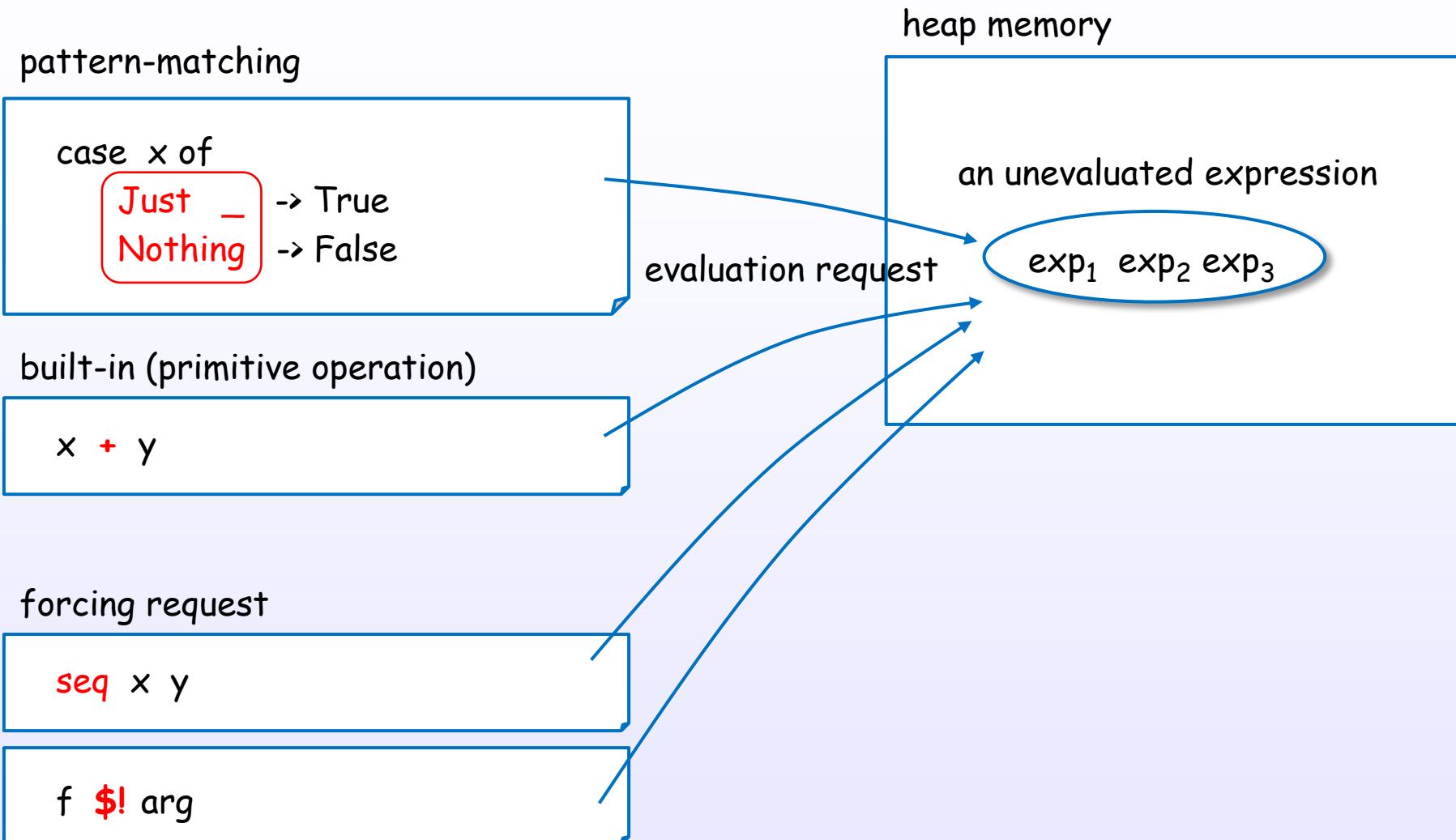
build/allocate

heap memory



To postpone the evaluation, an unevaluated expression is built in the heap memory.

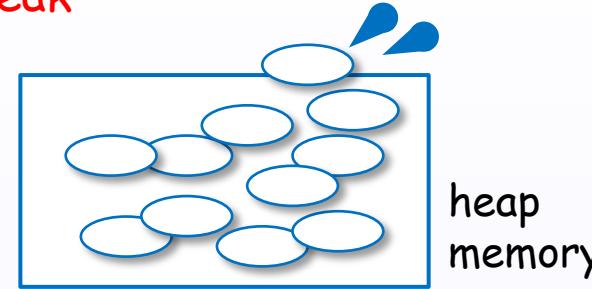
# When needed?



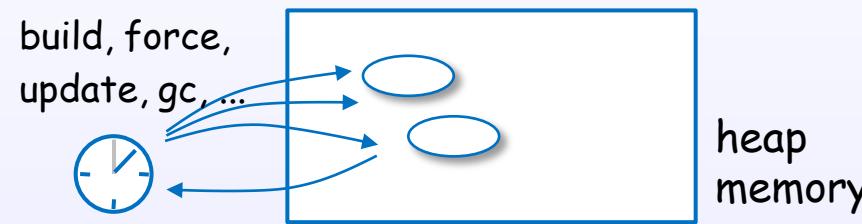
Pattern-matching or forcing request drive the evaluation.

# What to be careful about?

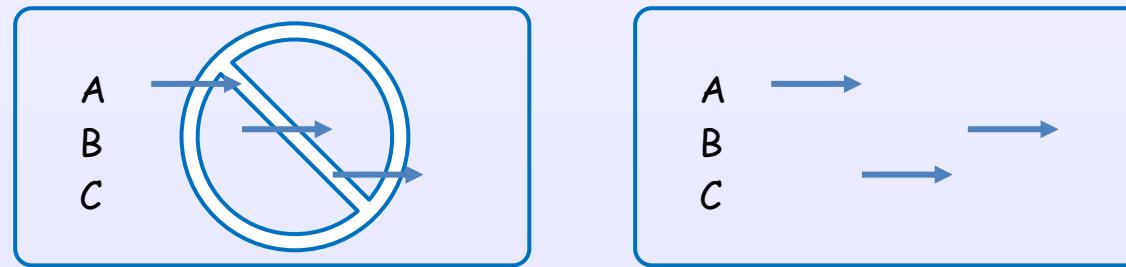
To consider hidden **space leak**



To consider **performance cost** to postpone unevaluated expressions



To consider evaluation (execution) **order** and **timing** in real world



You can avoid the pitfalls by controlling the evaluation.

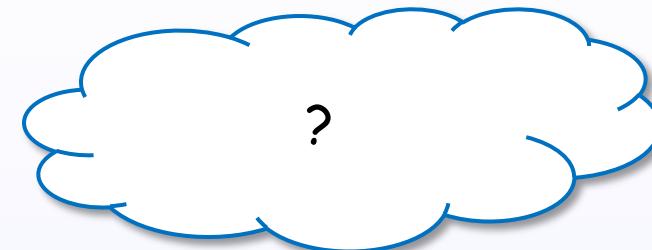
## 2. Expressions

## 2. Expressions

Expression and value

# What is an expression?

An expression

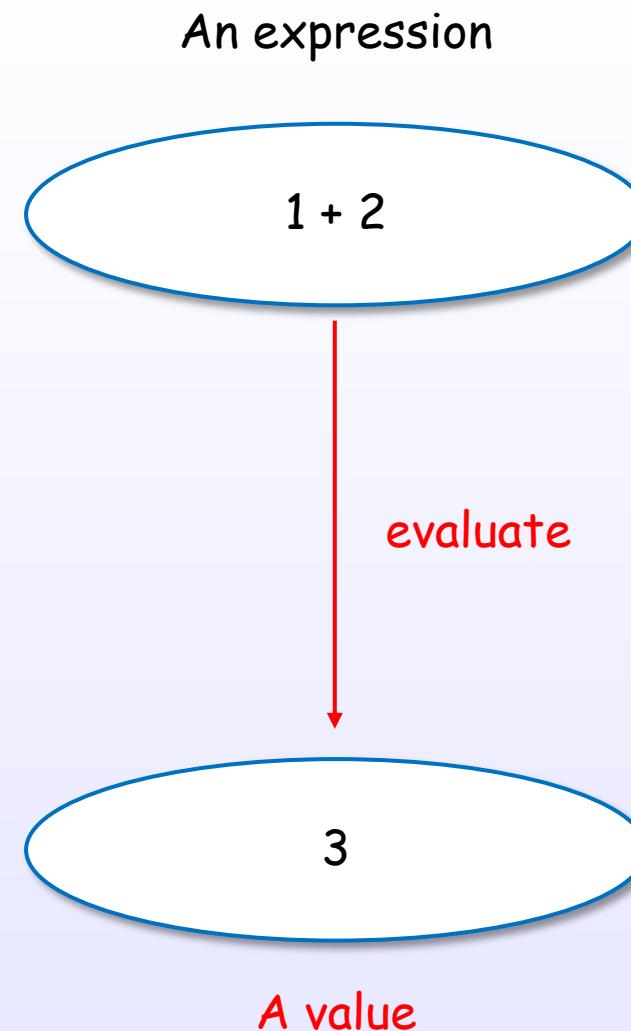


# An expression denotes a value

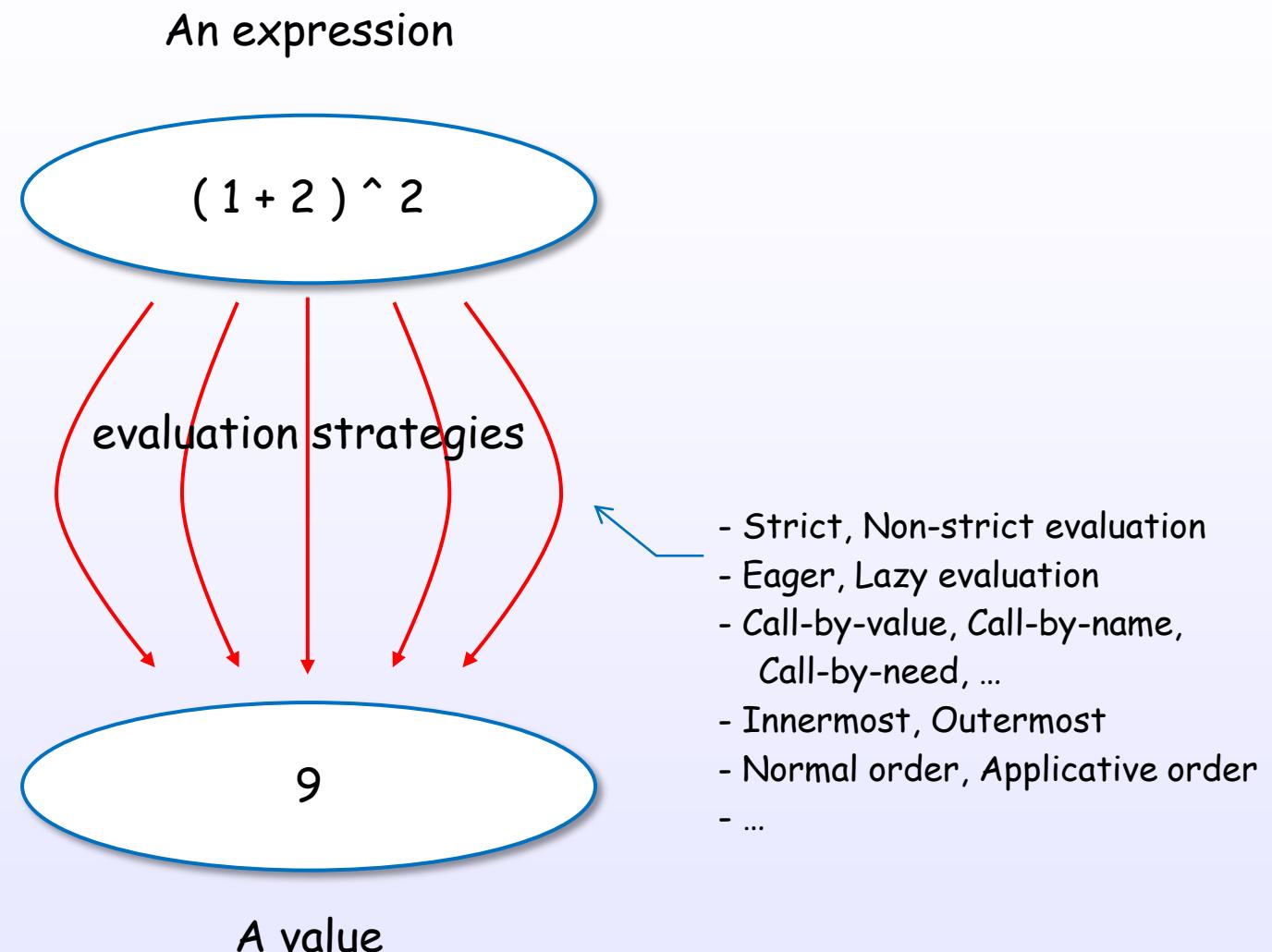
An expression

$$1 + 2$$

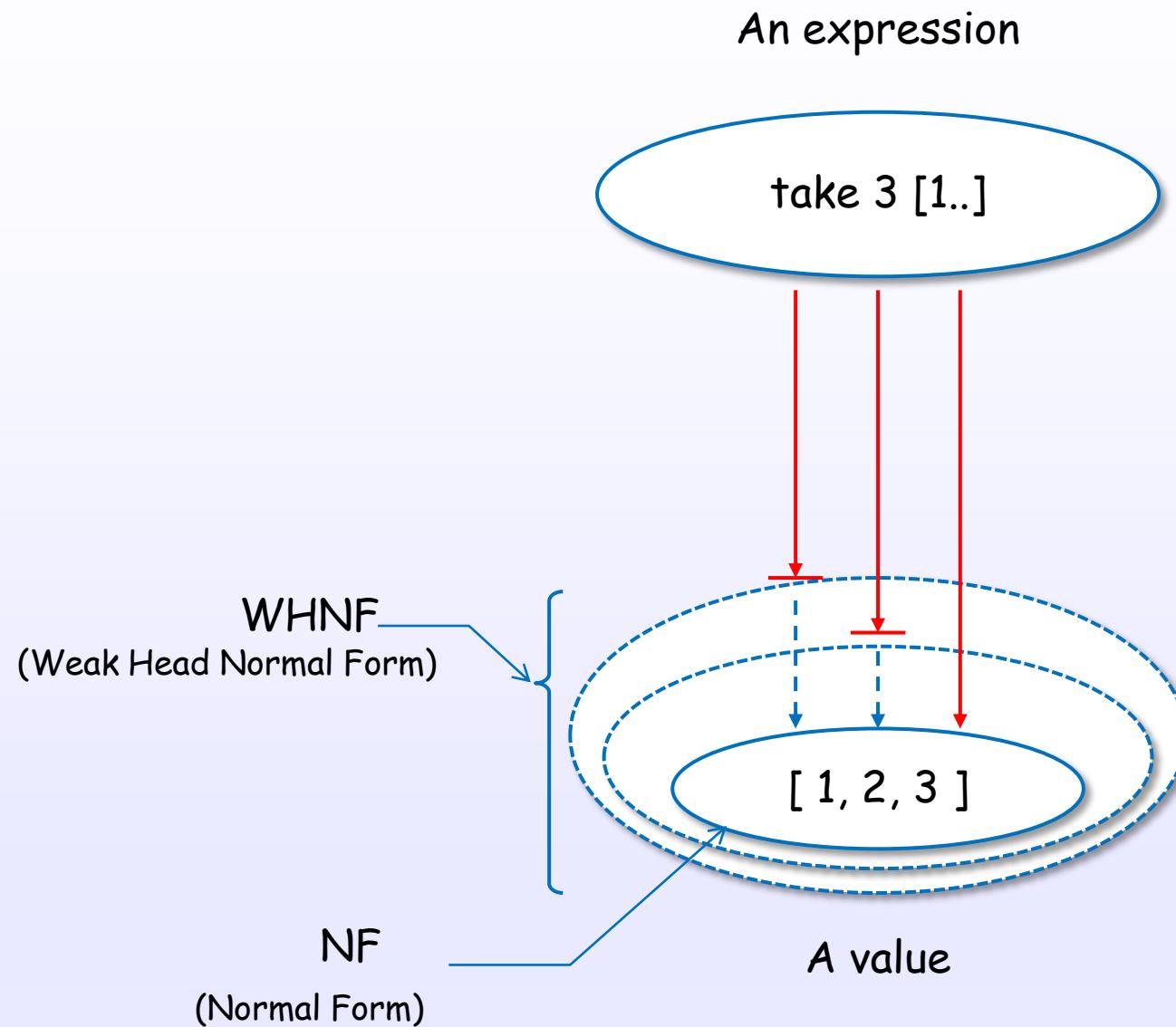
# An expression evaluates to a value



# There are many evaluation approaches



# There are some evaluation levels

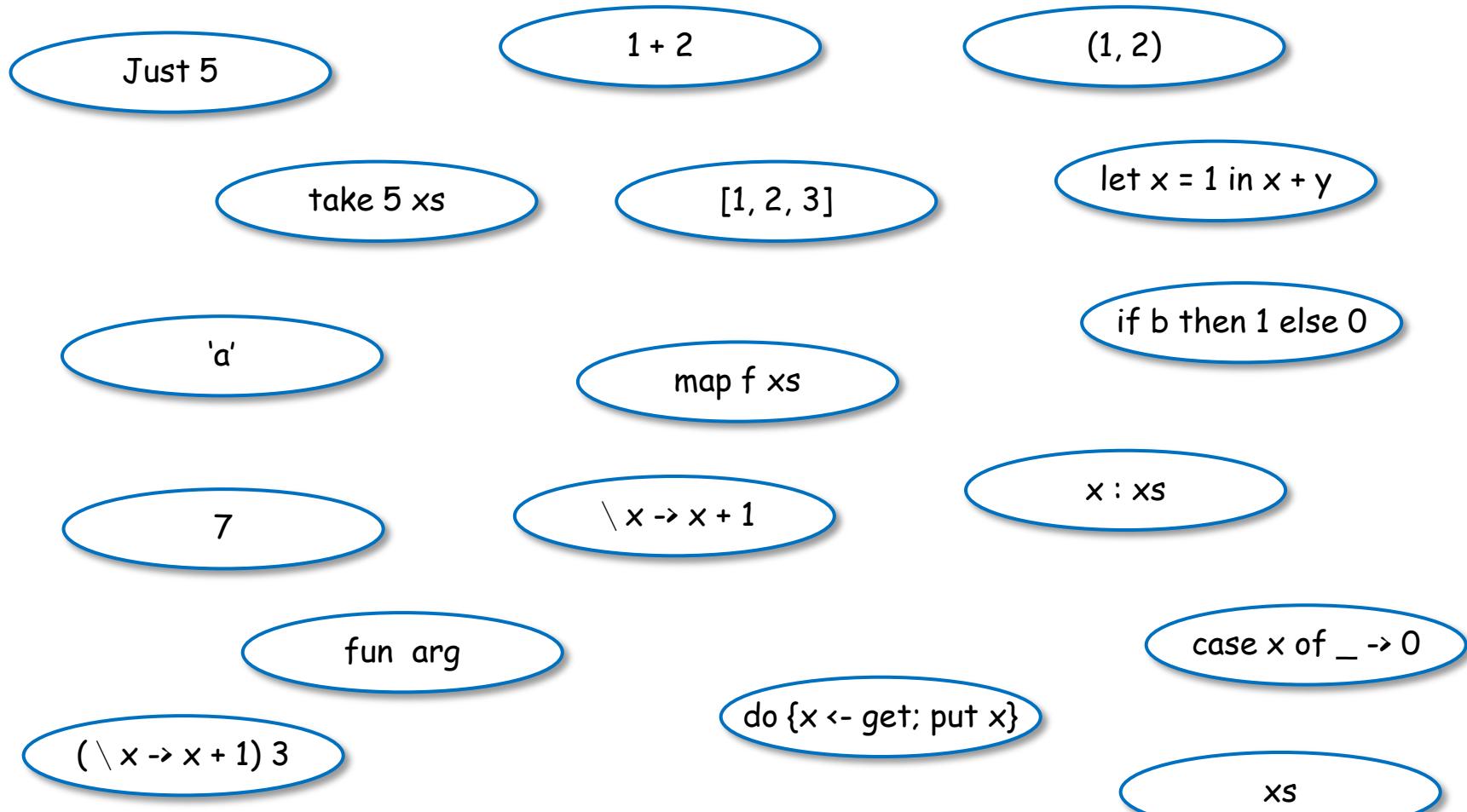


## 2. Expressions

Expressions in Haskell

# There are many expressions in Haskell

## Expressions



↓ categorizing

# Expression categories in Haskell

lambda abstraction

 $\lambda x \rightarrow x + 1$ 

let expression

 $\text{let } x = 1 \text{ in } x + y$ 

conditional

 $\text{if } b \text{ then } 1 \text{ else } 0$ 

case expression

 $\text{case } x \text{ of } _ \rightarrow 0$ 

do expression

 $\text{do } \{x \leftarrow \text{get}; \text{put } x\}$ 

function application

 $\text{take } 5 \text{ xs}$ 
 $(\lambda x \rightarrow x + 1) 3$ 
 $1 + 2$ 
 $\text{map } f \text{ xs}$ 

fun arg

general constructor, literal and some forms

 $7$ 
 $[1, 2, 3]$ 
 $(1, 2)$ 
 $'a'$ 
 $x : xs$ 
 $\text{Just } 5$ 

variable

 $xs$

# Specification is defined in Haskell 2010 Language Report

## "Haskell 2010 Language Report, Chapter 3 Expressions" [H1]

<i>exp</i>	$\rightarrow$	<i>infixexp</i> :: [ <i>context</i> =>] <i>type</i>	(expression type signature)
		<i>infixexp</i>	
<i>infixexp</i>	$\rightarrow$	<i>lexp qop infixexp</i>	(infix operator application)
		<i>- infixexp</i>	(prefix negation)
		<i>lexp</i>	
<i>lexp</i>	$\rightarrow$	$\lambda \ a_1 \dots \ a_n \rightarrow \ exp$	(lambda abstraction, $n \geq 1$ )
		<i>let decls in exp</i>	(let expression)
		<i>if exp [;] then exp [;] else exp</i>	(conditional)
		<i>case exp of { alts }</i>	(case expression)
		<i>do { stmts }</i>	(do expression)
		<i>fexp</i>	
<i>fexp</i>	$\rightarrow$	<i>[fexp] aexp</i>	(function application)
<i>aexp</i>	$\rightarrow$	<i>qvar</i>	(variable)
		<i>gcon</i>	(general constructor)
		<i>literal</i>	
		<i>( exp )</i>	(parenthesized expression)
		<i>( exp<sub>1</sub> , ... , exp<sub>k</sub> )</i>	(tuple, $k \geq 2$ )
		<i>[ exp<sub>1</sub> , ... , exp<sub>k</sub> ]</i>	(list, $k \geq 1$ )
		<i>[ exp<sub>1</sub> , exp<sub>2</sub> ] ... [ exp<sub>3</sub> ]</i>	(arithmetic sequence)
		<i>[ exp   qual<sub>1</sub> , ... , qual<sub>n</sub> ]</i>	(list comprehension, $n \geq 1$ )
		<i>( infixexp qop )</i>	(left section)
		<i>( qop(-) infixexp )</i>	(right section)
		<i>qcon { fbind<sub>1</sub> , ... , fbind<sub>n</sub> }</i>	(labeled construction, $n \geq 0$ )
		<i>aexp<sub>(qcon)</sub> { fbind<sub>1</sub> , ... , fbind<sub>n</sub> }</i>	(labeled update, $n \geq 1$ )

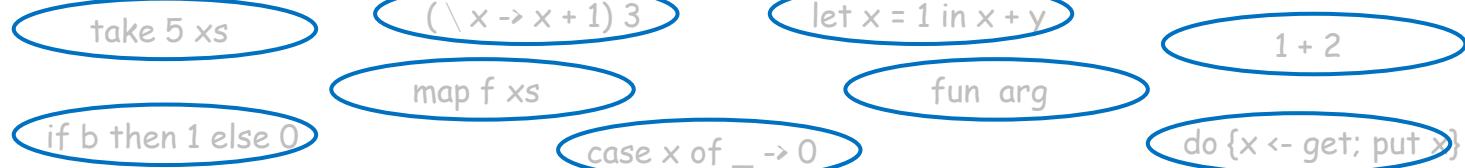
## 2. Expressions

Classification by values and forms

# Classification by values

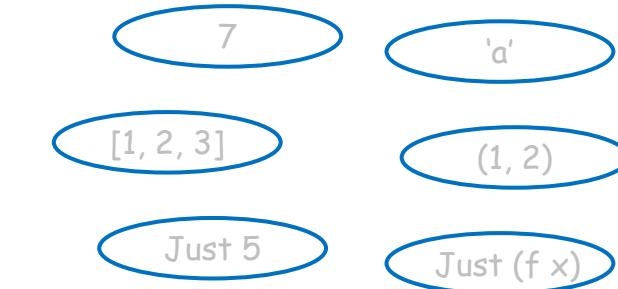
## Expressions

### unevaluated expressions



### values

#### data values



#### function values



#### bottom

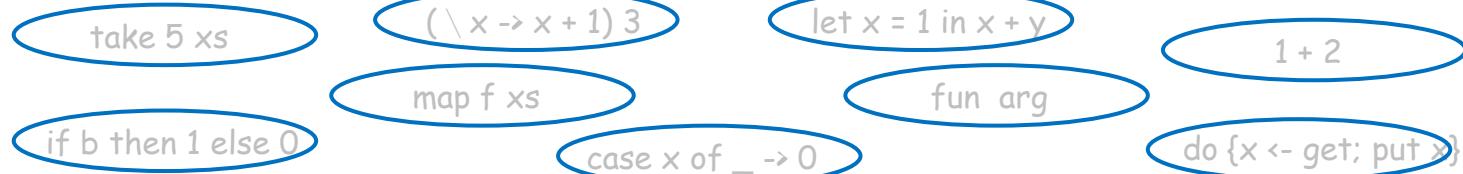


Values are data values or function values.

# Classification by forms

## Expressions

### unevaluated expressions



### values

#### WHNF

#### HNF

#### NF

$\lambda x \rightarrow \text{abs } 1$

$\lambda x \rightarrow x + (\text{abs } 1)$

$\lambda x \rightarrow x$

Just (f x)

[f x, g y]

7

'a'

[1, 2, 3]

(1, 2)

Just 5

#### bottom

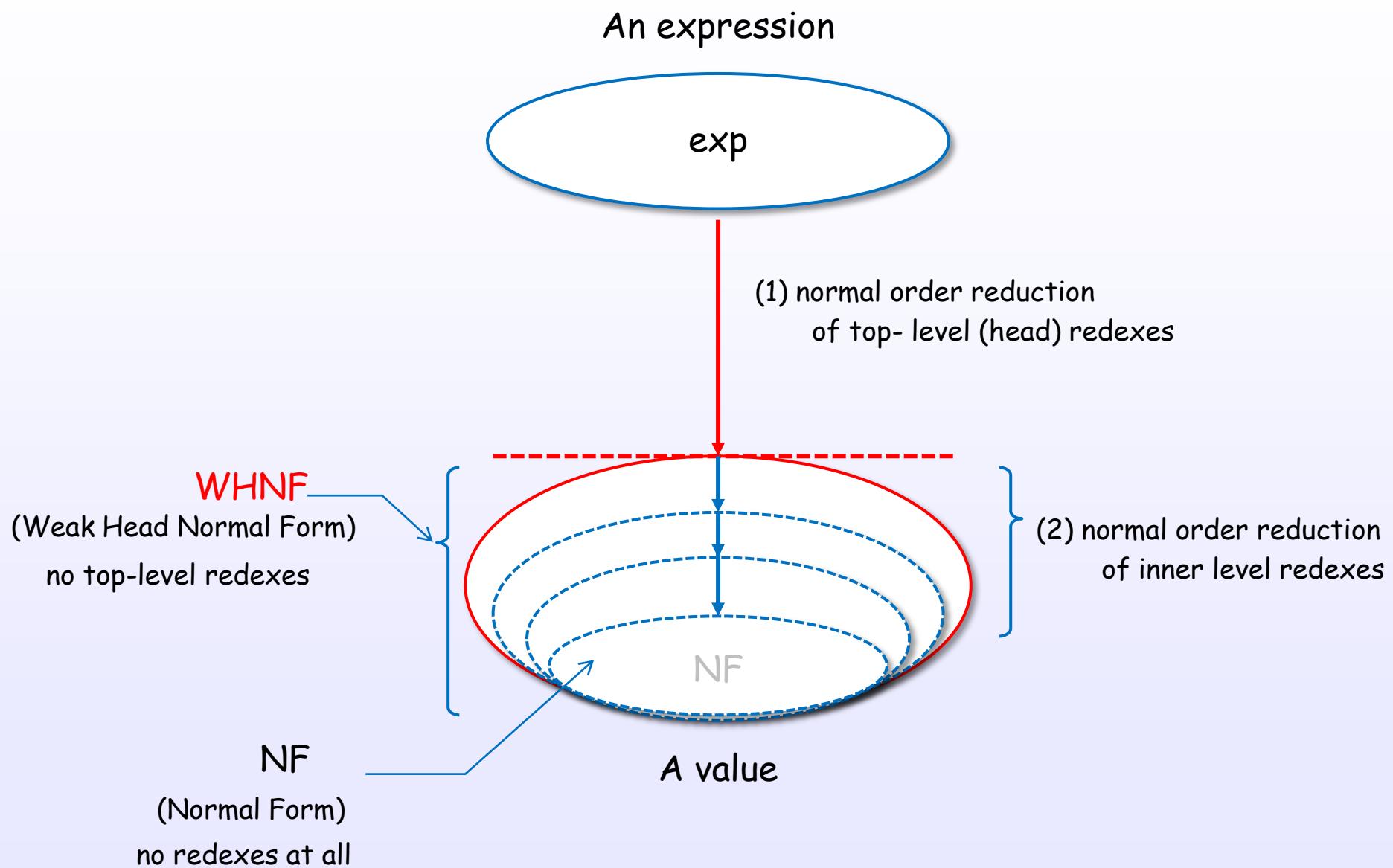
$\perp$

Values are WHNF, HNF or NF.

## 2. Expressions

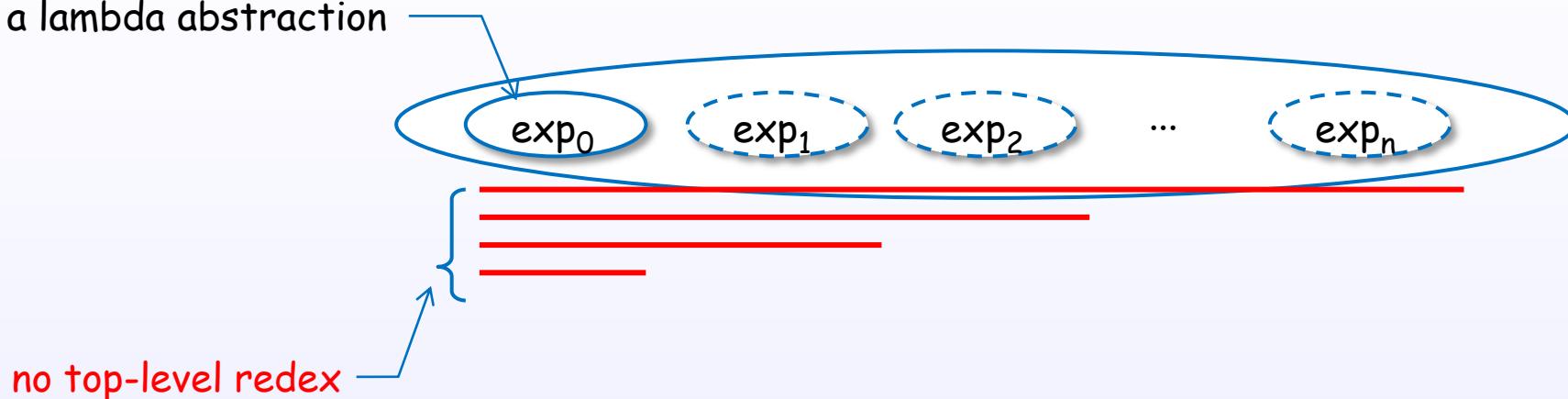
WHNF

# WHNF is one of the form for the evaluated values



# WHNF

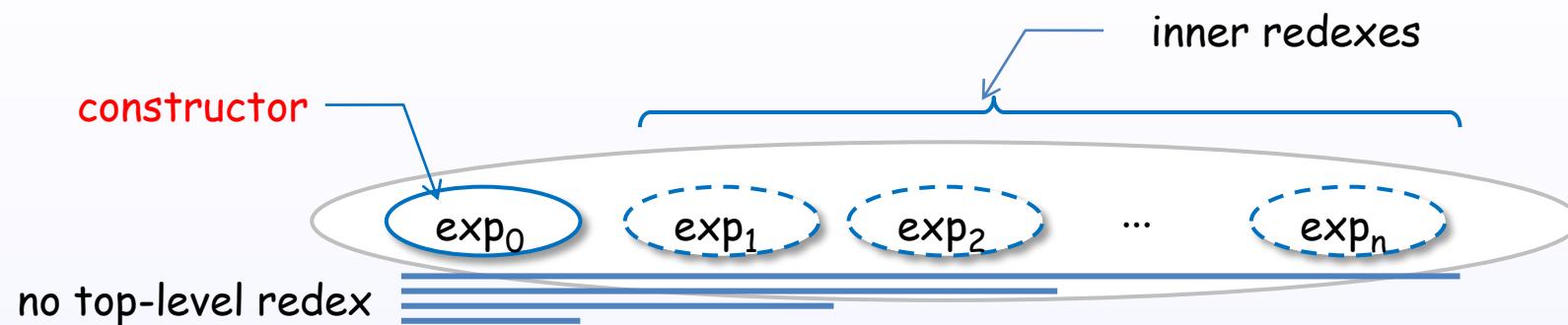
top-level (head) is  
a constructor or  
a lambda abstraction



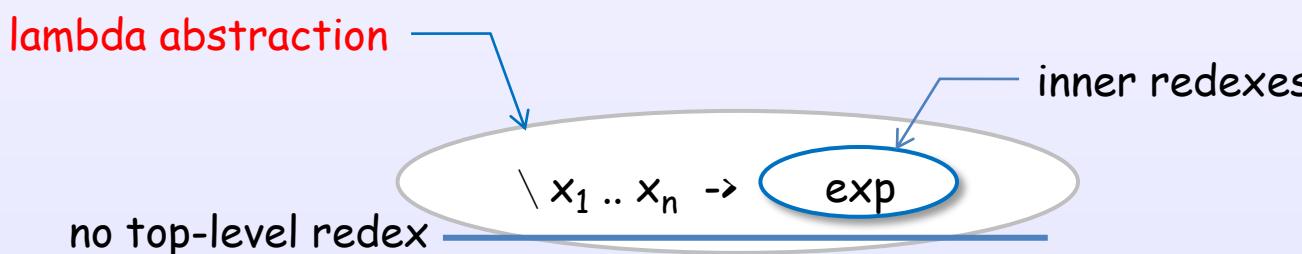
WHNF is a value which has evaluated top-level

# WHNF for a data value and a function value

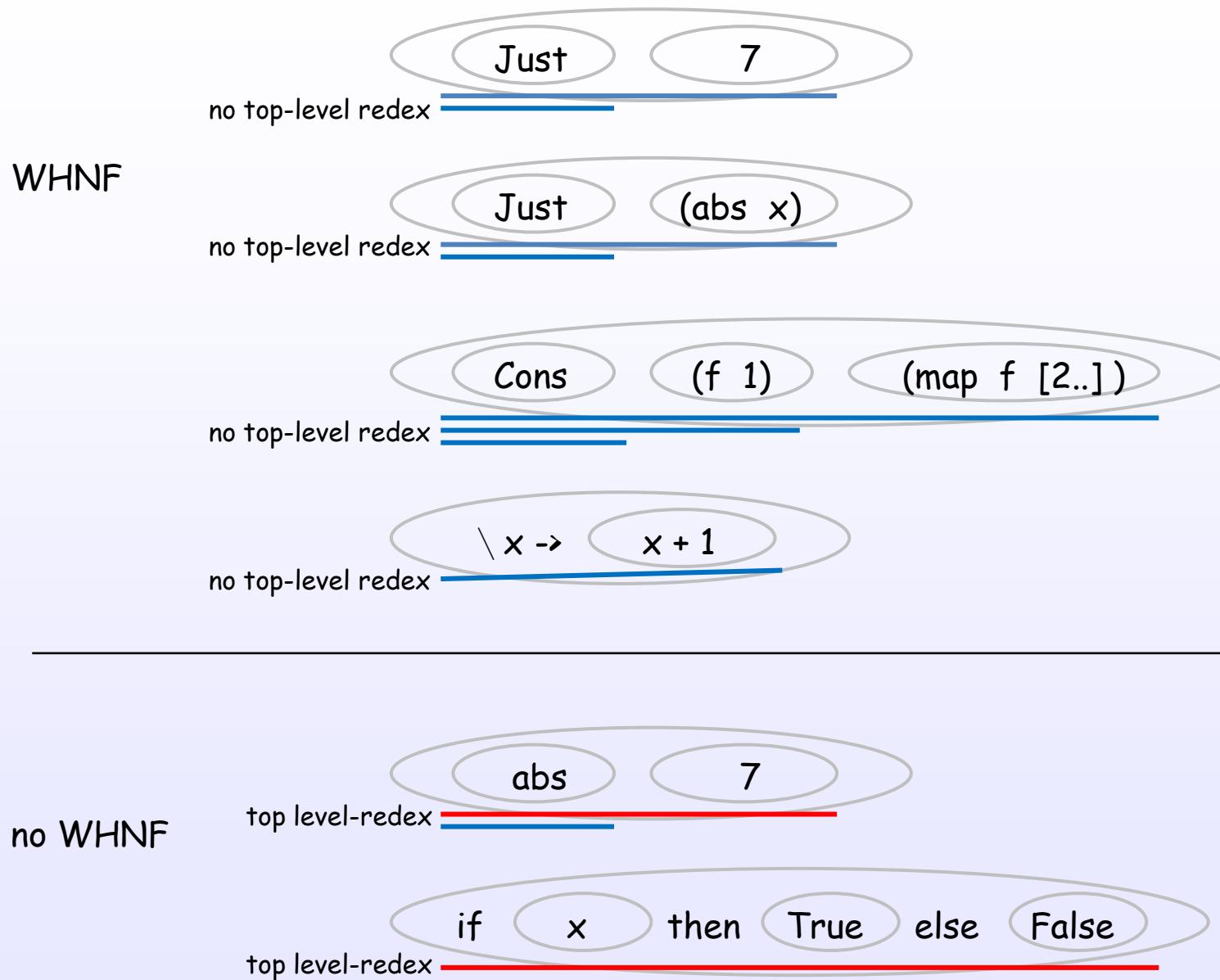
## a data value in WHNF



## a function value in WHNF

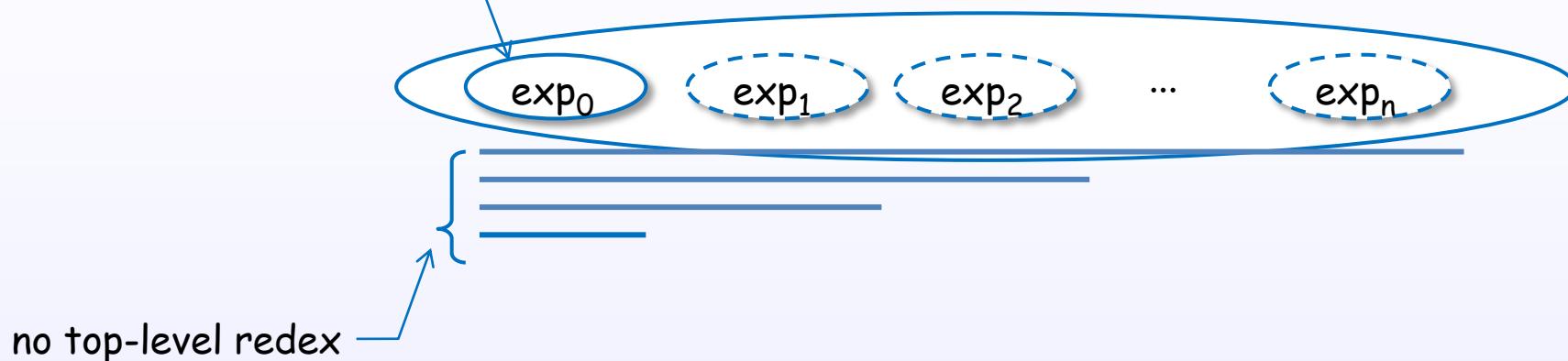


# Examples of WHNF



# HNF

top-level (head) is  
a constructor or  
**a lambda abstraction with no top-level redex**

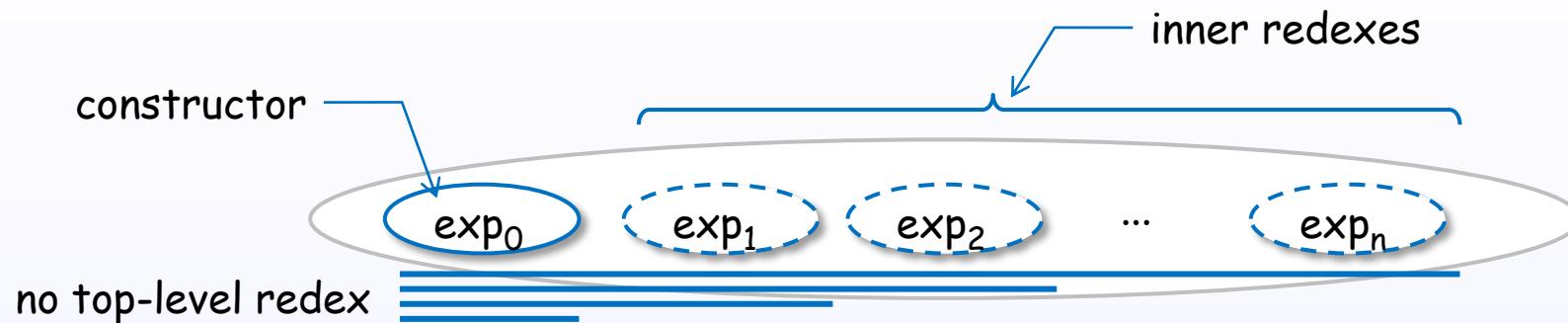


HNF is a value which has evaluated top-level

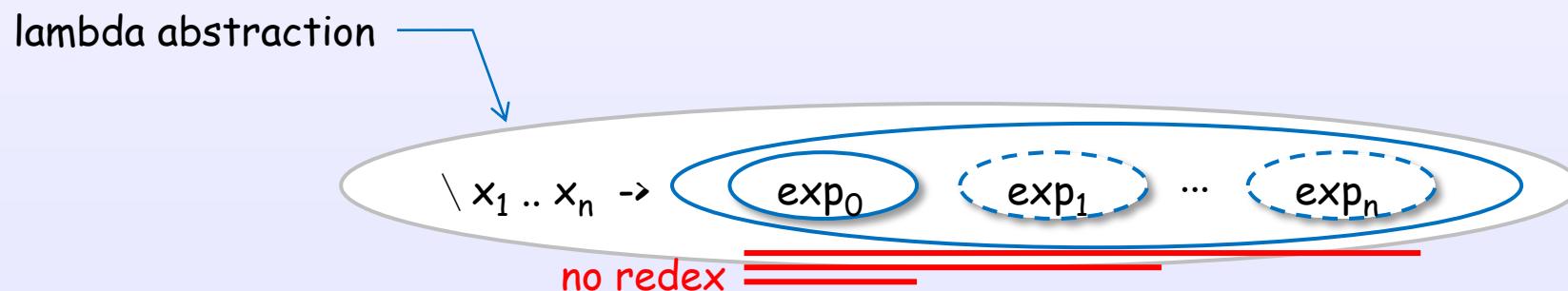
\* GHC uses WHNF rather than HNF.

