

Lazy evaluation illustrated

for Haskell divers

exploring some mental models and implementations

Takenobu T.

Lazy,... zzz

..., It's fun!

NOTE

- Meaning of terms are different by communities.
- There are a lot of good documents. Please see also references.
- This is written for GHC's Haskell.

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1. Introduction

1. Introduction

Basic mental models

How to evaluate a program in your brain ?

a program

```
code  
code  
code  
:  
?
```

How to evaluate (execute, reduce) the program in your brain?

What "mental model" do you have?

One of the mental models for C program

C program

A sequence of statements

```
main (...) {  
    code..  
    code.. } ?  
    code..  
    code..  
    code.. }
```

A nested structure

```
x = func1( func2( a ) );  
_____ ?
```

A sequence of arguments

```
y = func1( a(x), b(x), c(x) );  
_____ ? _____
```

A function and arguments

```
z = func1( m + n );  
_____ ? _____
```

How to evaluate (execute, reduce) the program in your brain?

What step, what order, ... ?

One of the mental models for C program

C program

A sequence of statements

```
main (...) {  
    code..  
    code..  
    code..  
    code..  
}
```

Statements are
executed downward.

A program is a collection of statements.

A nested structure

```
x = func1( func2( a ) );
```

from inner to outer

A sequence of arguments

```
y = func1( a(x), b(x), c(x) );
```

from left to right

A function and arguments

```
z = func1( m + n );
```

arguments first
apply second

Each programmers have some mental models in their brain.

One of the mental models for C program

Maybe, You have some implicit mental model in your brain for C program.

(1) A program is **a collection of statements**.

(2) There is **an order** between evaluations of elements.



(3) There is **an order** between termination and start of evaluations.



This is a **syntactically straightforward** model for programming languages.
(an implicit sequential order model)

One of the mental models for Haskell program

Haskell program

```
main = exp11 (exp12 exp13 exp14 )  
  
exp13 = exp131 exp132  
  
exp14 = exp141 exp142 exp143  
:
```



How to evaluate (execute, reduce) the program in your brain?
What step, what order, ... ?

One of the mental models for Haskell program

Haskell program

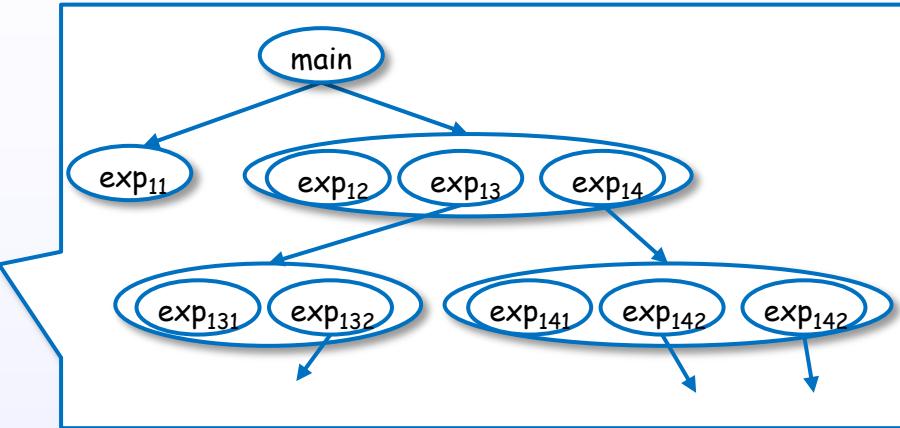
A program is a collection of expressions.

main = exp₁₁ (exp₁₂ exp₁₃ exp₁₄)

exp₁₃ = exp₁₃₁ exp₁₃₂

exp₁₄ = exp₁₄₁ exp₁₄₂ exp₁₄₃

:



main = exp₁₁ (exp₁₂ (exp₁₃₁ exp₁₃₂) (exp₁₄₁ exp₁₄₂ exp₁₄₃))

The entire program is regarded as a single expression.

The subexpression is evaluated (reduced) in some order.

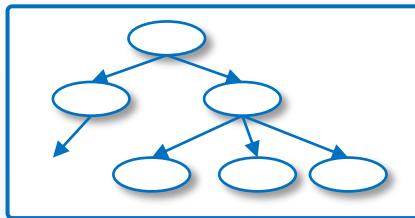
The evaluation is performed by replacement.

One of the mental models for Haskell program

(1) A program is a collection of expressions.

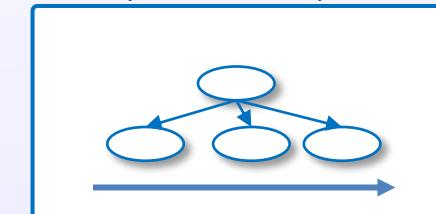
(2) A entire program is regarded as a single expression.

```
main = e (e (e (e e) e (e e e)) )
```

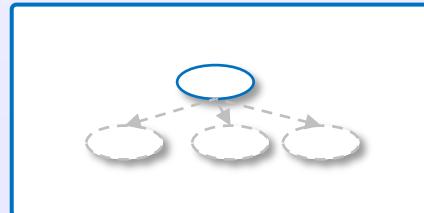
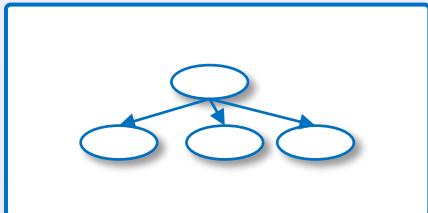


(3) The subexpressions are evaluated (reduced) in some order.

```
f = e (e (e (e e) e (e e e)) )
```



(4) The evaluation is performed by replacement.



This is an example of an expression reduction model for Haskell.

1. Introduction

Lazy evaluation

Why lazy evaluation?

To avoid unnecessary computation

To manipulate infinite data structures

modularity

To manipulate streams

abstraction

pure is order free

amortizing

To manipulate huge data structures

potentially parallelism

2nd Church-Rosser theorem

out-of-order optimization

To implement non-strict semantics

asynchronization

fun

reactive

...

There are various reasons ☺

Haskell(GHC) 's lazy evaluation

Lazy evaluation

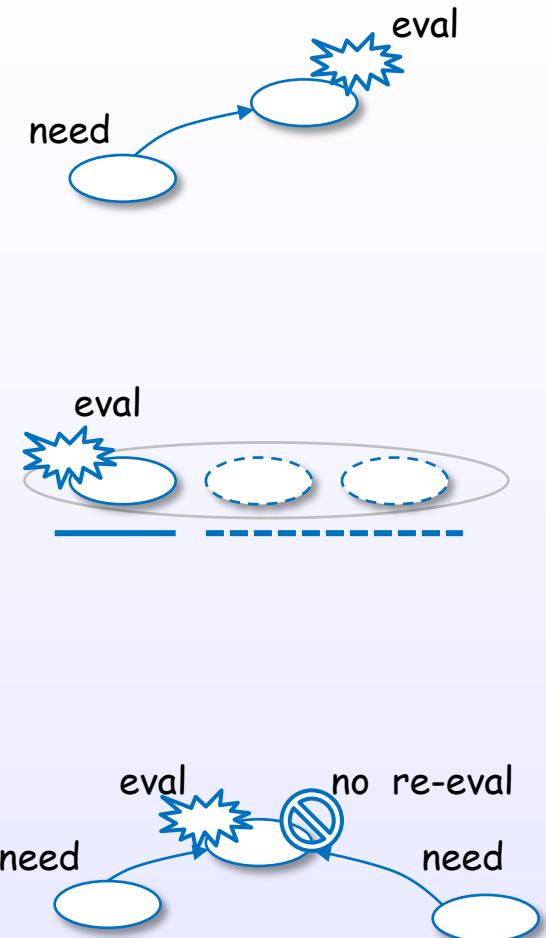
evaluate only if needed

+

evaluate only enough

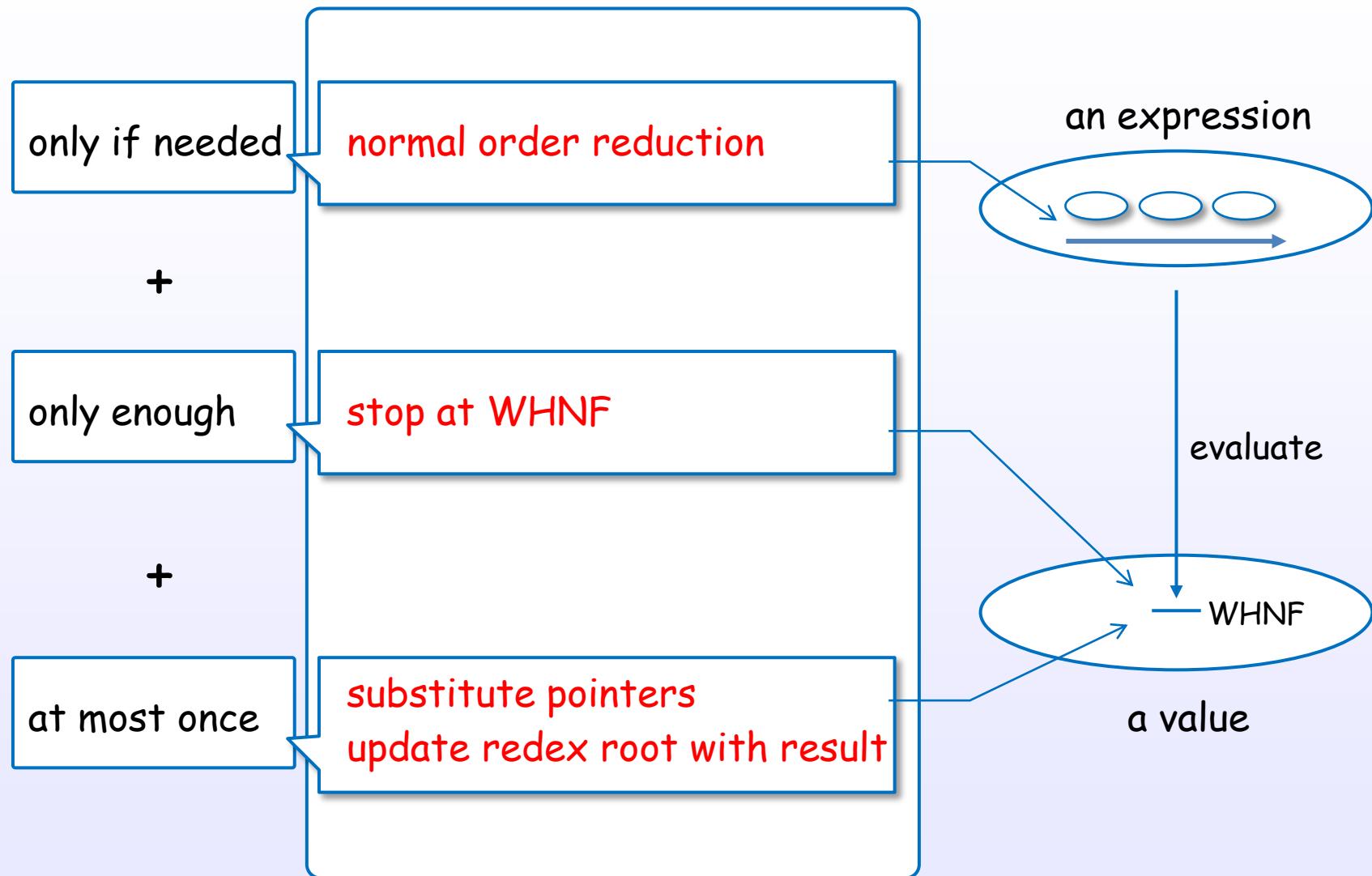
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evaluate at most once



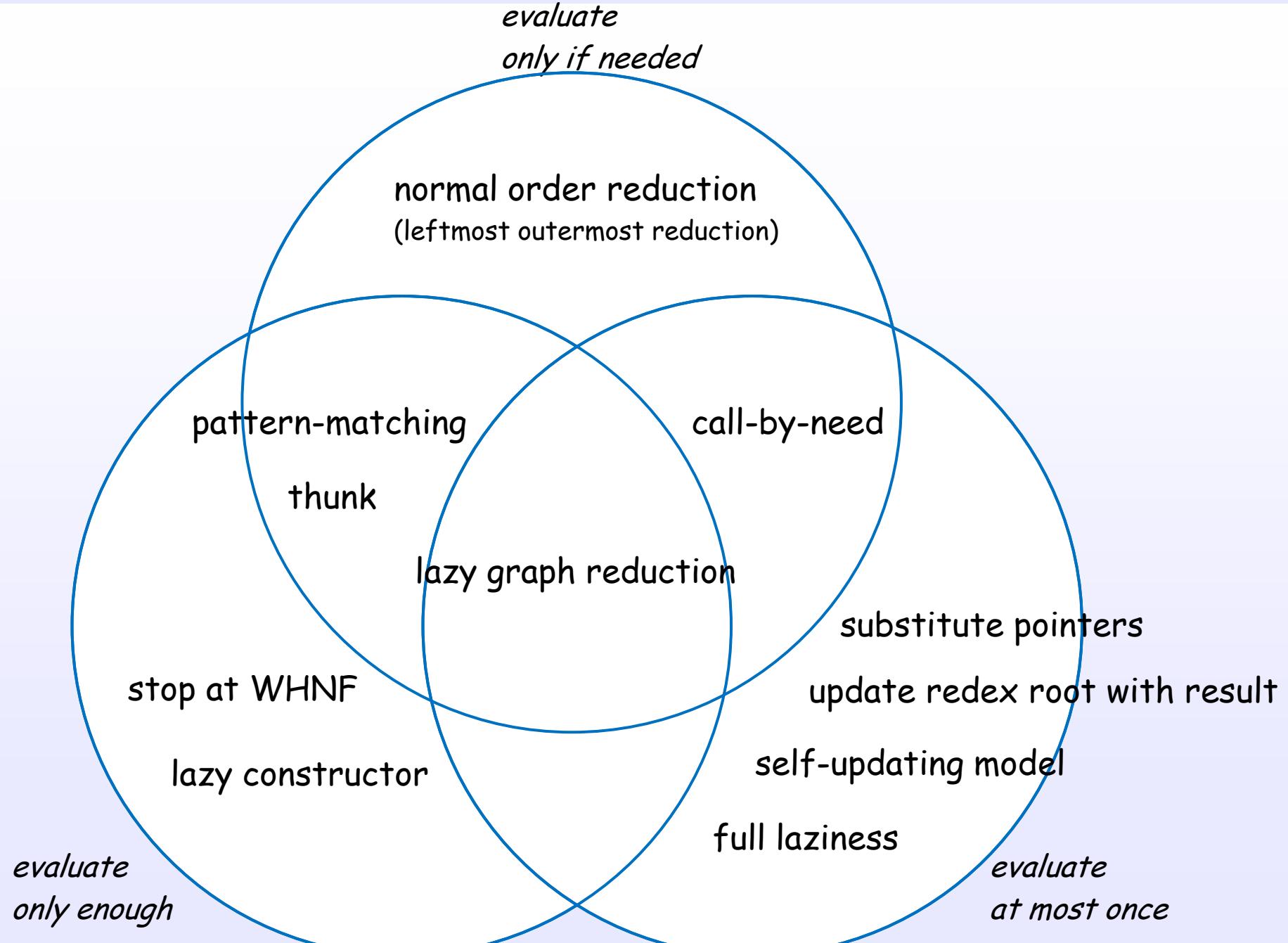
"Lazy" is "delay and avoidance" rather than "delay".

Ingredient of Haskell(GHC) 's lazy evaluation



This strategy is implemented by lazy graph reduction.