# HNF for a data value and a function value

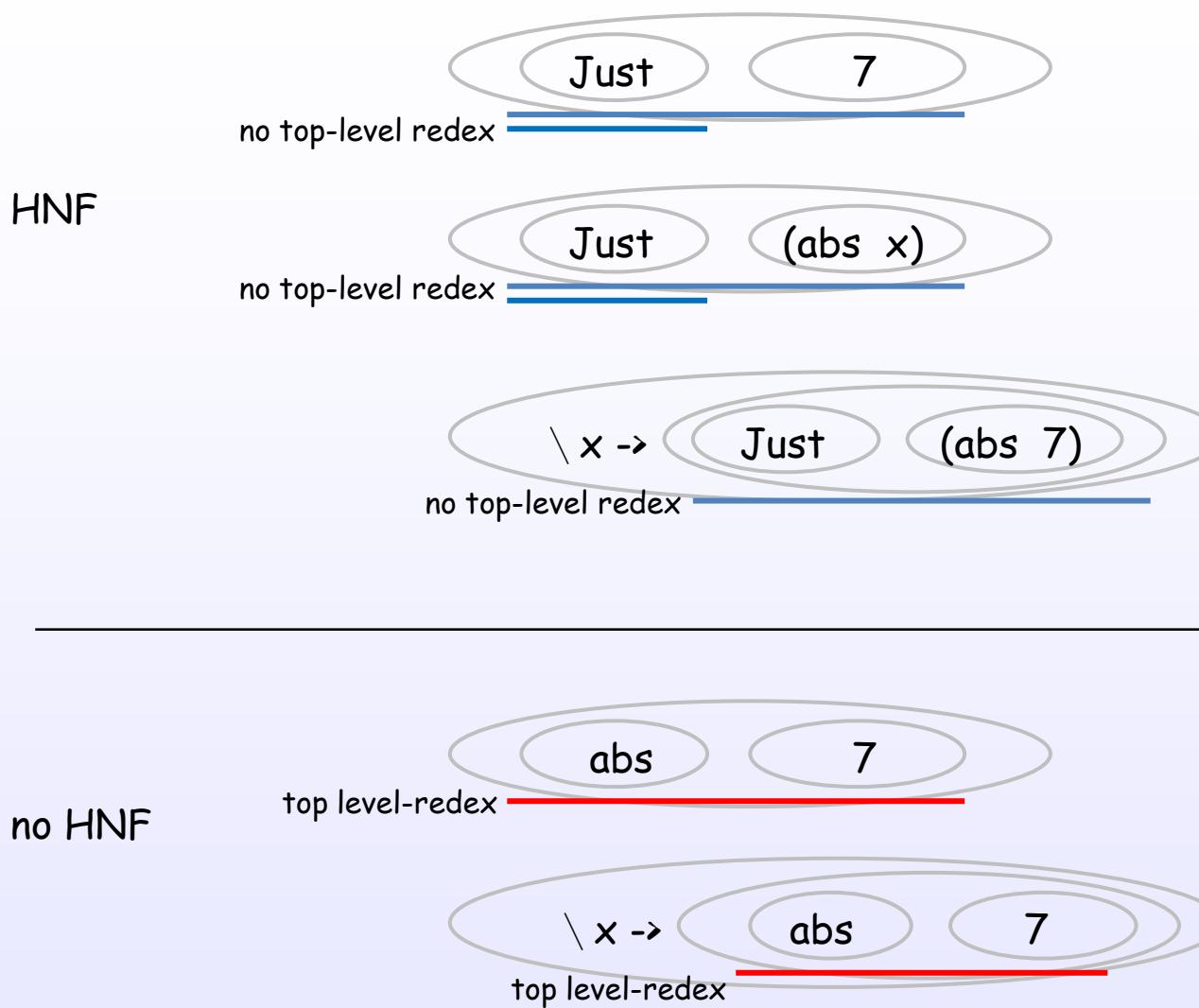
a data value in HNF (same as WHNF)



a function value in HNF

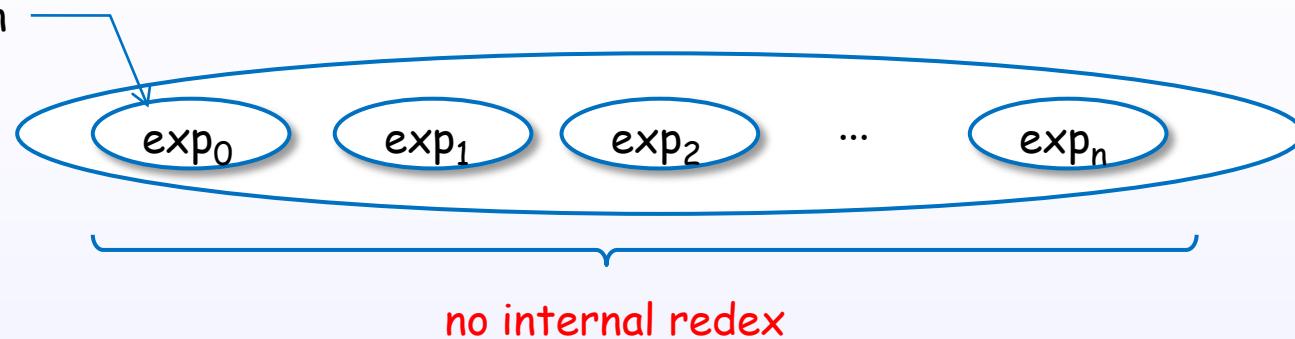


# Examples of HNF



## NF

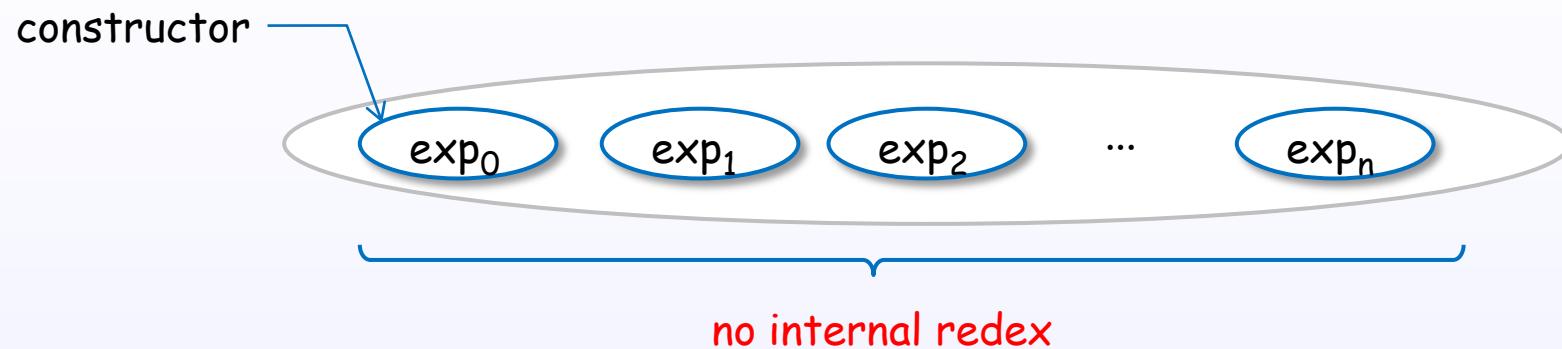
top-level (head) is  
a constructor or  
a lambda abstraction



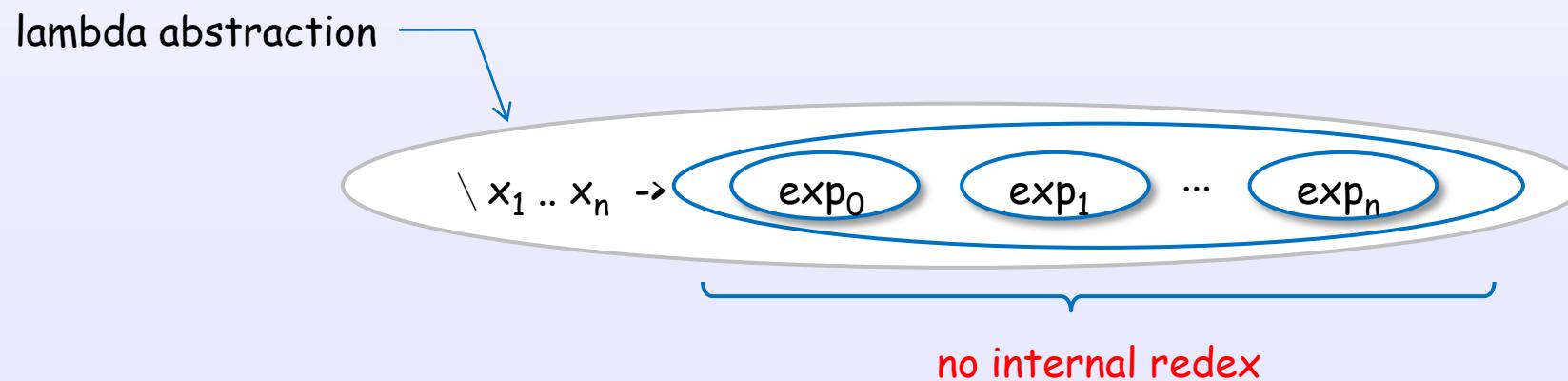
NF is a value which has no redex.

# NF for a data value and a function value

## a data value in NF

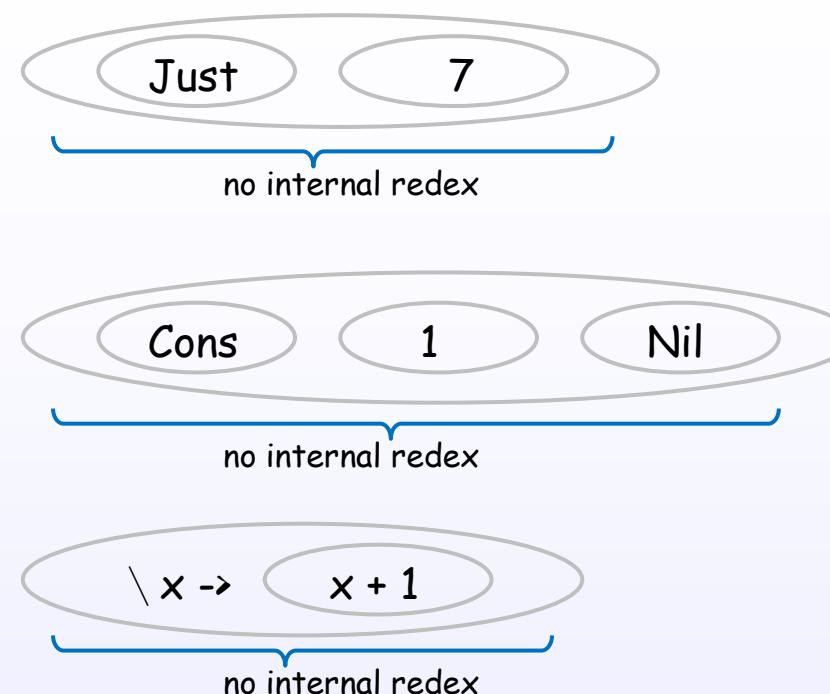


## a function value in NF

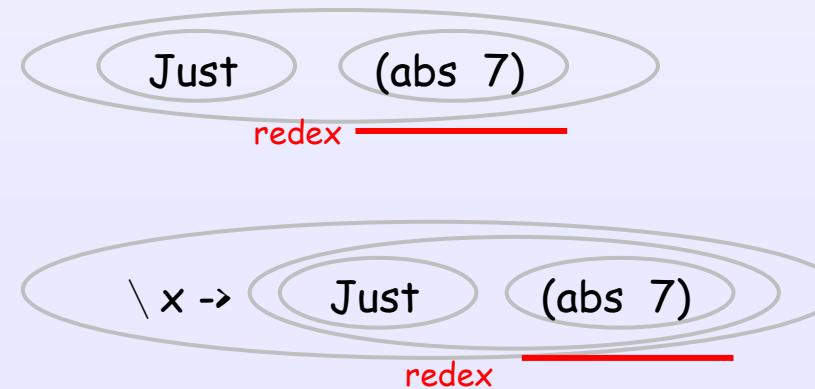


# Examples of NF

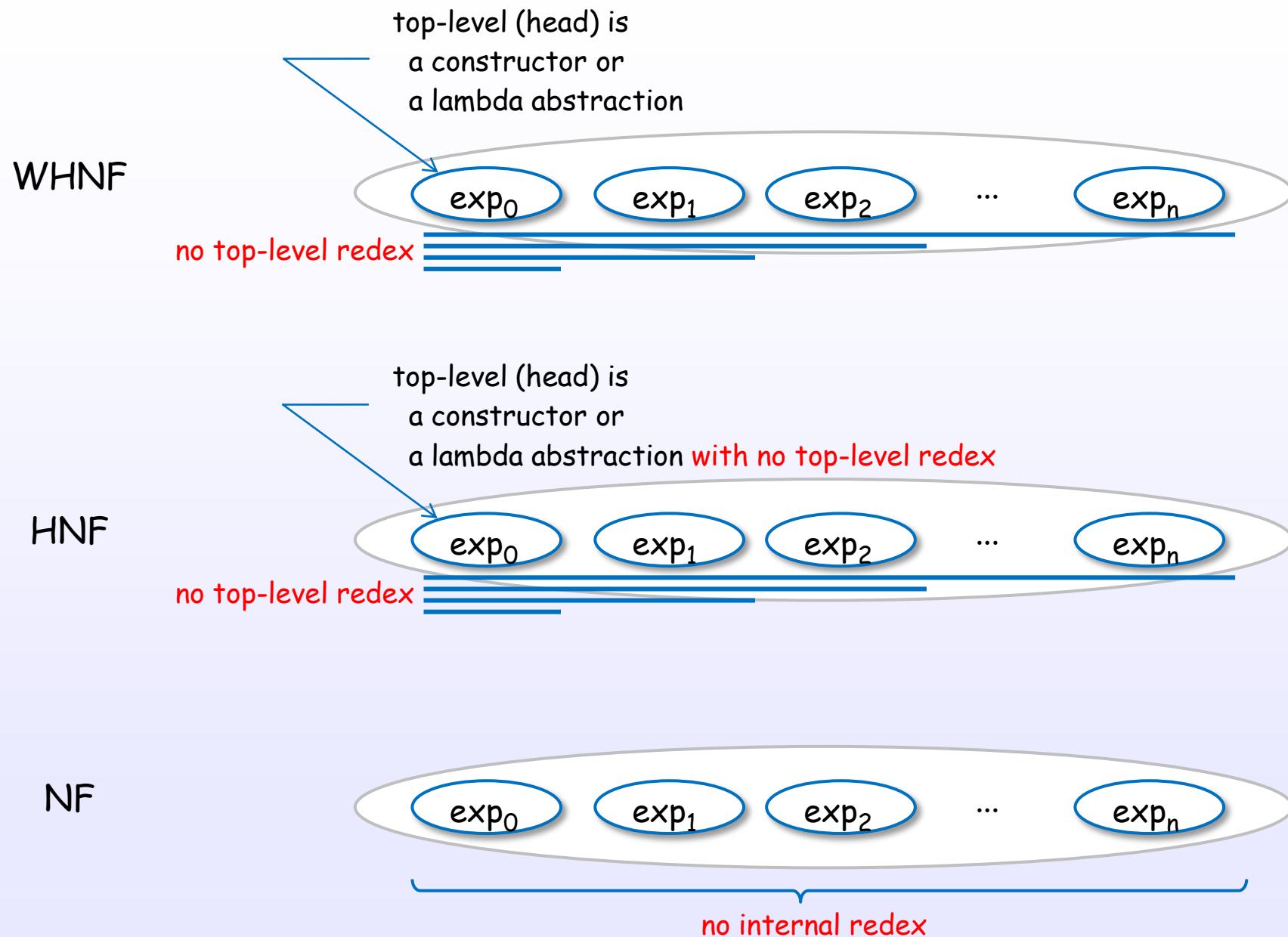
NF



no NF



# WHNF, HNF, NF



# Definition of WHNF and HNF

"The implementation of functional programming languages" [H4]

## 11.3.1 Weak Head Normal Form

To express this idea precisely we need to introduce a new definition:

### DEFINITION

A lambda expression is in *weak head normal form* (WHNF) if and only if it is of the form

$F E_1 E_2 \dots E_n$

where  $n \geq 0$ ;

and either  $F$  is a variable or data object

or  $F$  is a lambda abstraction or built-in function

and  $(F E_1 E_2 \dots E_m)$  is not a redex for any  $m \leq n$ .

An expression has no *top-level redex* if and only if it is in weak head normal form.

### DEFINITION

A lambda expression is in *head normal form* (HNF) if and only if it is of the form

$\lambda x_1. \lambda x_2. \dots \lambda x_n. (v M_1 M_2 \dots M_m)$

where  $n, m \geq 0$ ;

$v$  is a variable ( $x_i$ ), a data object, or a built-in function;

and  $(v M_1 M_2 \dots M_p)$  is not a redex for any  $p \leq m$ .

### 3. Internal representation of expressions

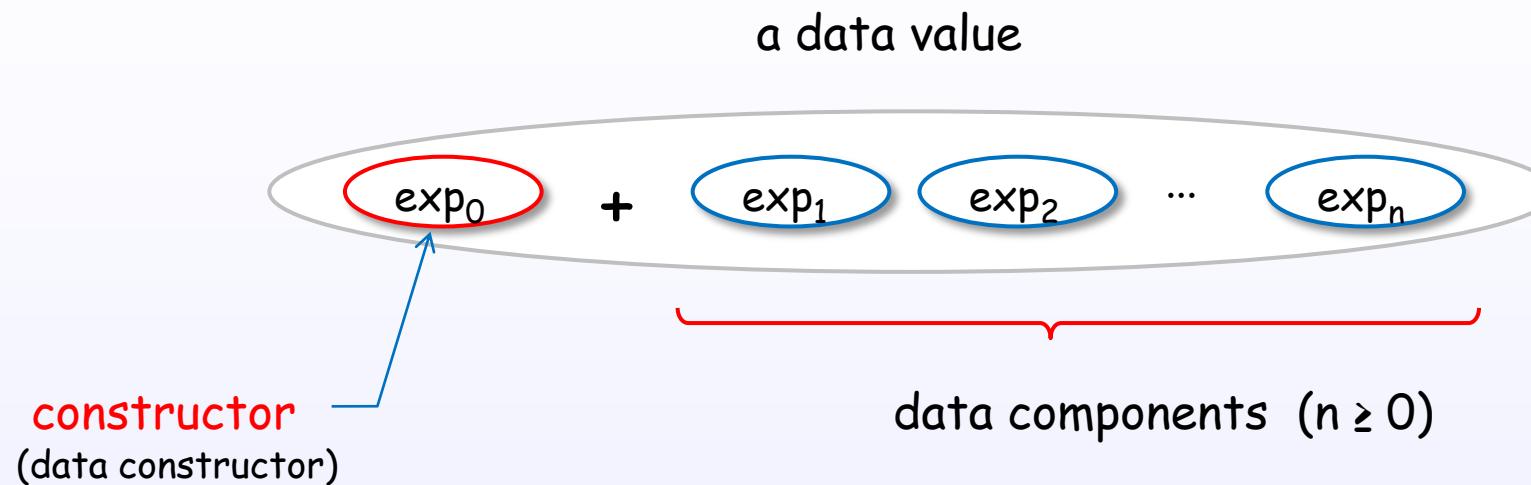
### 3. Internal representation of expressions

Constructor

# Constructor

Constructor is one of the key elements  
to understand WHNF and lazy evaluation in Haskell.

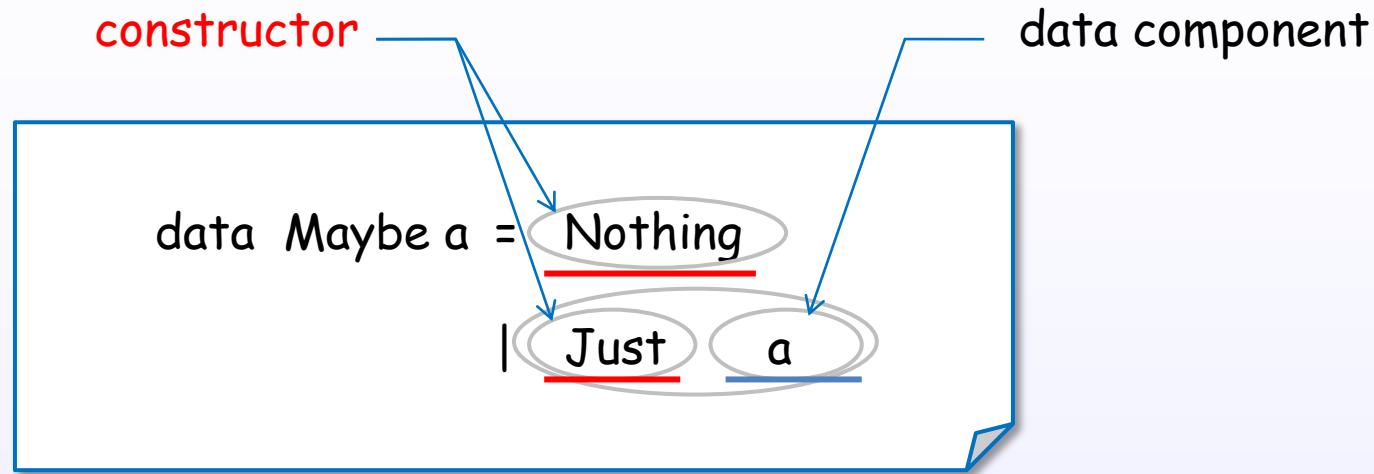
# Constructor



A constructor builds a structured data value.

A constructor distinguishes the data value in expressions.

# Constructors and data declaration



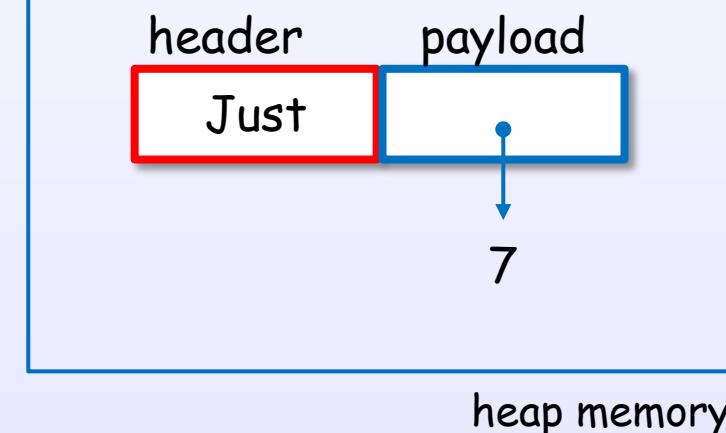
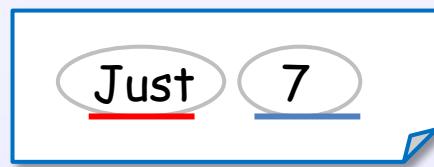
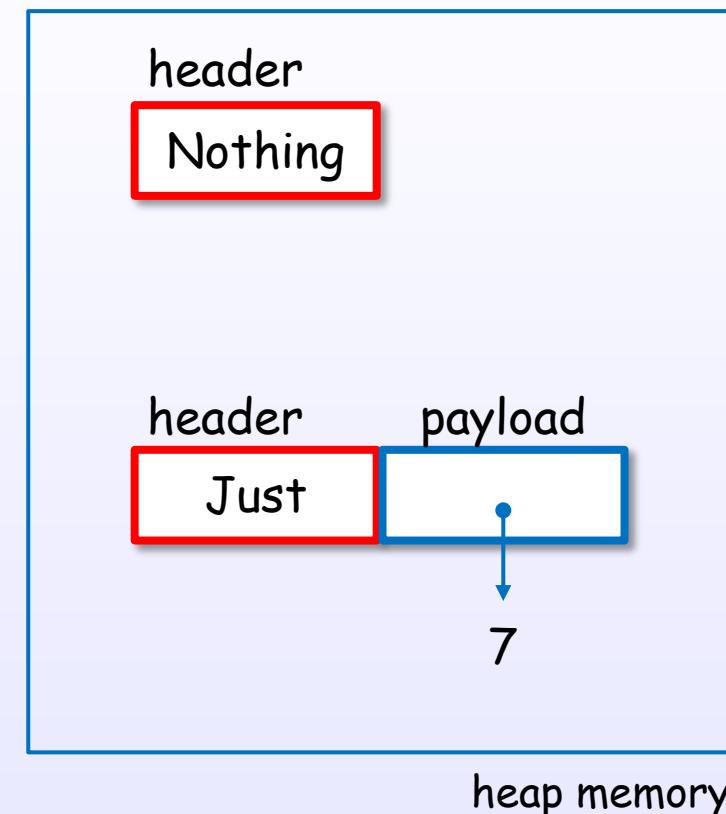
Constructors are defined by data declaration.

# Internal representation of Constructors for data values

Haskell code

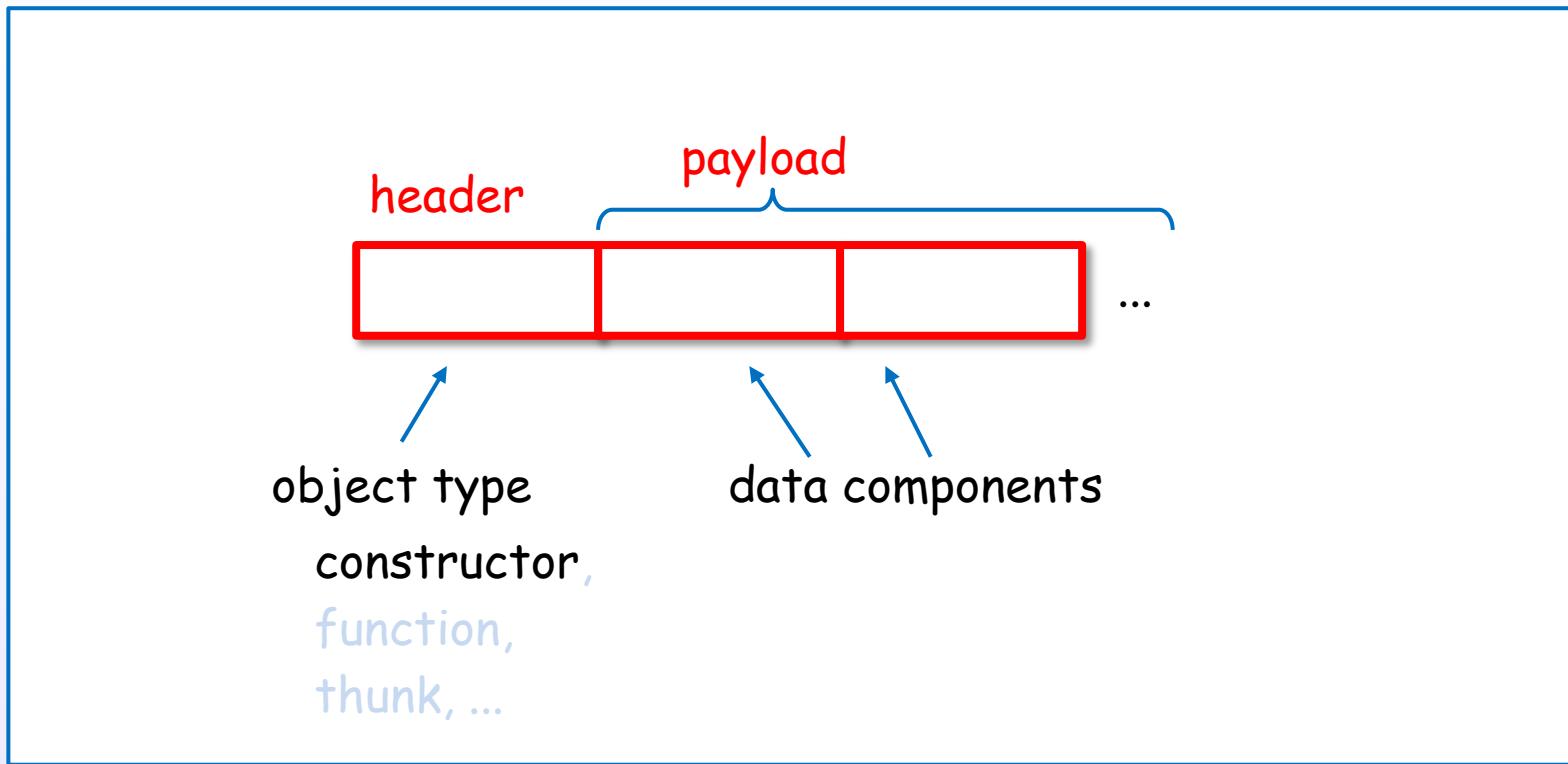


GHC's internal representation



# Constructors are represented uniformly

GHC's internal representation



A data value is represented with header(constructor) + payload(components).

# Representation of various constructors

Haskell code

```
data Bool = False
          | True
```

```
data Maybe a = Nothing
              | Just a
```

```
data Either a b = Left a
                  | Right b
```

GHC's internal representation

False

True

Nothing

Just

Left

Right

# Primitive data types are also represented with constructors

Haskell code

```
data Int = I# Int#
```

boxed integer

unboxed integer

```
data Char = C# Char#
```

I# 0#

I# 1#

⋮ 1 :: Int

C# 'a'#

C# 'b'#

⋮ 'a' :: Char

heap memory

# List is also represented with constructors

List

```
[ 1, 2, 3 ]
```

syntactic desugar

```
1 : ( 2 : ( 3 : [] ) )
```

prefix notation by section

```
(:) 1 ( (:) 2 ( (:) 3 [] ) )
```

equivalent data constructor

```
Cons 1 ( Cons 2 ( Cons 3 Nil ) )
```

constructor

# List is also represented with constructors

List

[ 1, 2, 3 ]

syntactic desugar

1 : ( 2 : ( 3 : [] ) )

prefix notation by section

(:) 1 ( (: ) 2 ( (: ) 3 [] ) )

equivalent data constructor

Cons 1 ( Cons 2 ( Cons 3 Nil ) )

type declaration

\* pseudo code

```
data List a = [] | a : (List a)
```

```
data List a = Nil | Cons a (List a)
```

# List is also represented with constructors

Haskell code

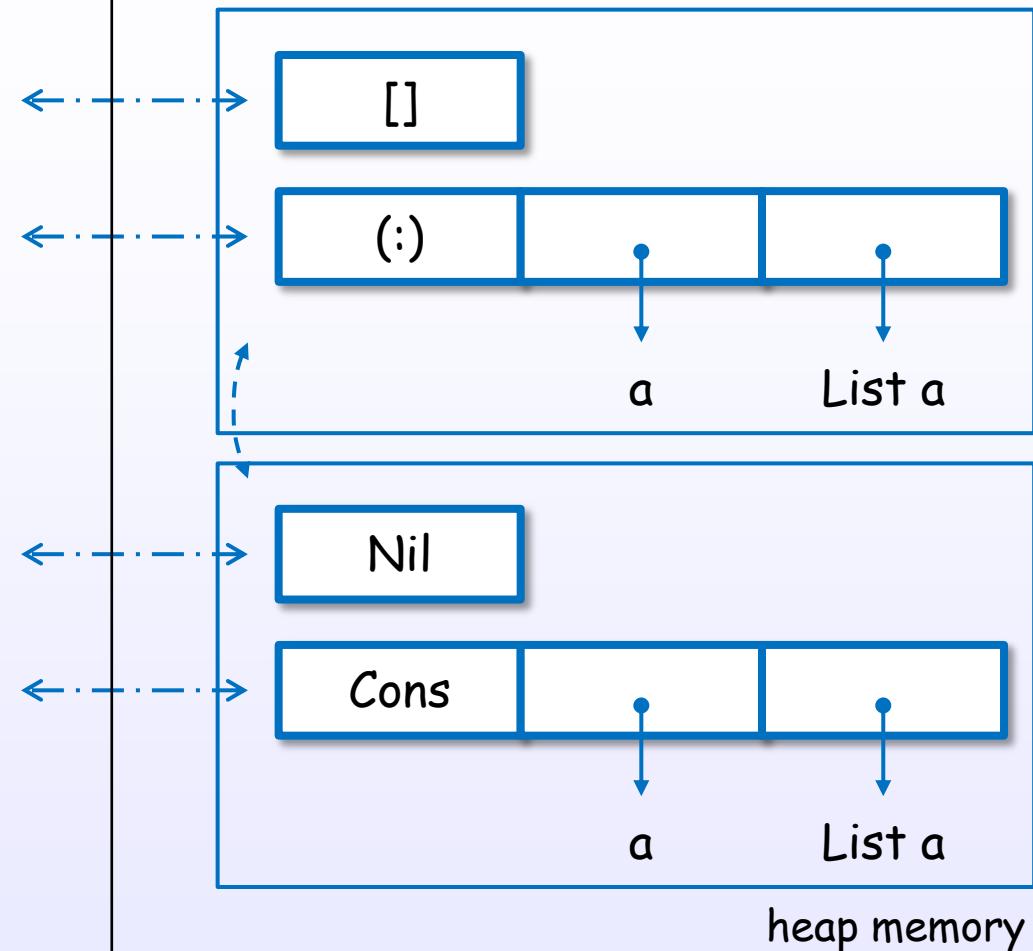
\* pseudo code

```
data List a = []  
| : a (List a)
```

equivalent data constructor

```
data List a = Nil  
| Cons a (List a)
```

GHC's internal representation



# List is also represented with constructors

Haskell code

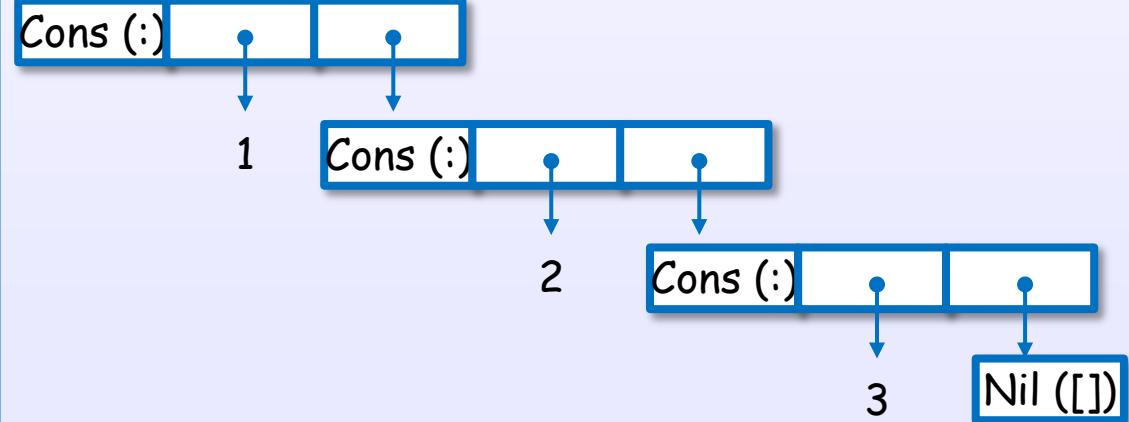
```
[ 1, 2, 3 ]
```

```
1 : ( 2 : ( 3 : [] ) )
```

```
(:) 1 ( (:) 2 ( (:) 3 [] ) )
```

```
Cons 1 ( Cons 2 ( Cons 3 Nil ) )
```

GHC's internal representation



# Tuple is also represented with constructor

Tuple (Pair)

( 7, 8 )

prefix notation by section

(.) 7 8

equivalent data constructor

Pair 7 8

constructor

type declaration

\* pseudo code

data Pair a = (.) a a

data Pair a = Pair a a

# Tuple is also represented with constructor

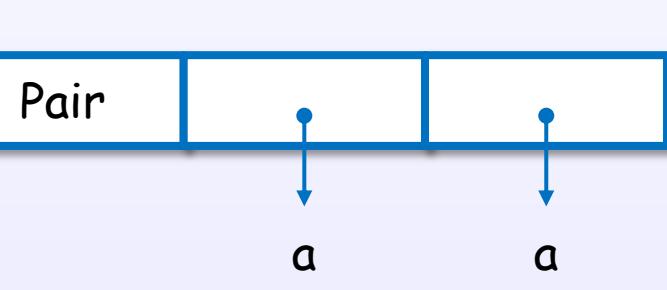
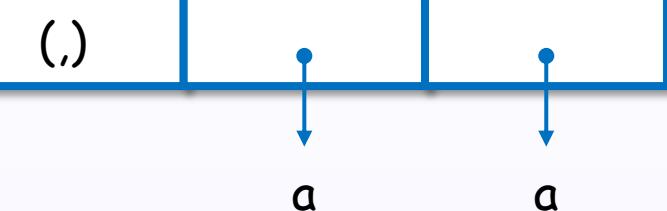
Haskell code

```
data Pair a = (,) a a
```

equivalent data constructor

```
data Pair a = Pair a a
```

GHC's internal representation



heap memory

# Tuple is also represented with constructor

Haskell code

( 7, 8 )

(.,) 7 8

Pair 7 8

GHC's internal representation

Pair (.)

7

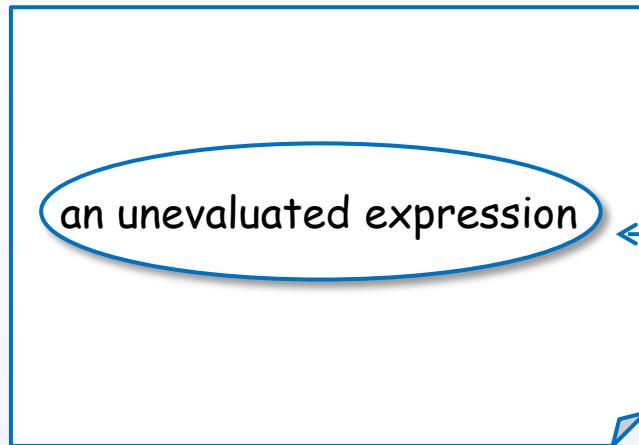
8

### 3. Internal representation of expressions

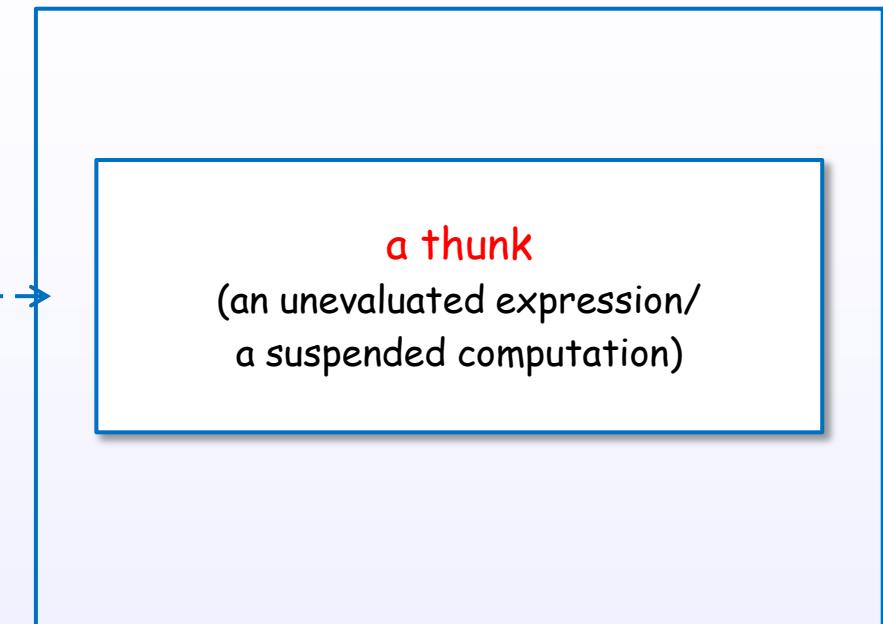
Thunk

# Thunk

Haskell code



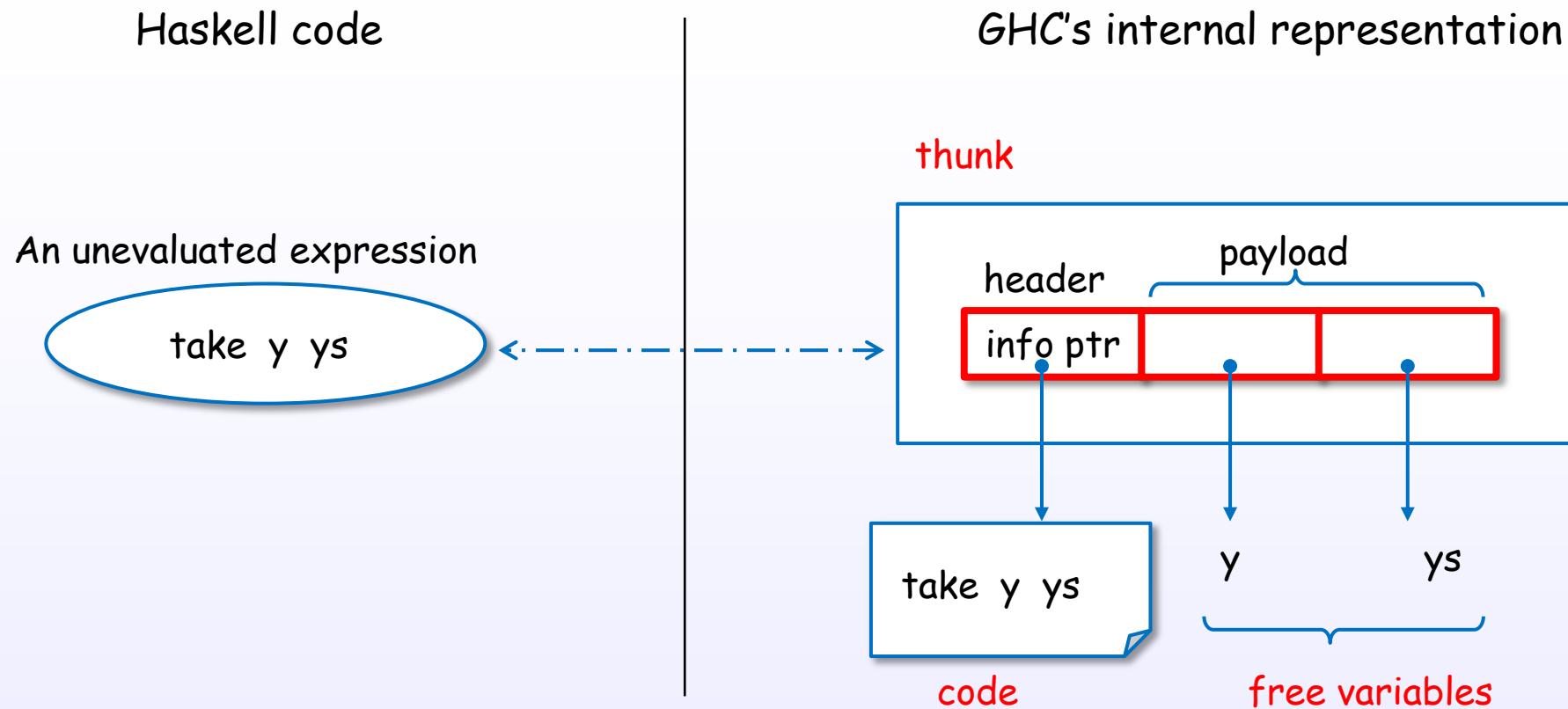
GHC's internal representation



heap memory

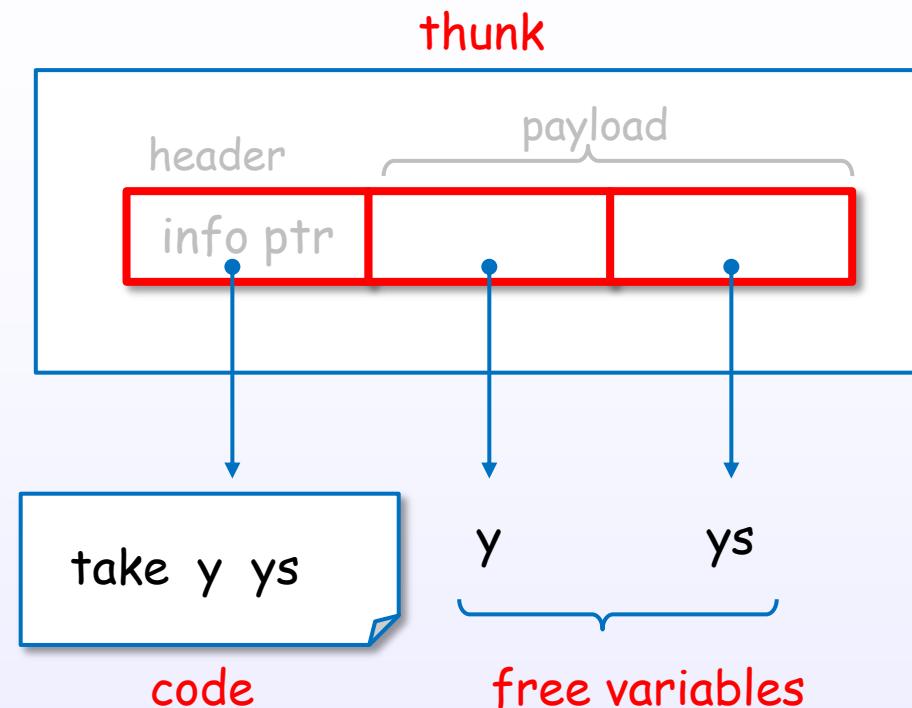
A thunk is an **unevaluated** expression in heap memory.  
A thunk is built to **postpone** the evaluation.

# Internal representation of thunk



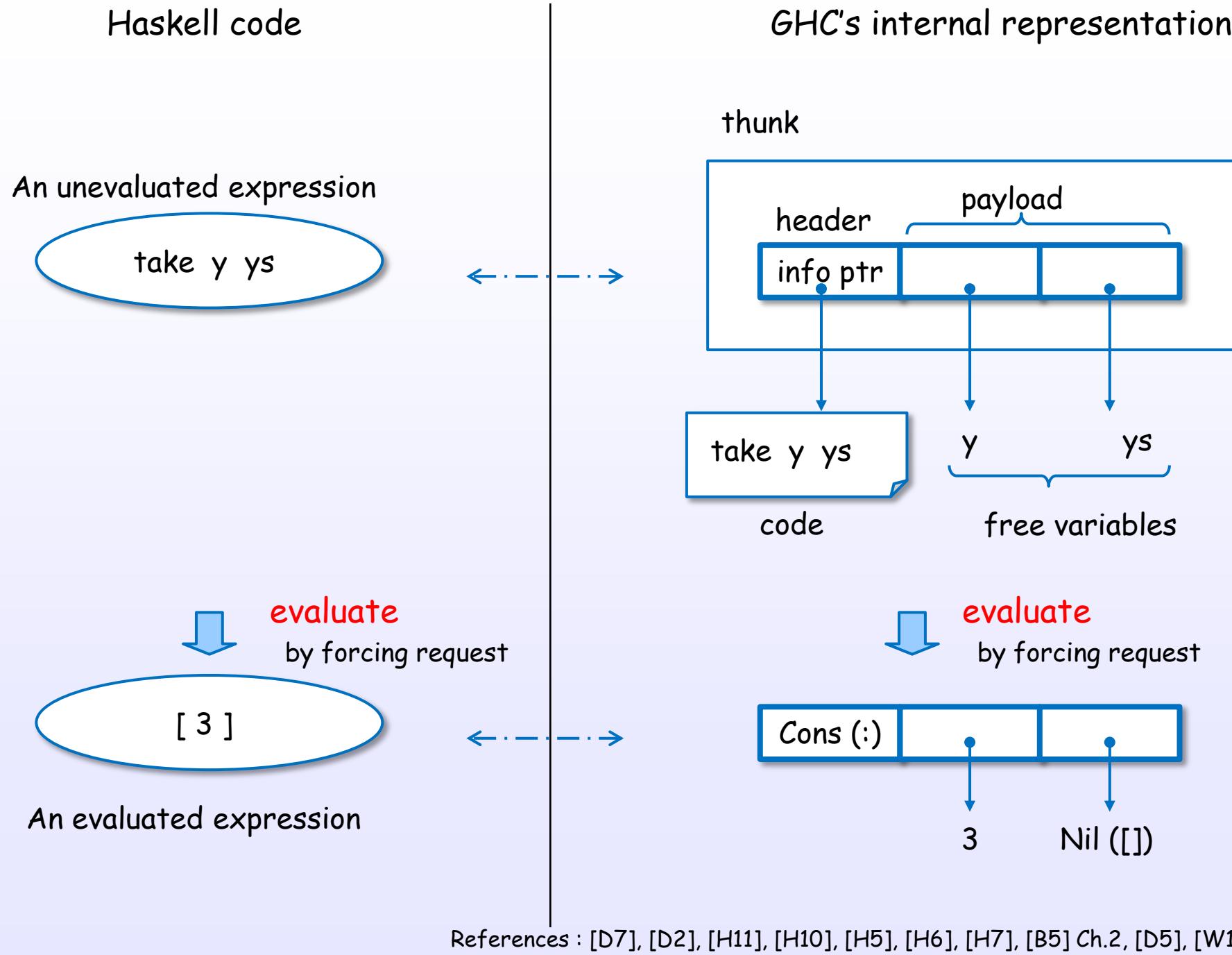
A thunk is represented with header(code) + payload(free variables).

# A thunk is a package



A thunk is a package of code + free variables.

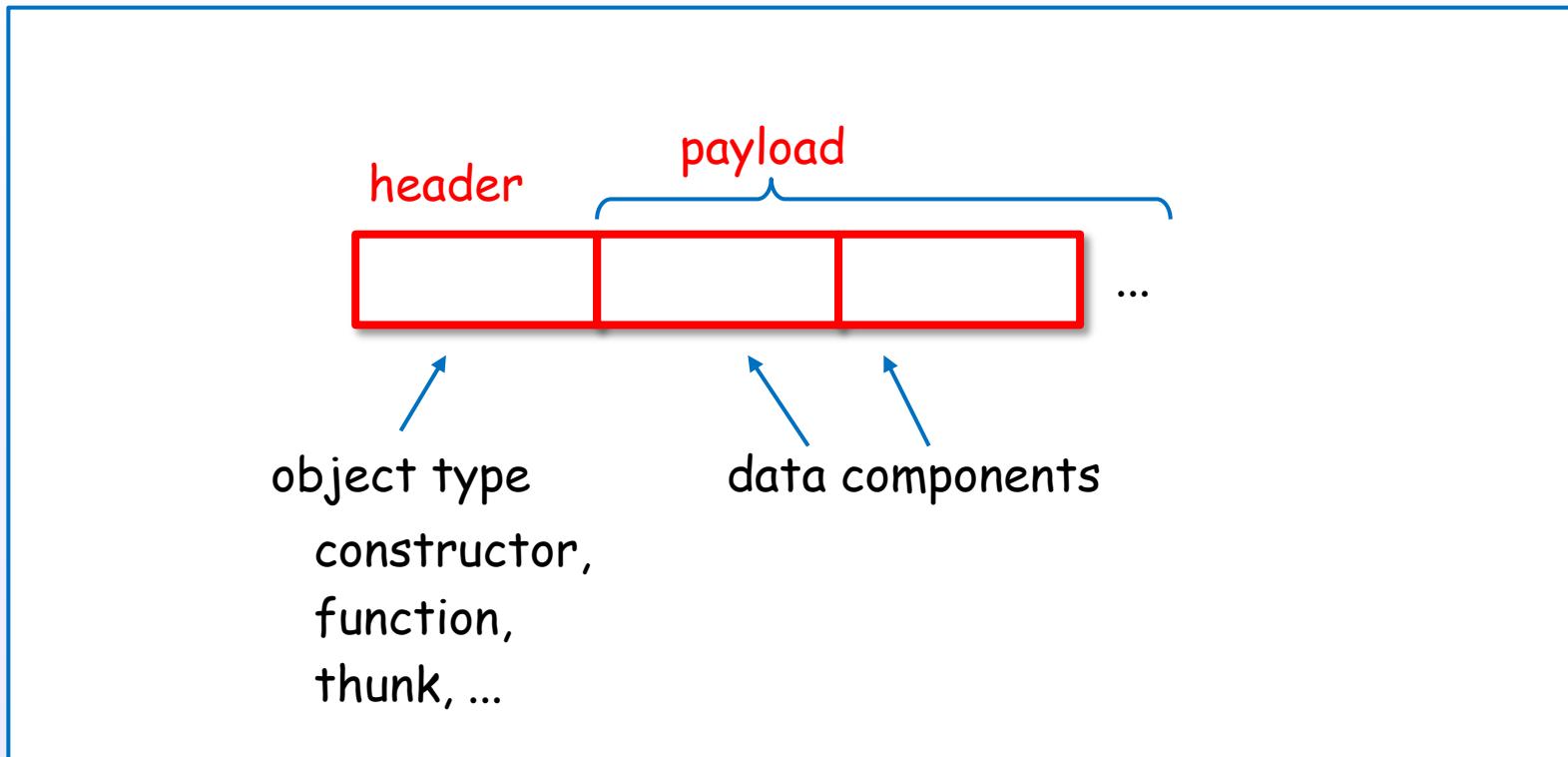
# A thunk is evaluated by forcing request



### 3. Internal representation of expressions

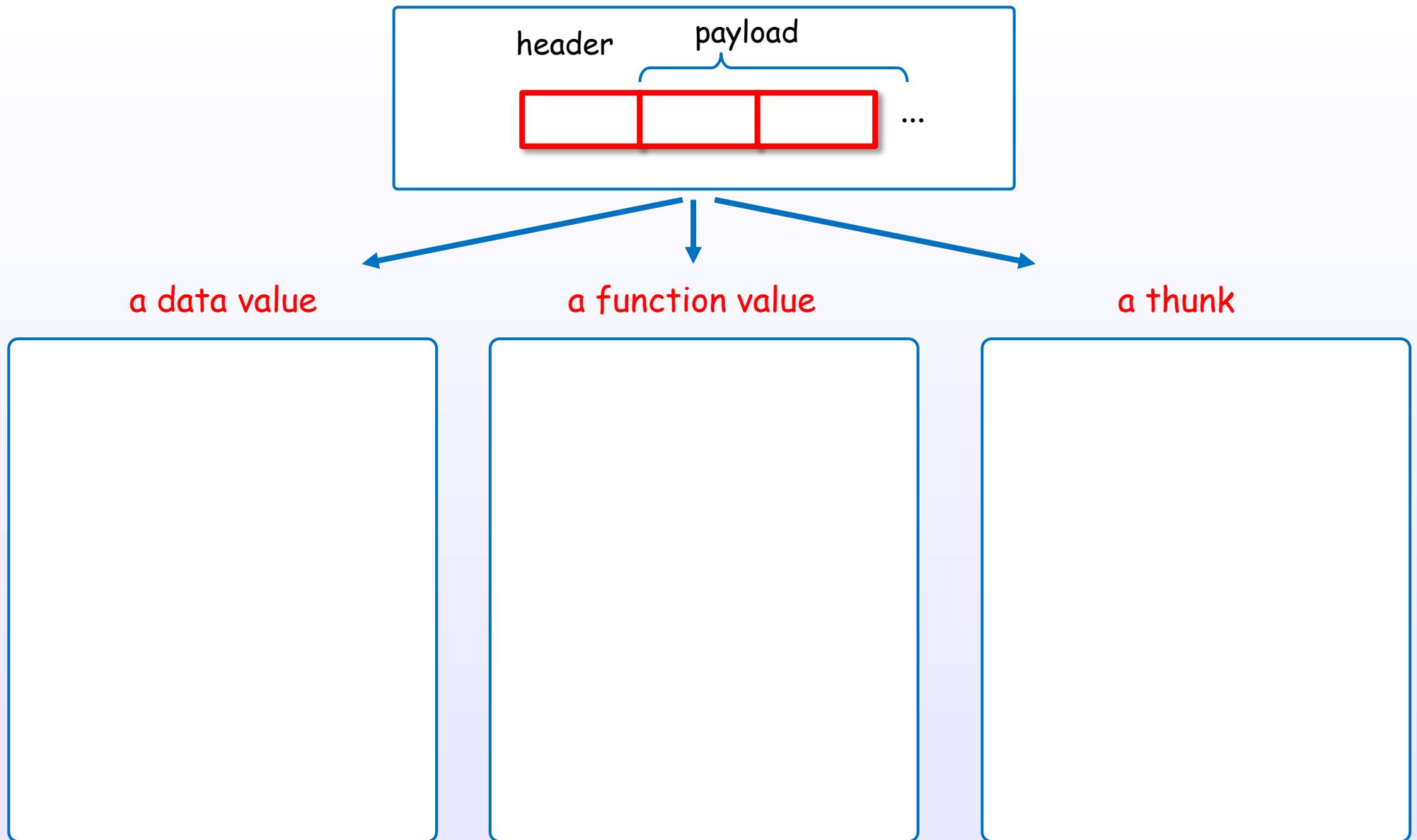
Uniform representation

# Every object is uniformly represented in memory

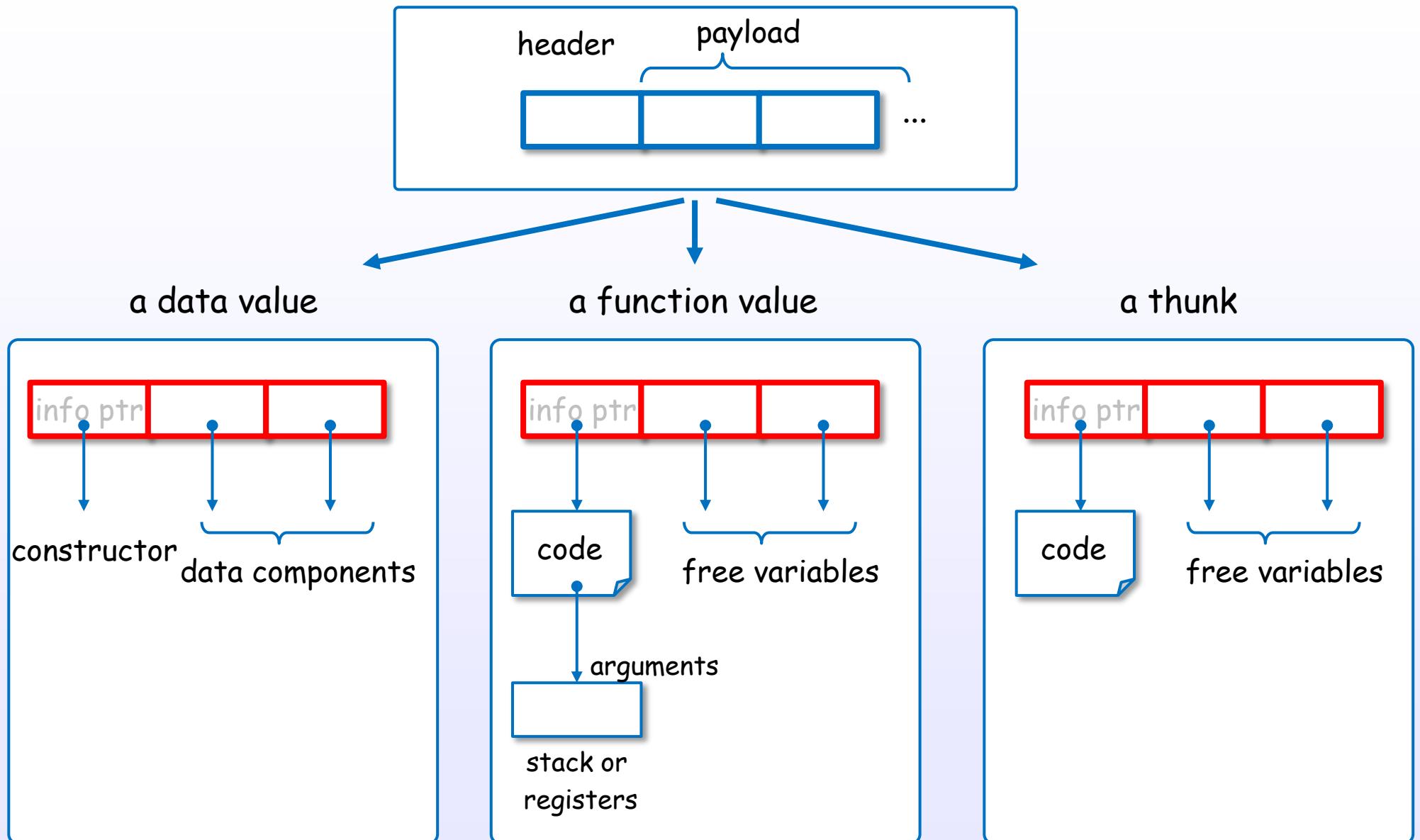


in heap memory, stack or static memory

# Every object is uniformly represented in memory



# Every object is uniformly represented in memory



\* At exactly, a thunk object has a reserved field in second.

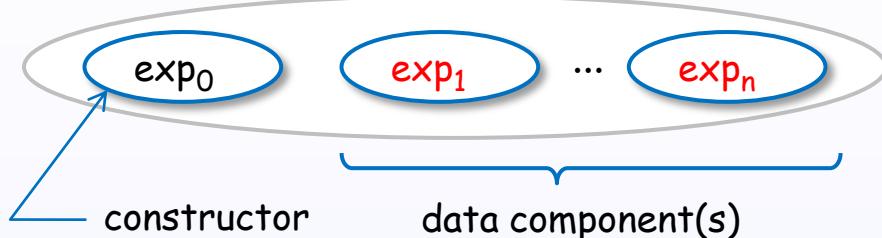
### 3. Internal representation of expressions

WHNF

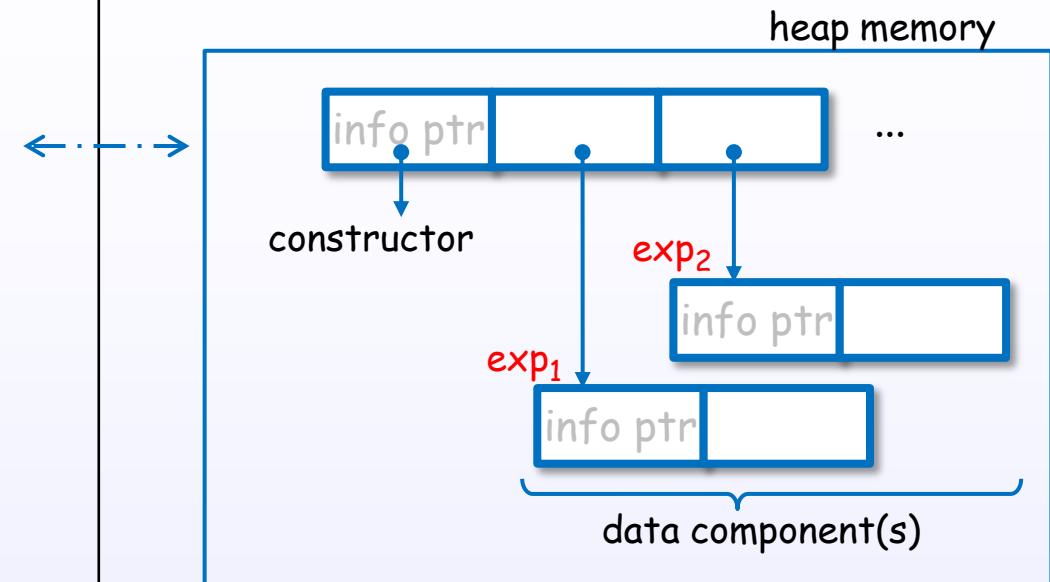
# Internal representation of WHNF

## Haskell code

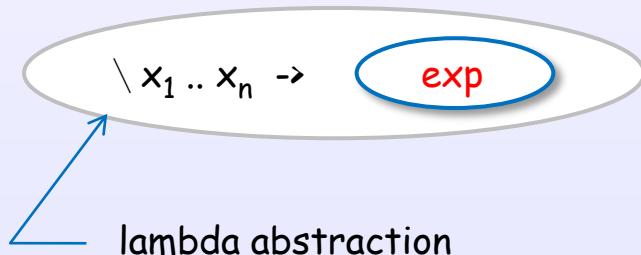
### a data value in WHNF



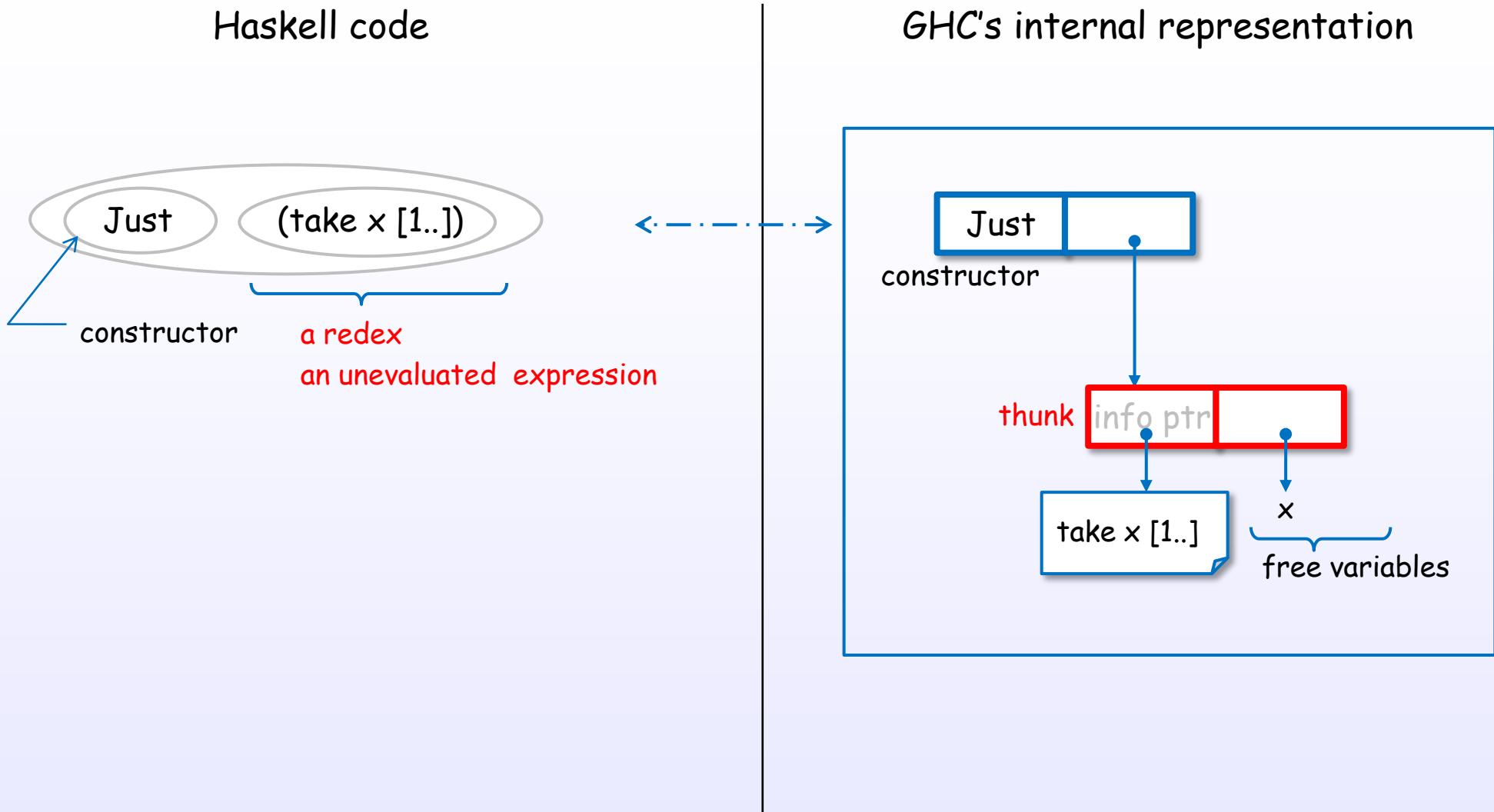
## GHC's internal representation



### a function value in WHNF

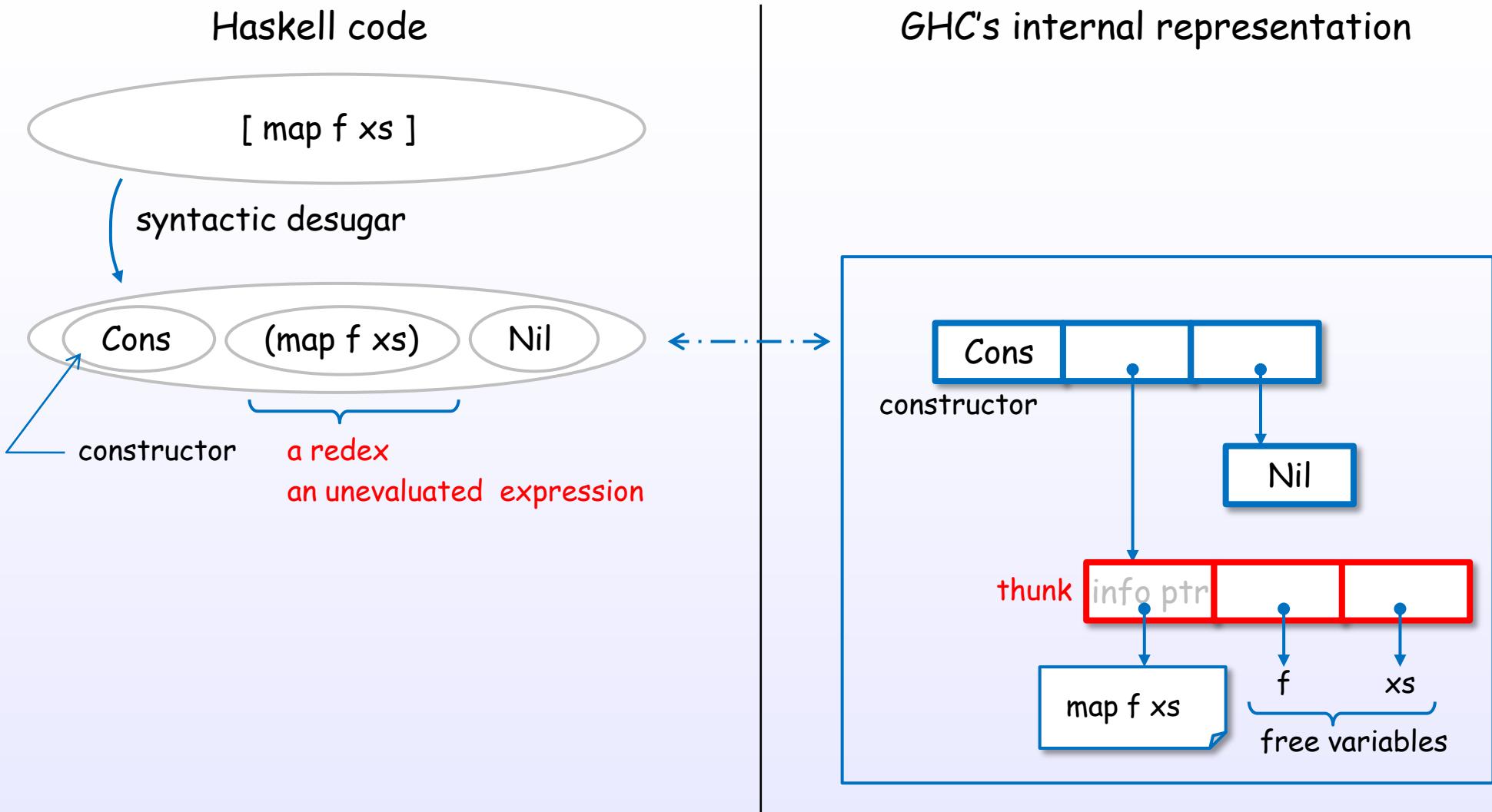


# Example of WHNF for a data value



Constructors can contain unevaluated expressions by thunks.  
Haskell's constructors are lazy constructors.

# Example of WHNF for a data value



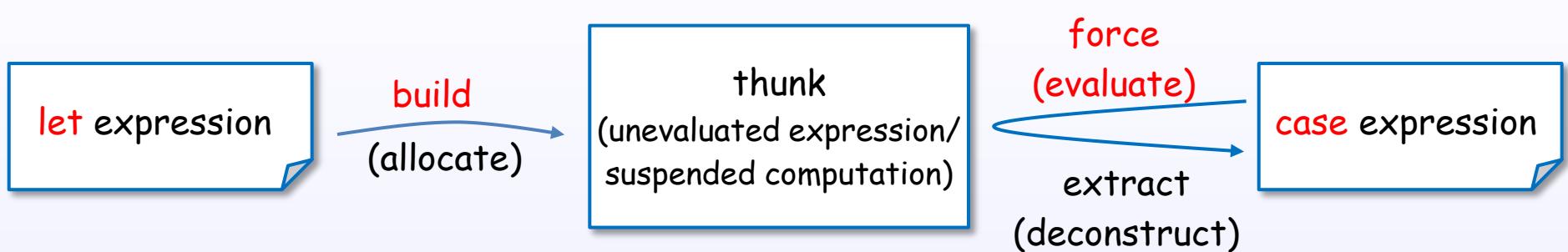
### 3. Internal representation of expressions

let, case expression

## let, case expression

let and case expressions are special role in the evaluation

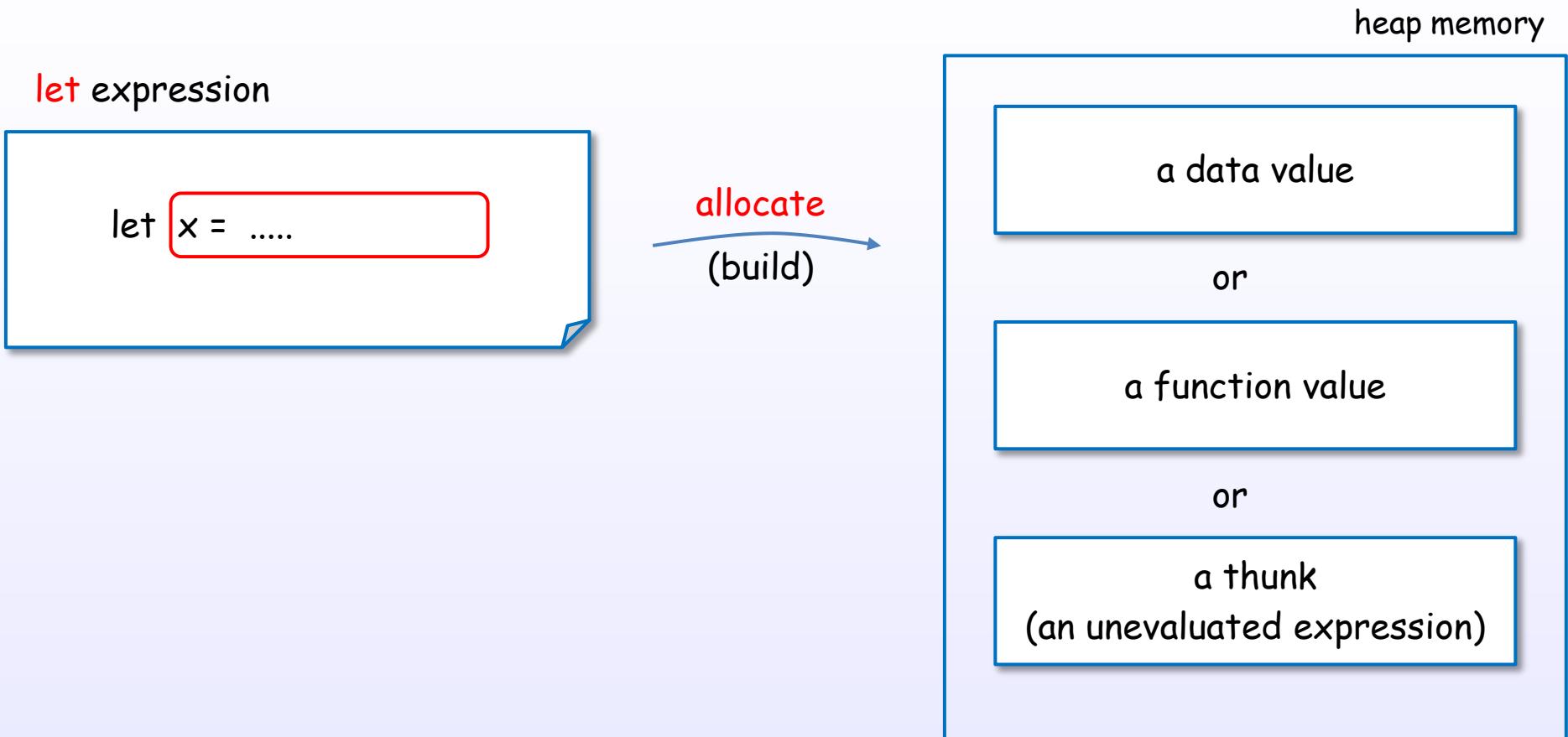
# let/case expressions and thunk



A let expression may build a thunk.

A case expression evaluates (forces) and deconstructs the thunk.

# A let expression may allocates a heap object



A let expression may allocates an object in the heap.  
(If GHC can optimize it, the let expression may not allocate.)

\* At exactly, STG language's let expression rather than Haskell's let expression

# Example of let expressions

## Haskell code

```
let x = Just 5
```

allocate

```
let x = \y -> y + z
```

allocate

```
let x = take y ys
```

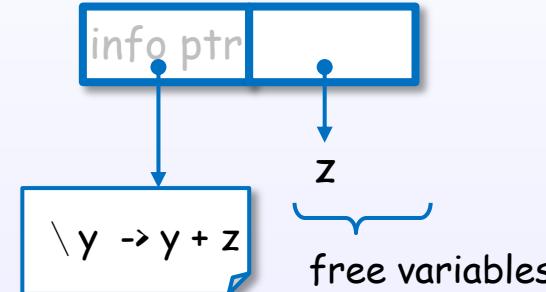
allocate  
(build)

## GHC's internal representation

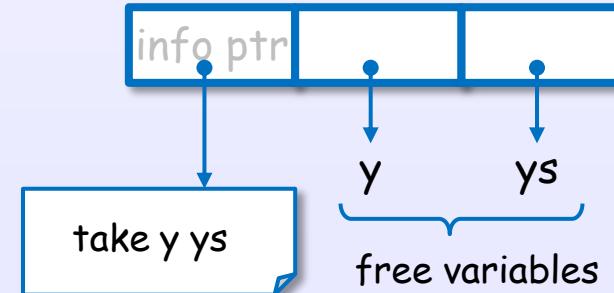
a data value



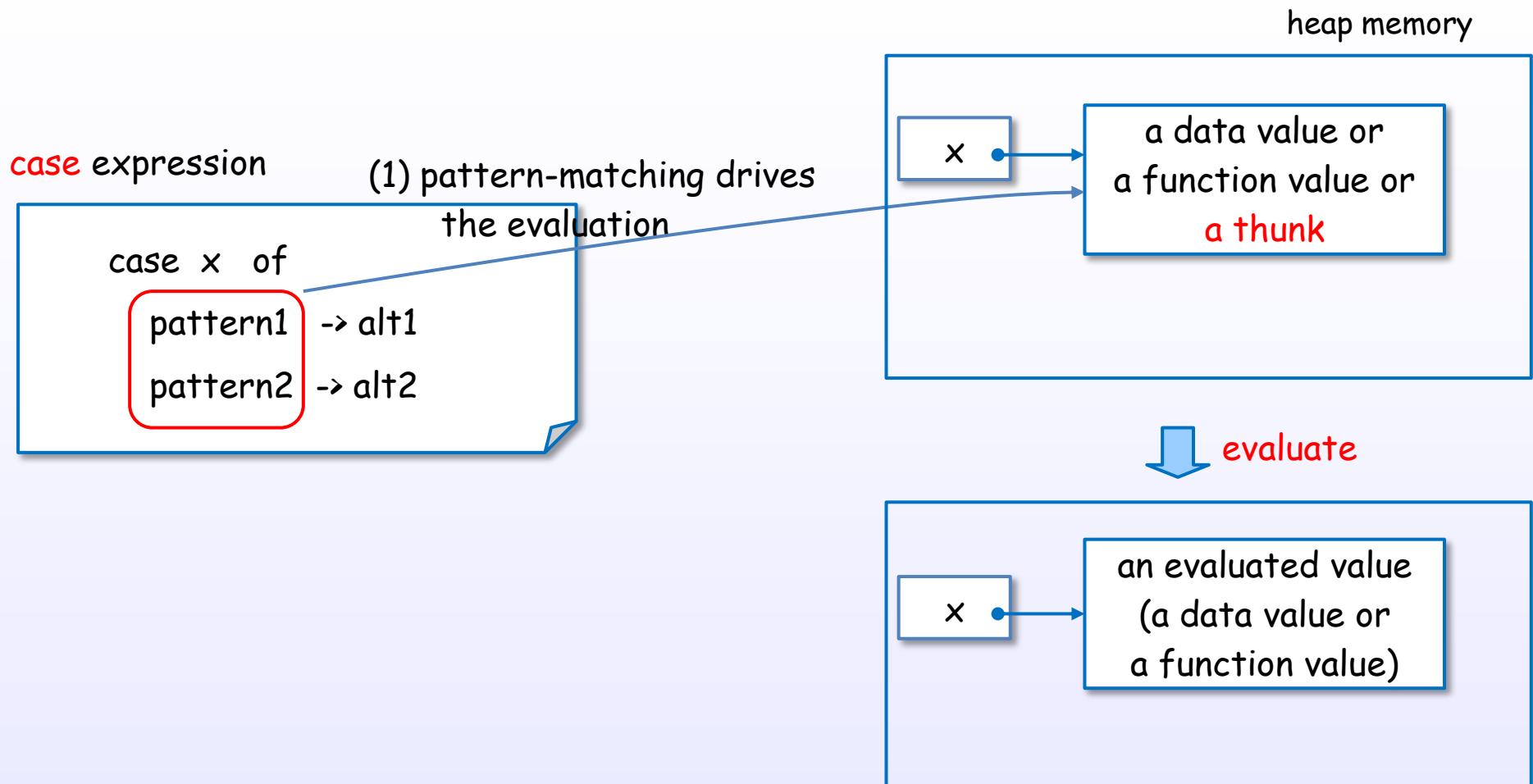
a function value



a thunk



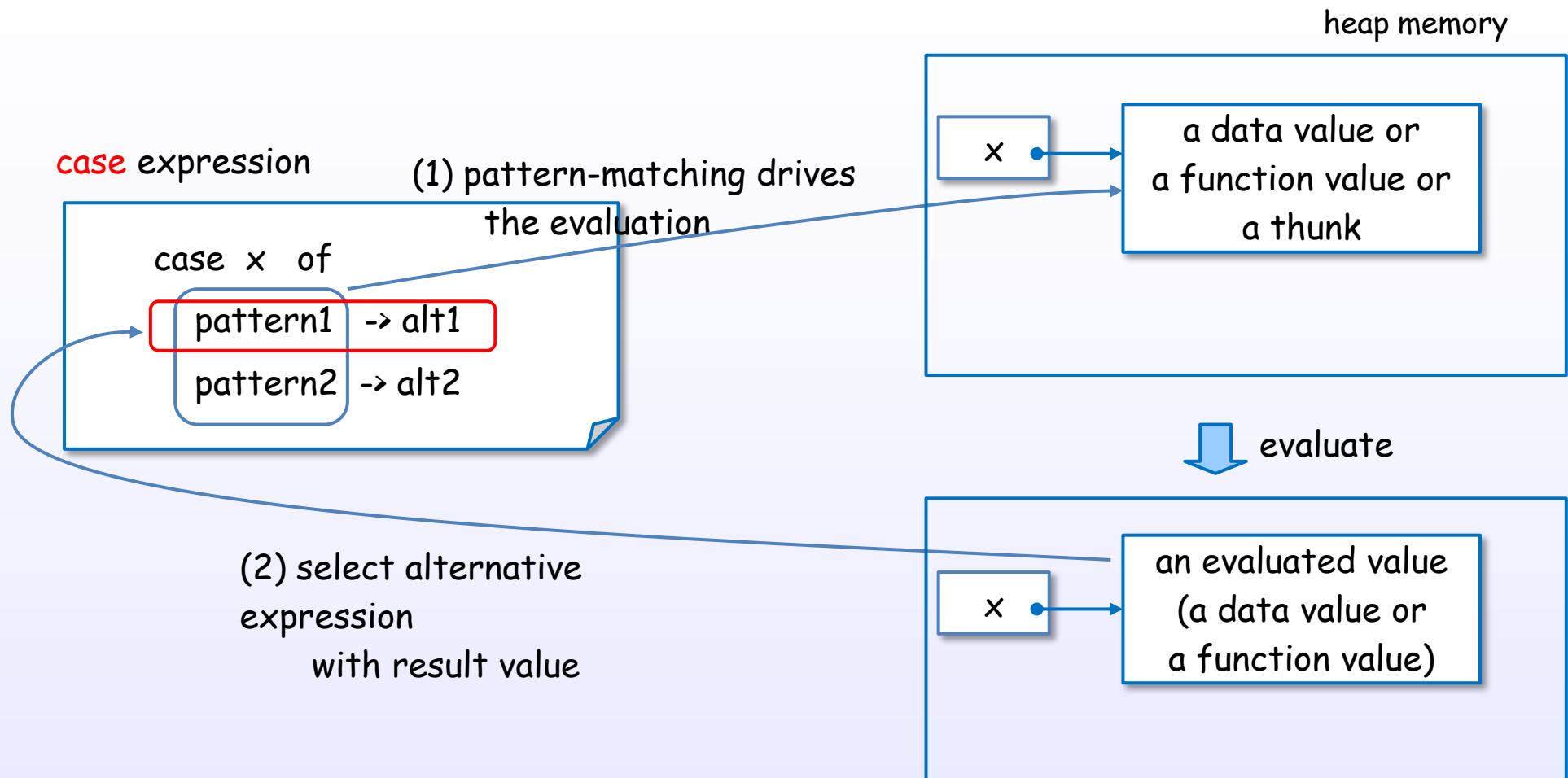
# A case expression evaluates a subexpression



Pattern-matching drives the evaluation.