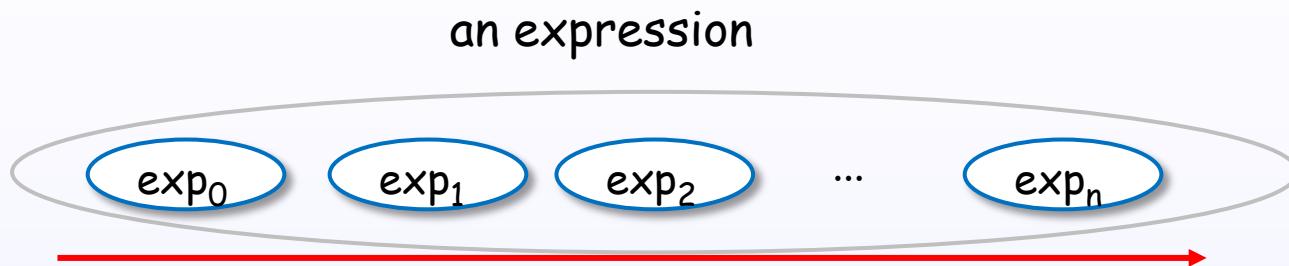
Techniques of Haskell(GHC) 's lazy evaluation



1. Introduction

Simple questions

What order?



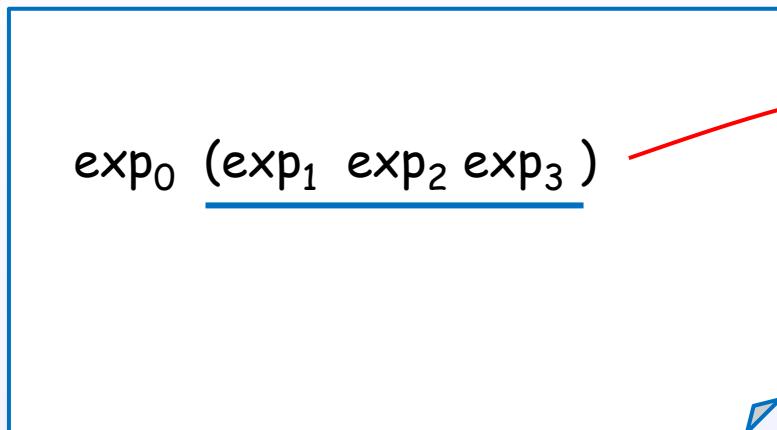
An expression is evaluated by normal order (leftmost outermost redex first).

Normal order reduction guarantees to find a normal form (if one exists)

To avoid unnecessary computation, normal order reduction chooses to apply the function rather than first evaluating the argument.

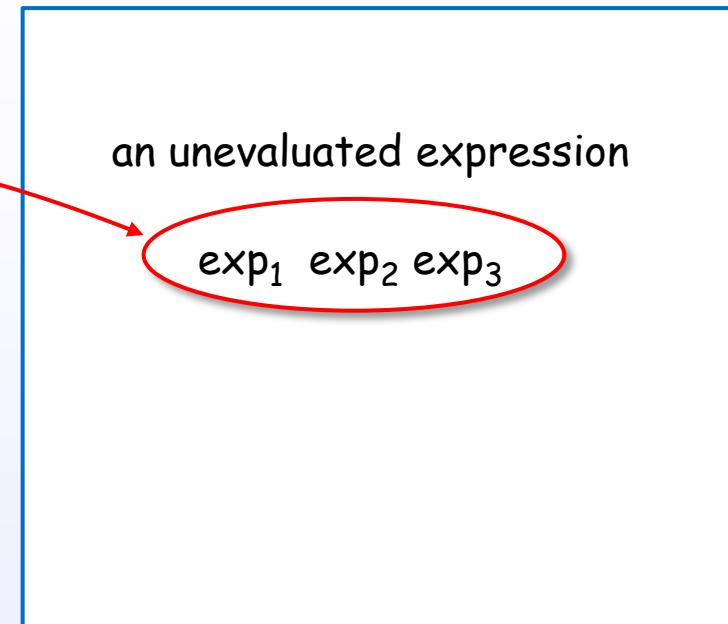
How to postpone?

Haskell code



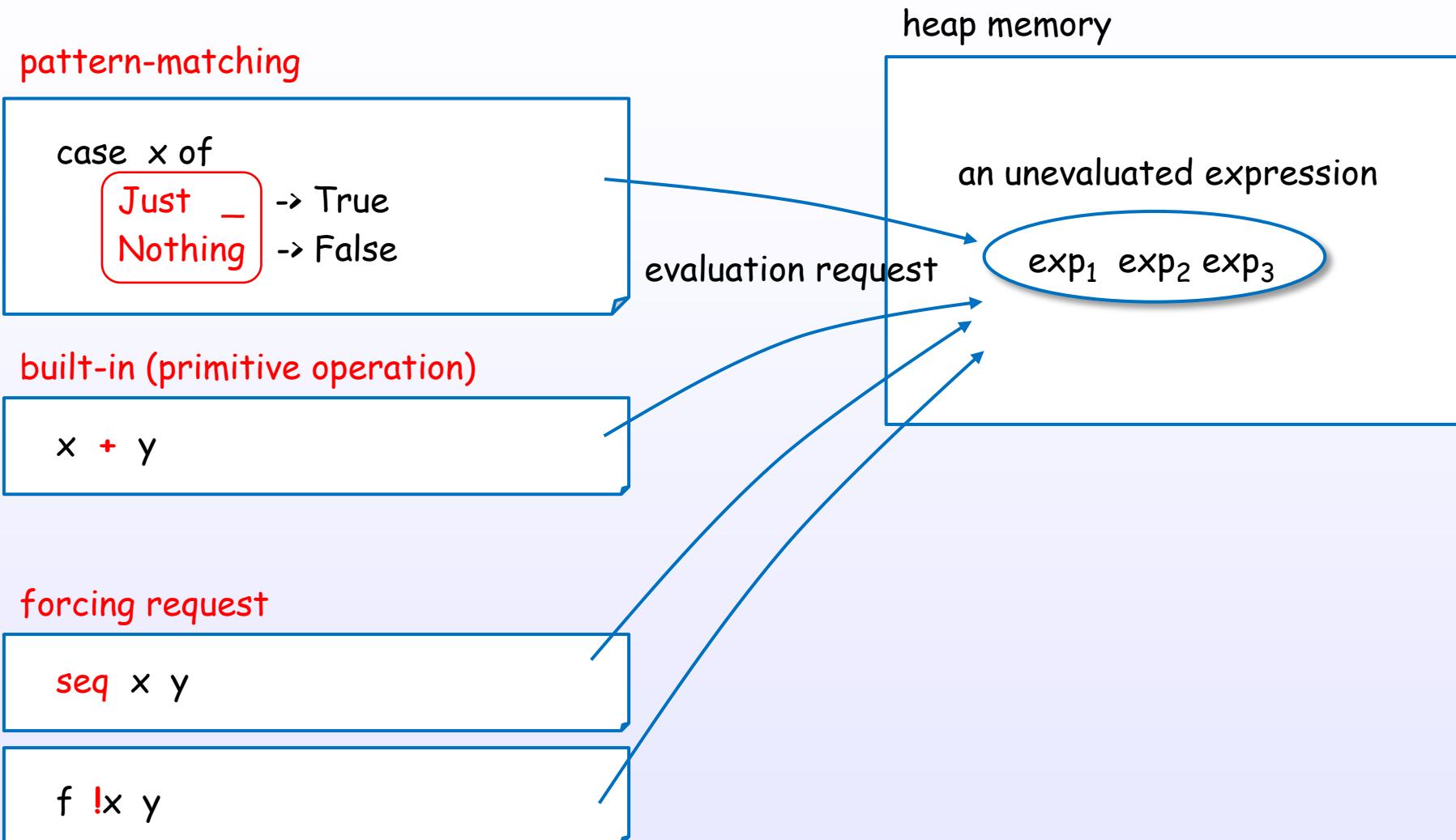
build/allocate

heap memory



To postpone the evaluation, an unevaluated expression is built in the heap memory.

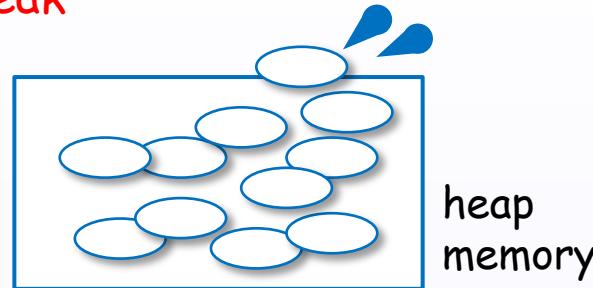
When needed?



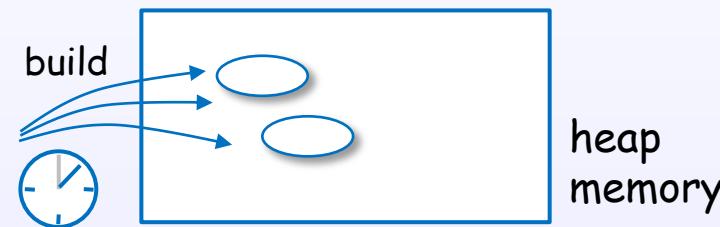
Pattern-matching or forcing request drive the evaluation.

What to be careful about?

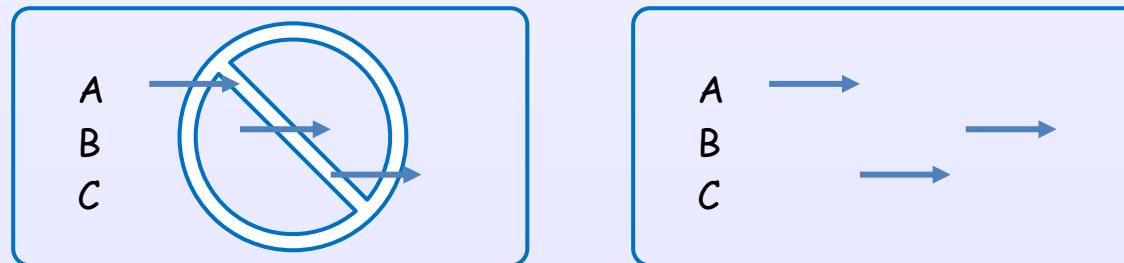
To consider hidden **space leak**



To consider **performance cost** to postpone unevaluated expressions



To consider evaluation (execution) **order** and **timing** in real world



You can avoid the pitfalls by controlling the evaluation.

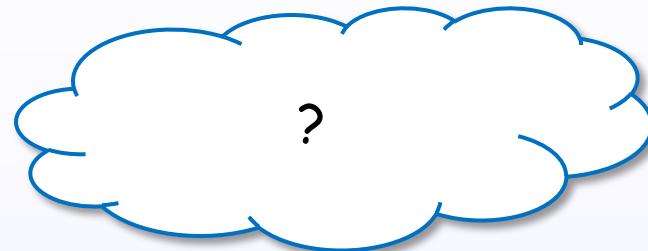
2. Expressions

2. Expressions

Expression and value

What is an expression?