\* At exactly, STG language's case expression rather than Haskell's case expression

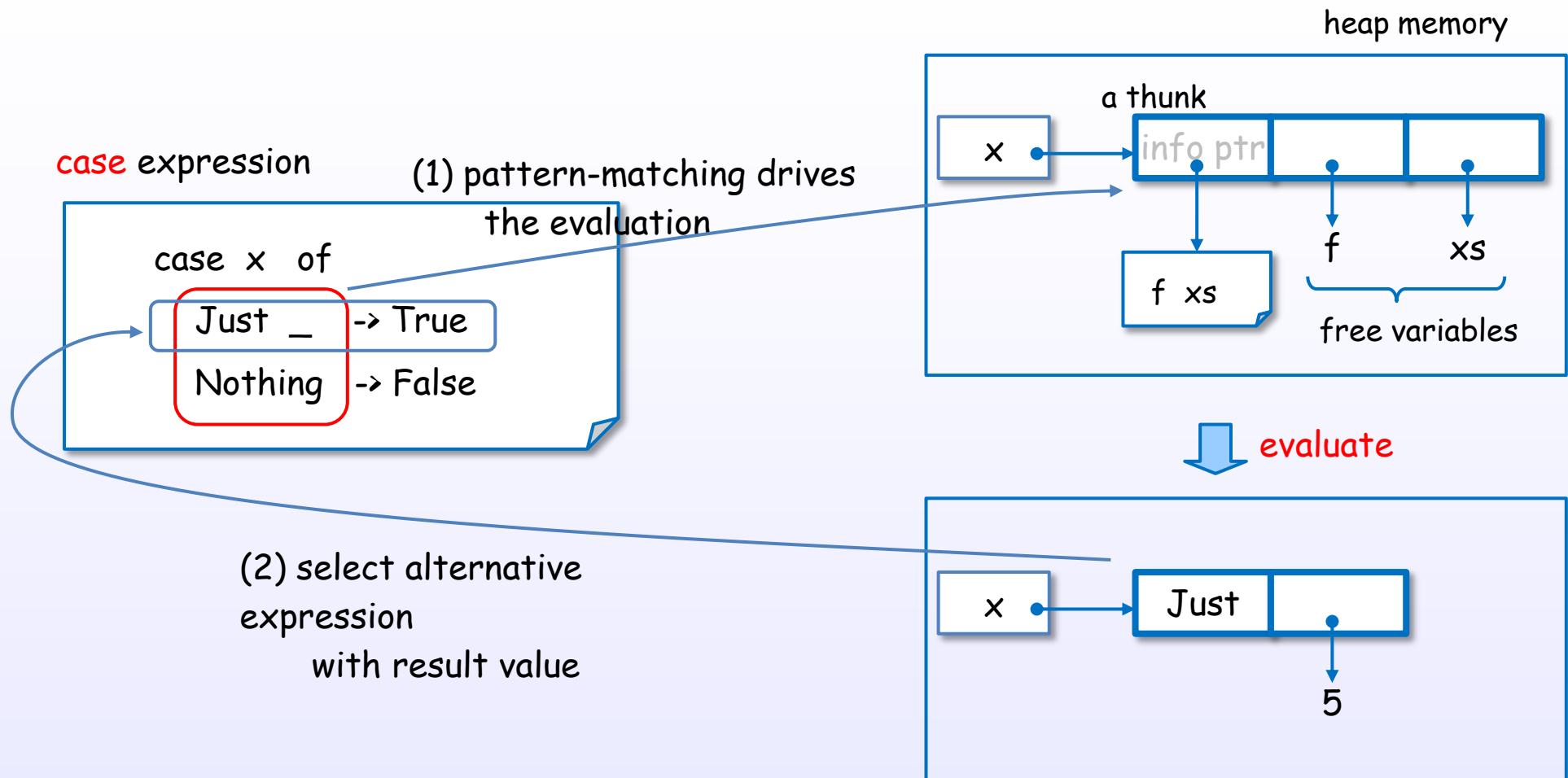
# A case expression also performs case analysis



A case expression evaluates a subexpression and optionally performs case analysis on its value.

\* At exactly, STG language's case expression rather than Haskell's case expression

# Example of a case expression



A case expression's pattern-matching says "I **need** the value".

# Pattern-matching in function definition

pattern-matching in **function definition**

$$\begin{aligned} f \text{ } \boxed{\text{Just } \_} &= \text{True} \\ f \text{ } \boxed{\text{Nothing}} &= \text{False} \end{aligned}$$

pattern-matching in **case expression**

$$\begin{aligned} f \text{ } x = \text{case } x \text{ of} \\ \boxed{\text{Just } \_} &\rightarrow \text{True} \\ \boxed{\text{Nothing}} &\rightarrow \text{False} \end{aligned}$$

syntactic desugar

A function's pattern-matching is syntactic sugar of case expression.

A function's pattern-matching also drives the evaluation.

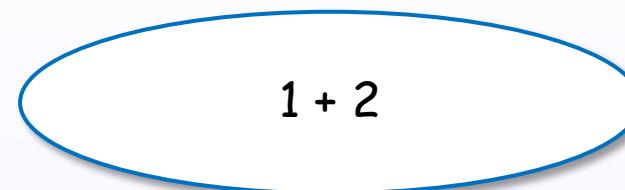
## 4. Evaluation

## 4. Evaluation

Evaluation strategies

# Evaluation

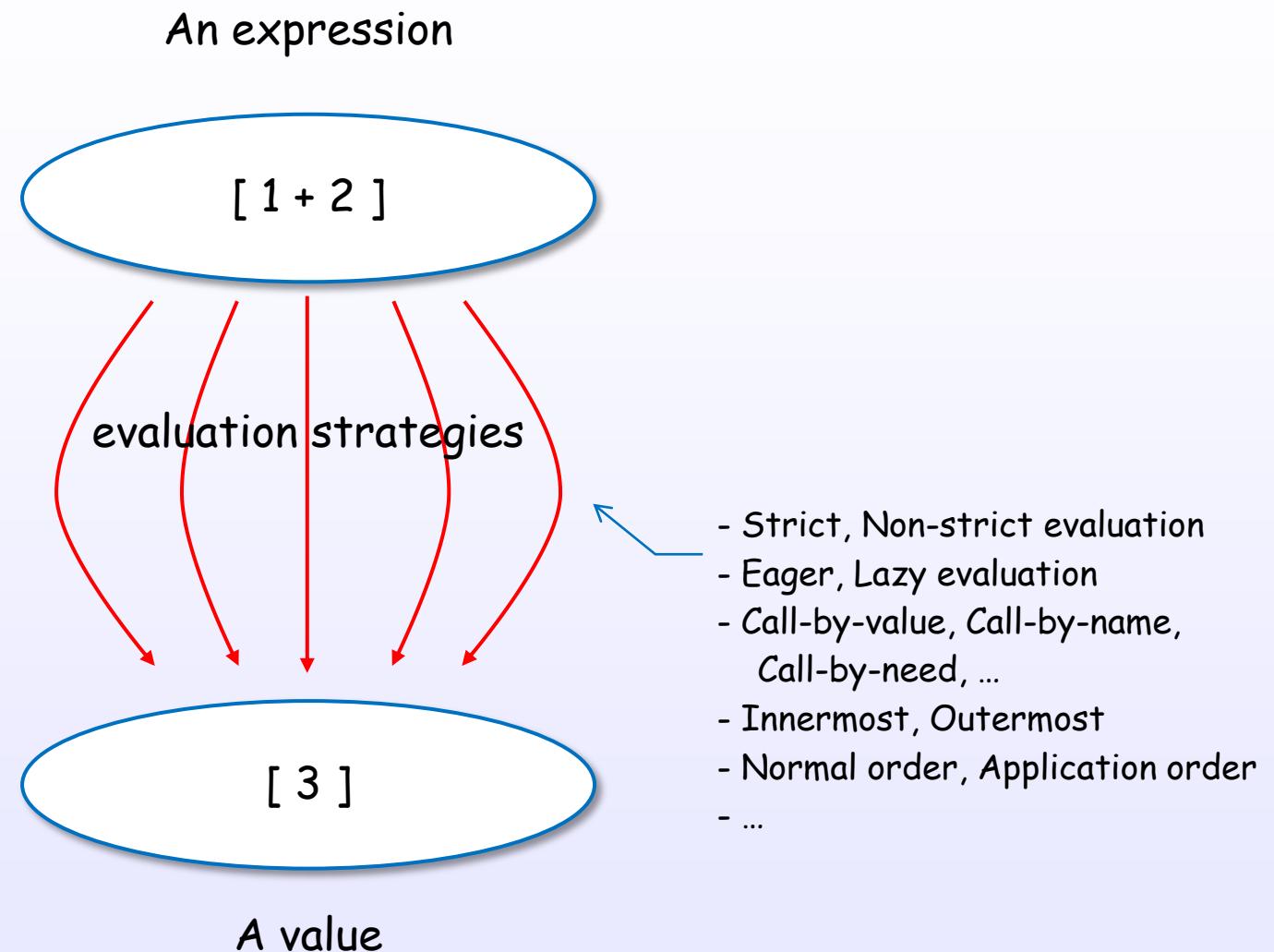
An expression



A value

The evaluation produces a value from an expression.

# There are many evaluation approaches



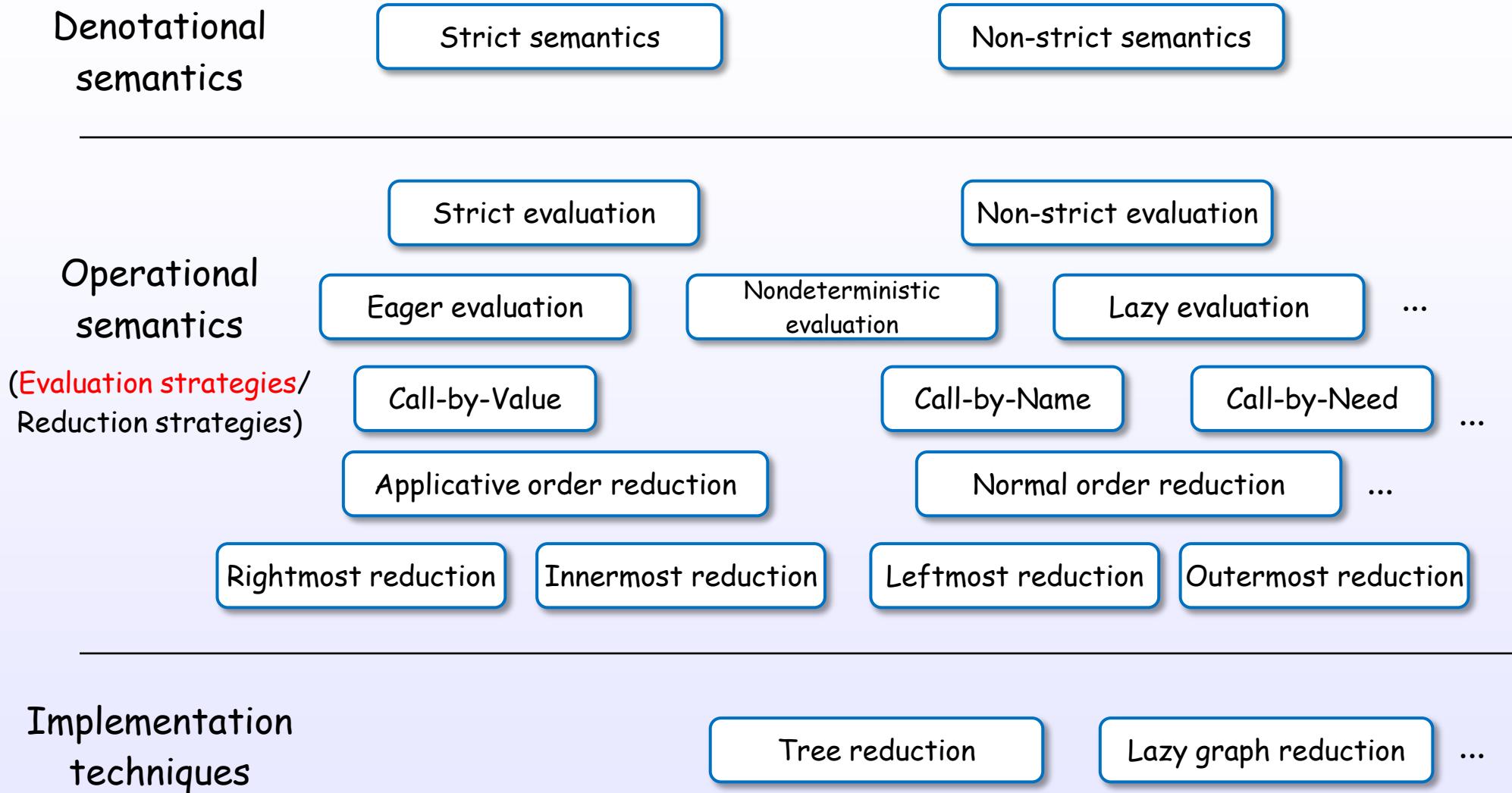
## Evaluation concept layer

Denotational semantics

Operational semantics  
(Evaluation strategies / Reduction strategies)

Implementation techniques

# Evaluation layer for GHC's Haskell



# Evaluation layer for GHC's Haskell

Denotational semantics

Strict semantics

Non-strict semantics

Operational semantics  
(Evaluation strategies/  
Reduction strategies)

Strict evaluation

Non-strict evaluation

Eager evaluation

Nondeterministic evaluation

Lazy evaluation

Call-by-Value

Call-by-Name

Call-by-Need

Applicative order reduction

Normal order reduction

Rightmost reduction

Innermost reduction

Leftmost reduction

Outermost reduction

Implementation techniques

Tree reduction

Lazy graph reduction

Haskell 2010 specification

GHC's strategy

GHC's strategy

...

GHC's strategy

GHC's strategy

...

GHC's strategy

...

GHC's implementation

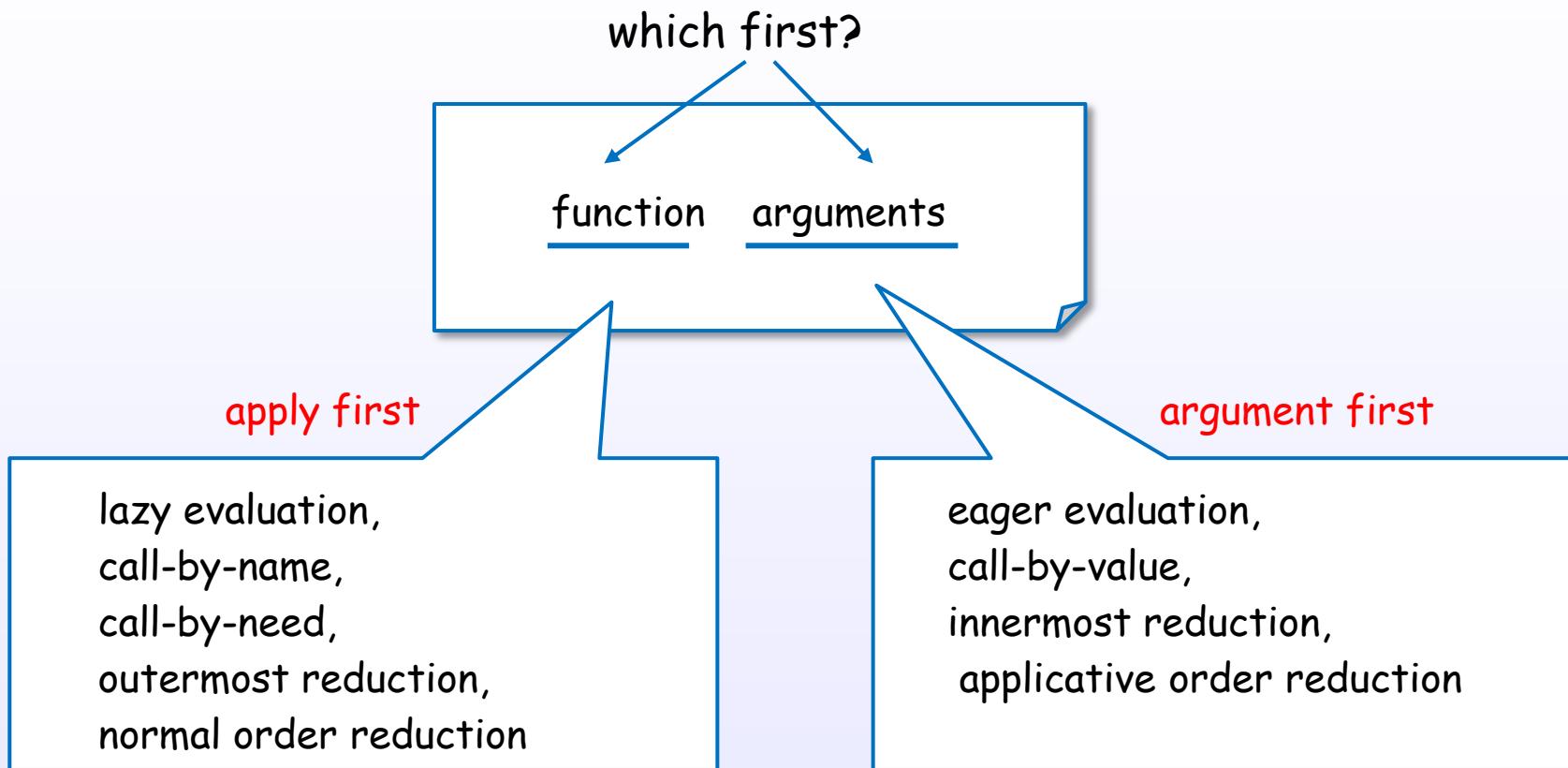
...

## Evaluation strategies

Each evaluation strategy decides how to operate the evaluation about

- ordering,
- region,
- trigger condition,
- termination condition,
- re-evaluation, ...

# One of the important points is the order



# Simple example of typical evaluations

call-by-value

default

C, Java, JavaScript,  
Python, OCaml, Scheme, ...

square ( 1 + 2 )

argument  
evaluation  
first



call-by-need

default

Haskell (GHC), ...

square ( 1 + 2 )

apply  
first



# Simple example of typical evaluations

## call-by-value

square ( 1 + 2 )



square ( 3 )



3 \* 3



9

## call-by-need

square ( 1 + 2 )



( 1 + 2 ) \* ( 1 + 2 )



( 3 ) \* ( 3 )



9

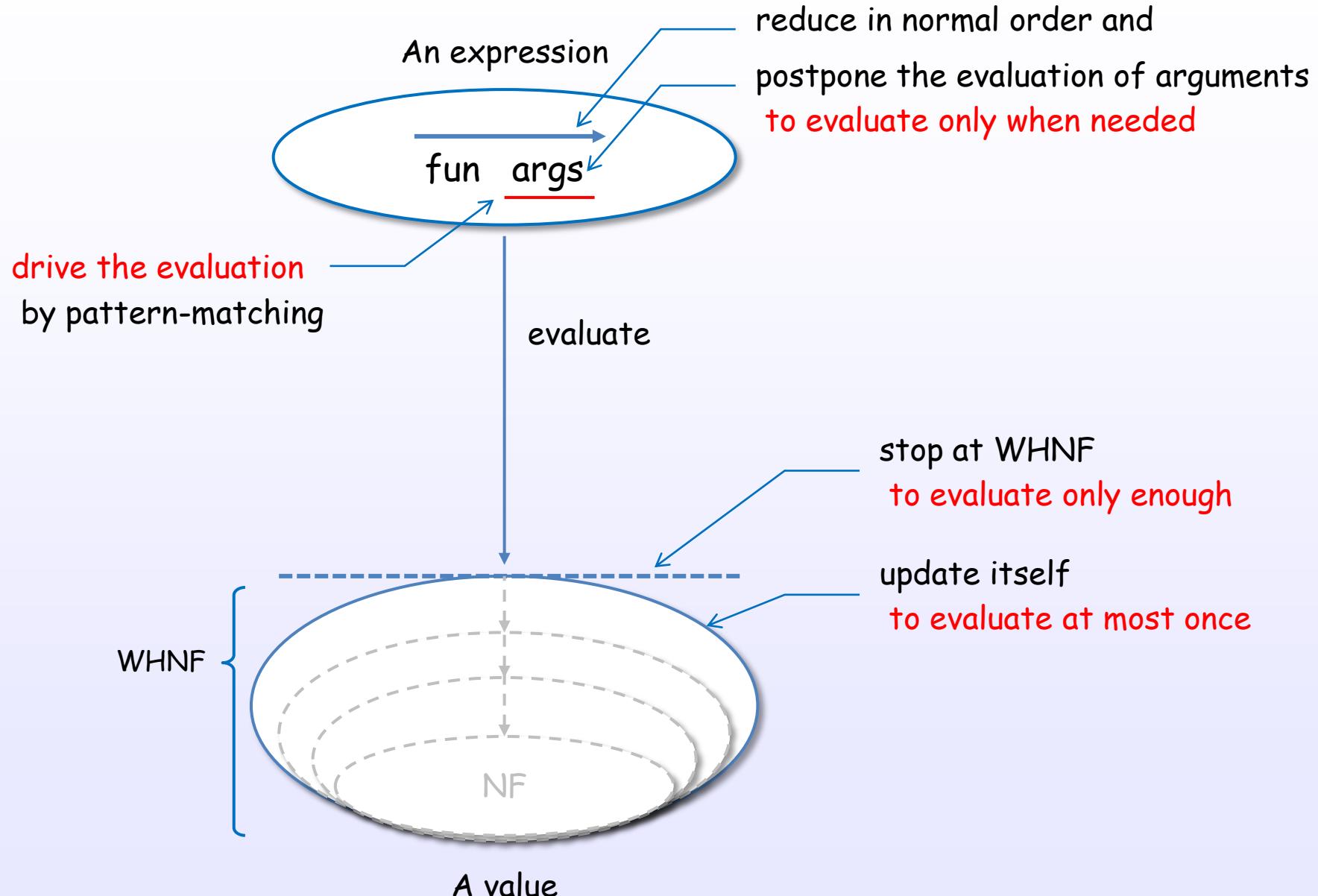
evaluation is  
performed

evaluation is  
delayed !

## 4. Evaluation

Evaluation in Haskell (GHC)

# Key concepts of Haskell's lazy evaluation



# Postpone the evaluation of arguments

Haskell code

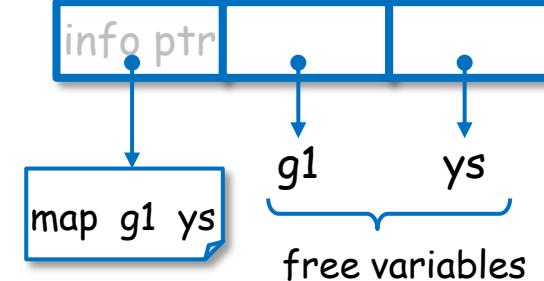
```
fun (map g1 ys)
```

internal translation

```
let thunk0 = map g1 ys
in fun thunk0
```

postpone  
(build)

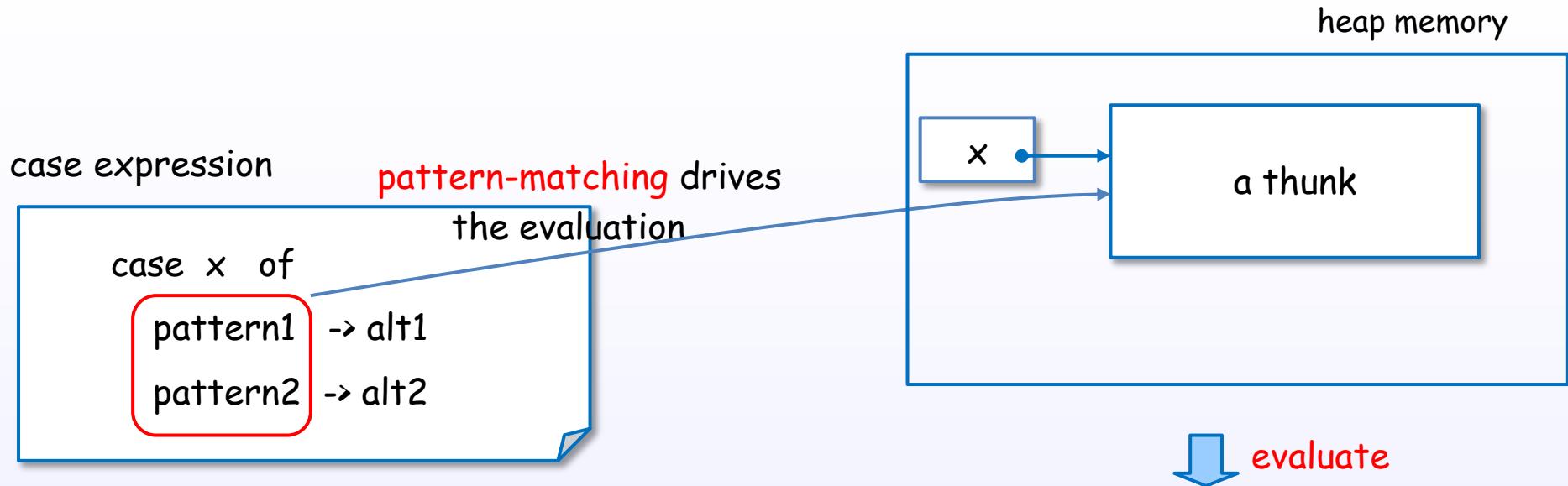
a thunk



heap memory

postpone the evaluation by a thunk which build with let expression  
(When GHC can optimize it by analysis, the thunk may not be build.)

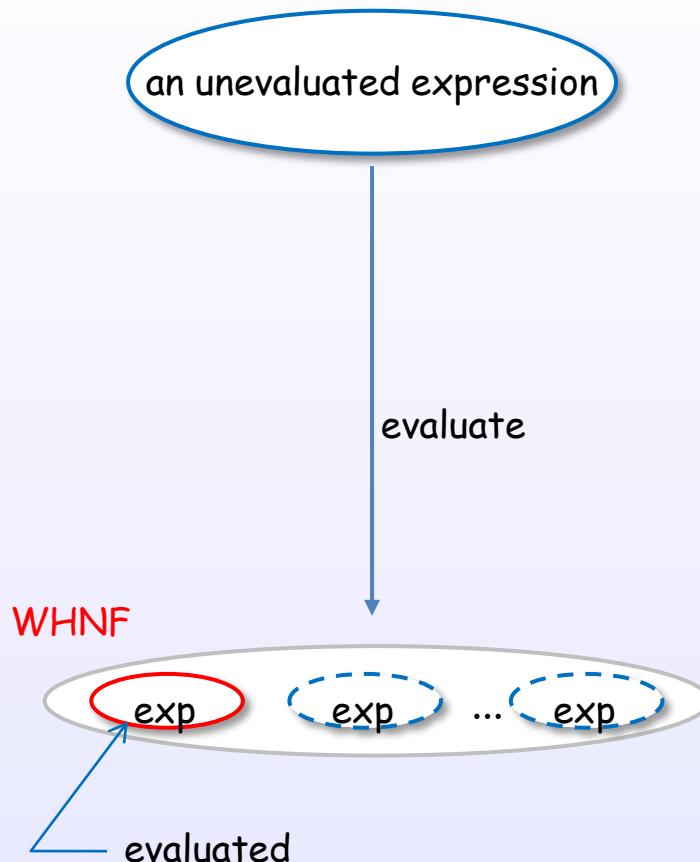
# Pattern-matching drives the evaluation



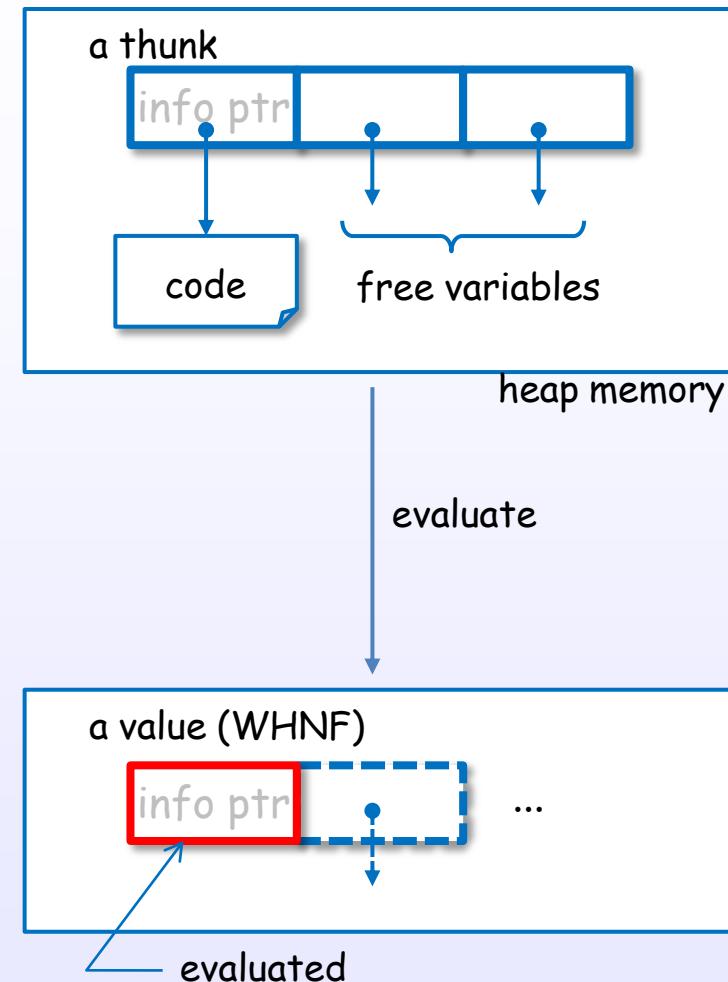
drive the evaluation by pattern-matching

# Stop at WHNF

Haskell code



GHC's internal representation

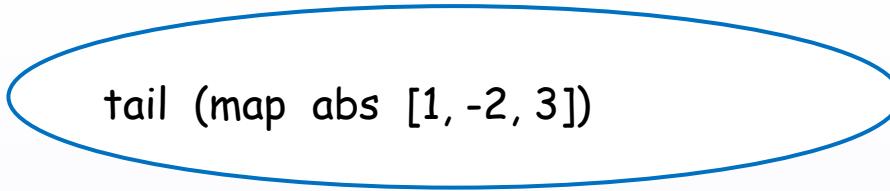


stop the evaluation at WHNF

## 4. Evaluation

Examples of evaluation steps

## (1) Example of GHC's evaluation

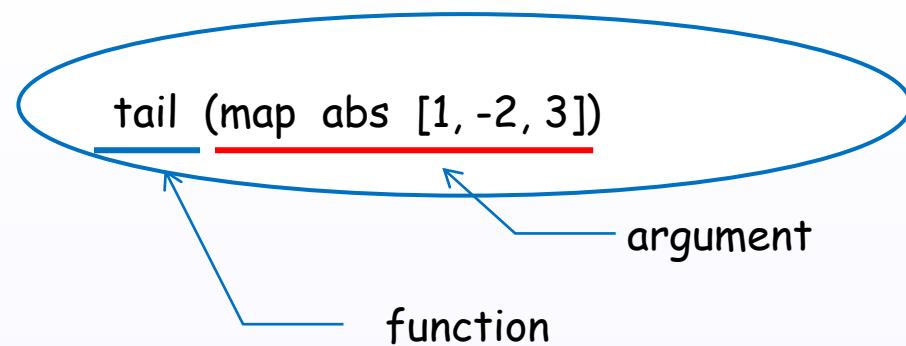


```
tail (map abs [1, -2, 3])
```

Let's evaluate. It's time to magic!

\* no optimizing case (without -O)

## (2) How to postpone the evaluation of arguments?



### (3) GHC internally translates the expression

tail (map abs [1, -2, 3])

*internal translation*

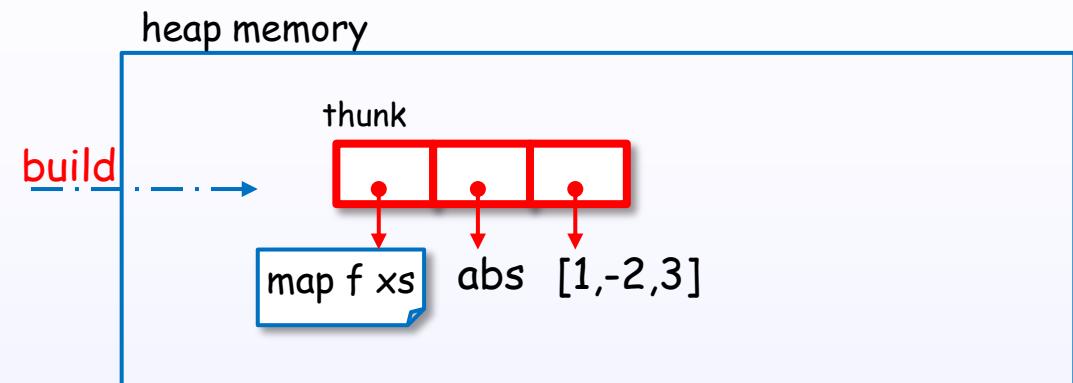
let thunk0 = map abs [1, -2, 3]  
in tail thunk0

## (4) a let expression builds a thunk

tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]  
in tail thunk0



## (5) function apply to argument

tail (map abs [1, -2, 3])

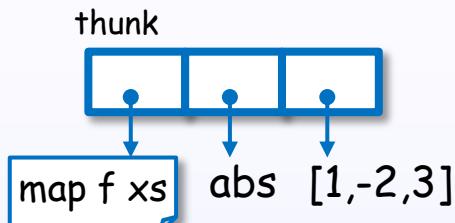
internal translation

let thunk0 = map abs [1, -2, 3]

in tail thunk0

apply

heap memory



## (6) tail is defined here

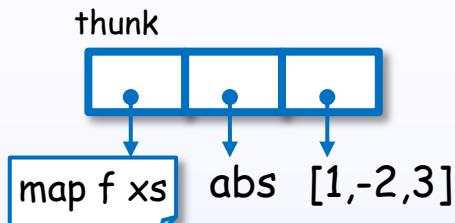
tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]  
in tail thunk0

tail (\_:xs) = xs *definition*

heap memory



# (7) function is syntactic sugar

tail (map abs [1, -2, 3])

internal translation

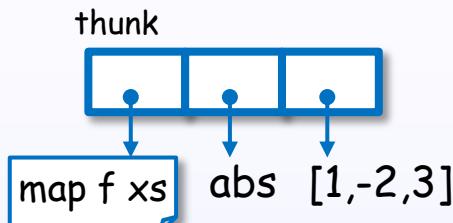
let thunk0 = map abs [1, -2, 3]  
in tail thunk0

syntactic  
desugar

tail (\_:xs) = xs

tail y = case y of  
(\_:xs) -> xs

heap memory



## (8) substitute function body (beta reduction)

tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]  
in tail thunk0

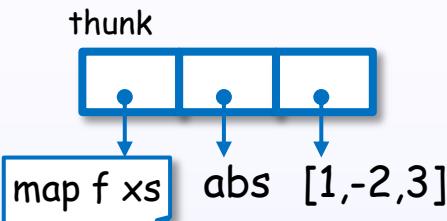
tail (\_:xs) = xs

tail y = case y of  
(\_:xs) -> xs

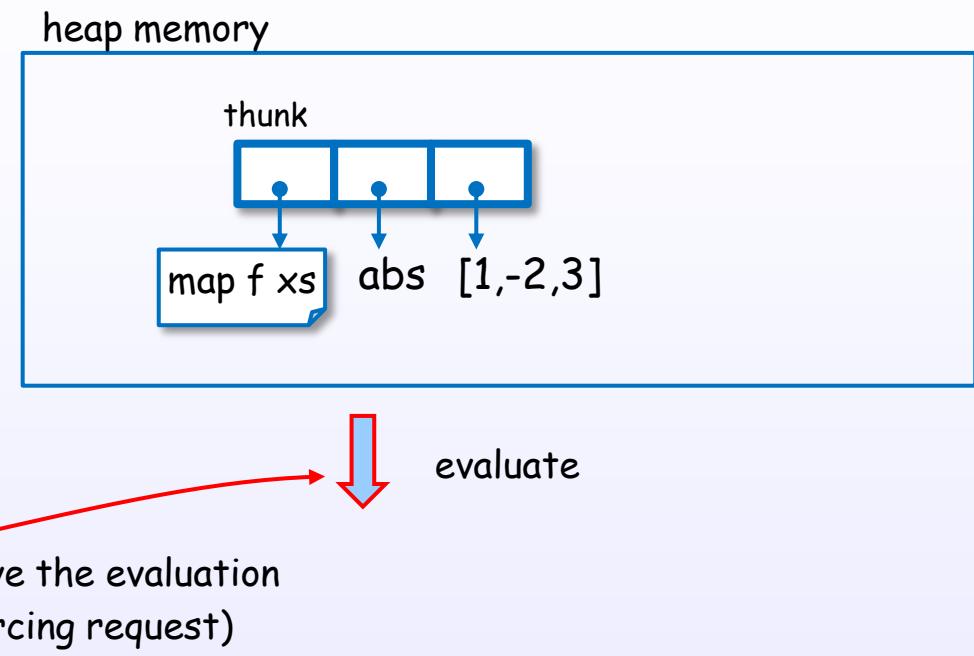
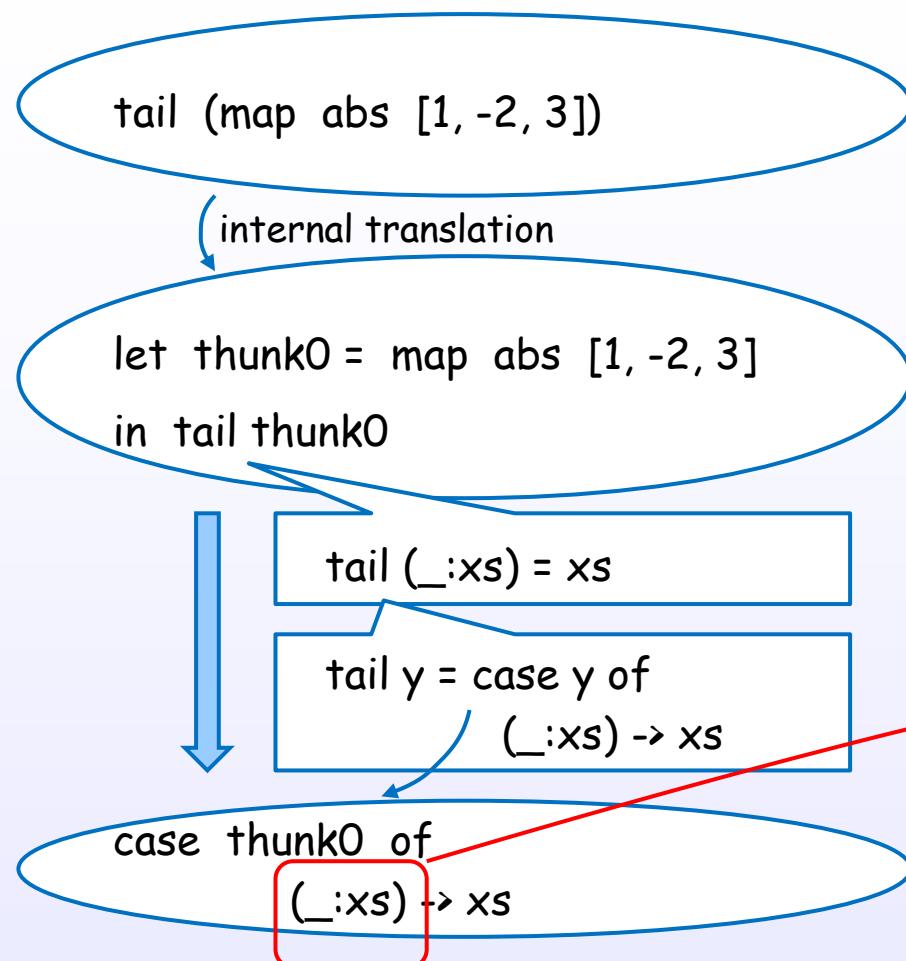
case thunk0 of  
(\_:xs) -> xs

reduction

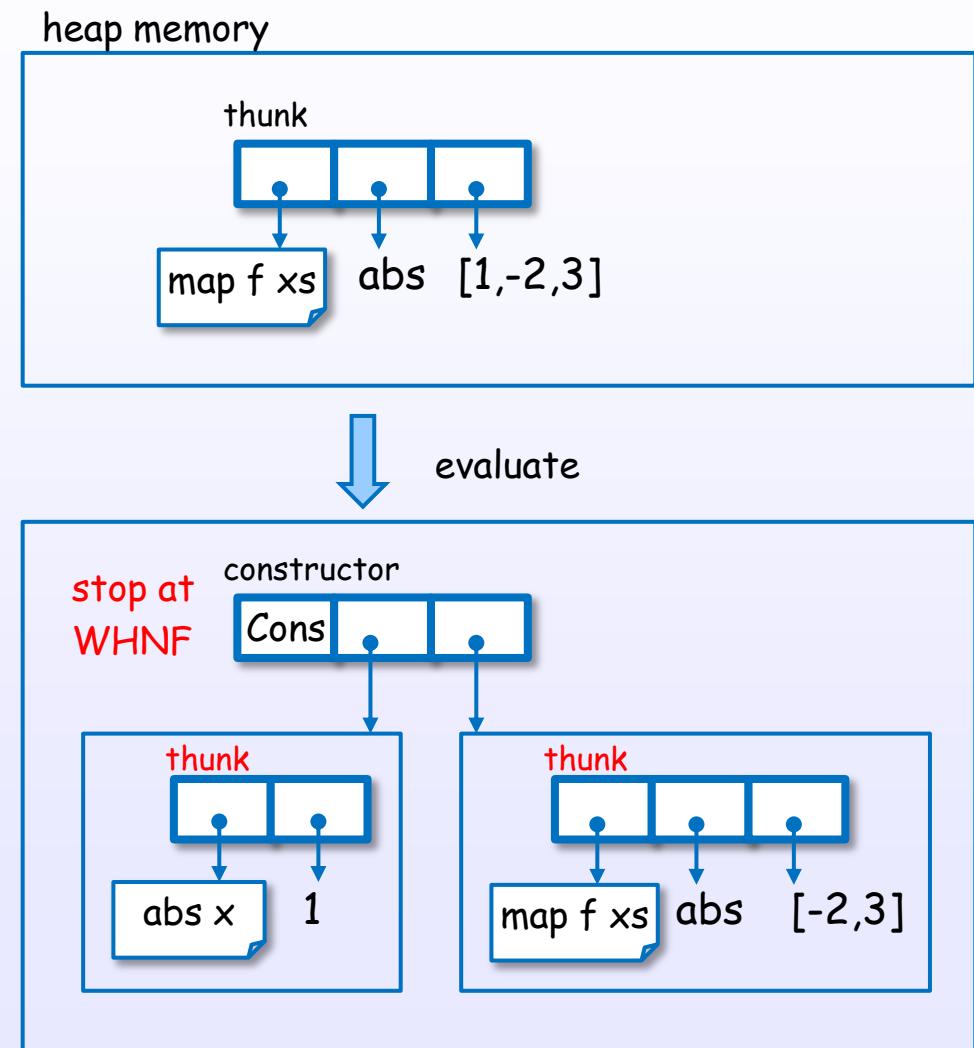
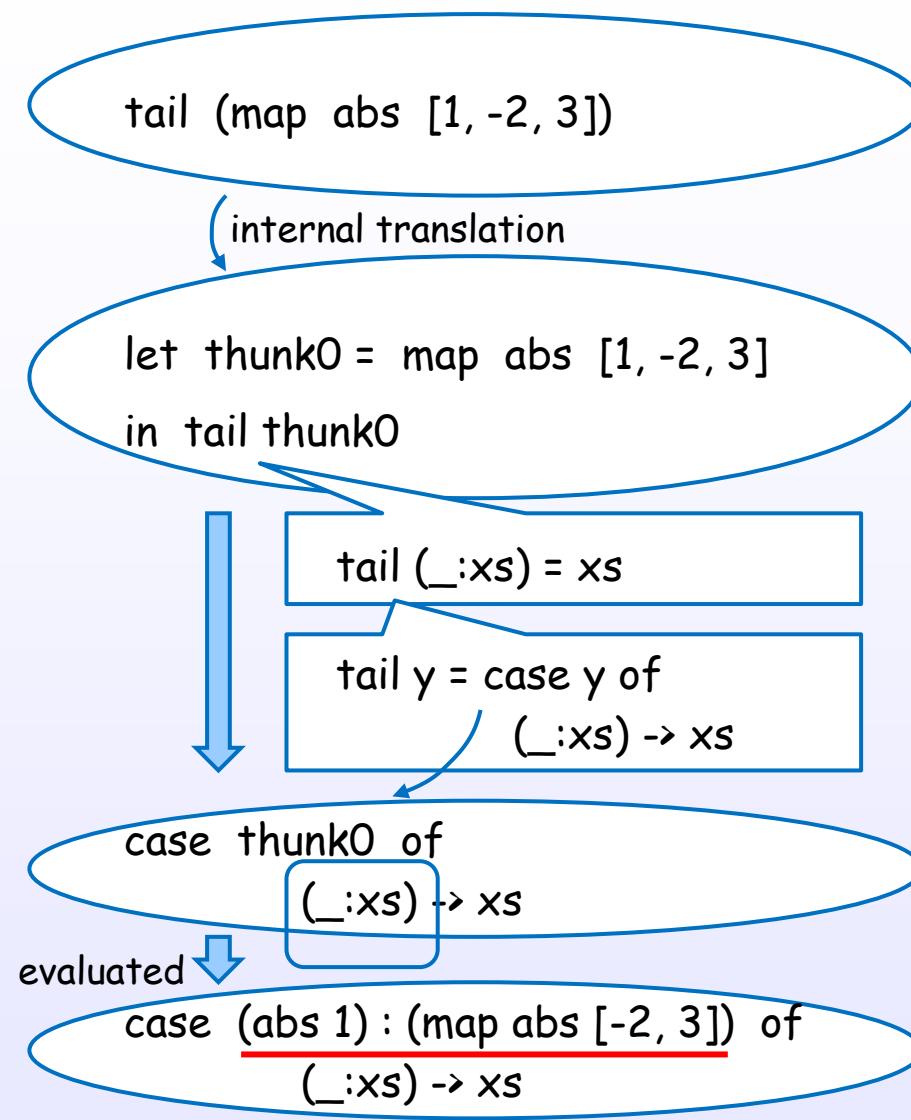
heap memory



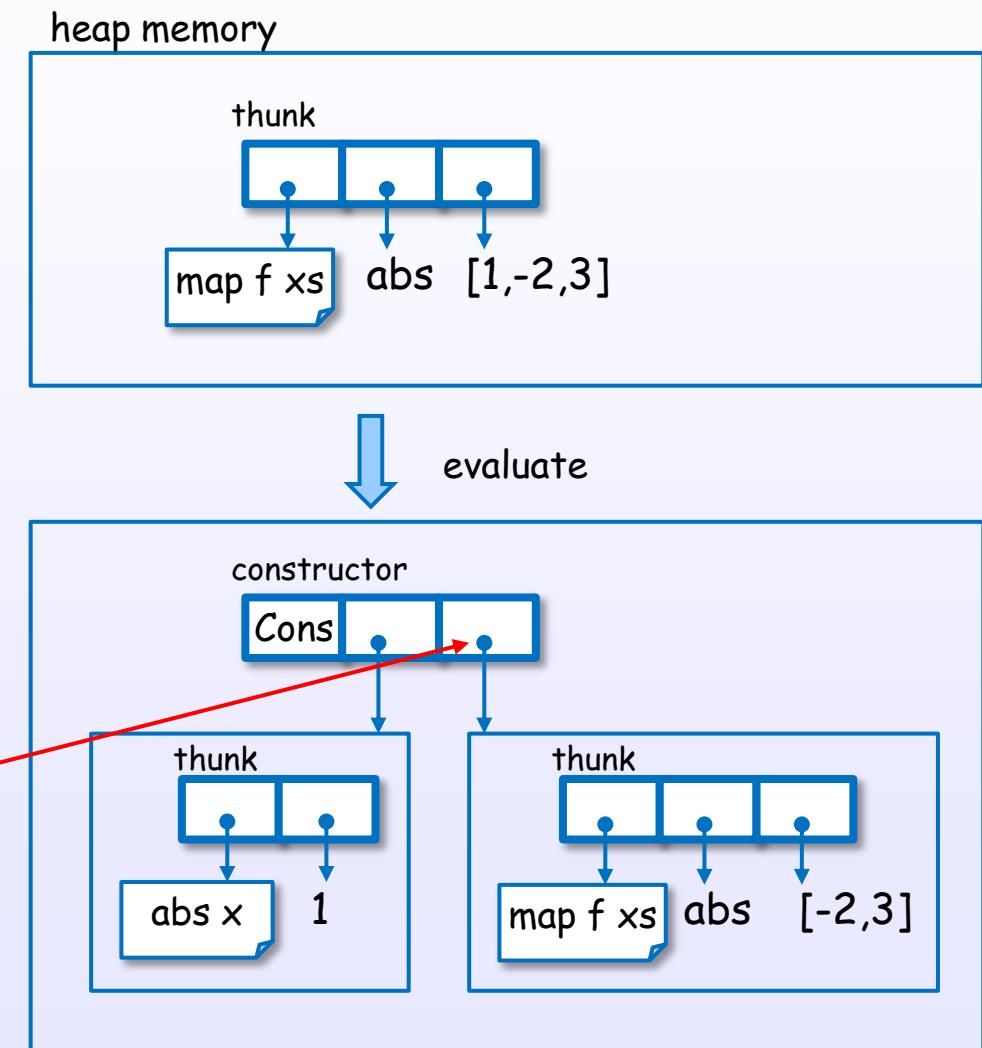
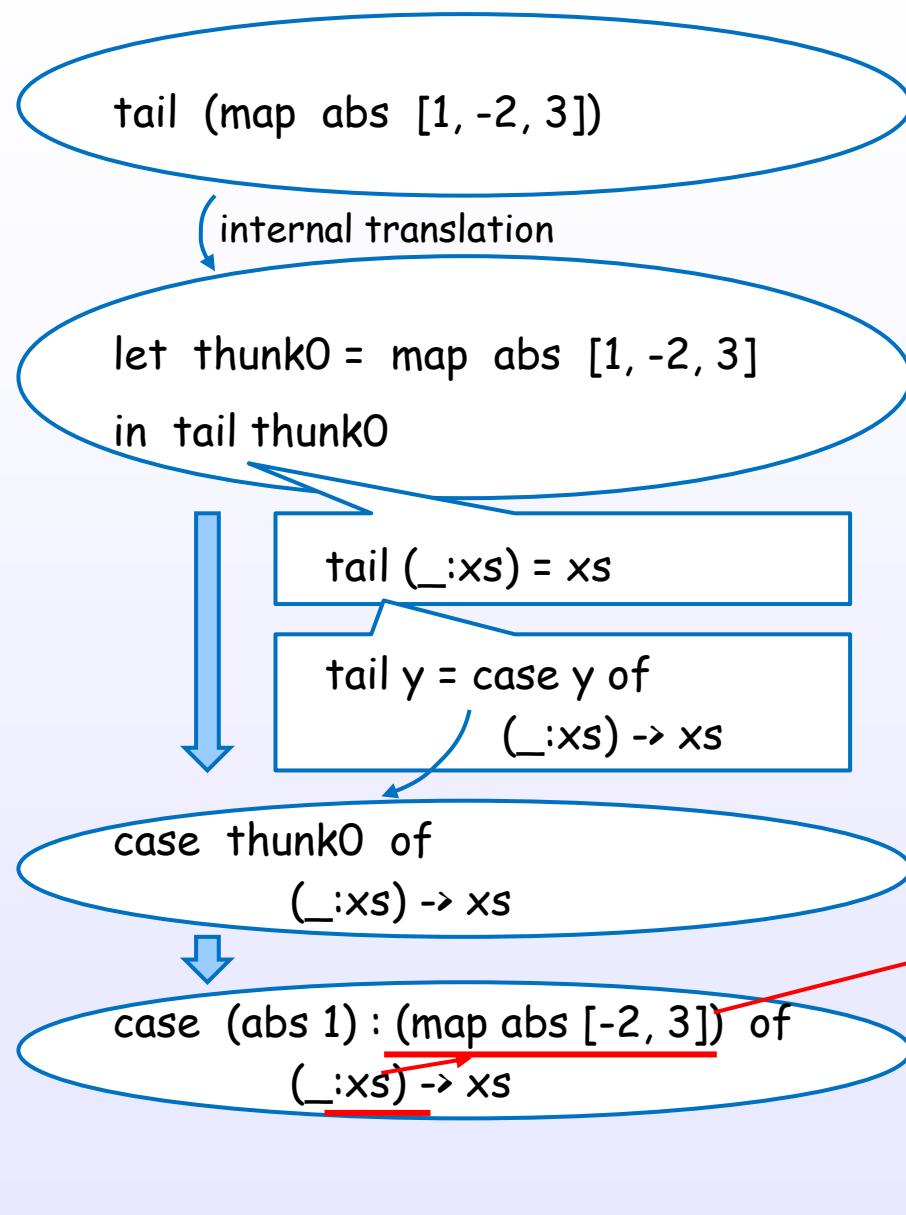
## (9) case pattern-matching drives the evaluation



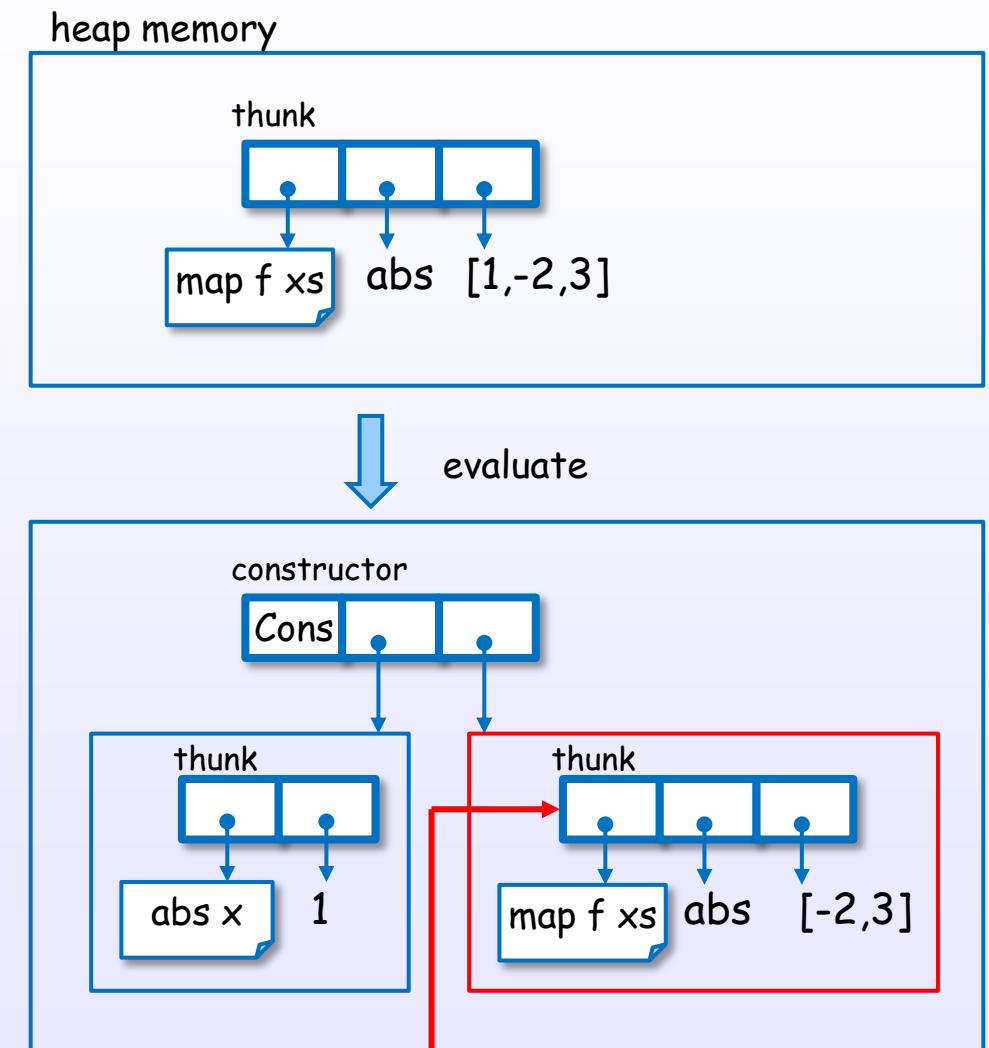
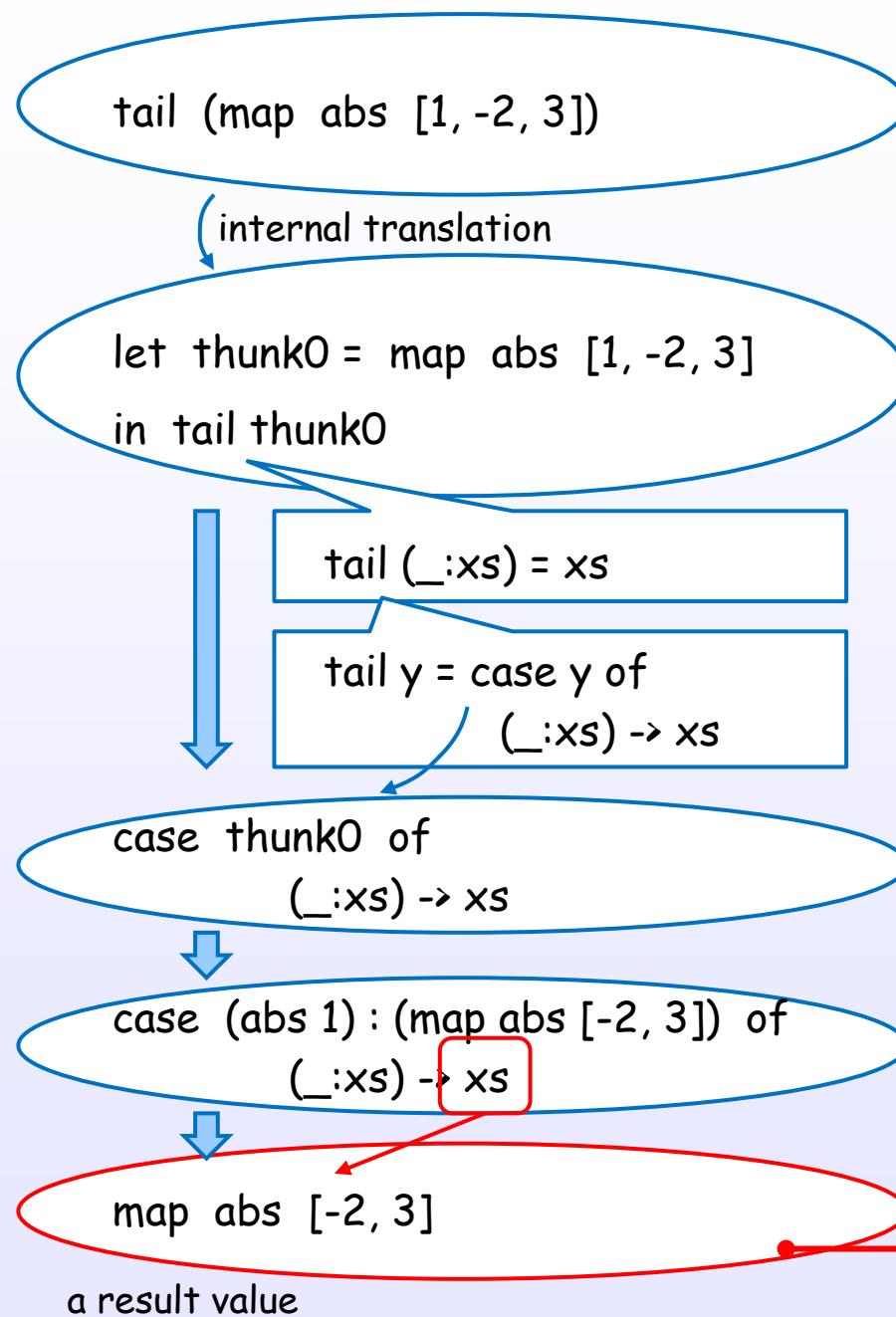
## (10) but, stop at WHNF



## (11) bind variables to result



## (12) return the value



# Key points

tail (map abs [1, -2, 3])

internal translation

postpone by thunk

let thunk0 = map abs [1, -2, 3]  
in tail thunk0

tail (\_:xs) = xs

tail y = case y of  
(\_:xs) -> xs

case thunk0 of  
(\_:xs) -> xs

case (abs 1) : (map abs [-2, 3]) of  
(\_:xs) -> xs

map abs [-2, 3]

a result value

to memory

thunk

map f xs

abs [1,-2,3]

Cons

thunk

abs x

1

map f xs

abs [-2,3]

constructor

stop at WHNF

evaluate

pattern-match drive the evaluation

## 4. Evaluation

### Examples of evaluations

\* no optimizing case (without -O)

# Example of repeat

repeat 1



1 : repeat 1



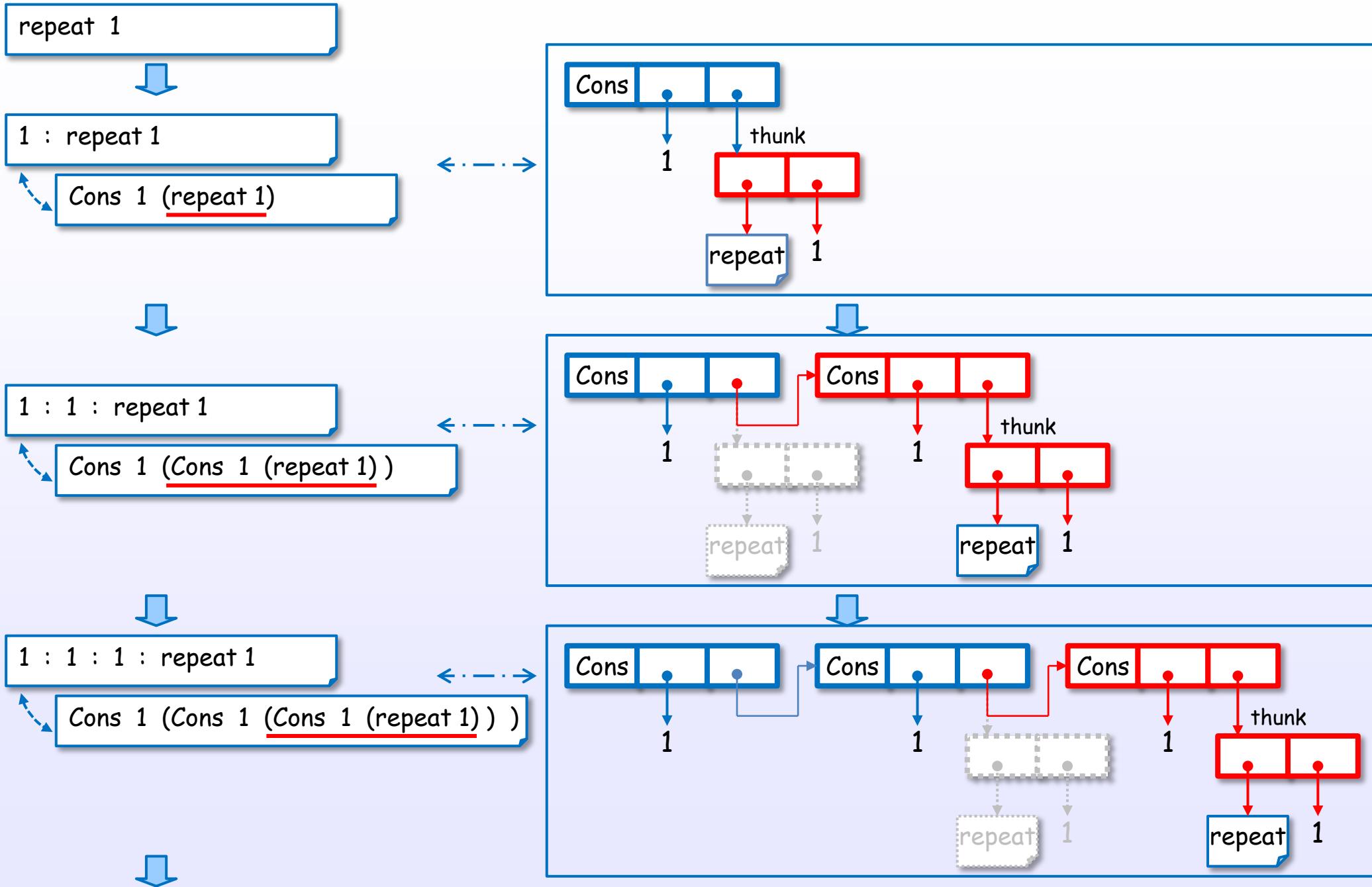
1 : 1 : repeat 1



1 : 1 : 1 : repeat 1



# Example of repeat



# Example of map

```
map f [1, 2, 3]
```



```
f 1 : map f [2, 3]
```



```
f 1 : f 2 : map f [3]
```

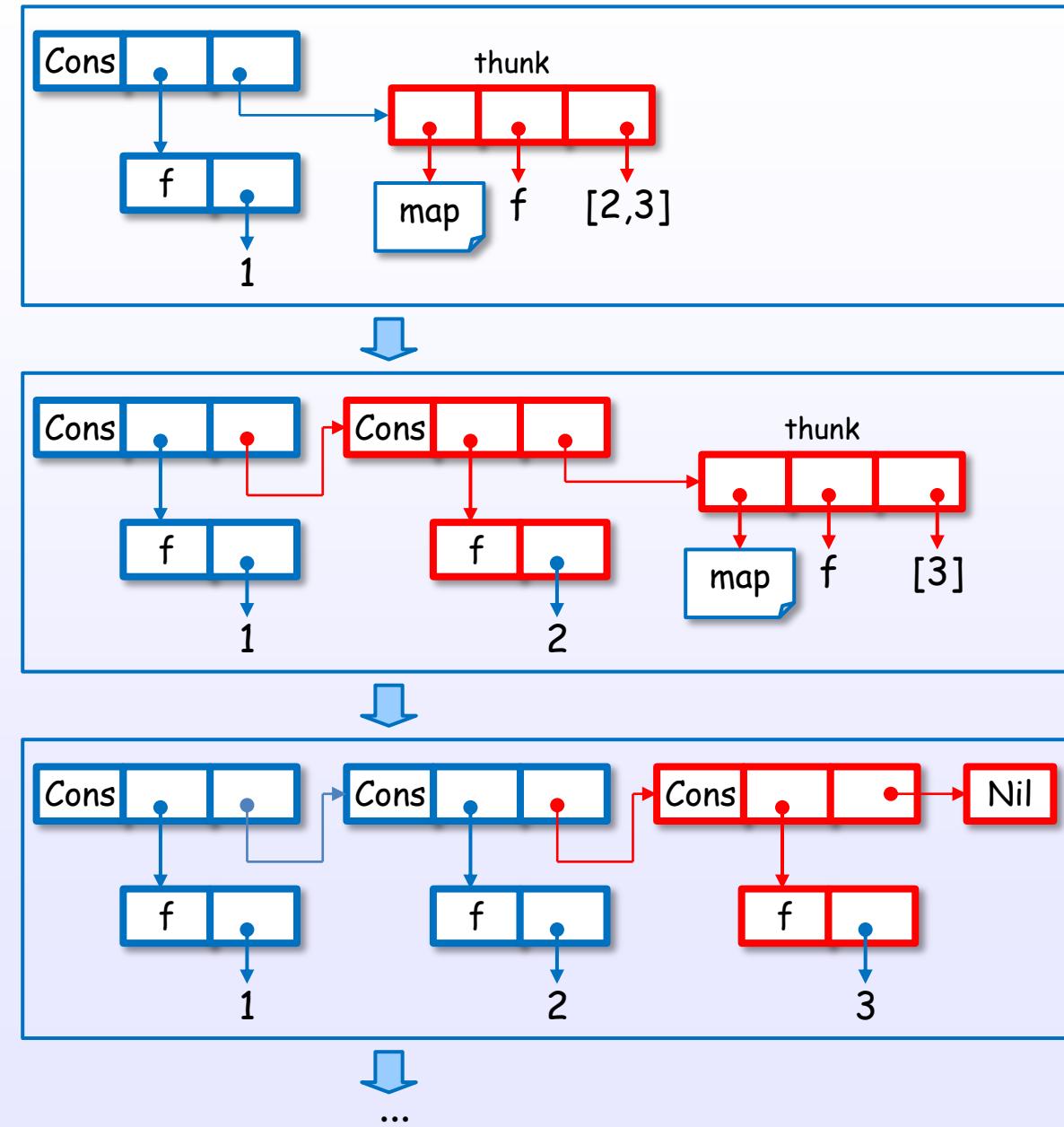
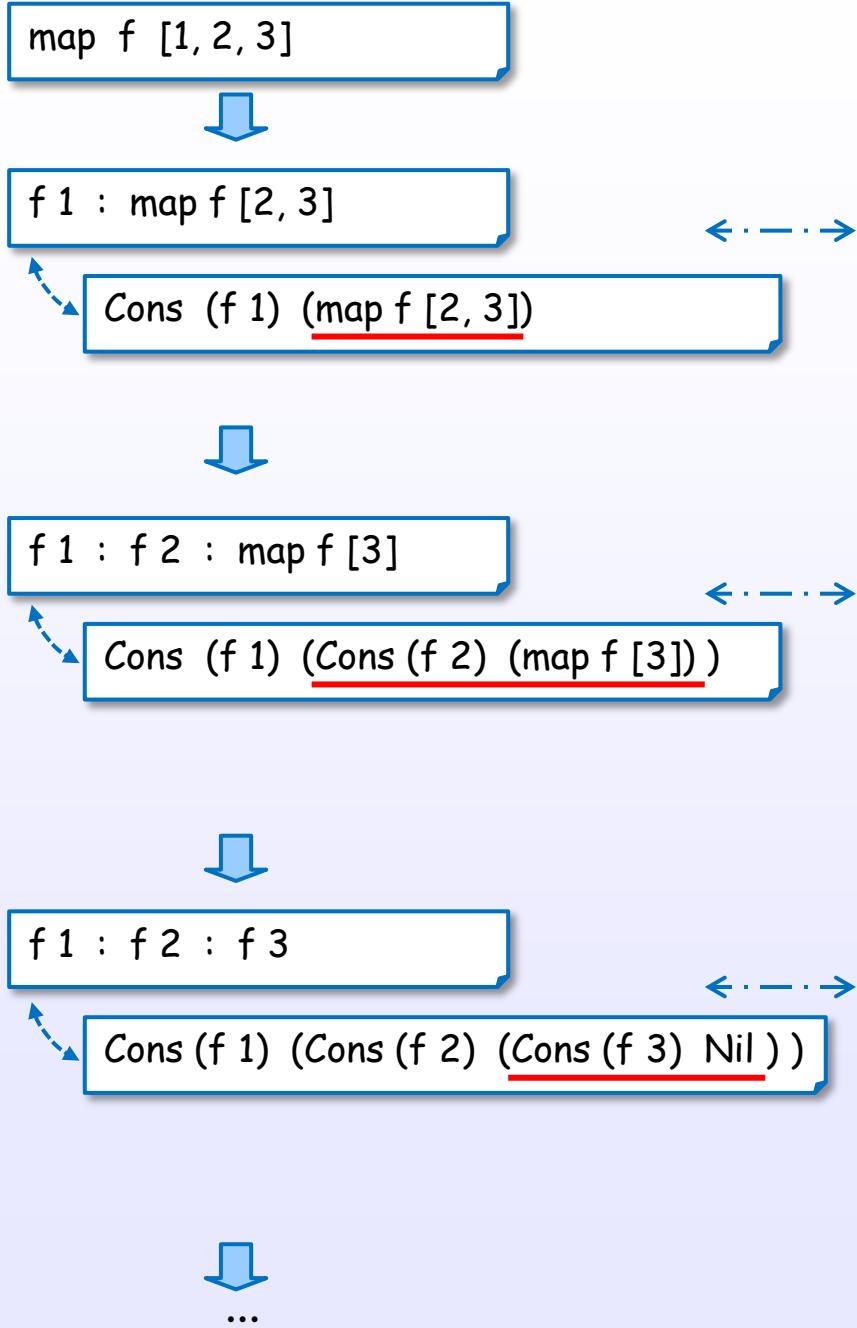


```
f 1 : f 2 : f 3
```



...

# Example of map



## Example of foldl (non-strict)

```
foldl (+) 0 [1 .. 100]
```



```
foldl (+) (0 + 1) [2 .. 100]
```



```
foldl (+) (((0 + 1) + 2) [3 .. 100]
```



```
foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]
```



...

# Example of foldl (non-strict)

`foldl (+) 0 [1 .. 100]`



`foldl (+) (0 + 1) [2 .. 100]`

`let thunk1 = (0 + 1)  
in foldl (+) thunk1 [2 .. 100]`



`foldl (+) ((0 + 1) + 2) [3 .. 100]`

`let thunk2 = (thunk1 + 2)  
in foldl (+) thunk2 [3 .. 100]`



`foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]`

`let thunk3 = (thunk2 + 3)  
in foldl (+) thunk3 [4 .. 100]`

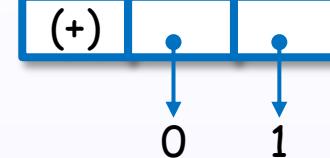


...

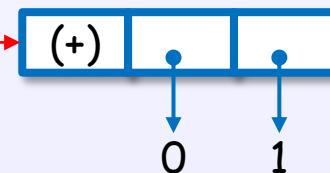
heap memory

\*show only accumulation value

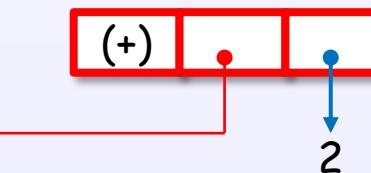
thunk1



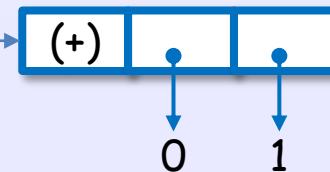
thunk1



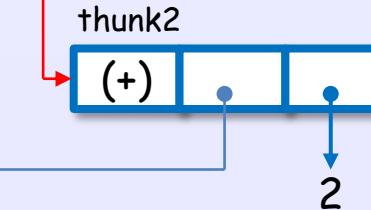
thunk2



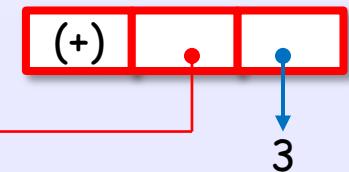
thunk1



thunk2



thunk3



increasing heap ...



References : [D5], [D6], [D8], [D9], [D10], [H10]

## Example of foldl' (strict)

```
foldl' (+) 0 [1 .. 100]
```



```
foldl' (+) (0 + 1) [2 .. 100]
```



```
foldl' (+) (1 + 2) [3 .. 100]
```



```
foldl' (+) (3 + 3) [4 .. 100]
```



...

# Example of foldl' (strict)

`foldl' (+) 0 [1 .. 100]`



`foldl' (+) (0 + 1) [2 .. 100]`

```
let thunk1 = (0 + 1)
in thunk1 `seq`
  foldl' (+) thunk1 [2 .. 100]
```



`foldl' (+) (1 + 2) [3 .. 100]`

```
let thunk2 = (1 + 2)
in thunk2 `seq`
  foldl' (+) thunk2 [3 .. 100]
```

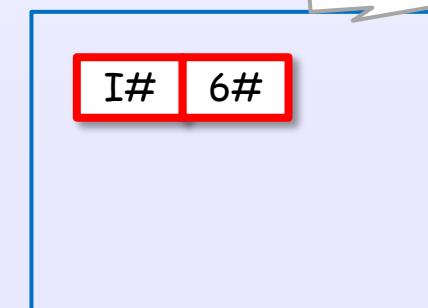
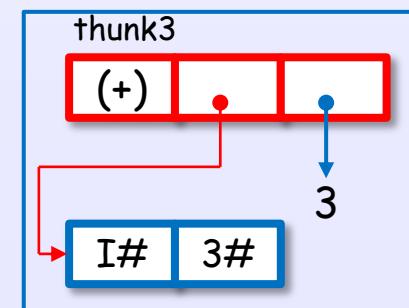
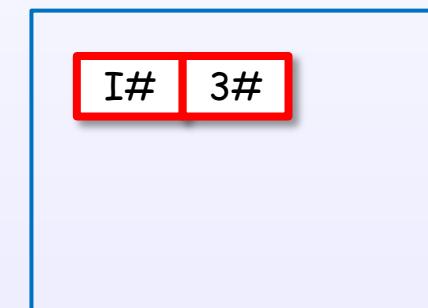
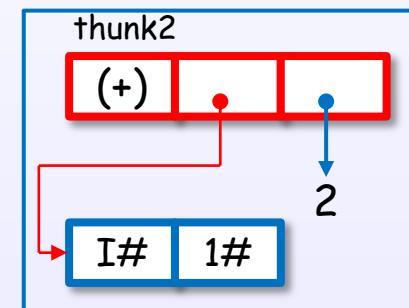
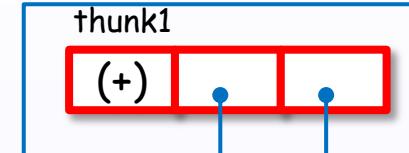


`foldl' (+) (3 + 3) [4 .. 100]`

```
let thunk3 = (3 + 3)
in thunk3 `seq`
  foldl' (+) thunk3 [4 .. 100]
```



heap memory



fixed heap size

...

References : [D5], [D6], [D8], [D9], [D10], [H10]

# Example of foldl (non-strict) and foldl' (strict)

foldl (+) (0 + 1) [2 .. 100]



foldl' (+) (0 + 1) [2 .. 100]

foldl (+) ((0 + 1) + 2) [3 .. 100]



foldl' (+) (1 + 2) [3 .. 100]



foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]



foldl' (+) (3 + 3) [4 .. 100]

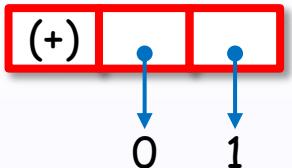


References : [D5], [D6], [D8], [D9], [D10], [H10]

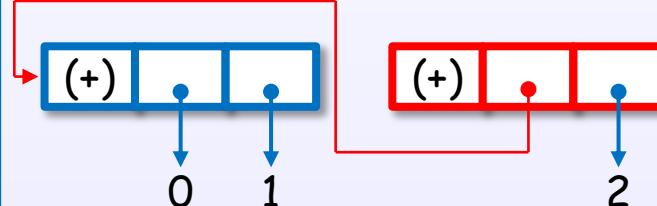
# Example of foldl (non-strict) and foldl' (strict)

**foldl (+) (0 + 1) [2 .. 100]**

heap memory

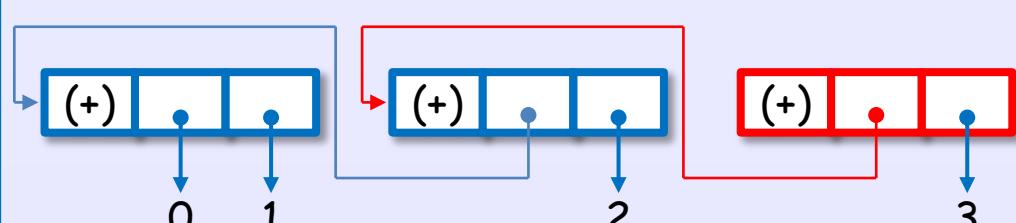


**foldl (+) ((0 + 1) + 2) [3 .. 100]**



**foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]**

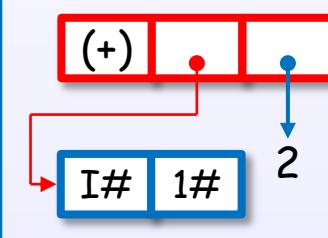
increasing heap ...