An expression

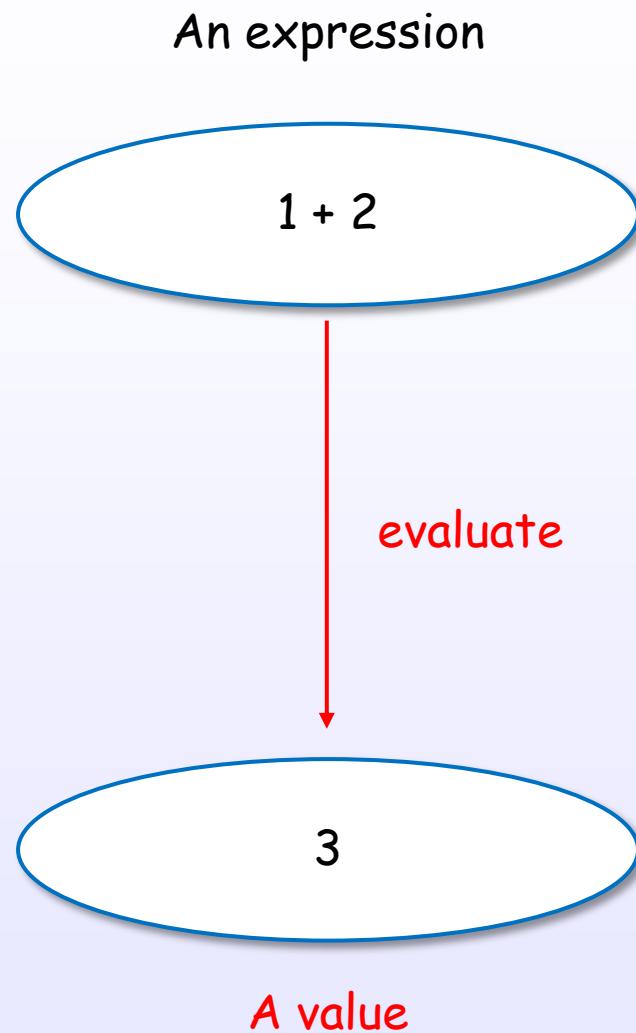


An expression denotes a value

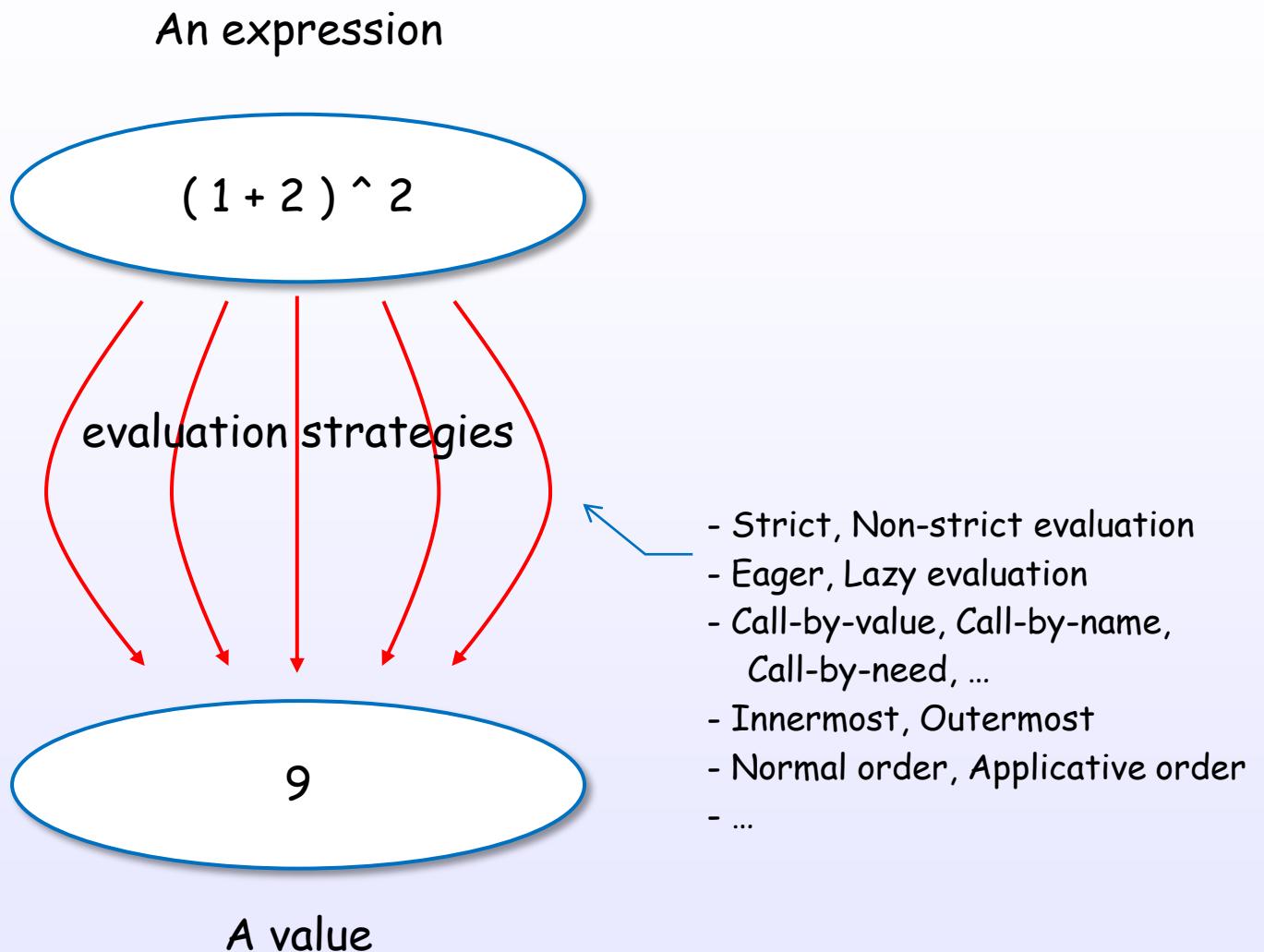
An expression

$$1 + 2$$

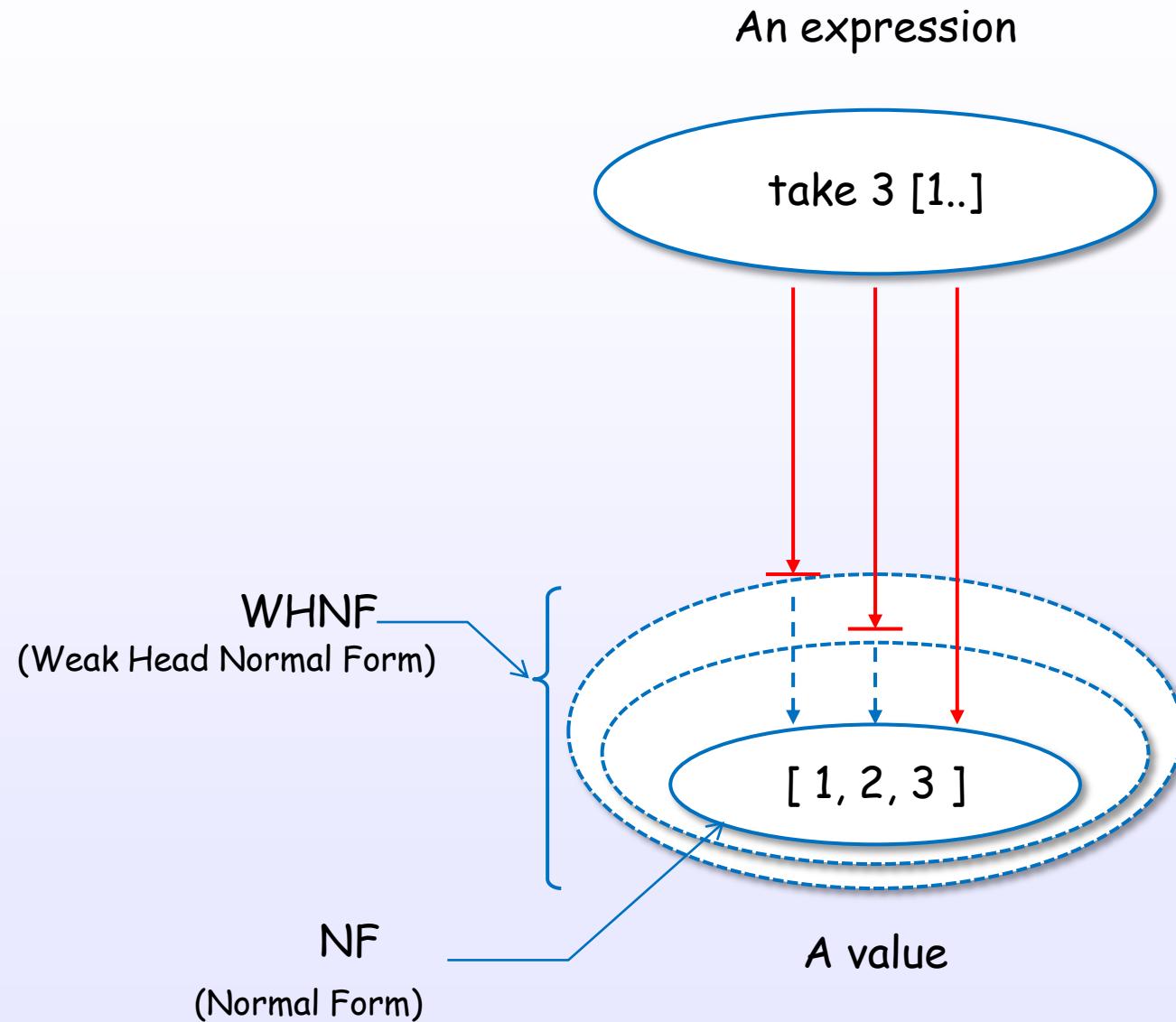
An expression evaluates to a value



There are many evaluation approaches



There are some evaluation levels

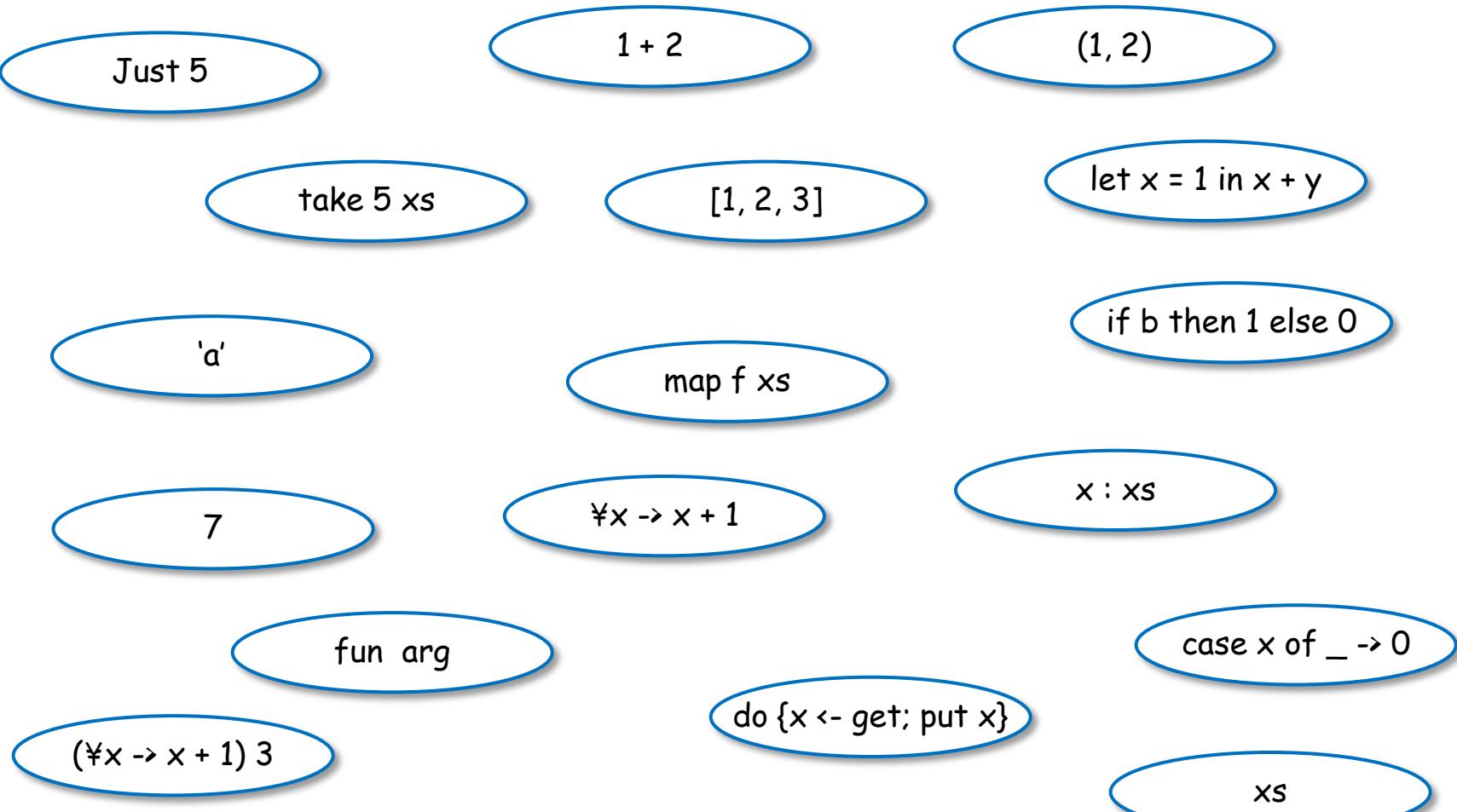


2. Expressions

Expressions in Haskell

There are many expressions in Haskell

Expressions



categorizing

Expression categories in Haskell

lambda abstraction

$\lambda x \rightarrow x + 1$

let expression

let $x = 1$ in $x + y$

conditional

if b then 1 else 0

case expression

case x of _ -> 0

do expression

do { $x \leftarrow \text{get}; \text{put } x$ }

function application

take 5 xs

$(\lambda x \rightarrow x + 1) 3$

1 + 2

map f xs

fun arg

general constructor, literal and some forms

7

[1, 2, 3]

(1, 2)

'a'

$x : xs$

Just 5

variable

xs

Specification is defined in Haskell 2010 Language Report

"Haskell 2010 Language Report, Chapter 3 Expressions" [H1]

<i>exp</i>	\rightarrow	<i>infixexp</i> :: [<i>context</i> =>] <i>type</i>	(expression type signature)
		<i>infixexp</i>	
<i>infixexp</i>	\rightarrow	<i>lexp qop infixexp</i>	(infix operator application)
		<i>- infixexp</i>	(prefix negation)
		<i>lexp</i>	
<i>lexp</i>	\rightarrow	$\lambda \ a_1 \dots \ a_n \rightarrow \ exp$	(lambda abstraction, $n \geq 1$)
		<i>let decls in exp</i>	(let expression)
		<i>if exp [;] then exp [;] else exp</i>	(conditional)
		<i>case exp of { alts }</i>	(case expression)
		<i>do { stmts }</i>	(do expression)
		<i>fexp</i>	
<i>fexp</i>	\rightarrow	<i>[fexp] aexp</i>	(function application)
<i>aexp</i>	\rightarrow	<i>qvar</i>	(variable)
		<i>gcon</i>	(general constructor)
		<i>literal</i>	
		<i>(exp)</i>	(parenthesized expression)
		<i>(exp₁ , ... , exp_k)</i>	(tuple, $k \geq 2$)
		<i>[exp₁ , ... , exp_k]</i>	(list, $k \geq 1$)
		<i>[exp₁ , exp₂] ... [exp₃]</i>	(arithmetic sequence)
		<i>[exp qual₁ , ... , qual_n]</i>	(list comprehension, $n \geq 1$)
		<i>(infixexp qop)</i>	(left section)
		<i>(qop(-) infixexp)</i>	(right section)
		<i>qcon { fbind₁ , ... , fbind_n }</i>	(labeled construction, $n \geq 0$)
		<i>aexp_(qcon) { fbind₁ , ... , fbind_n }</i>	(labeled update, $n \geq 1$)

2. Expressions

Classification by values and forms

Classification by values

Expressions

unevaluated expressions

take 5 xs $(\lambda x \rightarrow x + 1) 3$ let $x = 1$ in $x + y$
if b then 1 else 0 map f xs fun arg
case x of _ → 0 do {x ← get; put x}

values

data values

7 'a'
[1, 2, 3] (1, 2)
Just 5 Just (f x)

function values

$\lambda x \rightarrow x + 1$

bottom

\perp

Values are data values or function values.

Classification by forms

Expressions

unevaluated expressions

take 5 xs $(\lambda x \rightarrow x + 1) 3$ let $x = 1$ in $x + y$
if b then 1 else 0 map f xs fun arg
case x of _ → 0 do {x ← get; put x}

values

WHNF

HNF

NF

Just (f x) [f x, g y] $\lambda x \rightarrow \text{abs } 1$
[1, 2, 3] Just 5 $\lambda x \rightarrow x + (\text{abs } 1)$
7 'a' $\lambda x \rightarrow x$

bottom

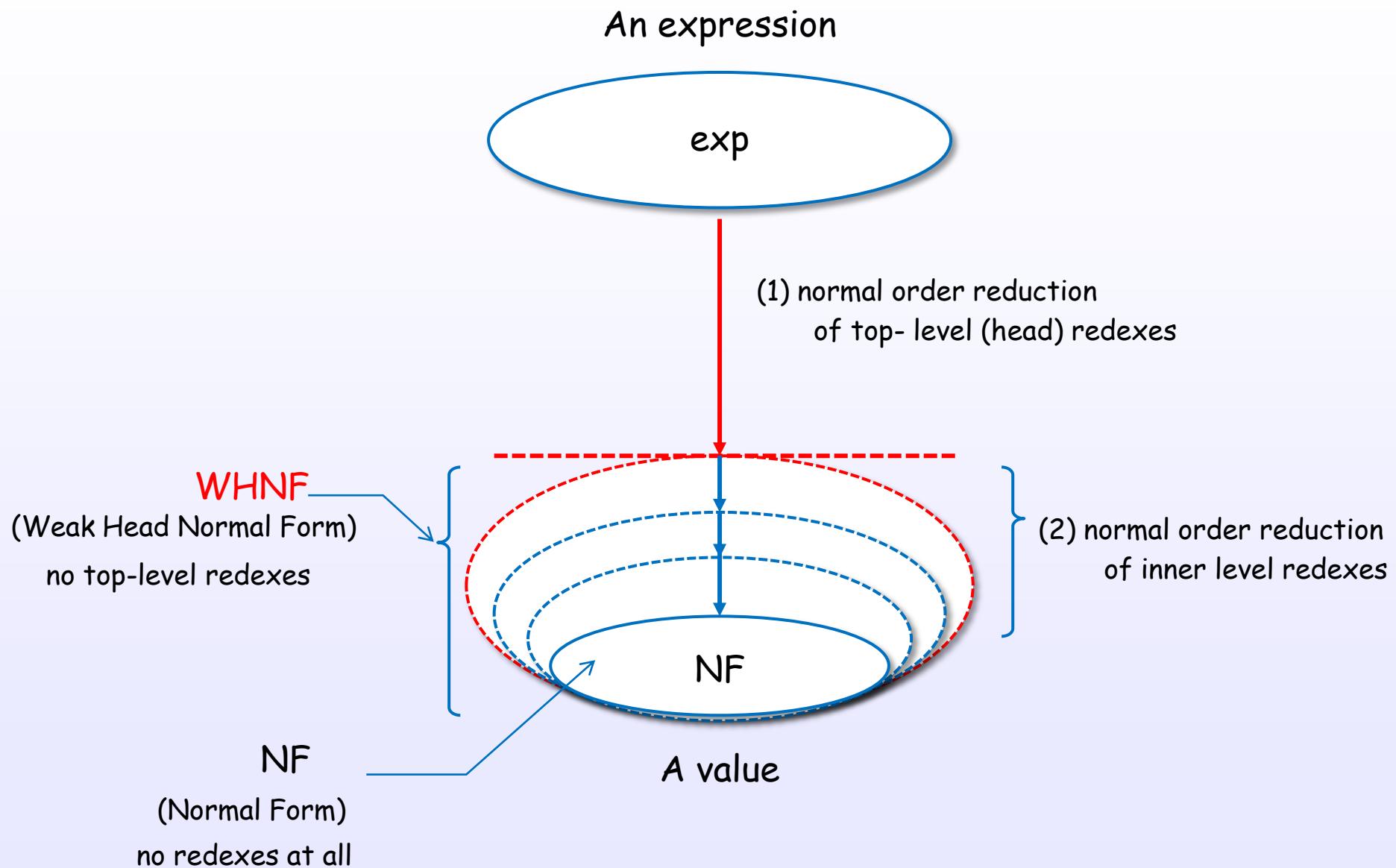
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Values are NF, HNF or WHNF.