**foldl' (+) (0 + 1) [2 .. 100]**

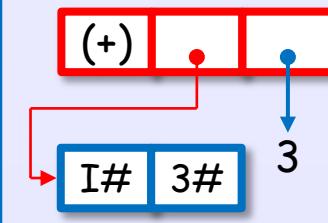


**foldl' (+) (1 + 2) [3 .. 100]**



**foldl' (+) (3 + 3) [4 .. 100]**

fixed heap size

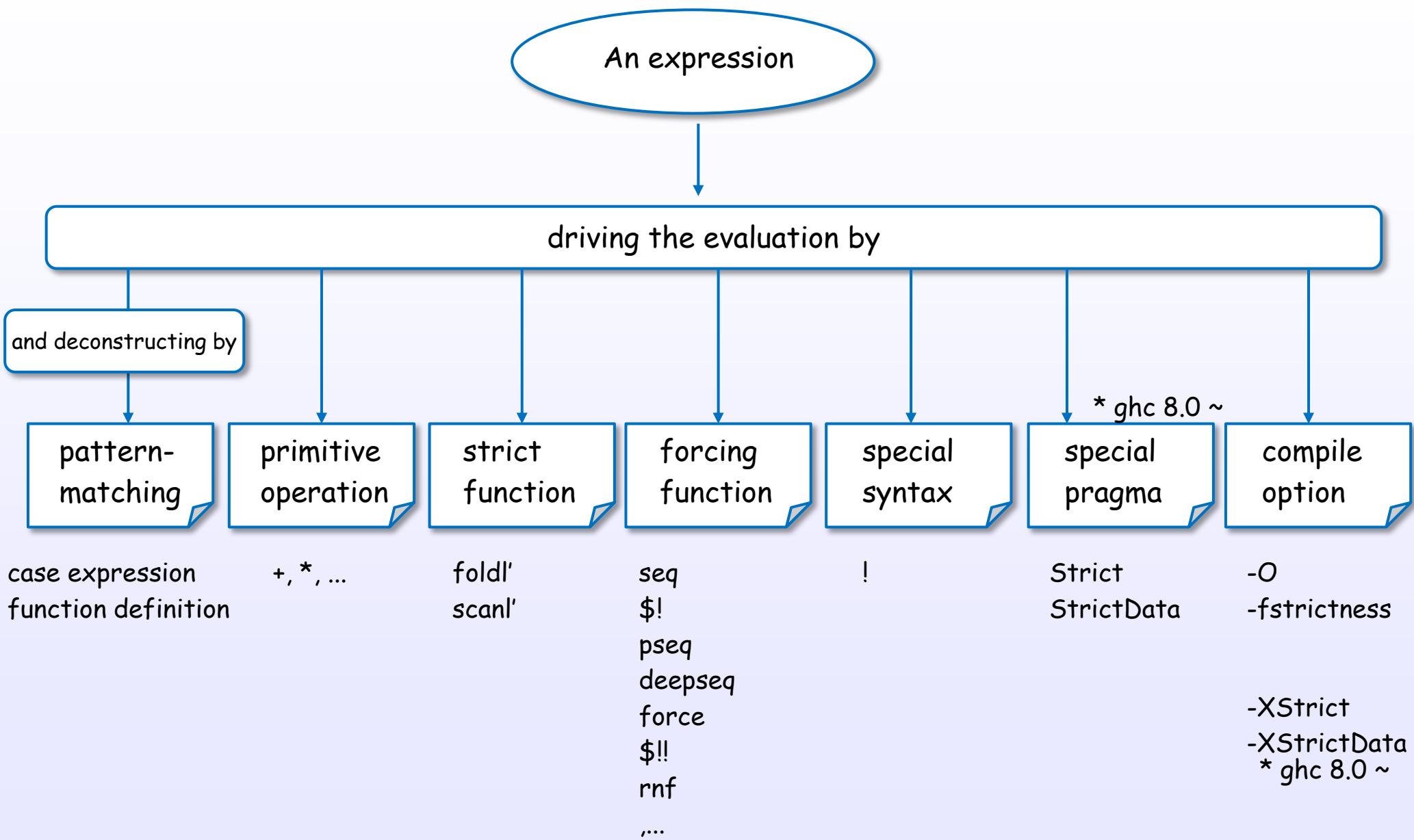


References : [D5], [D6], [D8], [D9], [D10], [H10]

## 4. Evaluation

Controlling the evaluation

# How to drive the evaluation



# (1) Evaluation by pattern-matching

pattern-matching in **case expression**

```
case ds of
  x:xs -> f x xs
  []      -> False
```

forcing  
(drive the evaluation of the thunk)

pattern-matching in **function definition**

```
f Just _ = True
f Nothing = False
```

forcing  
(drive the evaluation of the thunk)

# (1) Evaluation by pattern-matching

Strict patterns drive the evaluation

case expression

```
case ds of
  x:xs -> f x xs
  []      -> False
```

Lazy patterns postpone the evaluation.

let binding pattern

```
let (x:xs) = fun args
```

function definition

```
f Just _ = True
f Nothing = False
```

irrefutable patterns [H1] 3.17

```
f ~(Just _) = True
f ~(Nothing) = False
```

There are two kinds of pattern-matching.

## (2) Evaluation by primitive operation

primitive (built-in) operation

$$f \ x \ y = x + y$$

+ , \* , ...

forcing x and y  
(drive the evaluation of the thunks)

primitive operations are defined such as

\* pseudo code

$$(+)(I\# a)(I\# b) = I\# (a+b)$$

pattern-matching

### (3) Evaluation by strict version function

strict version function

foldl' (+) 0 xs

strict application of the operator

scanl' (+) 0 xs

## (4) Evaluation by forcing function

forcing functions to **WHNF**

`seq x y`

`f $! x`

`pseq x y`

forcing  
(drive the evaluation of the thunk)

forcing functions to **NF**

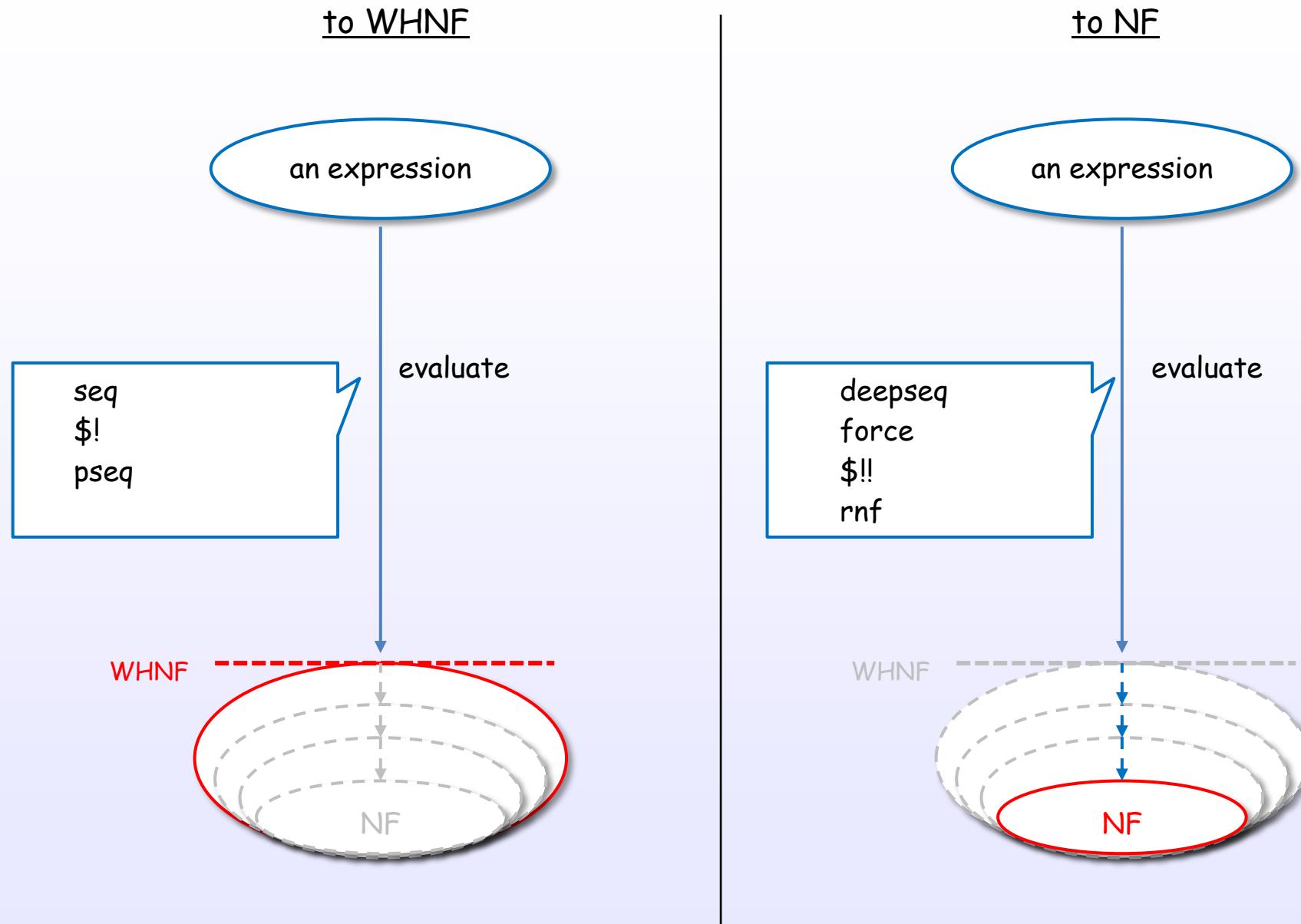
`deepseq x y`

`f $!! x`

`force x`

`rnf x`

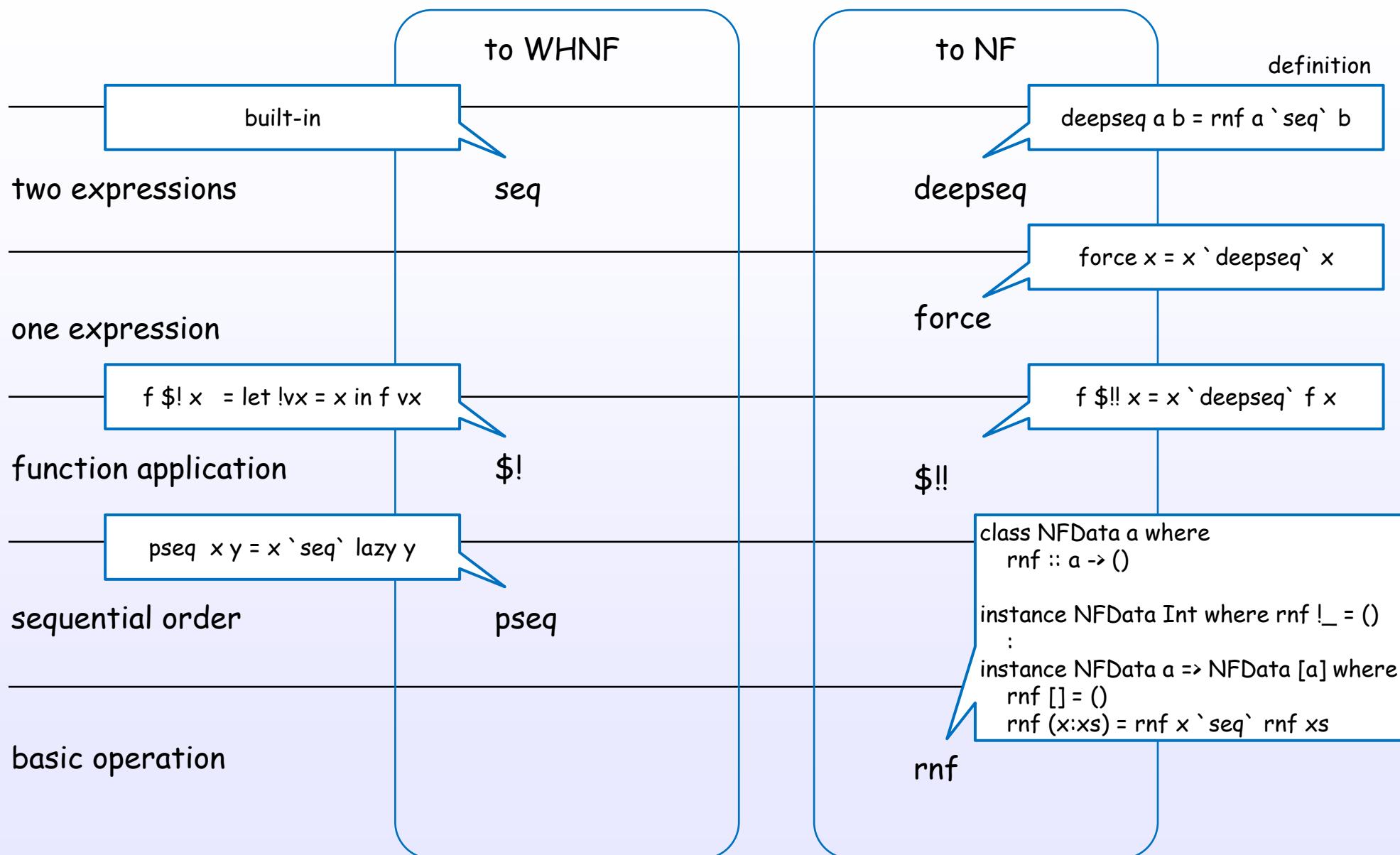
## (4) Evaluation by forcing function



## (4) Evaluation by forcing function

	to WHNF	to NF
two arguments	seq	deepseq
one argument		force
function application	\$!	\$!!
sequential order	pseq	
basic operation		rnf

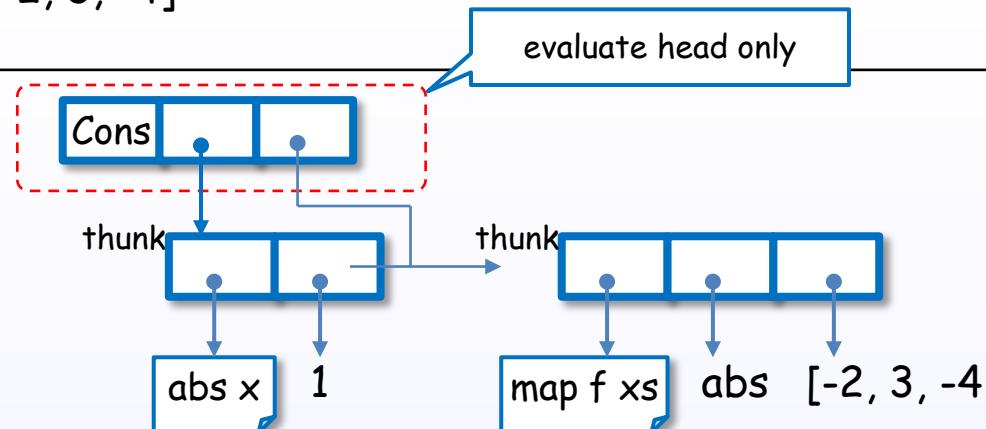
## (4) Evaluation by forcing function



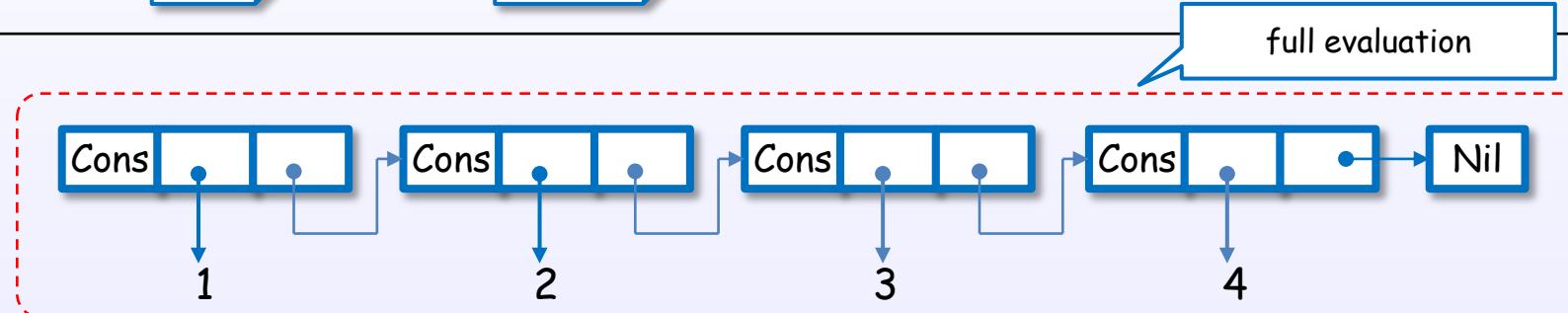
## (4) Evaluation by forcing function

`a = map abs [1, -2, 3, -4]`

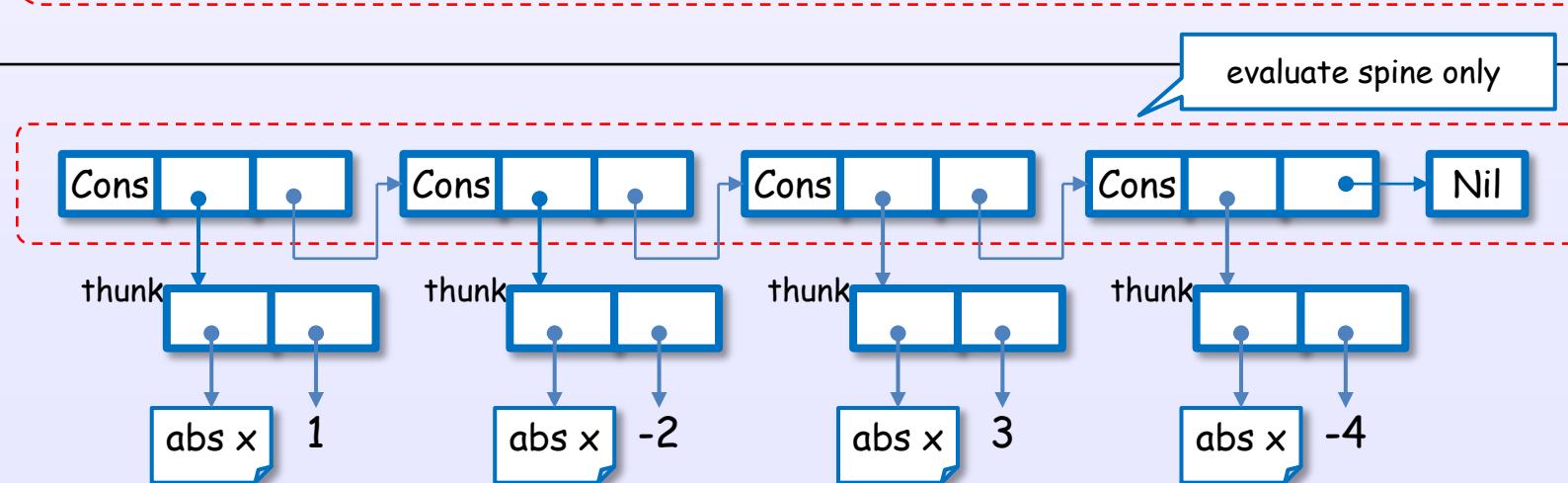
`seq a ()`



`deepseq a ()`



`length a`



## (5) Evaluation by special syntax

### Strictness annotation

Bang pattern [H2] 7.19

see also Strict pragma

{-# LANGUAGE BangPatterns #-}

f !xs = g xs

arguments are evaluated  
before function application

Strictness flag [H1] 4.2.1

see also StrictData and Strict pragma

data Pair = Pair !a !b

arguments are evaluated  
before constructor application

Strictness annotations assist strictness analysis.

## (6) Evaluation by special pragma

Special pragma for strictness language extension

Strict pragma

\* ghc 8.0 ~

see also bang pattern and strictness flag

{-# LANGUAGE Strict #-}

let f xs = g xs in f ys

data Pair = Pair a b

arguments are evaluated  
before application

StrictData pragma

\* ghc 8.0 ~

see also strictness flag

{-# LANGUAGE StrictData #-}

data Pair = Pair a b

Strict and StrictData pragmas are module level control.

These can use in ghc 8.0 or later.

## (7) Evaluation by compile option

### Compile option

#### strictness analysis

```
$ ghc -O
```

Turn on optimization.  
Implied by -O.

```
$ ghc -fstrictness
```

Turn on strictness analysis.  
Implied by -O.

#### strictness language extension \* ghc 8.0 ~

```
$ ghc -XStrict
```

apply Strict pragma

```
$ ghc -XStrictData
```

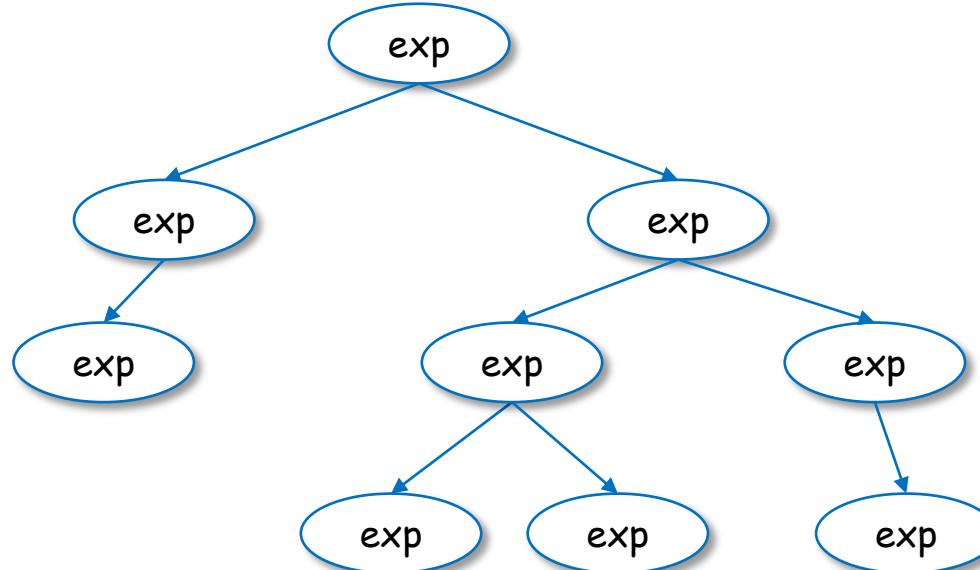
apply StrictData pragma

## 5. Implementation of evaluator

## 5. Implementation of evaluator

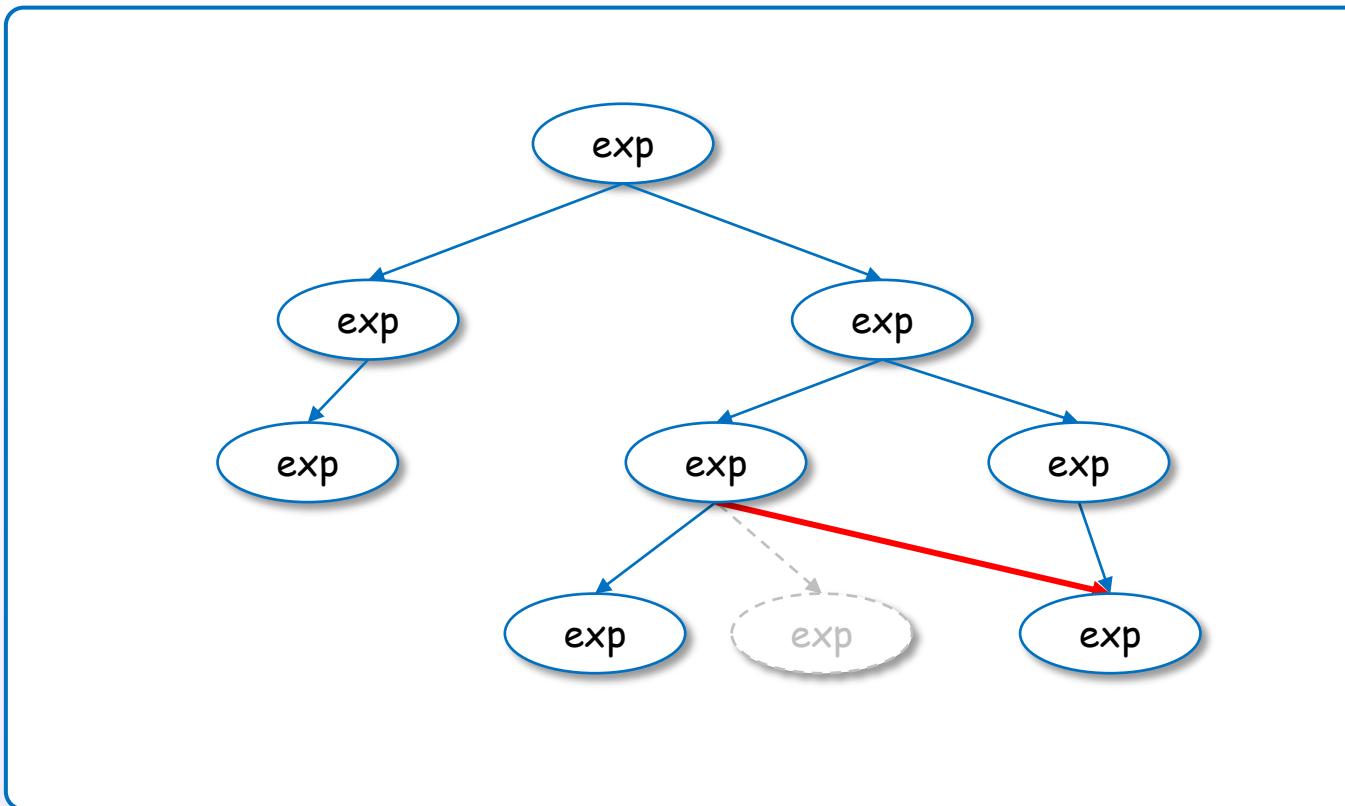
Lazy graph reduction

# Tree



An expression can be represented in the form of Abstract Syntax **Tree** (AST). AST is reduced using stack (sequential access memory).

# Graph

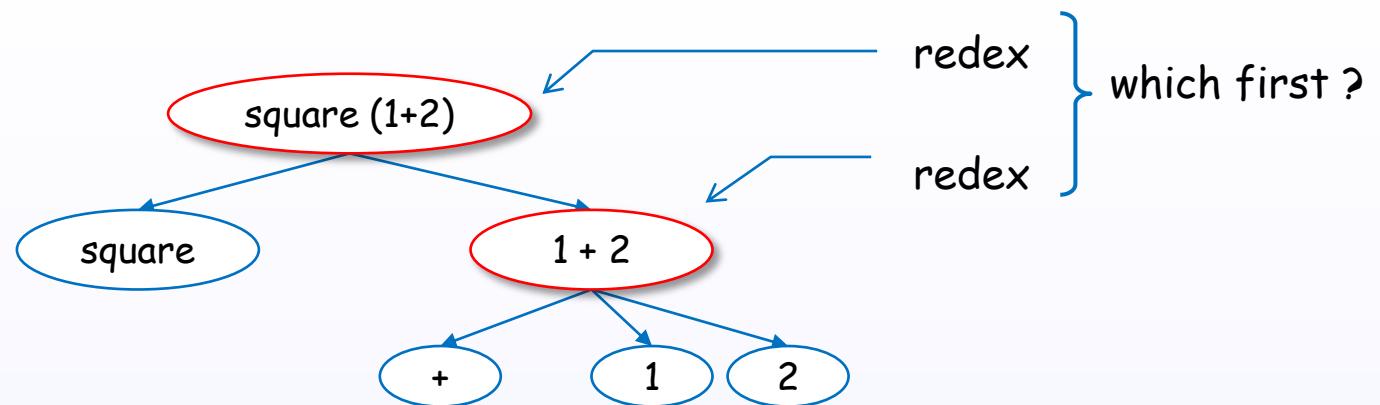


An expression can be also represented in the form of Graph.

Graph can share subexpressions to evaluate at once.

So, graph is reduced using heap (random access memory) rather than stack.

# Graph can be reduced some order

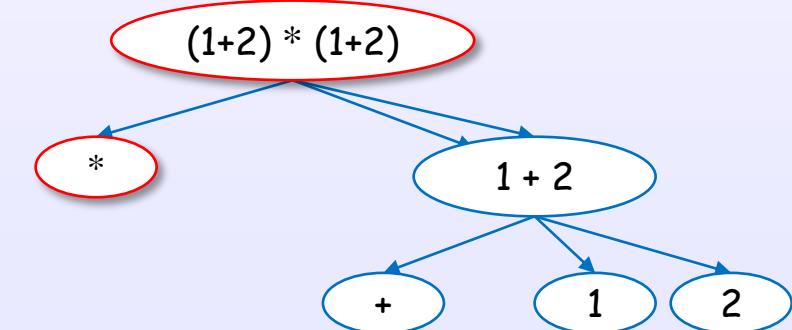
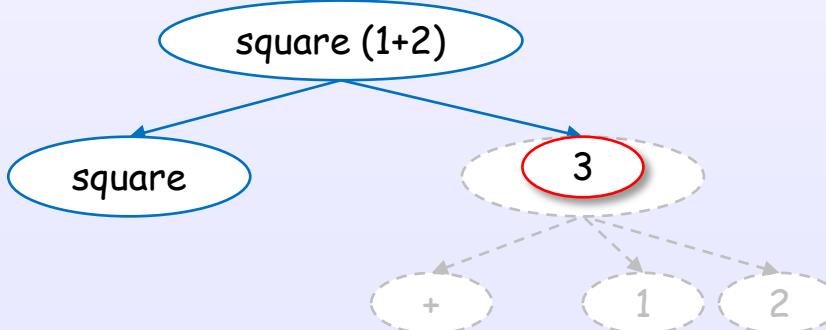


call-by-value

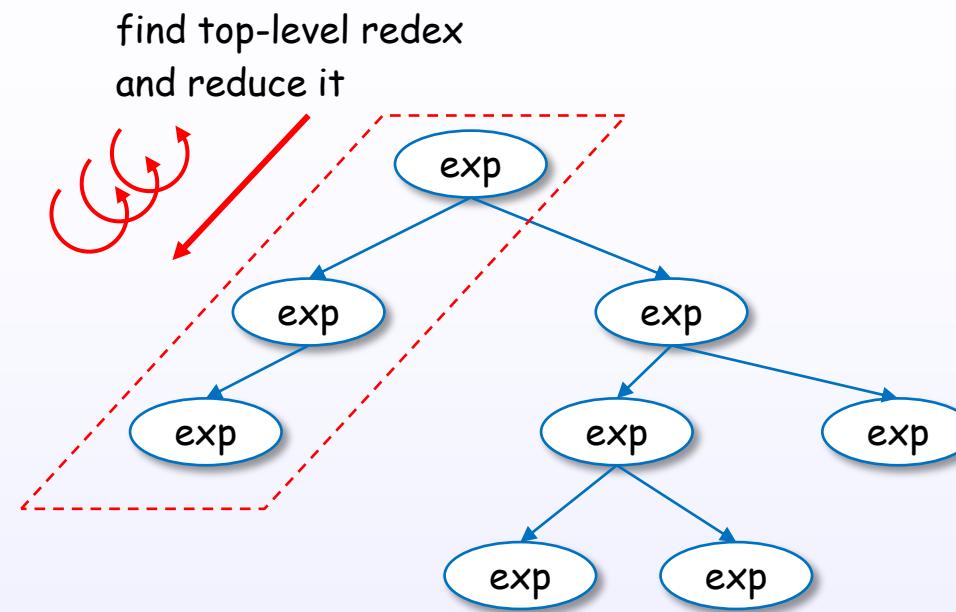
reduce inner most

call-by-need

reduce outer most (top most)



# Normal order reduction is implemented by lazy graph reduction



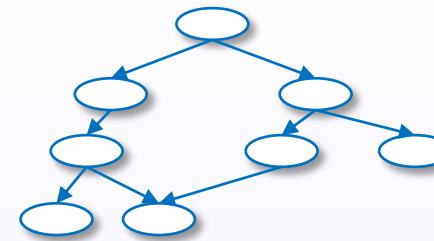
Normal order reduction specifies reducing the leftmost outermost redex (top-level redex). Given an application of a function, the outermost redex is the function application itself.

## 5. Implementation of evaluator

STG-machine

# Abstract machine

Graph  
(expression)



evaluate  
(reduce / execute)

STG-machine

Evaluator  
(abstract machine)

GHC uses abstract machine to reduce the expression.  
It's called "STG-machine".

# Concept layer

Haskell code

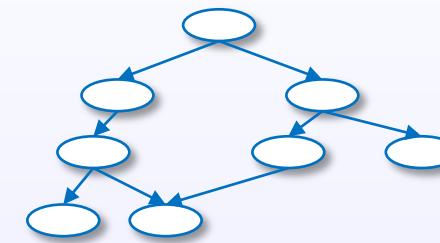
`take 5 [1..10]`

:

---

Graph  
(internal representation  
of the expression)

---



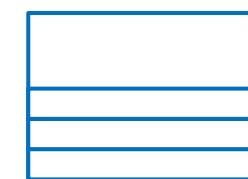
Evaluator (reducer, executer)  
(abstract machine)

STG-machine

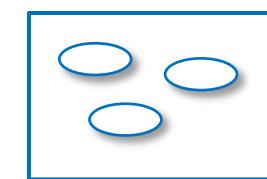
STG Registers

R1, ...

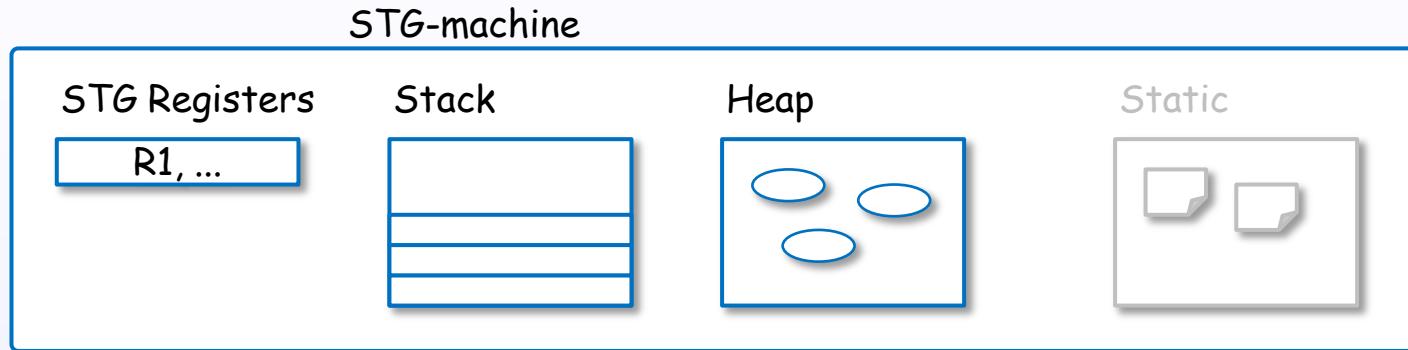
Stack



Heap



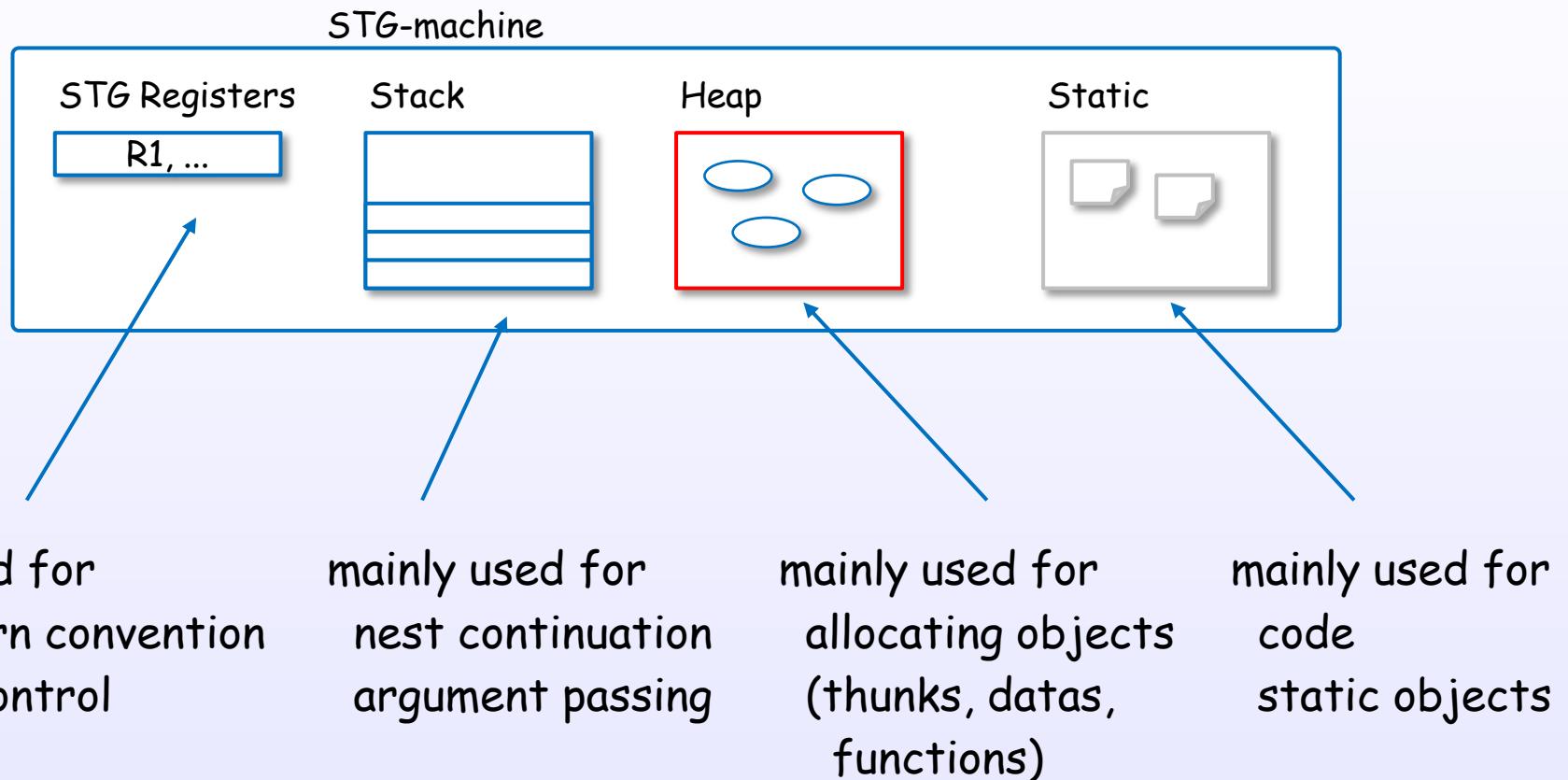
# STG-machine



STG-machine is abstraction machine  
which is defined by operational semantics.

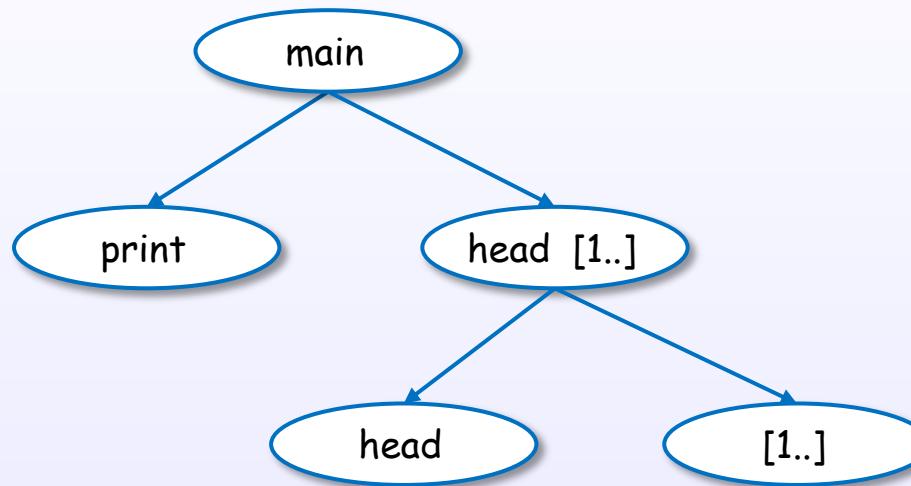
STG-machine efficiently performs lazy graph reduction.

# STG-machine



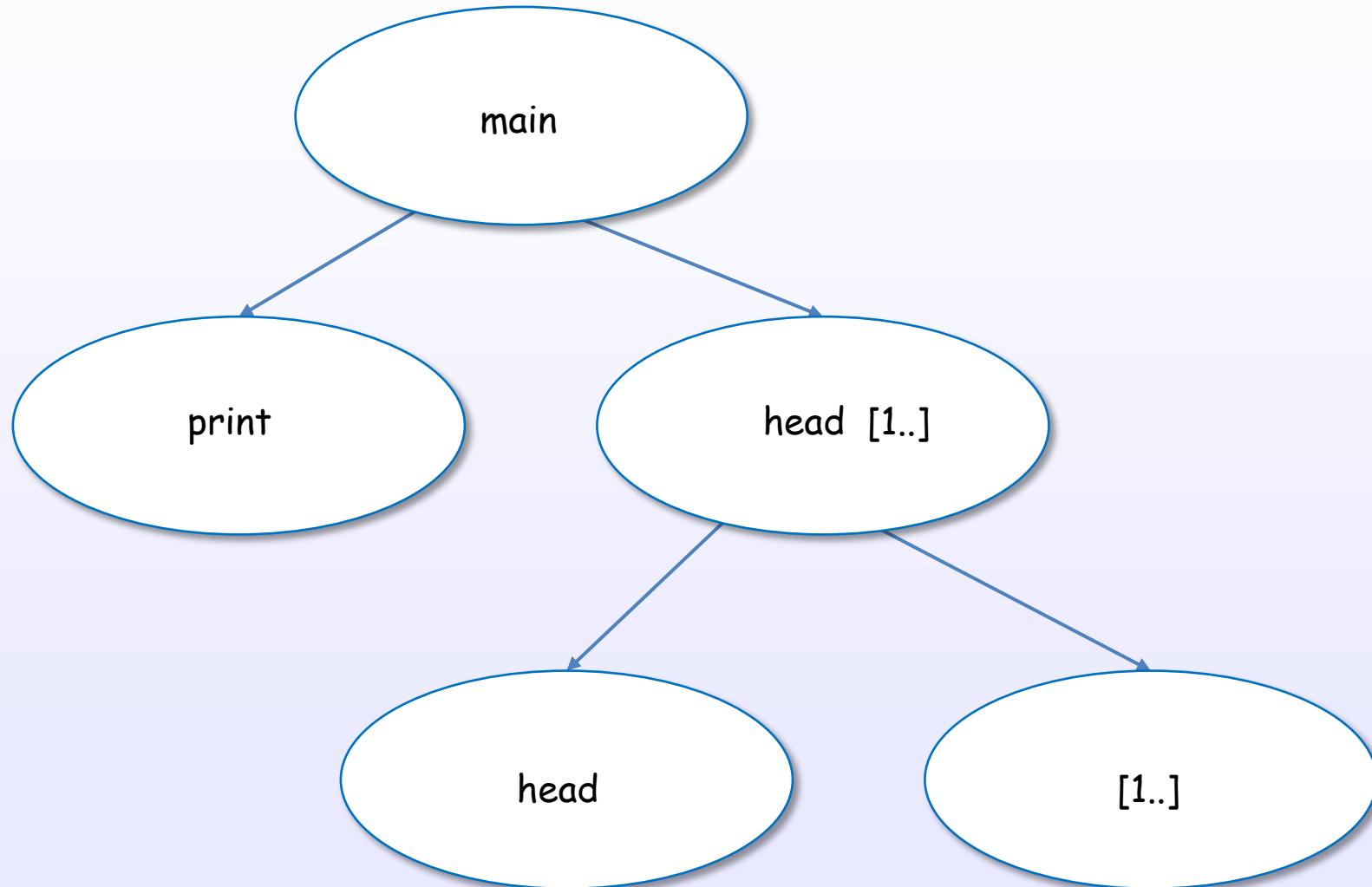
# Example of mapping a code to a graph

main = print (head [1..])



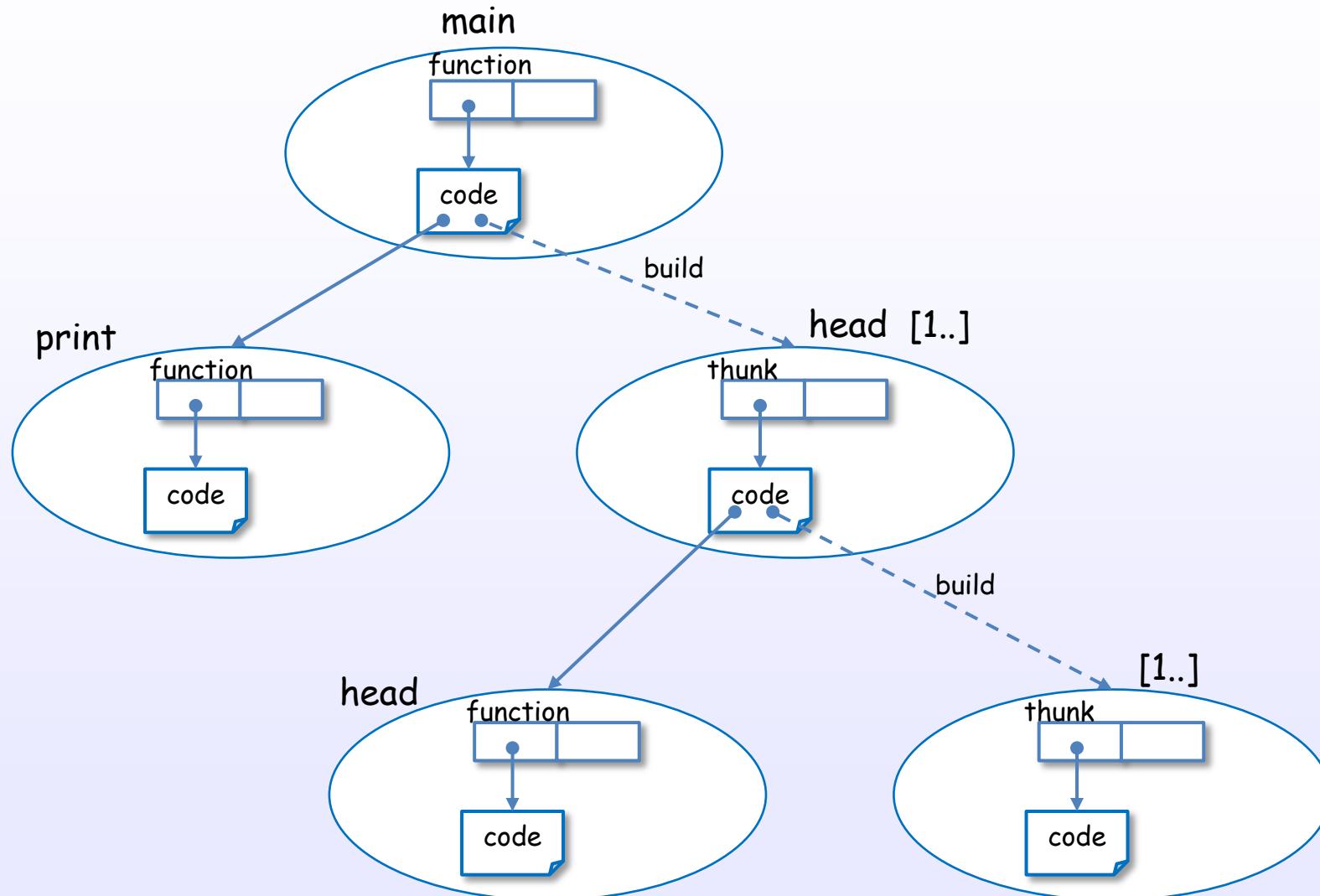
# Example of mapping a code to a graph

main = print (head [1..])



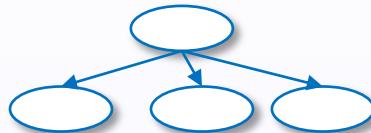
# Example of mapping a code to a graph

main = print (head [1..])



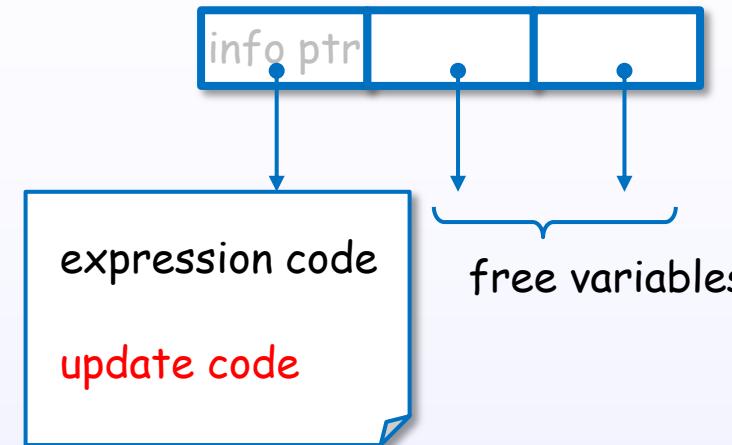
# Self-updating model

Expression



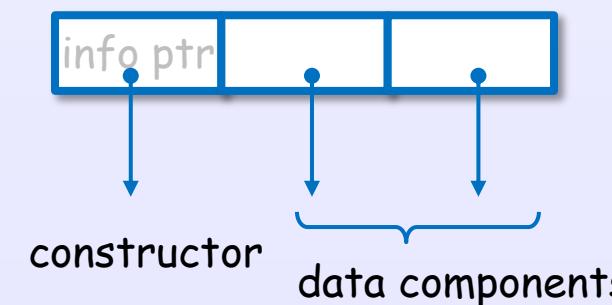
GHC's internal representation

a thunk



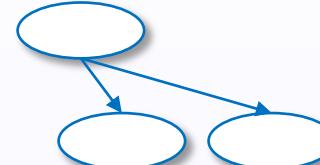
evaluate and update  
(replace myself to result value)

a data value

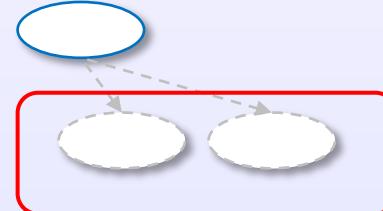


# Unreferenced expression (object) will be removed by GC

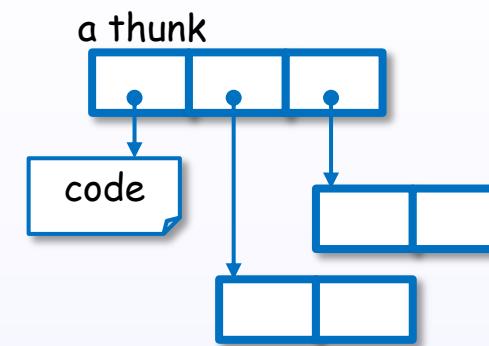
Expression



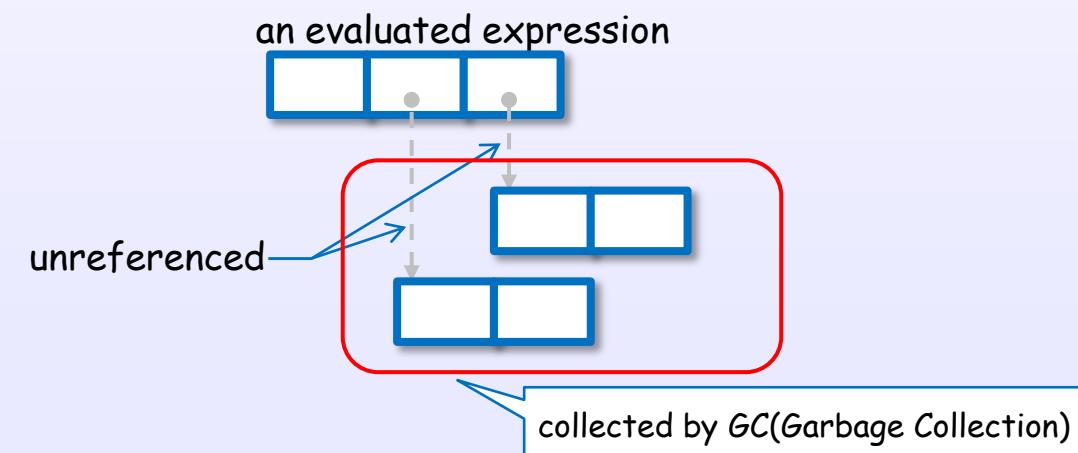
evaluate and update



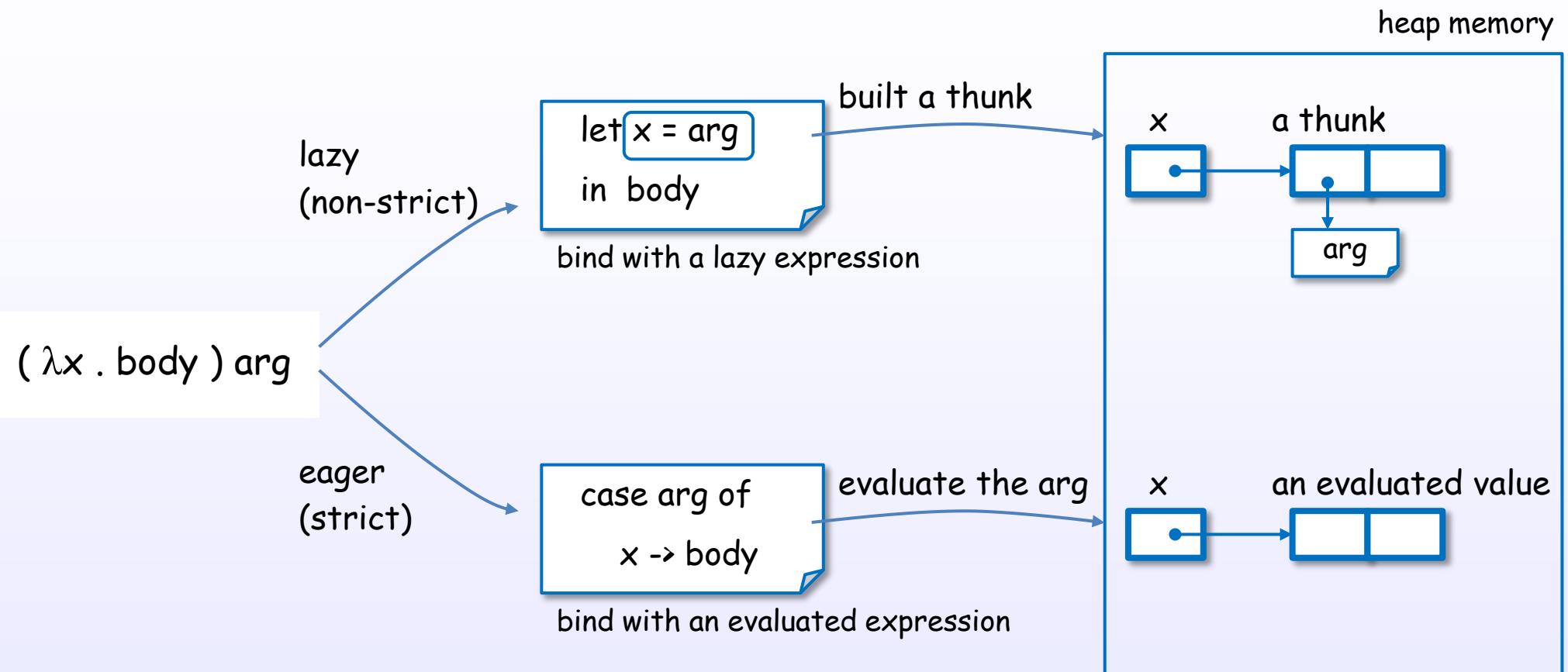
GHC's internal representation



evaluate and update

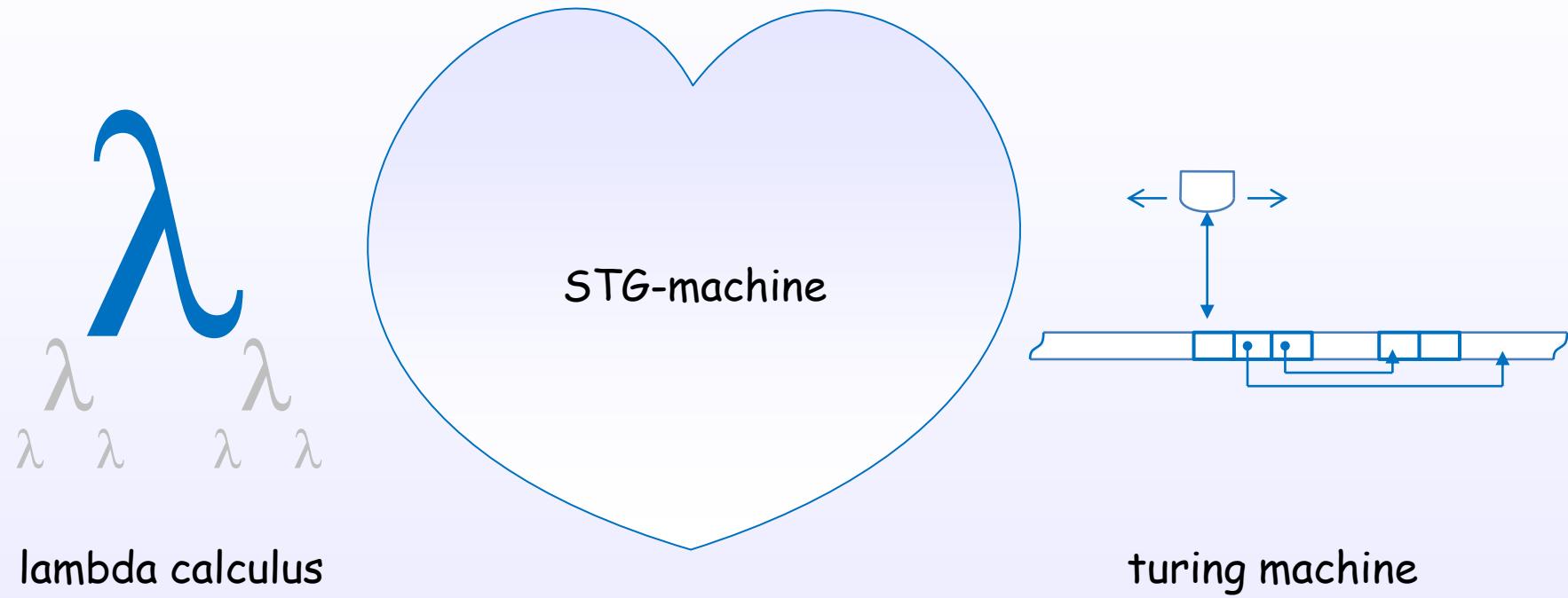


# STG-machine directly associate ...



STG-machine directly associate lambda calculus and physical machine.

# The STG-machine is ...



The STG-machine is the marriage of lambda calculus and turing machine.

# STG-dump shows which expression is build as thunks

[Example.hs]

```
module Example where
  fun f1 n = take 1 f1 n
```

STG code dump  
by "ghc -O -ddump-stg Example.hs"

Example.fun

```
:: forall a_aME t_aMF. (t_aMF -> [a_aME]) -> t_aMF ->
[a_aME]
[GblId,
Arity=2,
Caf=NoCafRefs,
Str=DmdType <L,1*C1(U)><L,U>,
Unf=OtherCon []]=
\ r srt:SRT:[] [f1_sQT n_sQU]
let {
  sat_sQV [Occ=Once, Dmd=<L,1*U>] :: [a_aMH]
  [LclId, Str=DmdType] =
    \ s srt:SRT:[] [] f1_sQT n_sQU;
} in GHC.List.take_unsafe_UInt 1 sat_sQV;
```

thunk

f1\_sQT n\_sQU

f1\_sQT n\_sQU

heap memory

build/allocate

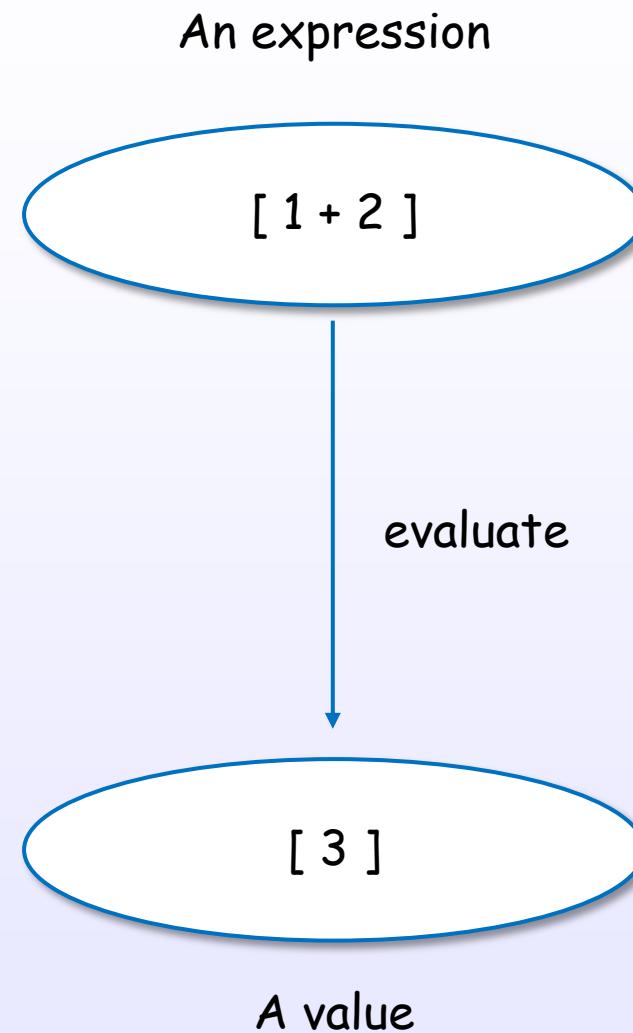
let expression in STG language

## 6. Semantics

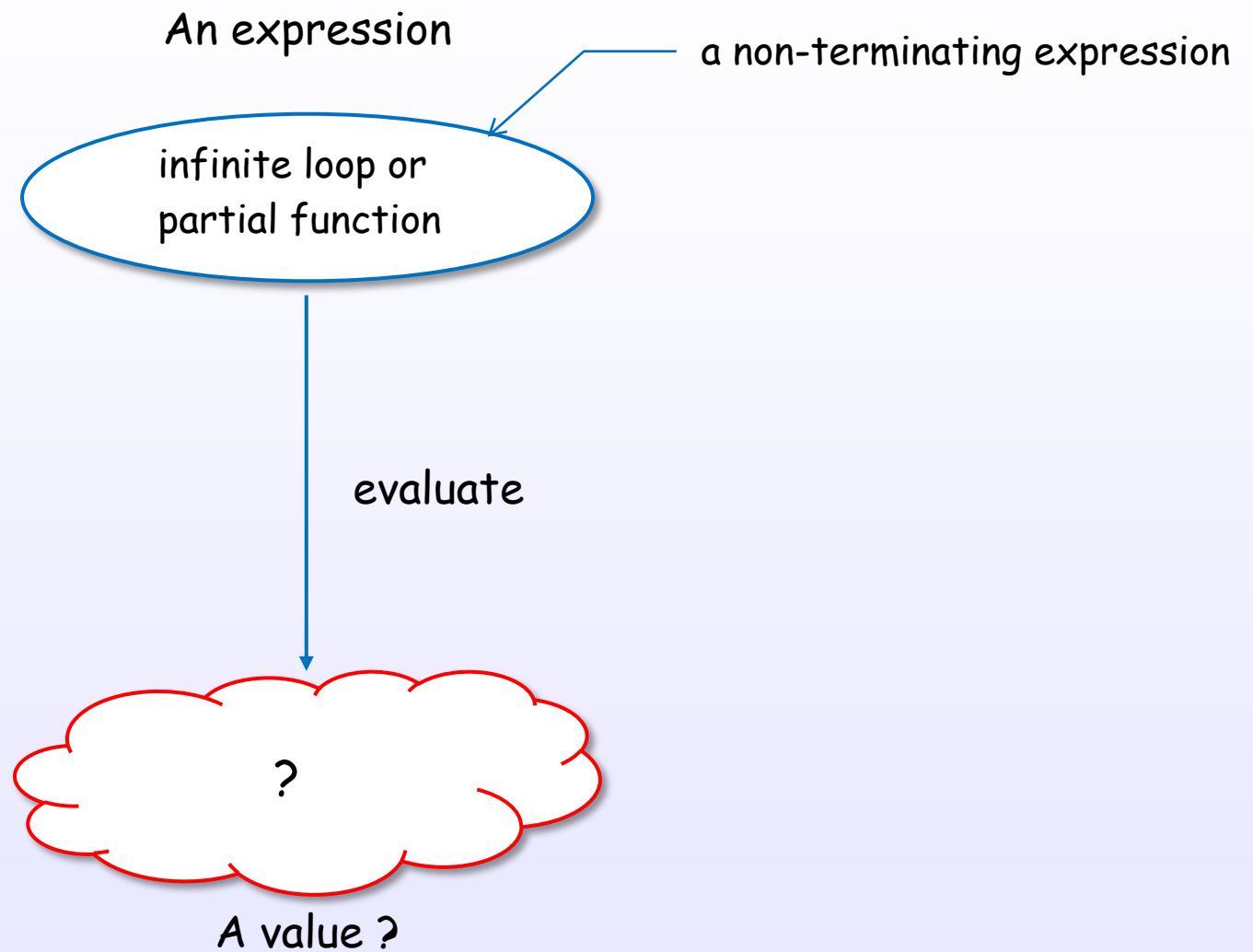
## 6. Semantics

Bottom

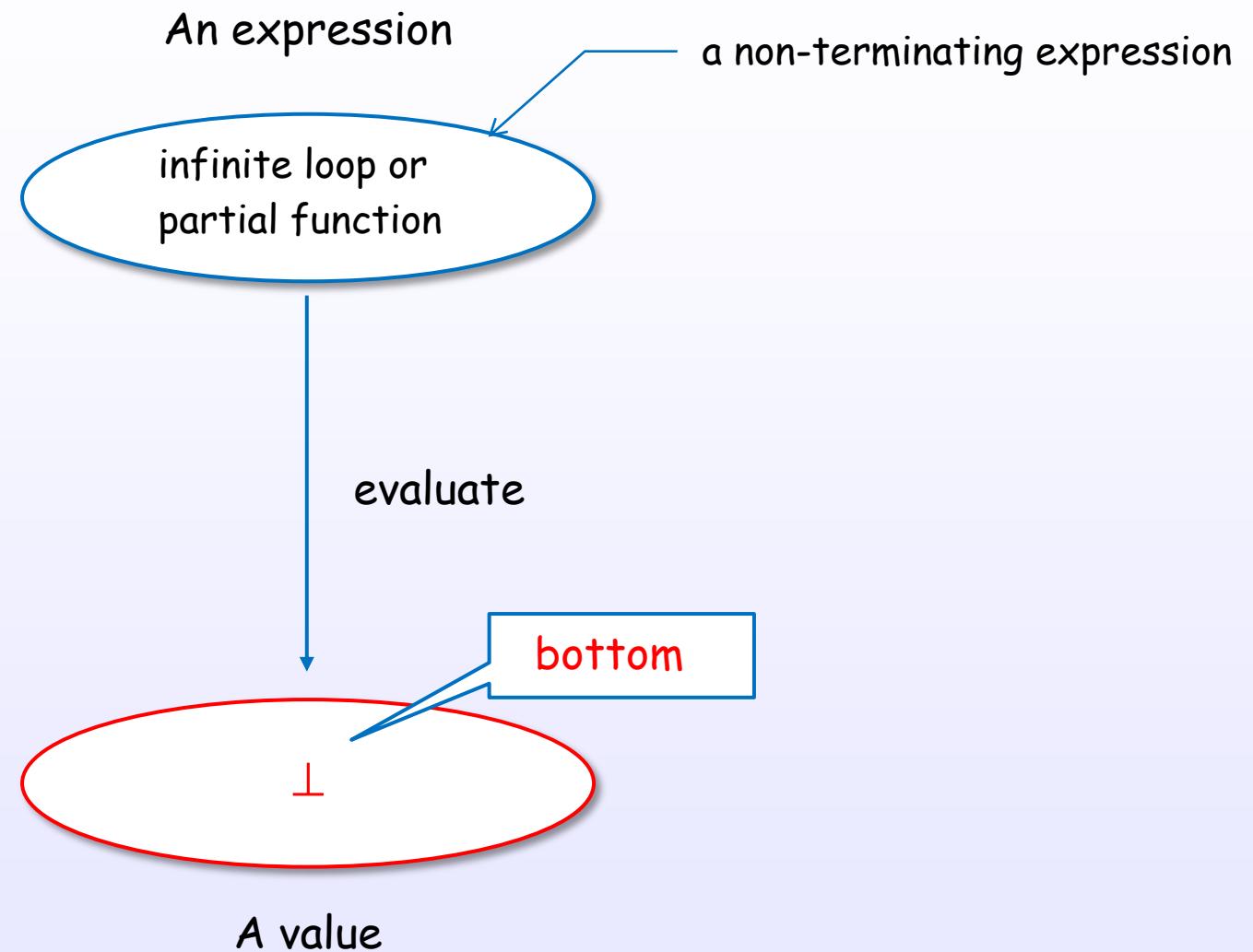
# Well formed expression should have a value



# What is a value in this case?

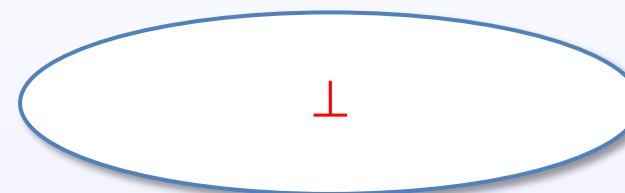


## A value “bottom” is introduced



# Bottom

A value



Bottom ( $\perp$ ) is “an undefined value”.

Bottom ( $\perp$ ) is “a non-terminating value”.

# Bottom is also represented by “undefined” in Haskell

Haskell code

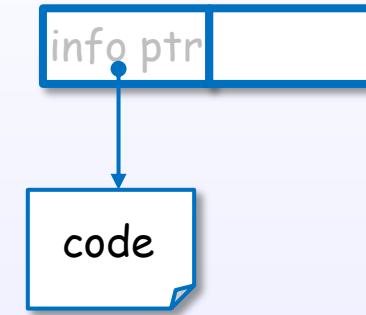
undefined :: a

Expression

$\perp$

GHC's internal representation

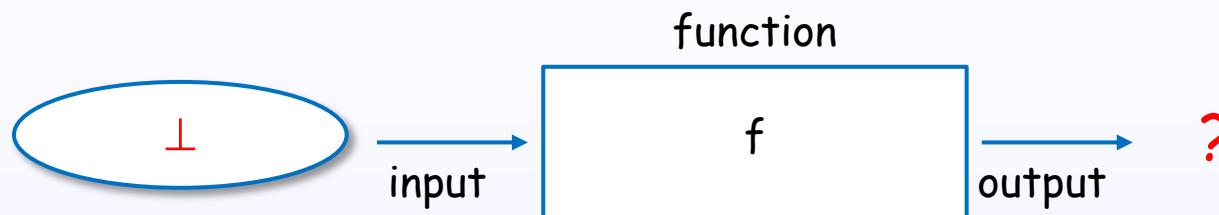
GHC.Err.undefined



## 6. Semantics

### Non-strict Semantics

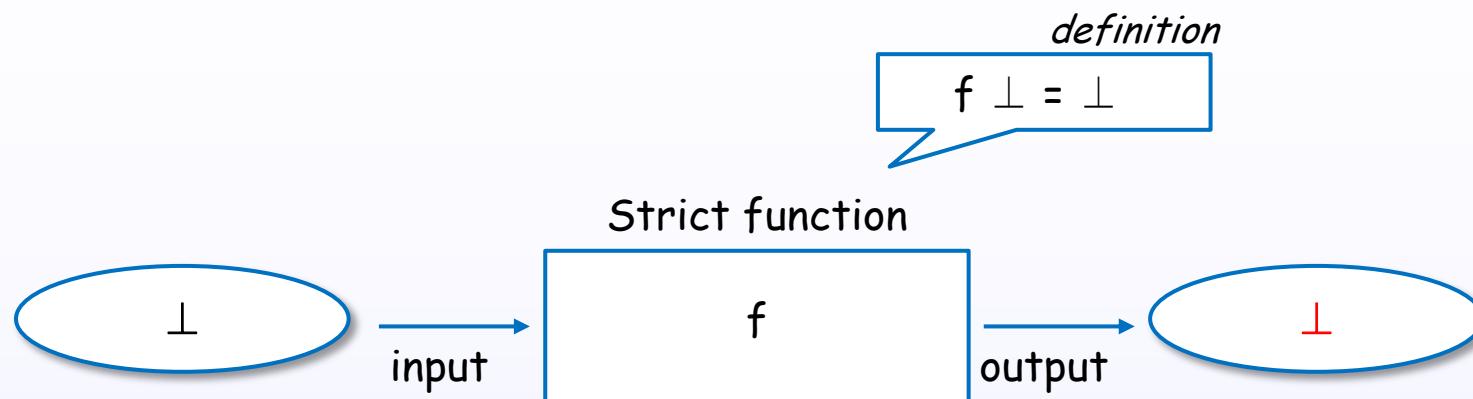
# Strictness



Strictness is property of the function.

"given a non-terminating arguments, the function will terminate?"

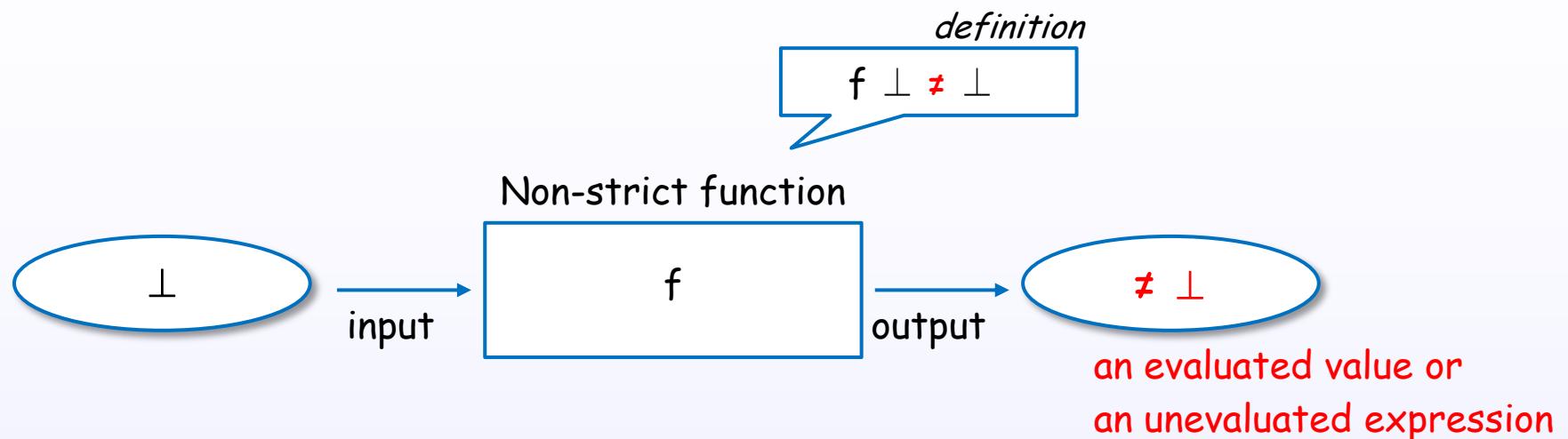
# Strict function



Strict function's output is bottom when input is bottom.

given a non-terminating arguments, strict function will **not** terminate.

# Non-strict function

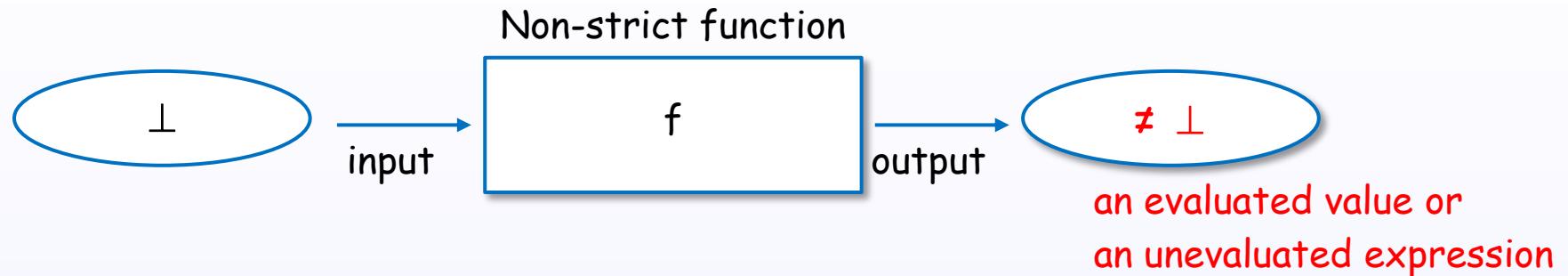


Non-strict function's output is **not** bottom when input is bottom.

given a non-terminating arguments, non-strict function will terminate.

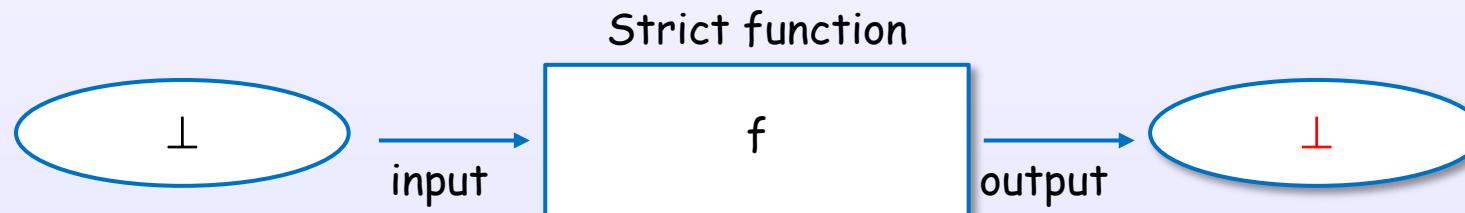
# Strict and Non-strict functions

## Non-strict



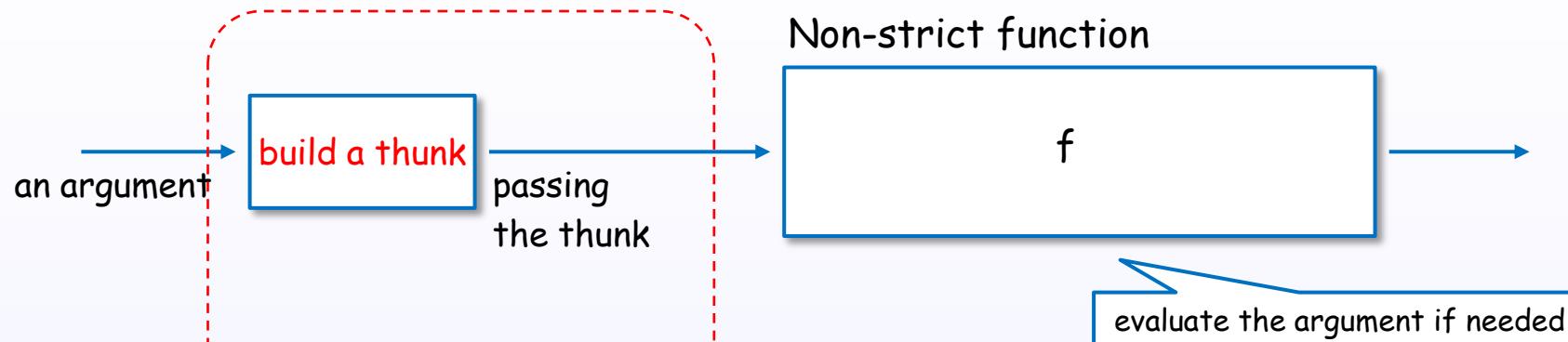
---

## Strict

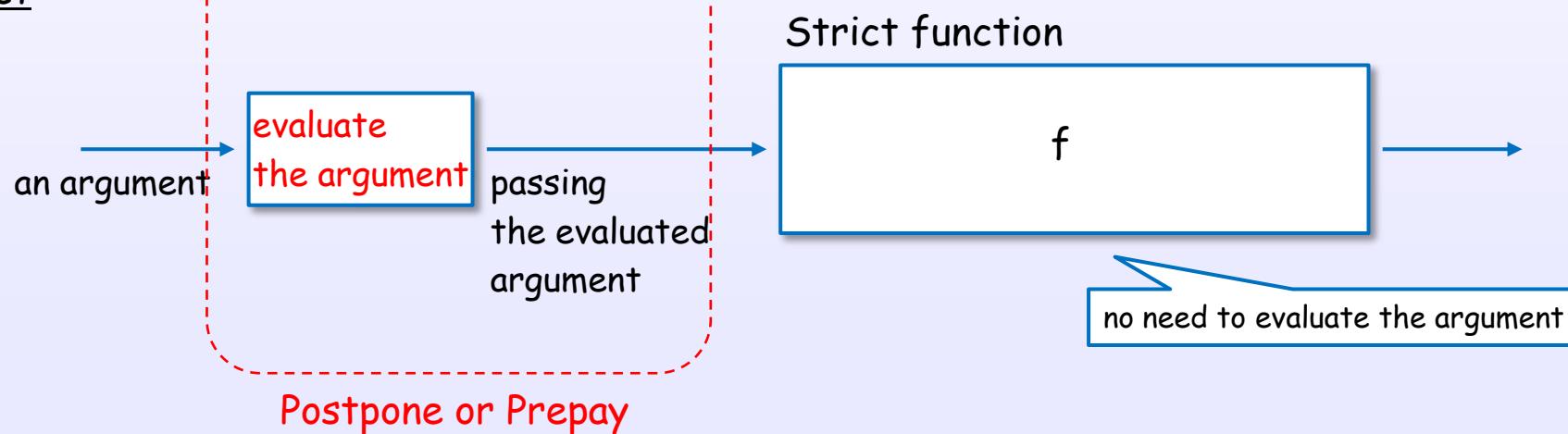


# Function application and strictness

## Non-strict



## Strict



The front stage is also important.

# Strict and normal form

## Example of function application

	to WHNF	to NF
Strict	$f \$! \text{arg}$ (seq)	$f \$!! \text{arg}$ (deepseq)
Non-strict	$f \$ \text{arg}$	

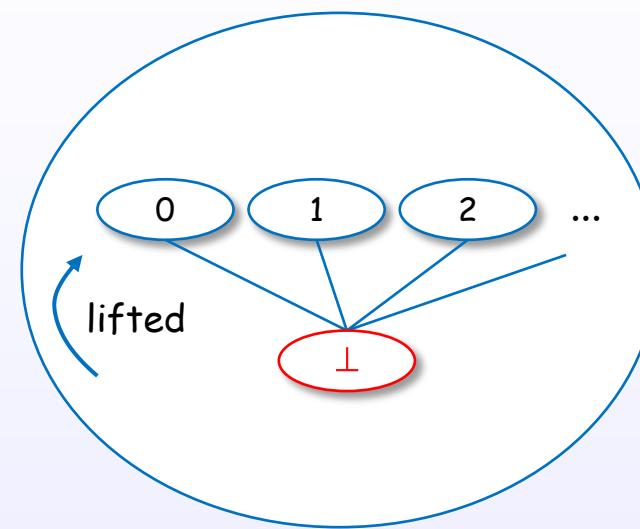
Strict  $\neq$  Normal form

## 6. Semantics

Lifted and boxed types

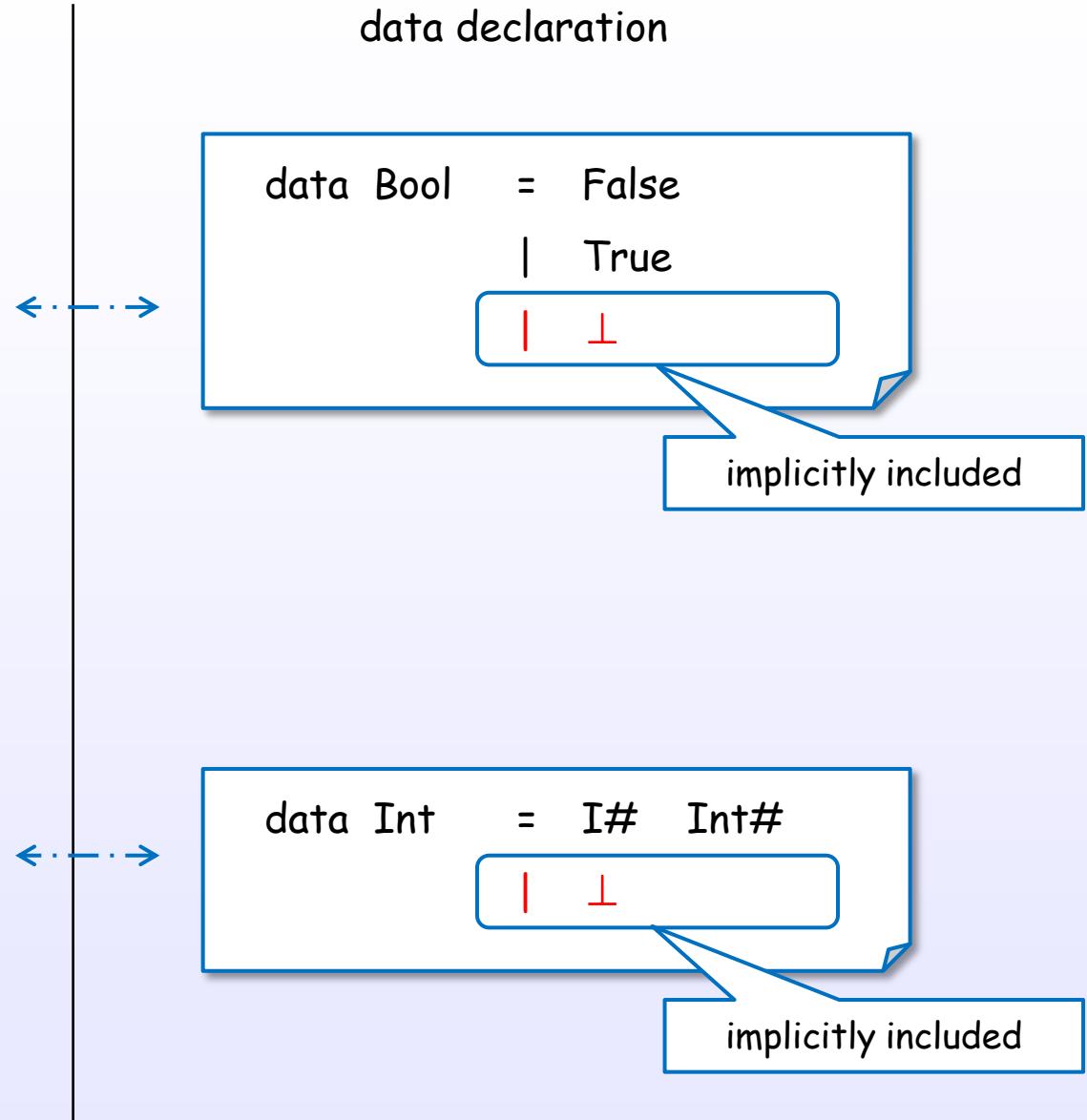
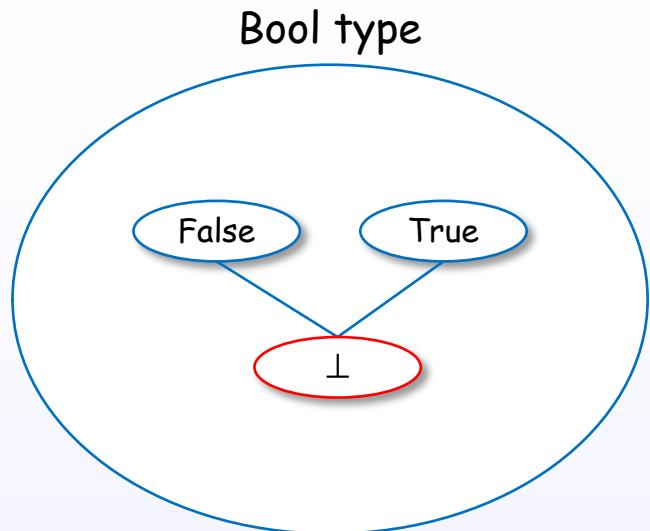
# Lifted types

Lifted type



Lifted types include bottom as an element.