2. Expressions

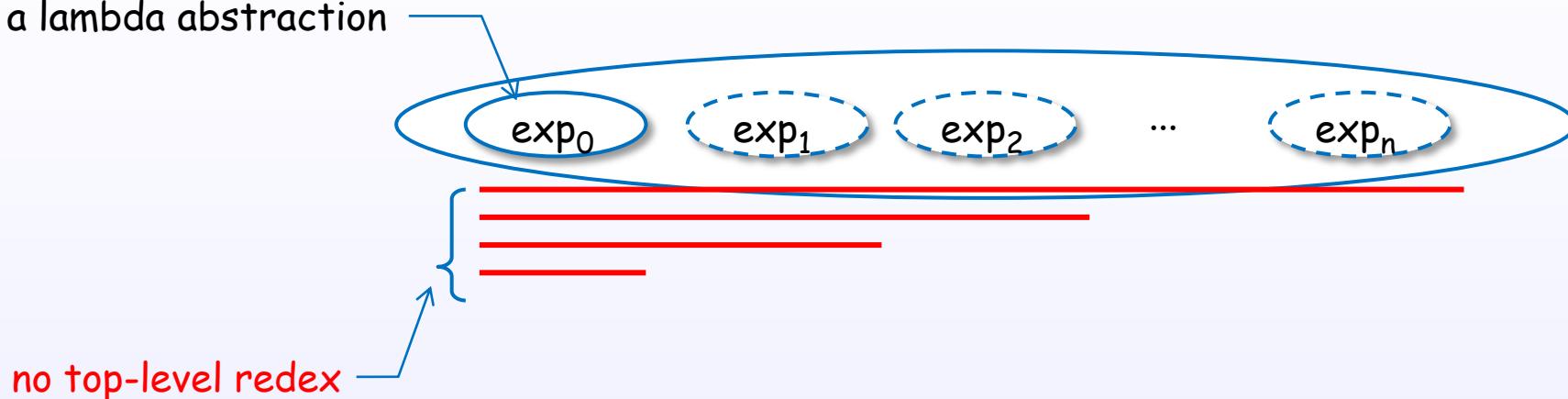
WHNF

WHNF is one of the evaluated values.



WHNF

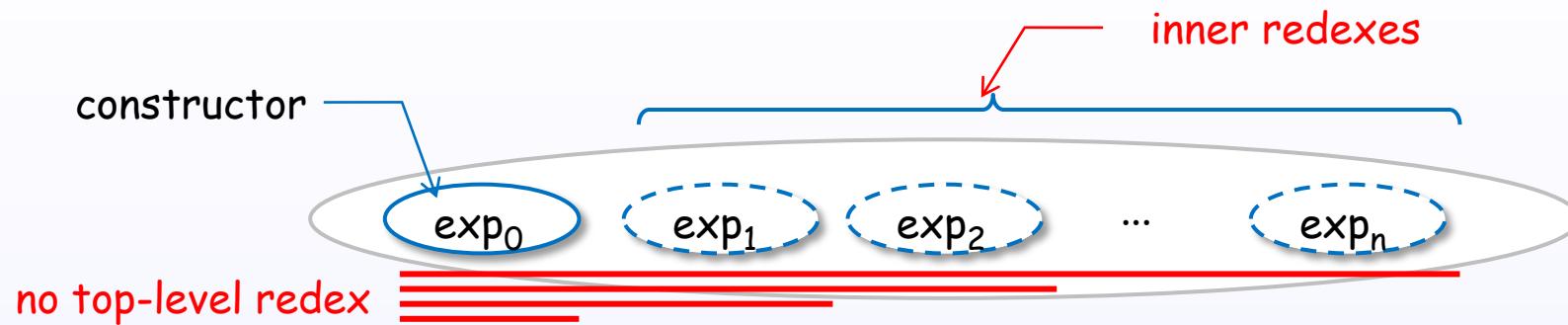
top-level (head) is
a constructor or
a lambda abstraction



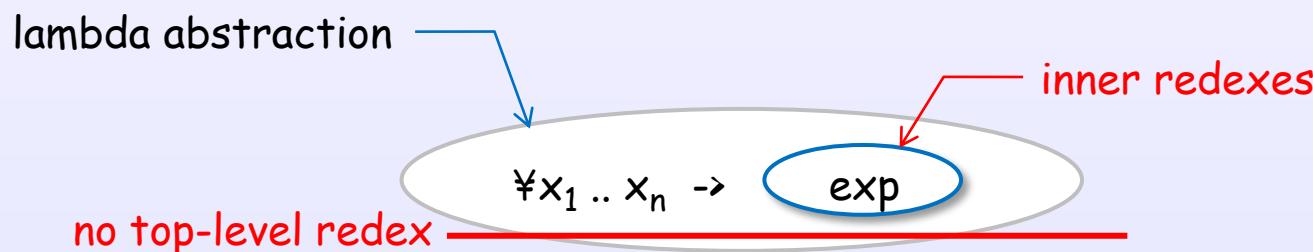
WHNF is a value which has evaluated top-level

WHNF for a data value and a function value

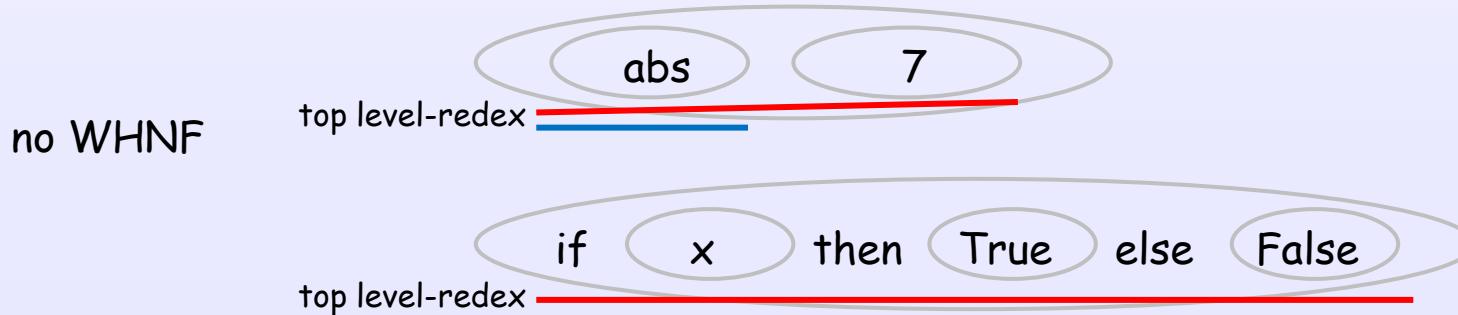
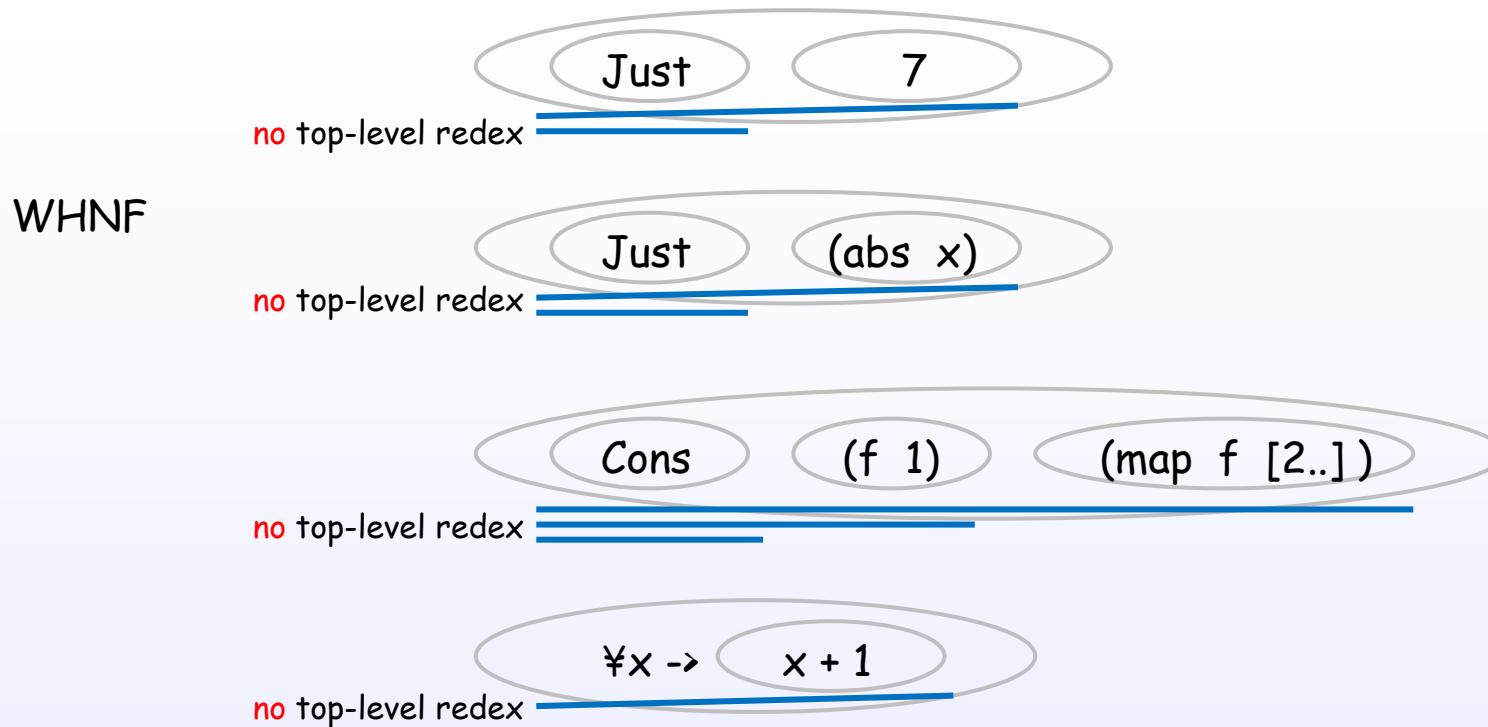
a data value in WHNF



a function value in WHNF

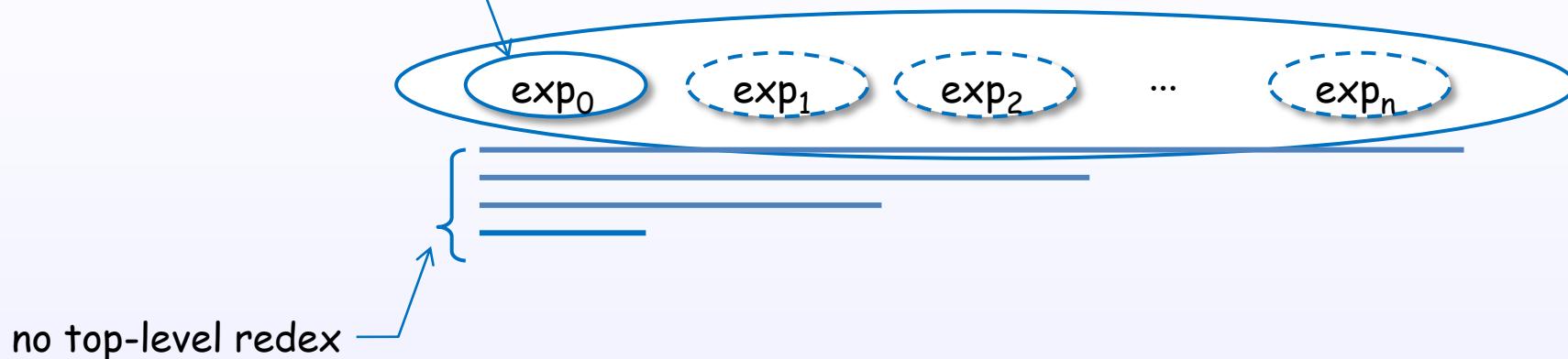


Examples of WHNF



HNF

top-level (head) is
a constructor or
a lambda abstraction with no top-level redex

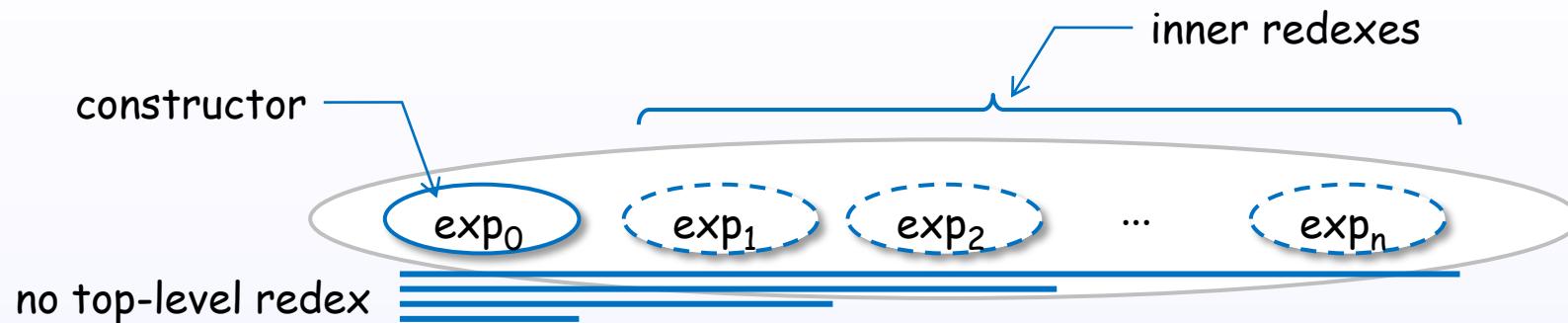


HNF is a value which has evaluated top-level

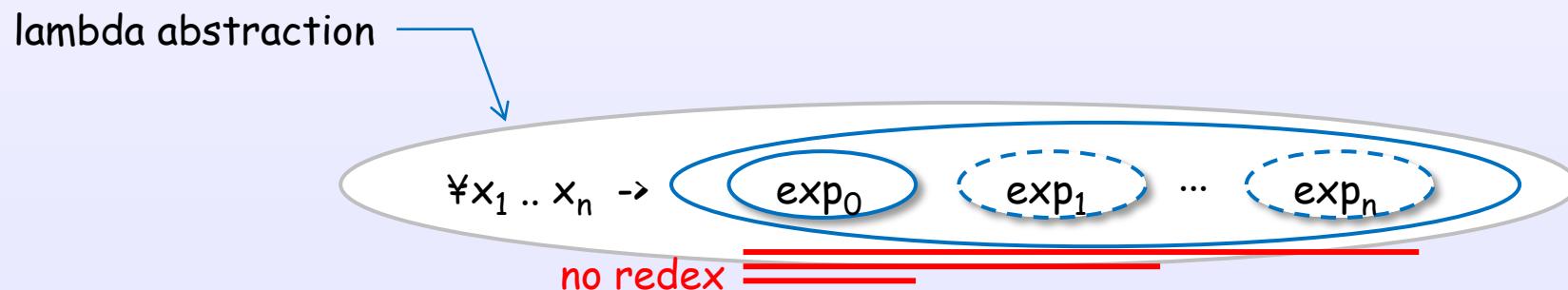
* GHC uses WHNF rather than HNF.

HNF for a data value and a function value

a data value in HNF (same as WHNF)

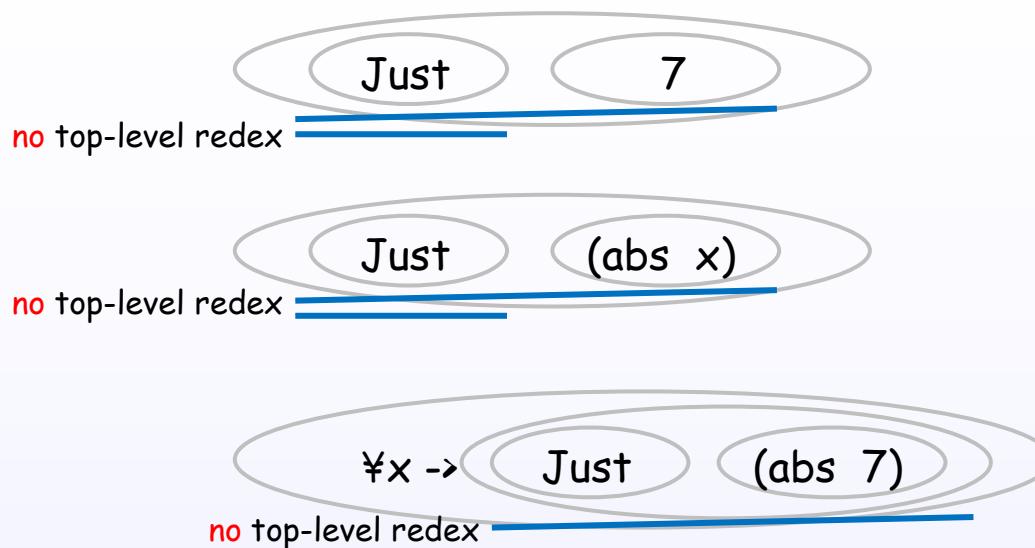


a function value in HNF



Examples of HNF

HNF

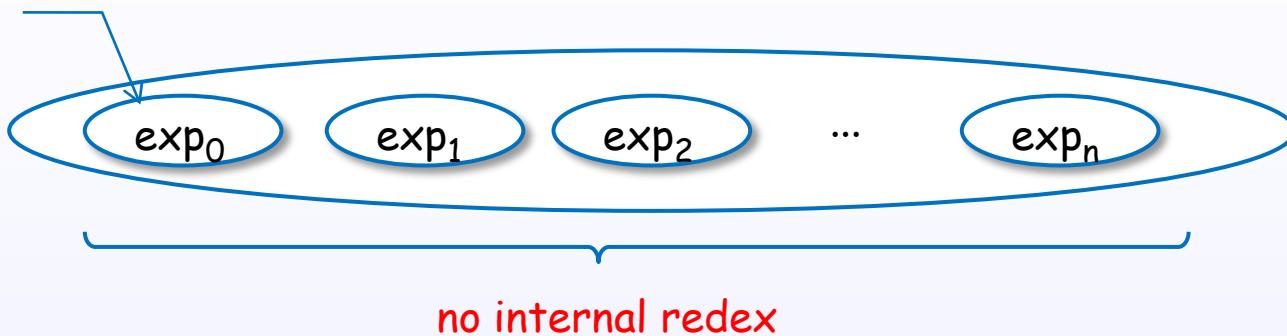


no HNF



NF

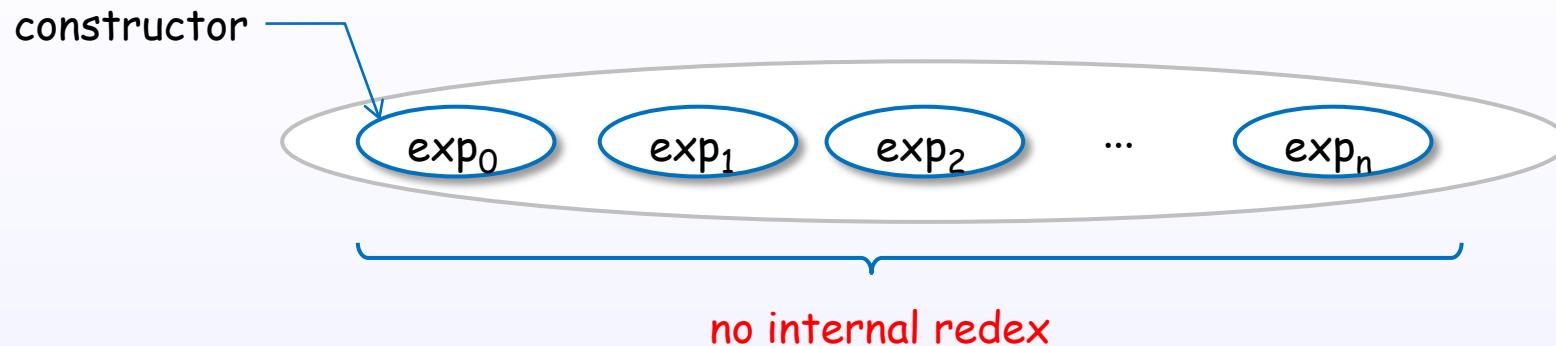
top-level (head) is
a constructor or
a lambda abstraction



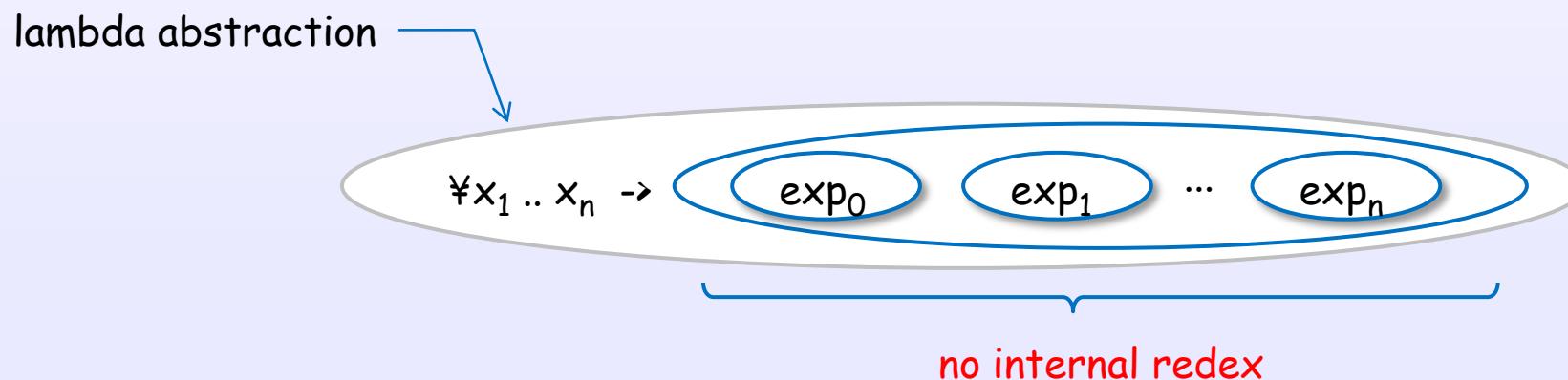
NF is a value which has no redex.

NF for a data value and a function value

a data value in NF

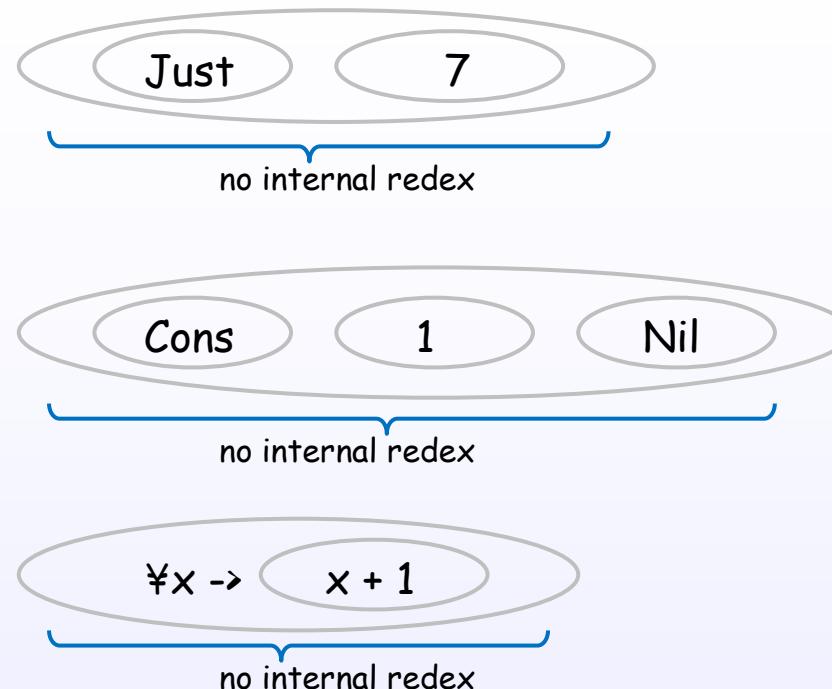


a function value in NF

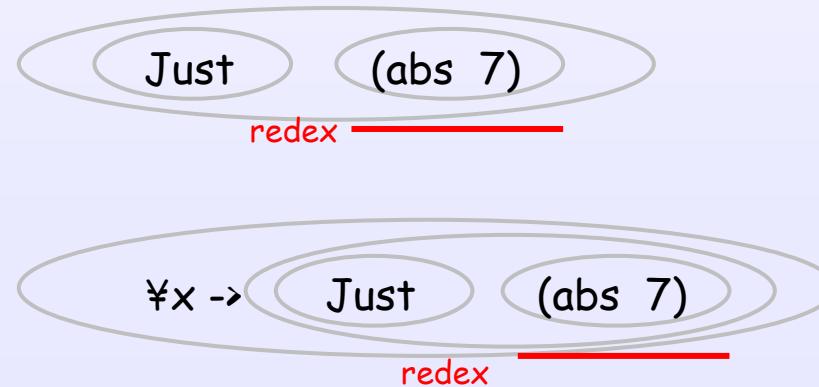


Examples of NF

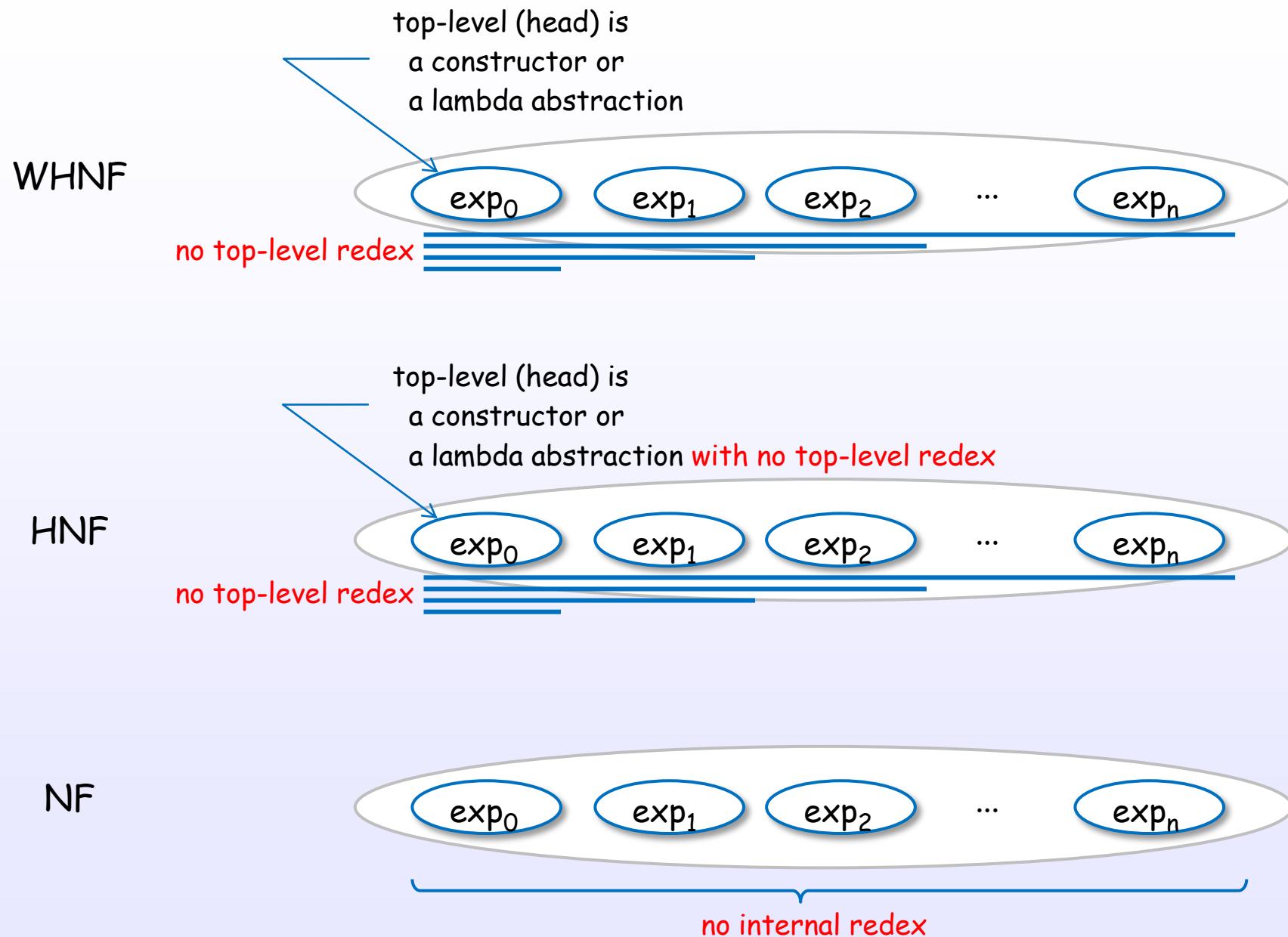
NF



no NF



WHNF, HNF, NF



Definition of WHNF and HNF

"The implementation of functional programming languages" [H4]

11.3.1 Weak Head Normal Form

To express this idea precisely we need to introduce a new definition:

DEFINITION

A lambda expression is in *weak head normal form* (WHNF) if and only if it is of the form

$F E_1 E_2 \dots E_n$

where $n \geq 0$;
and either F is a variable or data object
or F is a lambda abstraction or built-in function
and $(F E_1 E_2 \dots E_m)$ is not a redex for any $m \leq n$.

An expression has no *top-level redex* if and only if it is in weak head normal form.

DEFINITION

A lambda expression is in *head normal form* (HNF) if and only if it is of the form

$\lambda x_1. \lambda x_2. \dots \lambda x_n. (v M_1 M_2 \dots M_m)$

where $n, m \geq 0$;
 v is a variable (x_i), a data object, or a built-in function;
and $(v M_1 M_2 \dots M_p)$ is not a redex for any $p \leq m$.

3. Internal representation of expressions

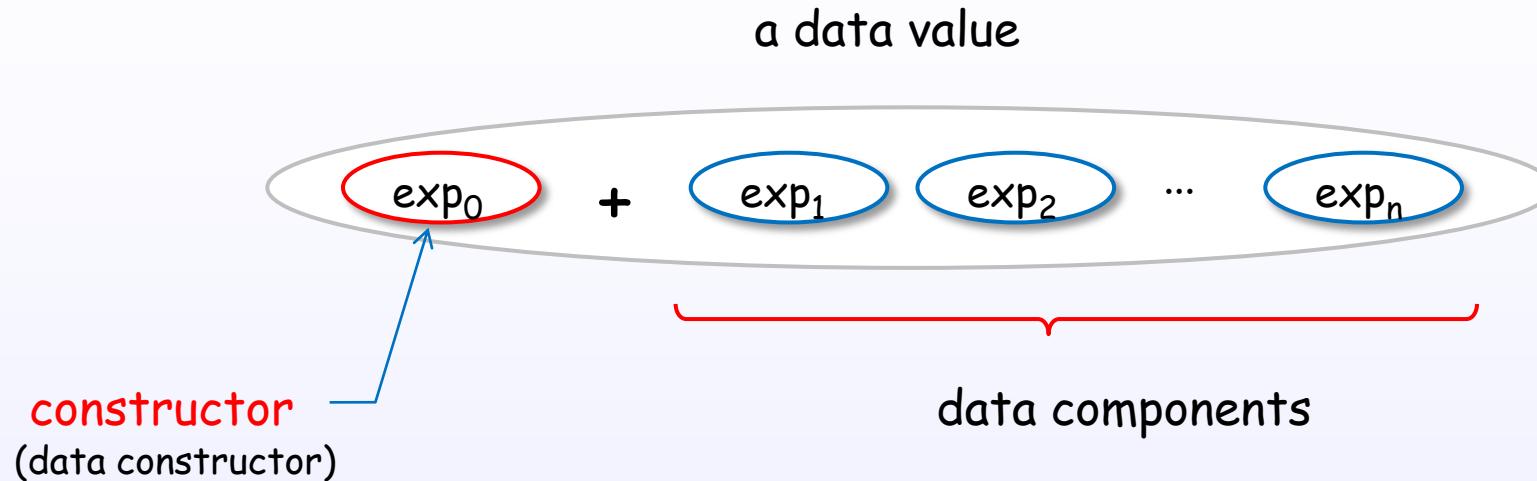
3. Internal representation of expressions

Constructor

Constructor

Constructor is one of the key elements
to understand WHNF and lazy evaluation in Haskell.