# Lifted type's declaration implicitly include bottom



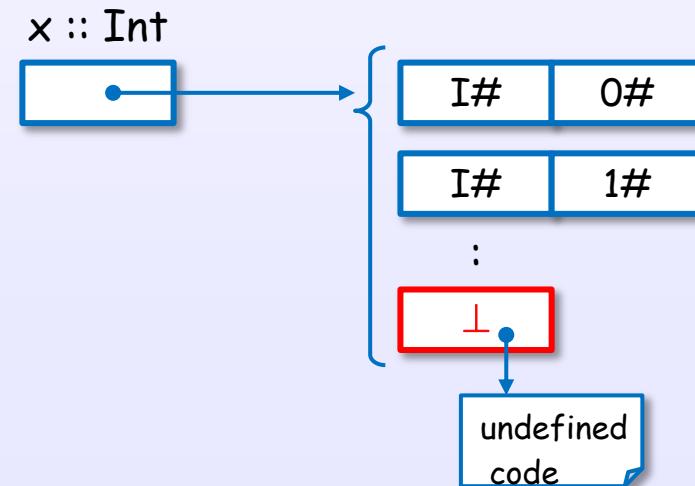
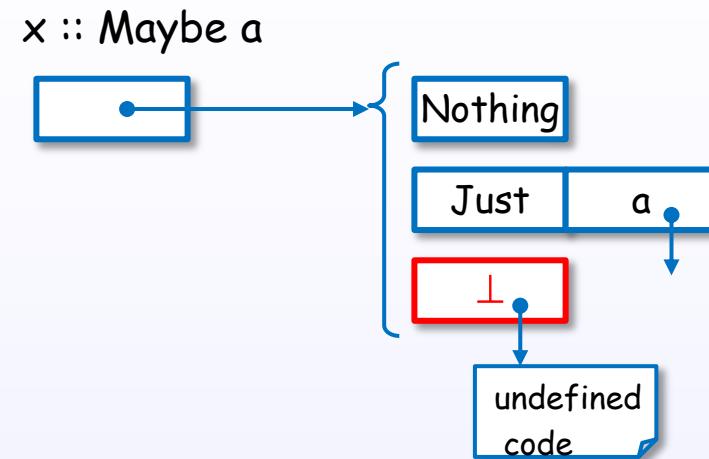
# Lifted type are also implemented by uniform representation

data declaration

```
data Maybe a = Nothing
             | Just a
             | ⊥
```

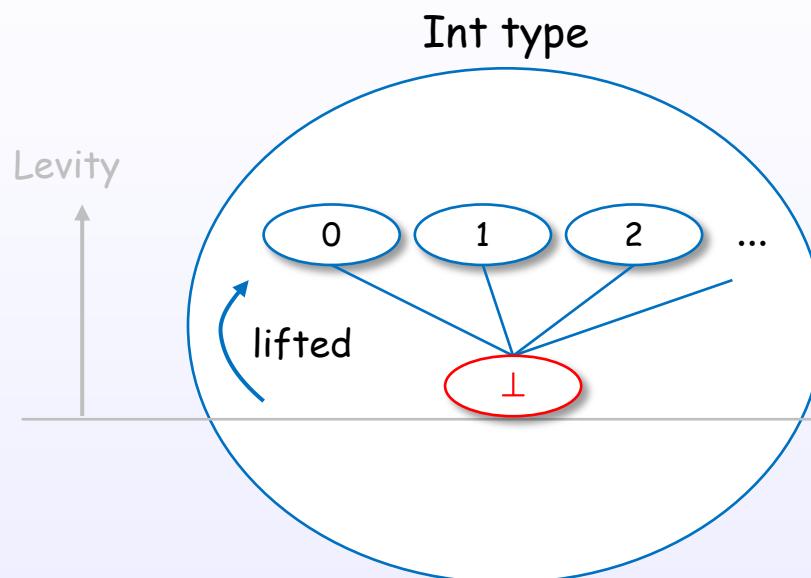
```
data Int    = I# Int#
             | ⊥
```

GHC's internal representation



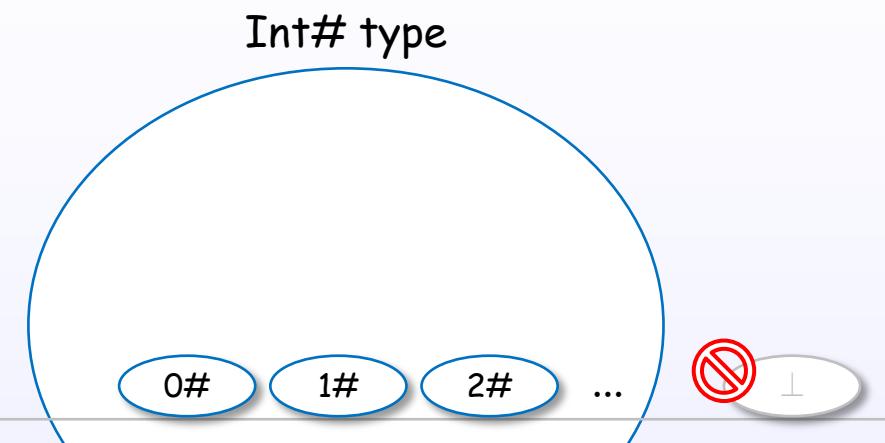
# Lifted and unlifted types

## Lifted types



Lifted types include bottom.  
(Bool, Int, Char, Maybe, List, ...)

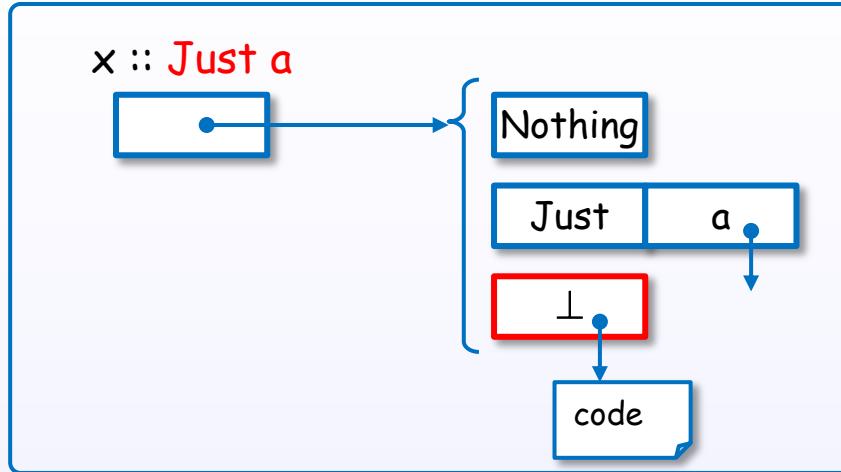
## Unlifted types



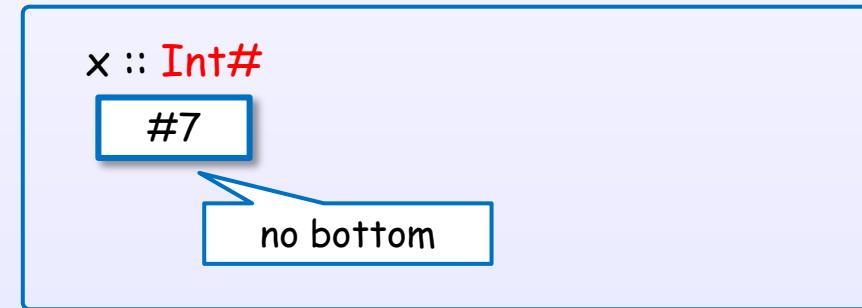
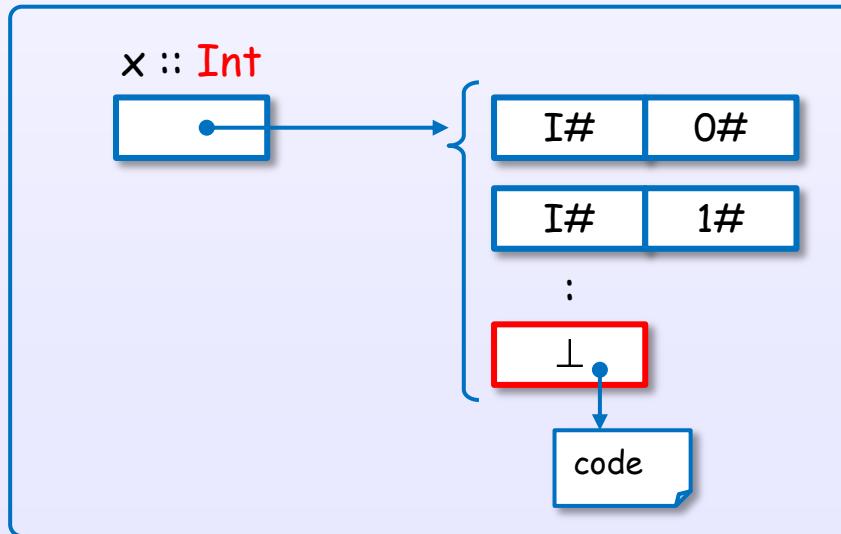
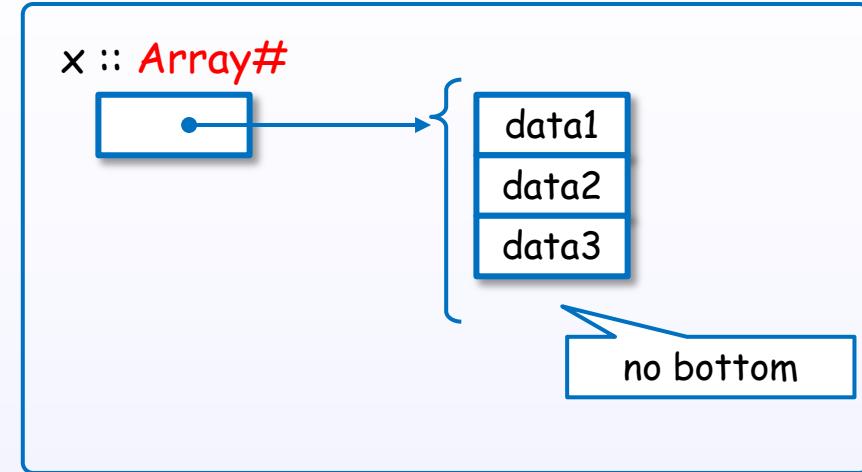
Unlifted types do not include bottom.  
(Int#, Char#, Addr#, Array#, ByteArray#, ...)

# Example of lifted and unlifted types

Lifted types



Unlifted types



# Boxed and unboxed type

Boxed  
types



Boxed types are represented as a pointer.

---

Unboxed  
types

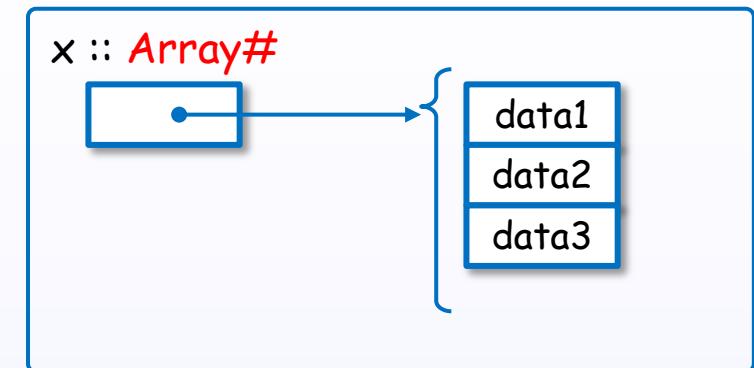
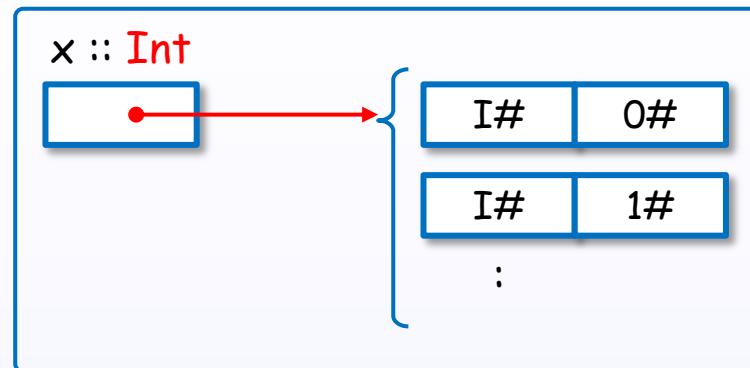


Unboxed types are represented other than a pointer.

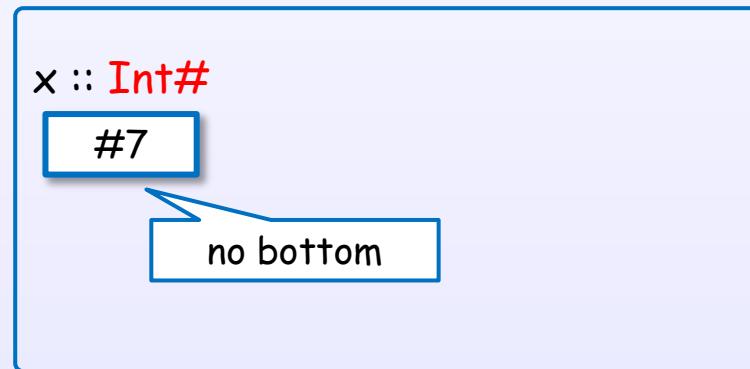
- no bottom (can't be lifted)
- no thunk (can't be postponed)
- no polymorphism (non-uniform size)
- + low cost memory size (no pointer)
- + high performance (no wrap/unwrap)

# Example of boxed and unboxed types

Boxed  
types



Unboxed  
types



# Lifted and boxed type

Boxed  
types

Lifted types

Int  
Char  
Float  
Maybe  
:

Unlifted types

Array#  
ByteArray#  
:

no bottom

Unboxed  
types

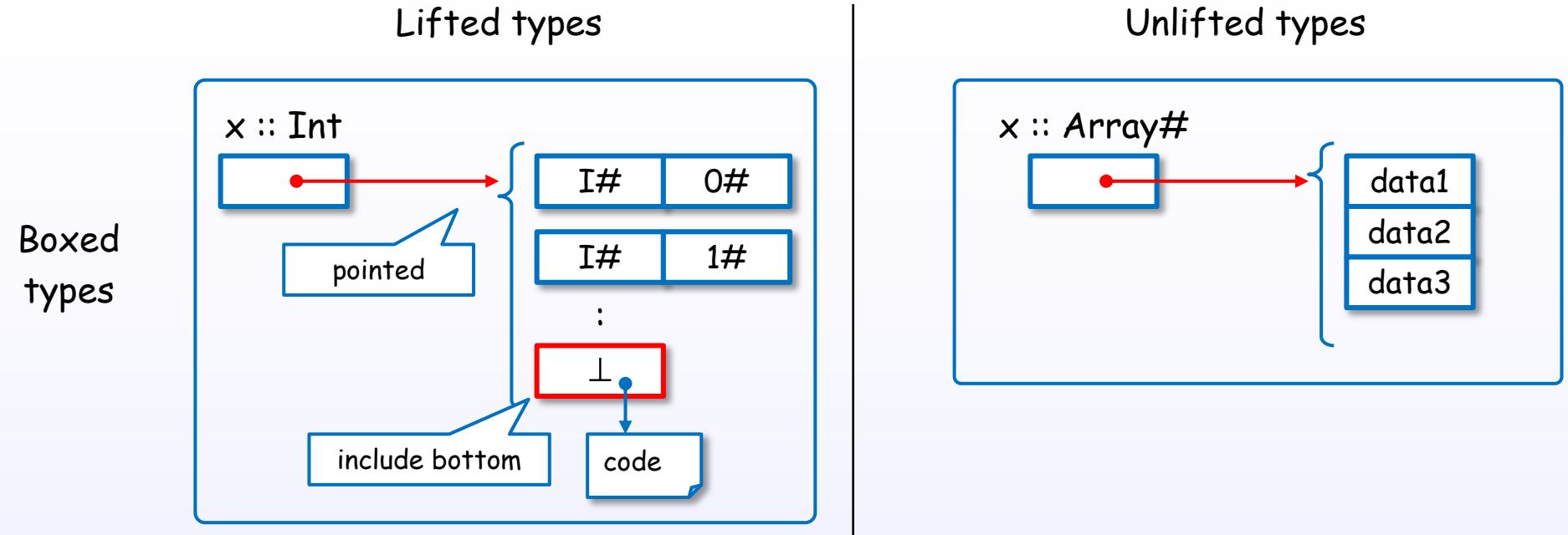


unboxed can't be lifted

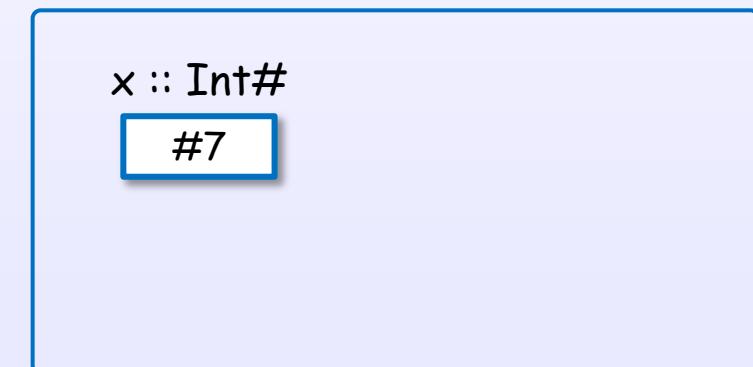
Int#  
Char#  
Float#  
:

no bottom  
no packed

# Example of lifted and boxed type



Unboxed  
types



# Types and kinds

Boxed  
types

Lifted types

kind '\*' 

Int  
Char  
Float  
Maybe  
:

Unlifted types

kind '#' 

Array#  
ByteArray#  
:

Unboxed  
types



Note:

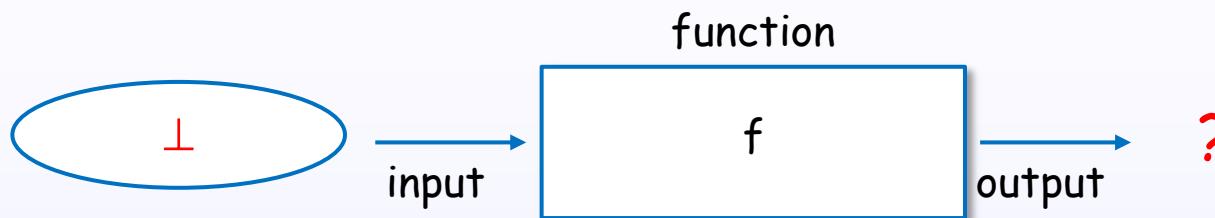
Identifier's '#' customarily means "primitive" rather than "unboxed" or "unlifted".

Kind's '#' means "unlifted".

## 6. Semantics

Strict analysis

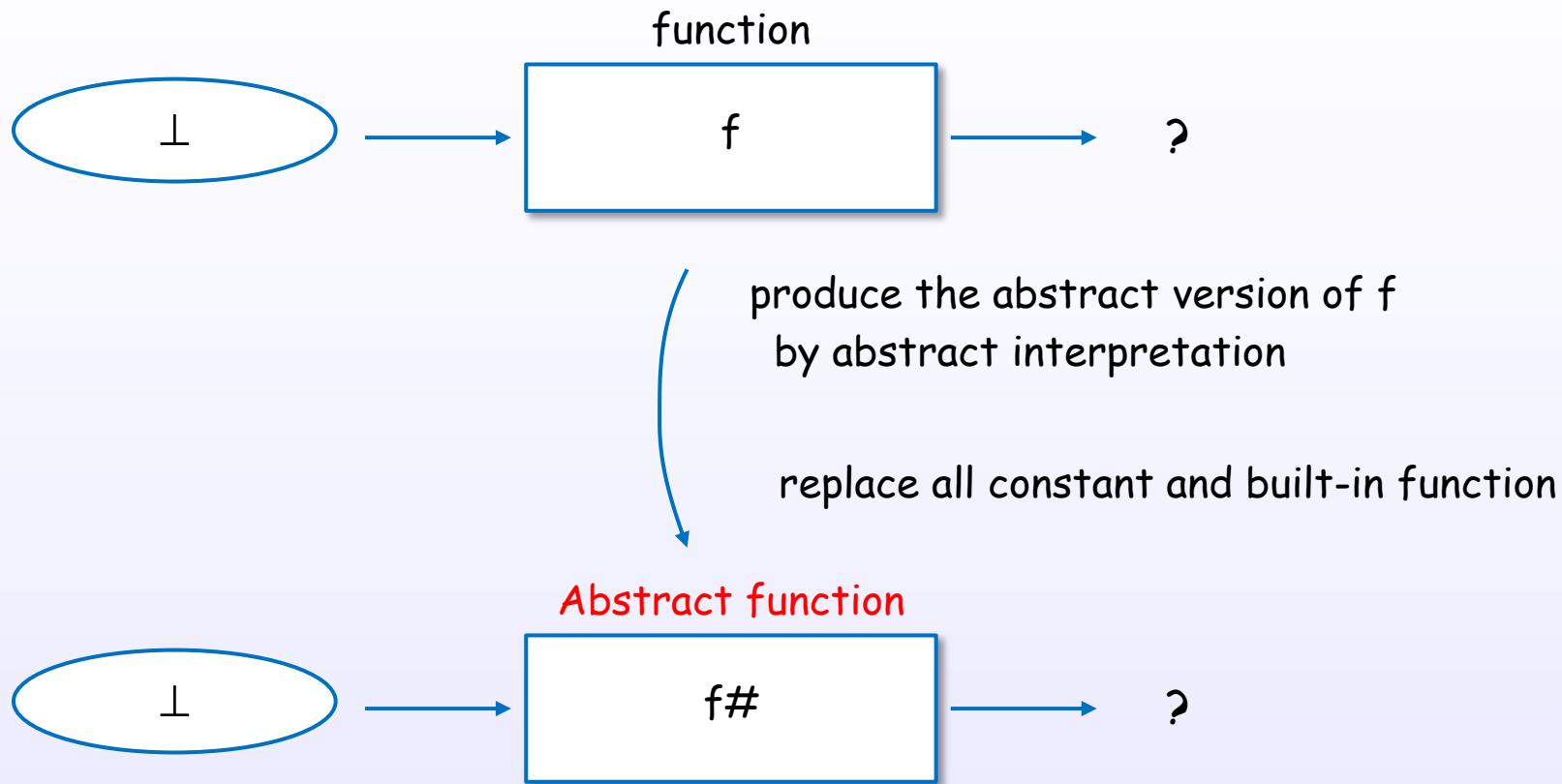
# Strict analysis



Strictness analysis analyzes whether a function is sure to evaluate its argument.

"-ddump-stranal" and "-ddump-strsigs" show strictness analysis information.

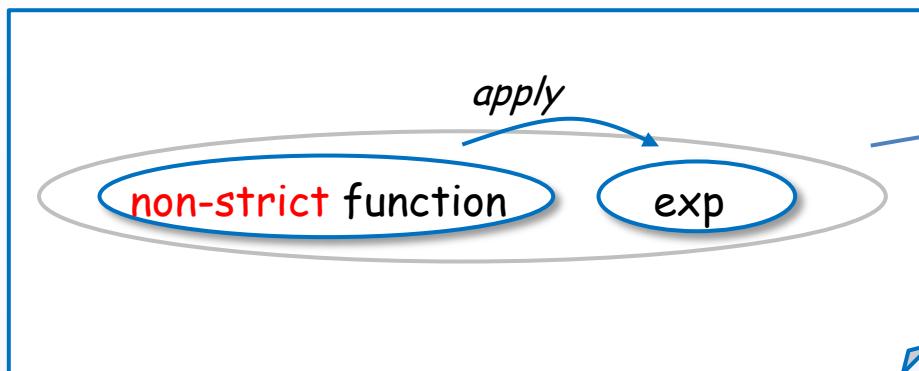
# Strict analysis using abstract function



Abstract function can decide strictness without going via full evaluation.

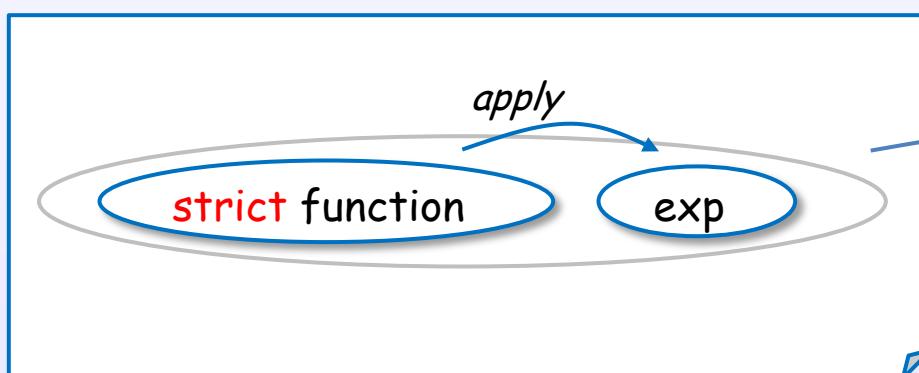
# Strictness analysis are used to avoid the thunk

non-strict function

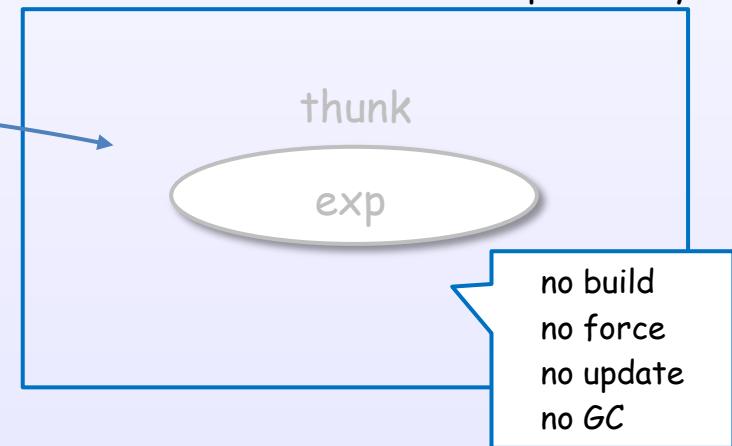


heap memory

strict function



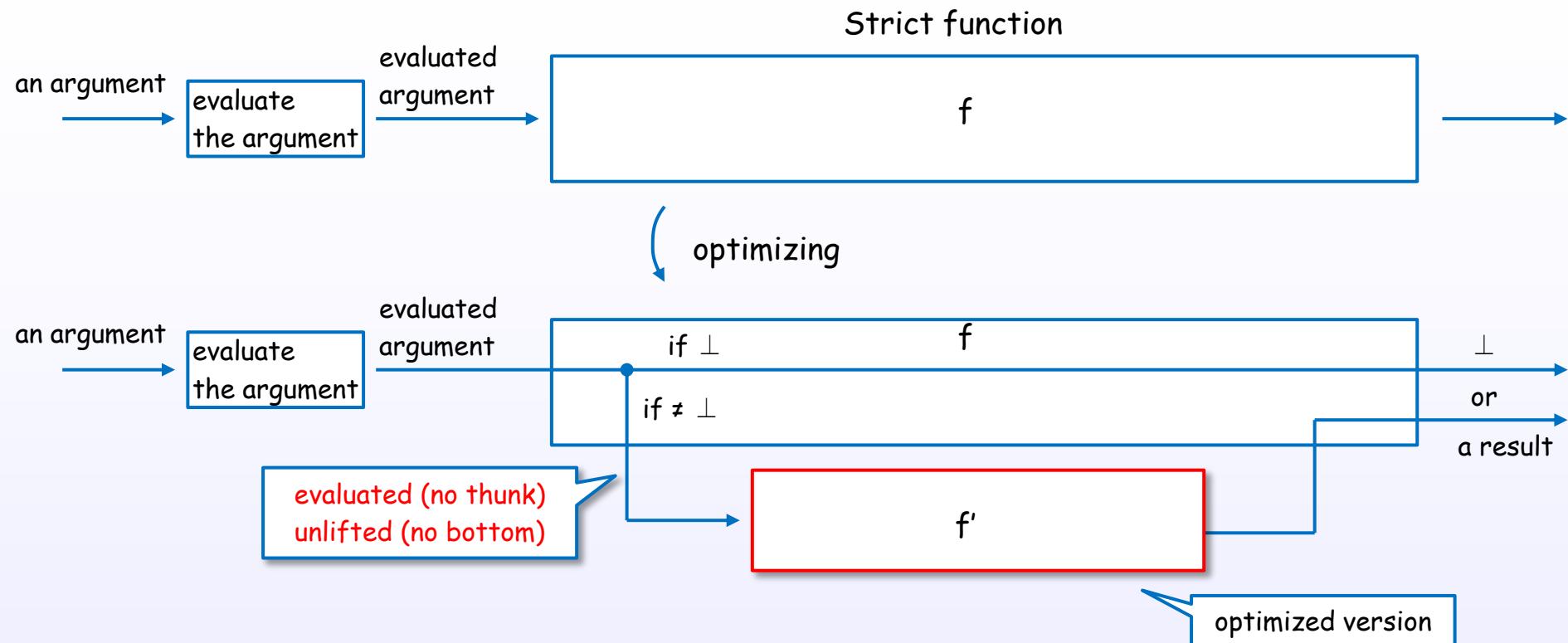
heap memory



If GHC knows that a function is strict, arguments are evaluated before application.

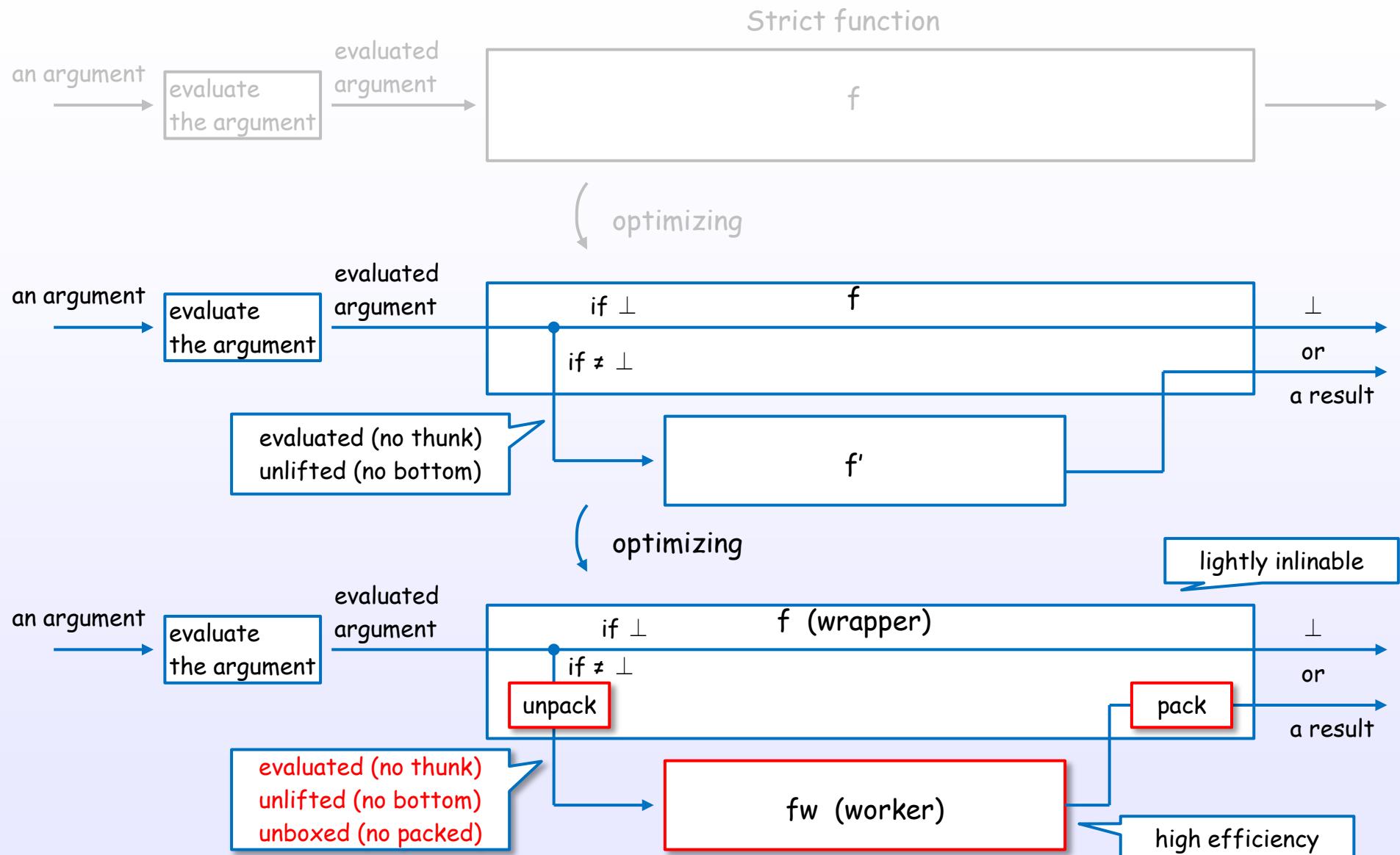
GHC finds strict functions by "strictness analysis (demand analysis)".

# Strictness analysis are also used to optimize



Strictness function can be optimized to assume no thunk, no bottom.

# Strictness analysis are also used to optimize



Strictness function can be optimized to assume no thunk, no bottom, no wrap.

## 6. Semantics

Sequential order

# "seq" doesn't guarantee the evaluation order

specification

```
seq a b = ⊥, if a = ⊥  
      = b, otherwise
```

strictness for each arguments

```
seq ⊥ b = ⊥ // a is strict  
seq a ⊥ = ⊥ // b is strict
```

"seq" function only guarantee that it is strict in both arguments.

This semantics property makes **no operational guarantee** about **order** of evaluation.

# "seq" and "pseq"

specification

```
seq a b = ⊥, if a = ⊥  
= b, otherwise
```

```
seq ⊥ b = ⊥ // a is strict  
seq a ⊥ = ⊥ // b is strict
```

specification

```
pseq a b = ⊥, if a = ⊥  
= b, otherwise
```

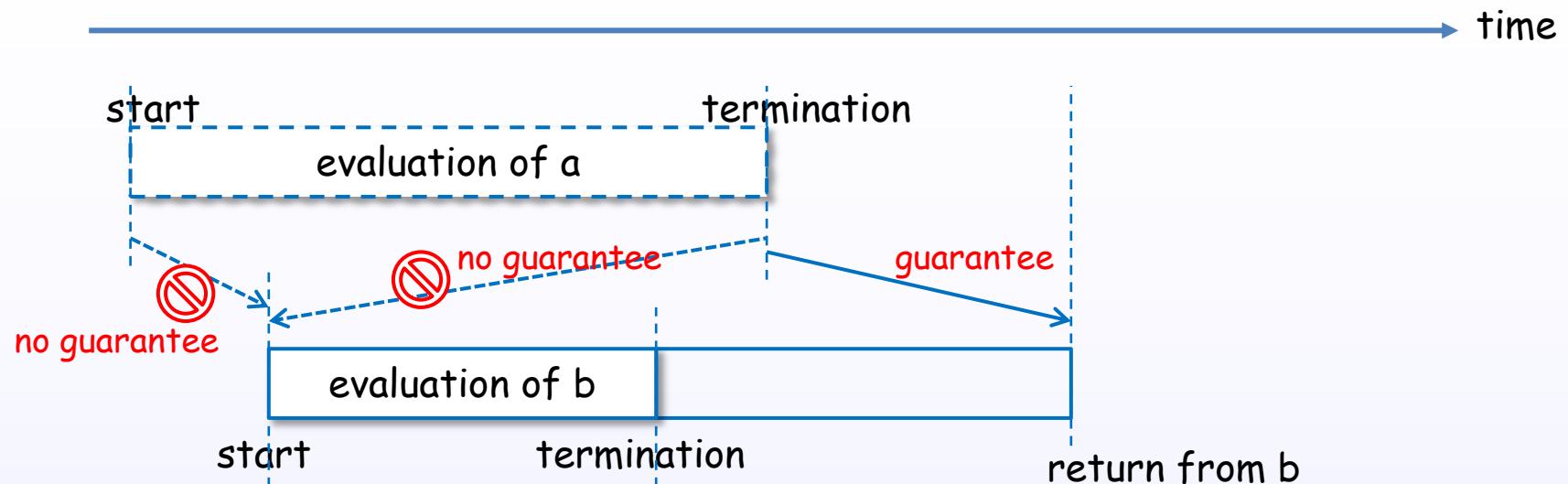
```
pseq ⊥ b = ⊥ // a is strict  
pseq a ⊥ = ⊥ // b is strict
```

Both of denotational semantics are the same.

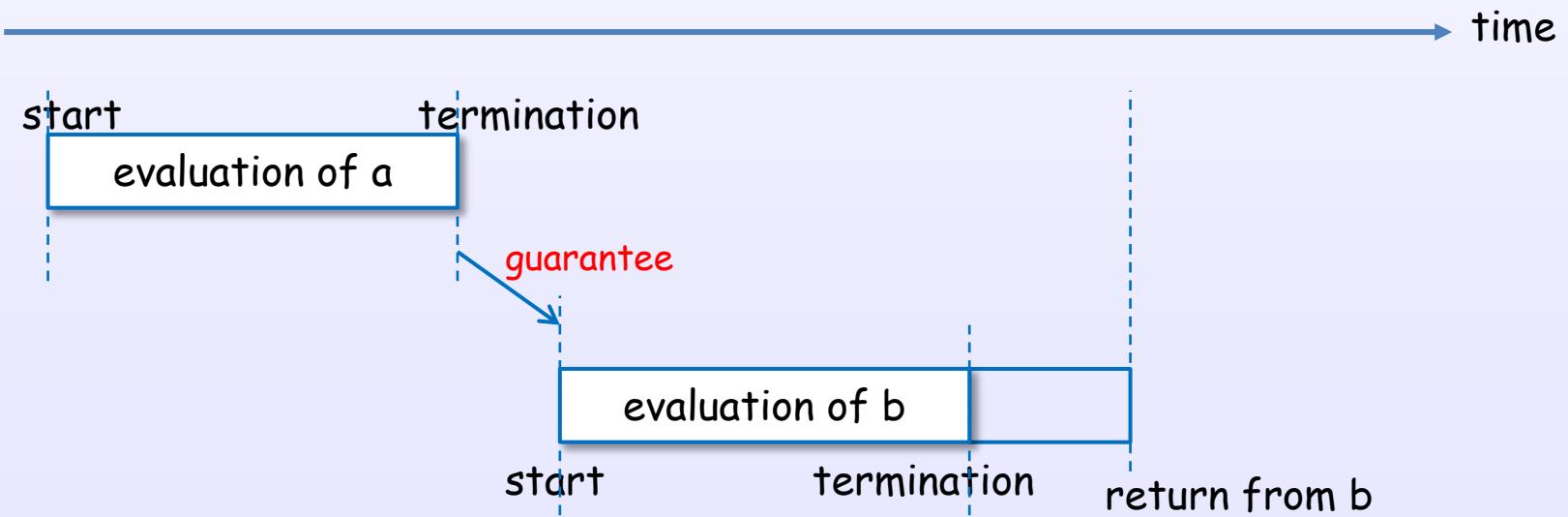
But "pseq" makes operational guarantee about order of evaluation.

# Evaluation order of "seq" and "pseq"

seq a b



pseq a b



# Implementation of "seq" and "pseq"

specification

```
seq a b = ⊥, if a = ⊥
          = b, otherwise
```

specification

```
pseq a b = ⊥, if a = ⊥
          = b, otherwise
```

Haskell's built-in

`pseq x y = x `seq` lazy y`

GHC's "lazy" function restrains  
the strictness analysis.

"seq" is built-in function.

"pseq" is implemented by built-in functions ("seq" and "lazy").

## 7. Appendix

## 7. Appendix

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Lazy,... zzz

*to be as lazy as possible...*