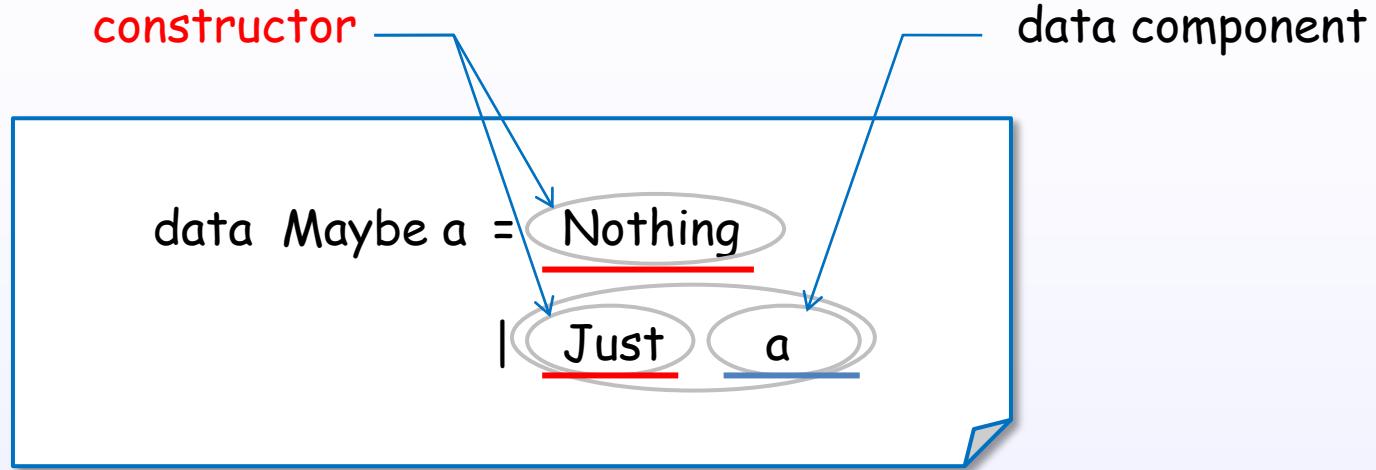
Constructor



A constructor builds a structured data value.

A constructor distinguishes the data value in expressions.

Constructors are defined by data declaration



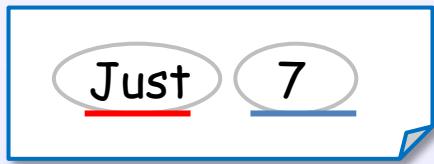
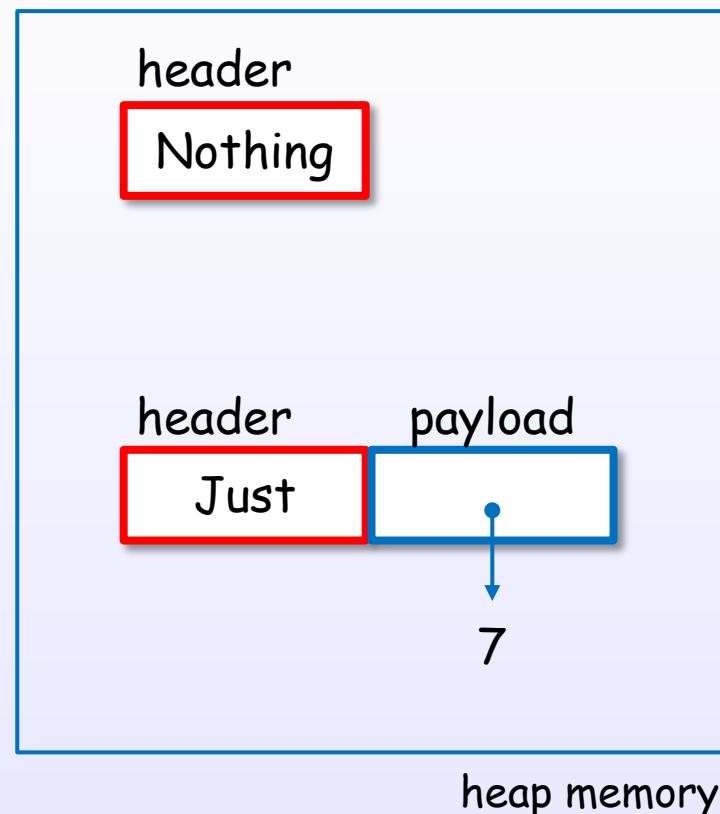
Constructors are defined by data declaration.

Internal representation of Constructors for data values

Haskell code

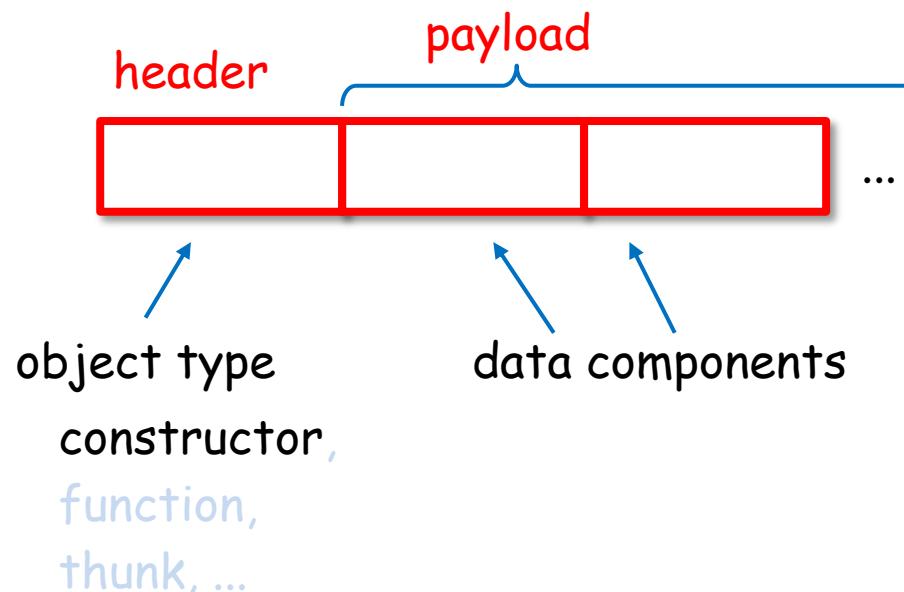


GHC's internal representation



Constructors are represented uniformly

GHC's internal representation



in heap memory, stack or static memory

A data value is represented with header(constructor) + payload(component).

Representation of various constructors

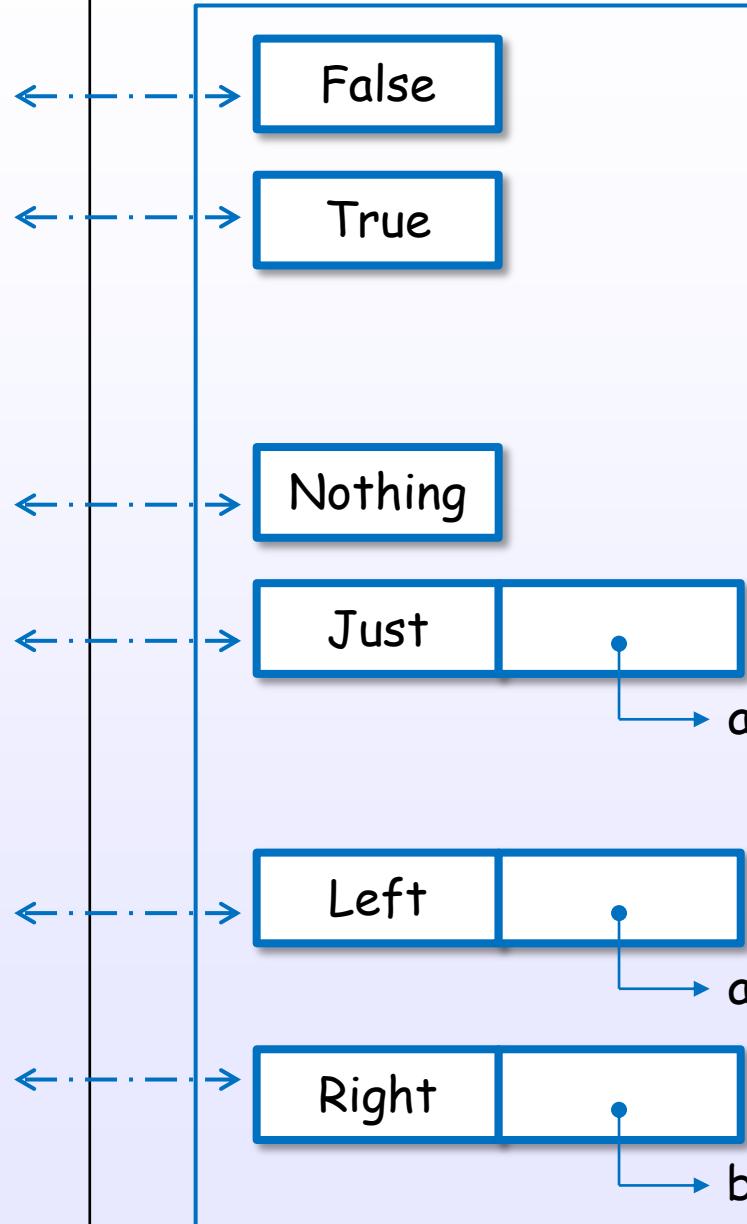
Haskell code

```
data Bool = False  
          | True
```

```
data Maybe a = Nothing  
             | Just a
```

```
data Either a b = Left a  
                 | Right b
```

GHC's internal representation



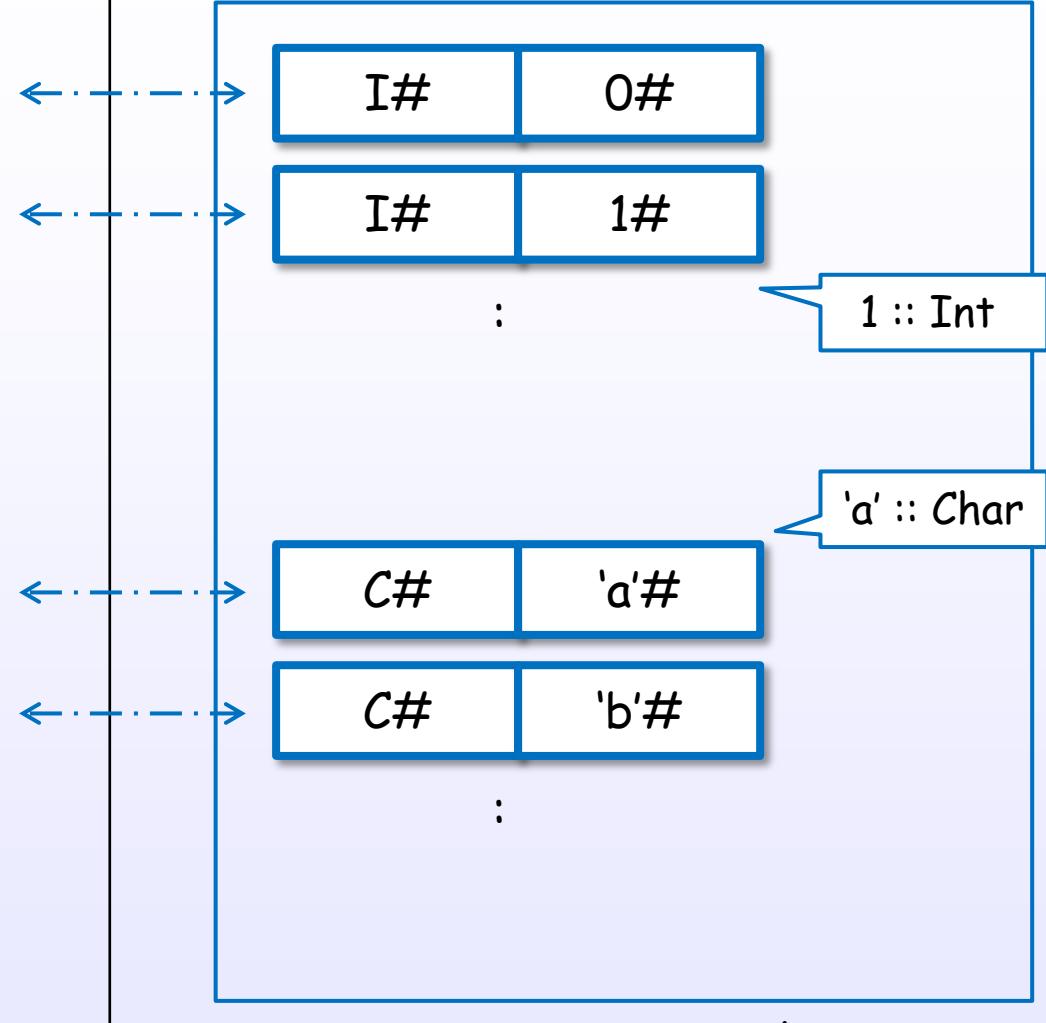
Primitive data types are also represented with constructor

Haskell code

```
data Int    = I#(0#)
              | I#(1#)
              |
              :;
```

```
data Char = C# 'a'#  
          | C# 'b'#  
          | :  
          |
```

GHC's internal representation



heap memory

「Tereit」

List is also represented with constructor

List

```
[ 1, 2, 3 ]
```

syntactic desugar

```
1 : ( 2 : ( 3 : [] ) )
```

prefix notation by section

```
(:) 1 ( (:) 2 ( (:) 3 [] ) )
```

equivalent data constructor

```
Cons 1 ( Cons 2 ( Cons 3 Nil ) )
```

constructor

List is also represented with constructor

List

[1, 2, 3]

syntactic desugar

1 : (2 : (3 : []))

prefix notation by section

(:) 1 ((:) 2 ((:) 3 []))

equivalent data constructor

Cons 1 (Cons 2 (Cons 3 Nil))

type declaration

* pseudo code
data List a = **[]** | **:** **a** (List a)

data List a = **Nil** | **Cons** **a** (List a)

List is also represented with constructor

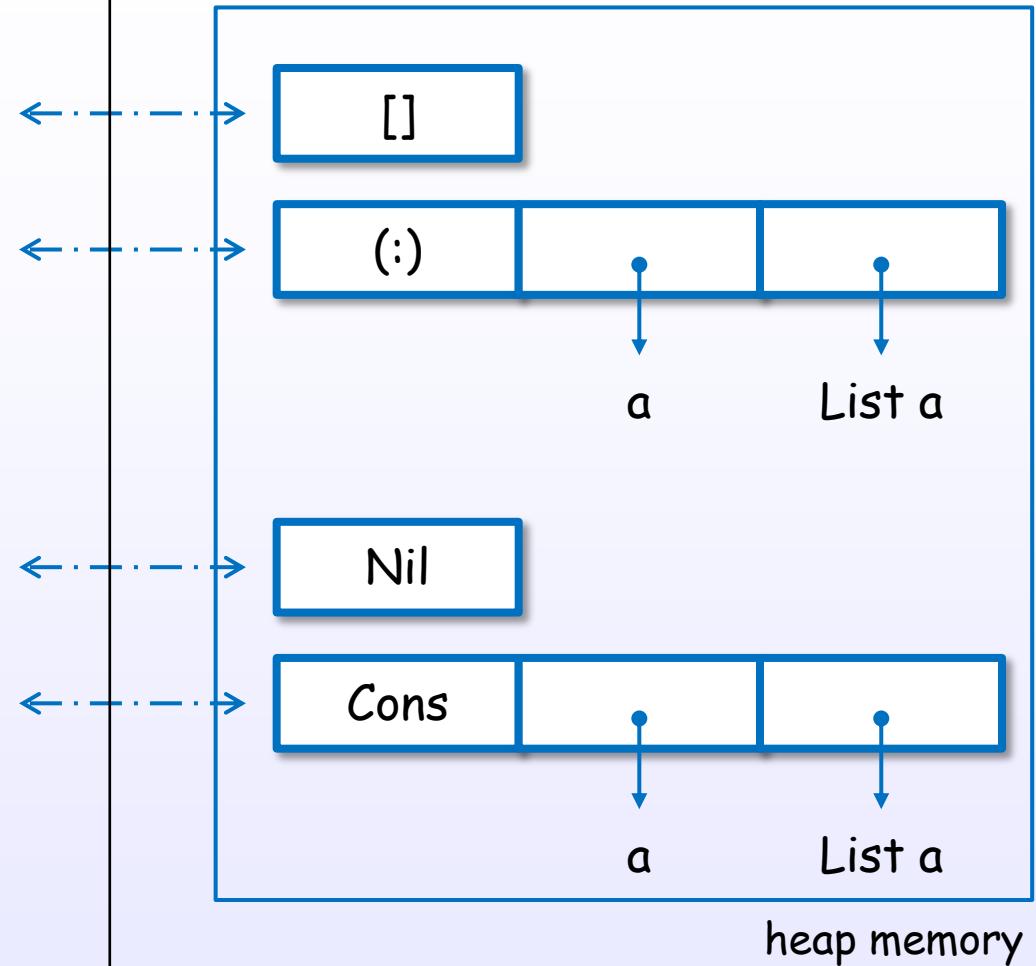
Haskell code

* pseudo code

```
data List a    = []  
               | : a (List a)
```

```
data List a    = Nil  
               | Cons a (List a)
```

GHC's internal representation



List is also represented with constructor

Haskell code

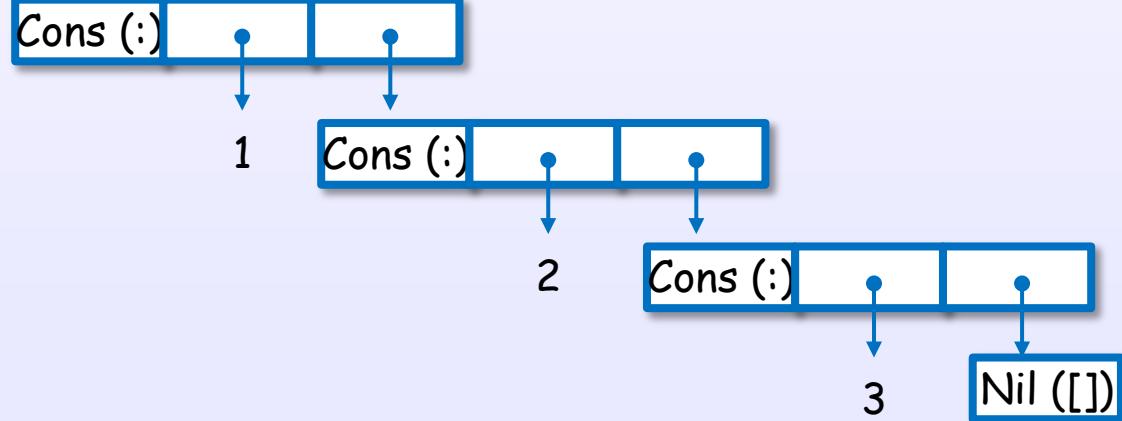
```
[ 1, 2, 3 ]
```

```
1 : ( 2 : ( 3 : [] ) )
```

```
(:) 1 ( (:) 2 ( (:) 3 [] ) )
```

```
Cons 1 ( Cons 2 ( Cons 3 Nil ) )
```

GHC's internal representation



Tuple is also represented with constructor

Tuple (Pair)

(7, 8)

prefix notation by section

(.) 7 8

equivalent data constructor

Pair 7 8

constructor

type declaration

* pseudo code

data Pair a = (.) a a

data Pair a = Pair a a

Tuple is also represented with constructor

Haskell code

```
data Pair a = (,) a a
```

```
data Pair a = Pair a a
```

GHC's internal representation

(,)

a

a

Pair

a

a

heap memory

Tuple is also represented with constructor

Haskell code

```
(7, 8)
```

```
(.) 7 8
```

```
Pair 7 8
```

GHC's internal representation

```
Pair (.)
```

```
7
```

```
8
```

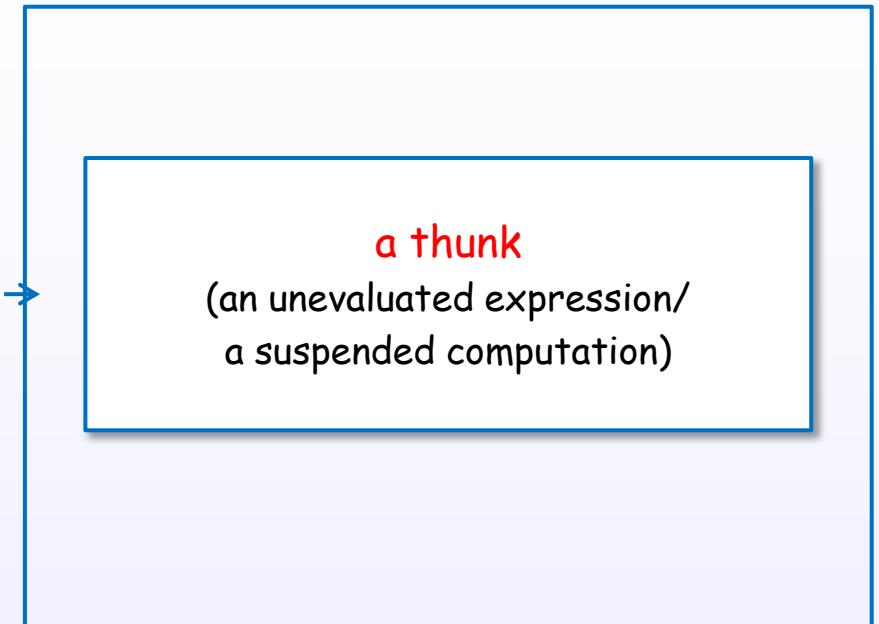
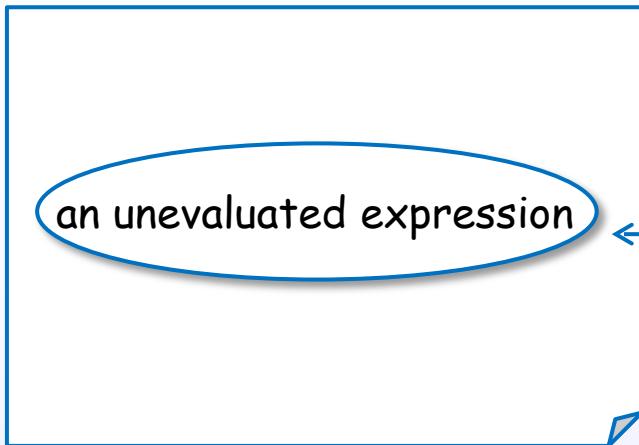
3. Internal representation of expressions

Thunk

Thunk

Haskell code

GHC's internal representation



A thunk is an **unevaluated** expression in heap memory.
A thunk is built to **postpone** the evaluation.

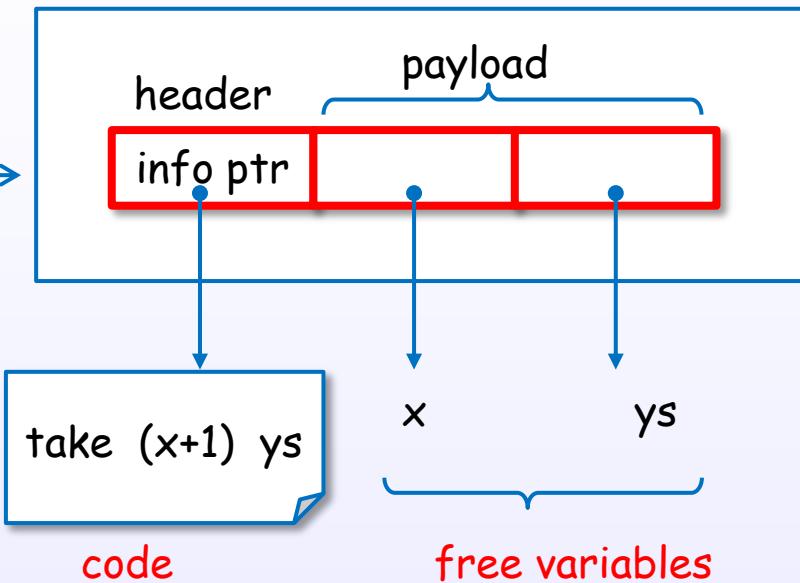
Internal representation of thunk

Haskell code

An unevaluated expression
`take (x+1) ys`

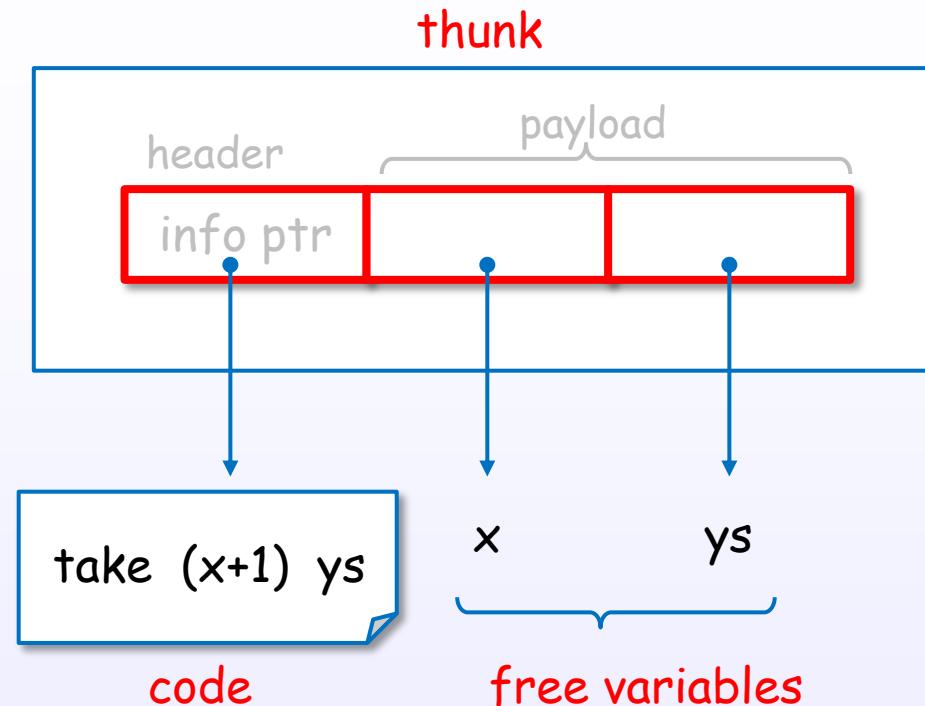
GHC's internal representation

thunk



A thunk is represented with header(code) + payload(free variables).

A thunk is a package of code and free variables



A thunk is a package of code + free variables.

A thunk is evaluated by forcing request

Haskell code

An unevaluated expression

`take (x+1) ys`



evaluate

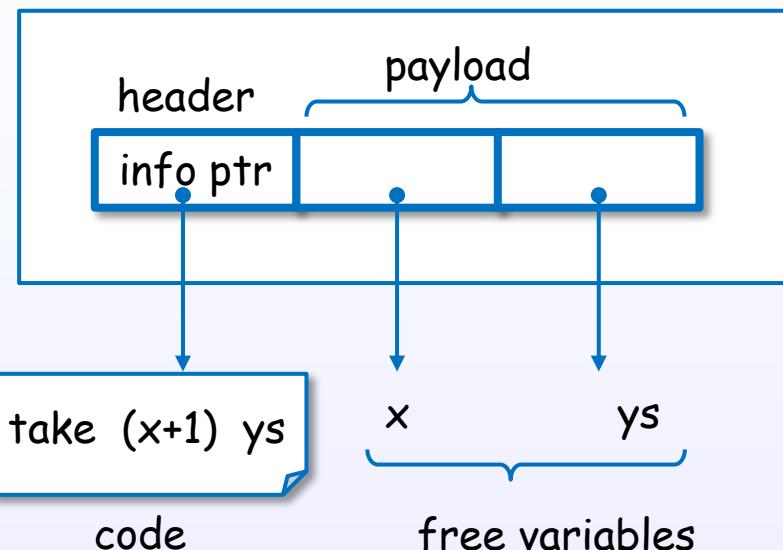
by forcing request

[3]

An evaluated expression

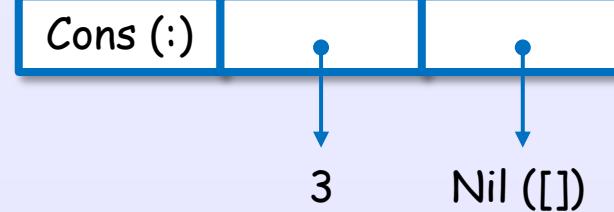
GHC's internal representation

thunk



evaluate

by forcing request

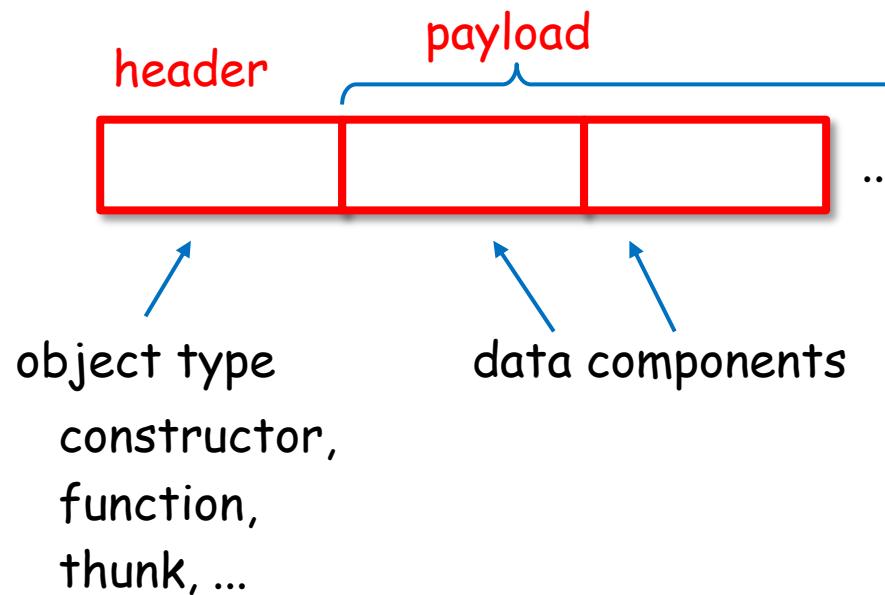


References : [D7], [D2], [H11], [H10], [H5], [H6], [H7], [B5] Ch.2, [D5], [W1]

3. Internal representation of expressions

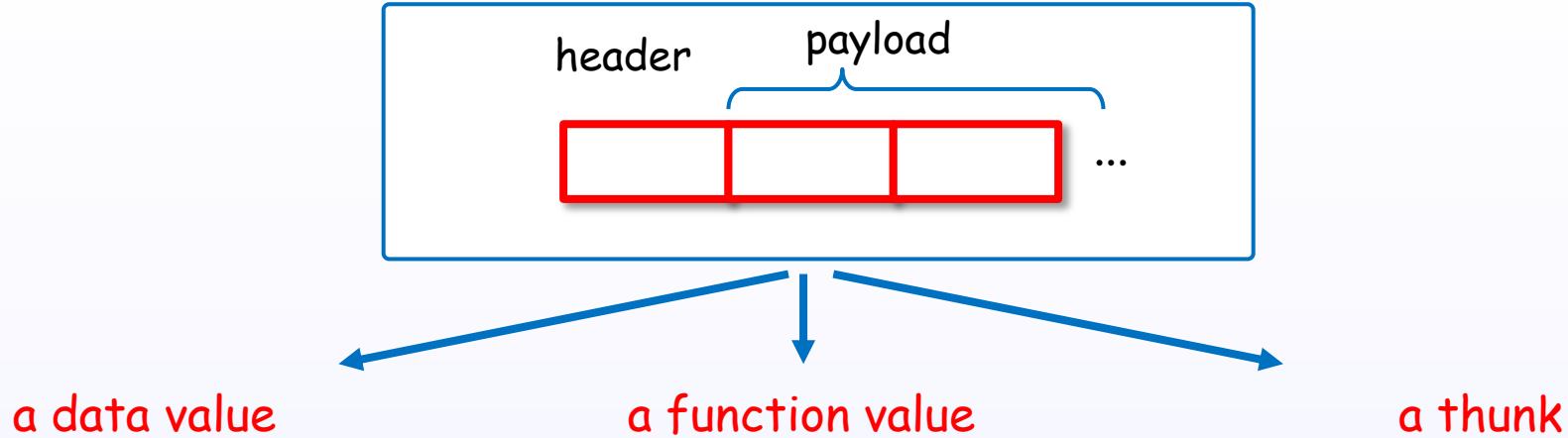
Uniform representation

Every object is represented uniformly in memory

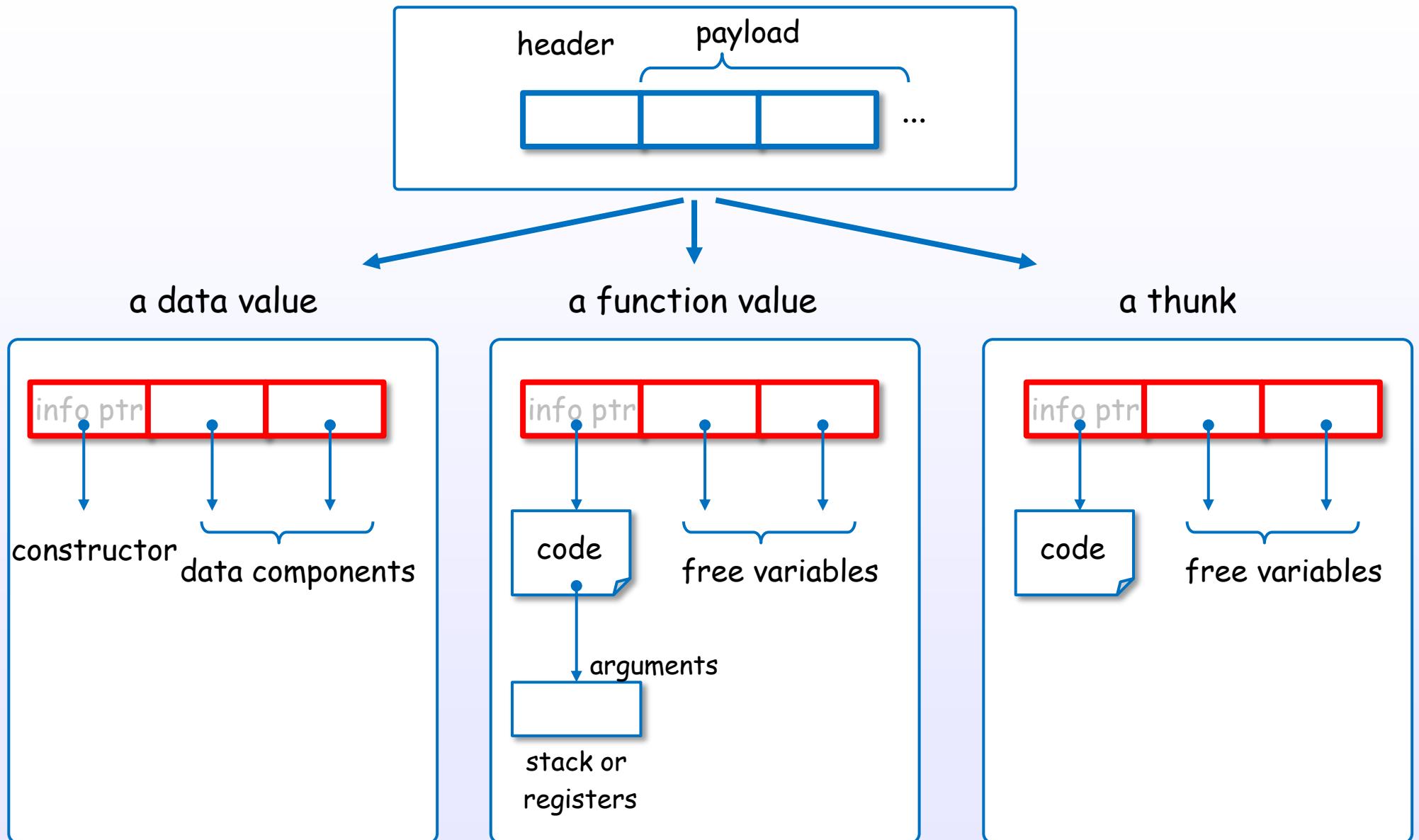


in heap memory, stack or static memory

Every object is represented uniformly



Every object is represented uniformly



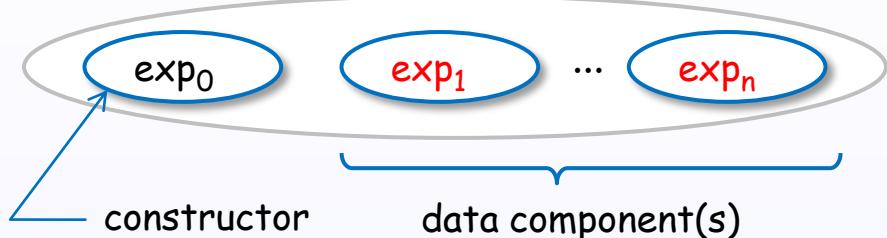
3. Internal representation of expressions

WHNF

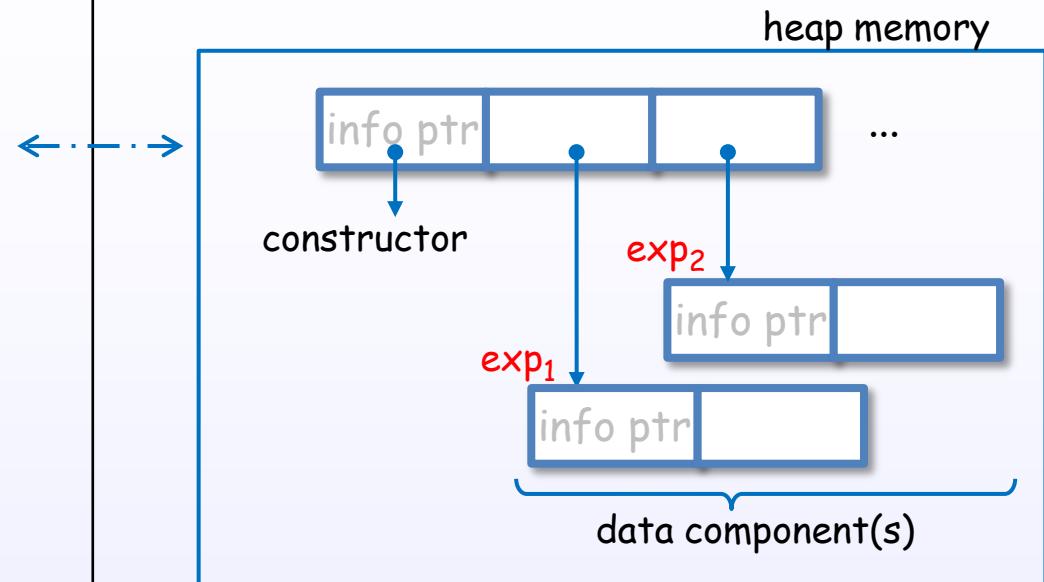
Internal representation of WHNF

Haskell code

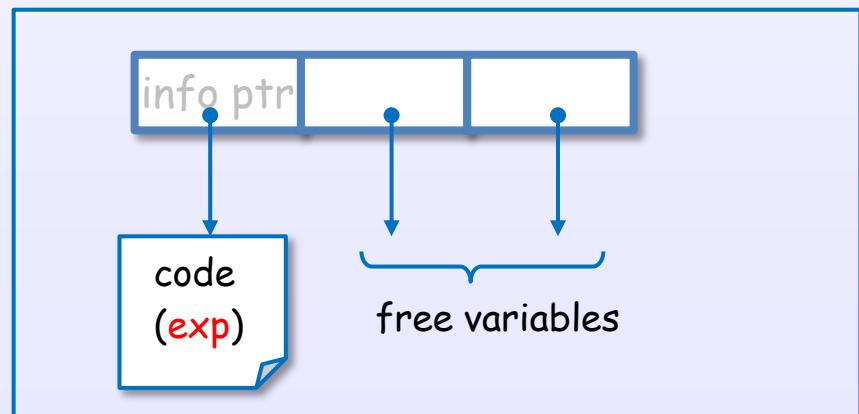
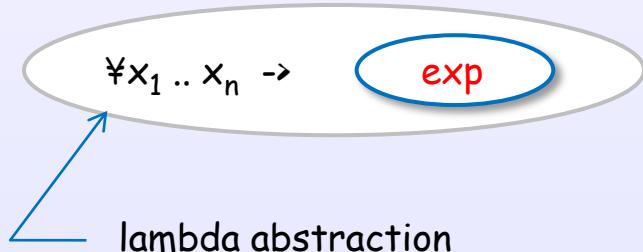
a data value in WHNF



GHC's internal representation

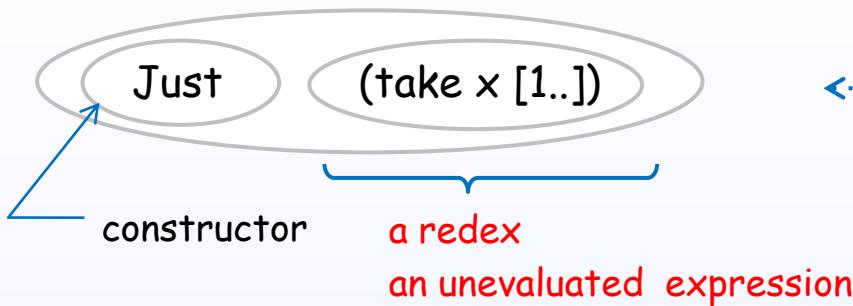


a function value in WHNF

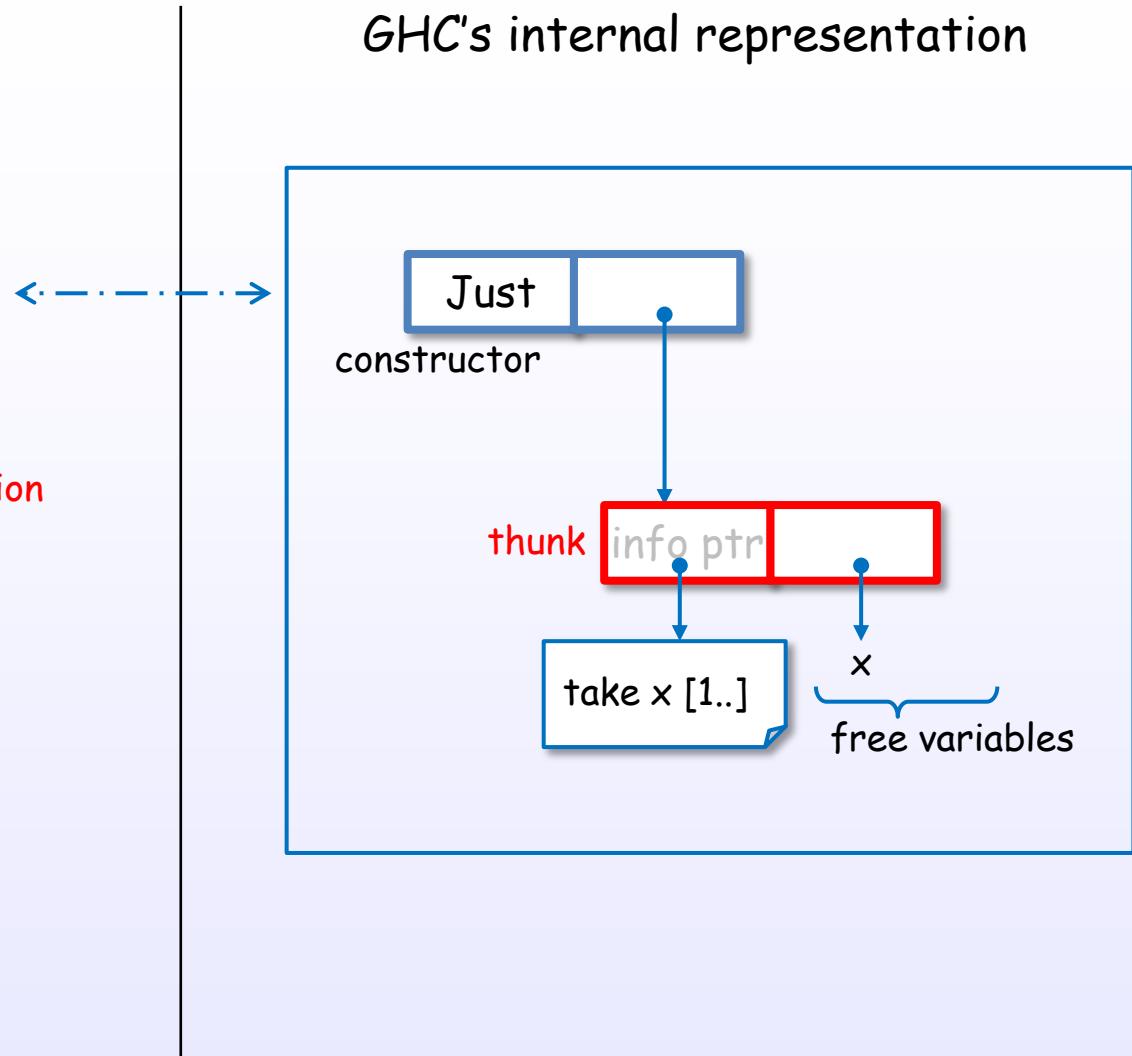


Example of WHNF for a data value

Haskell code



GHC's internal representation

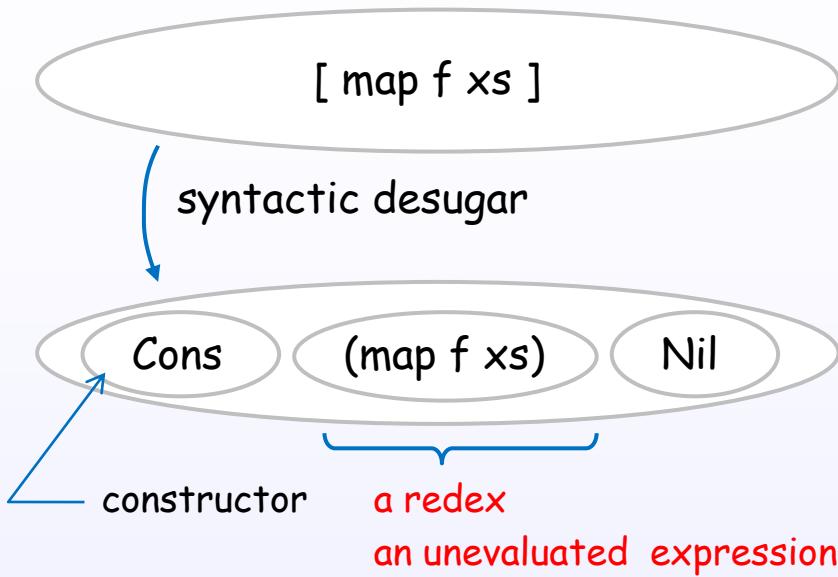


Constructors can contain unevaluated expressions by thunks.

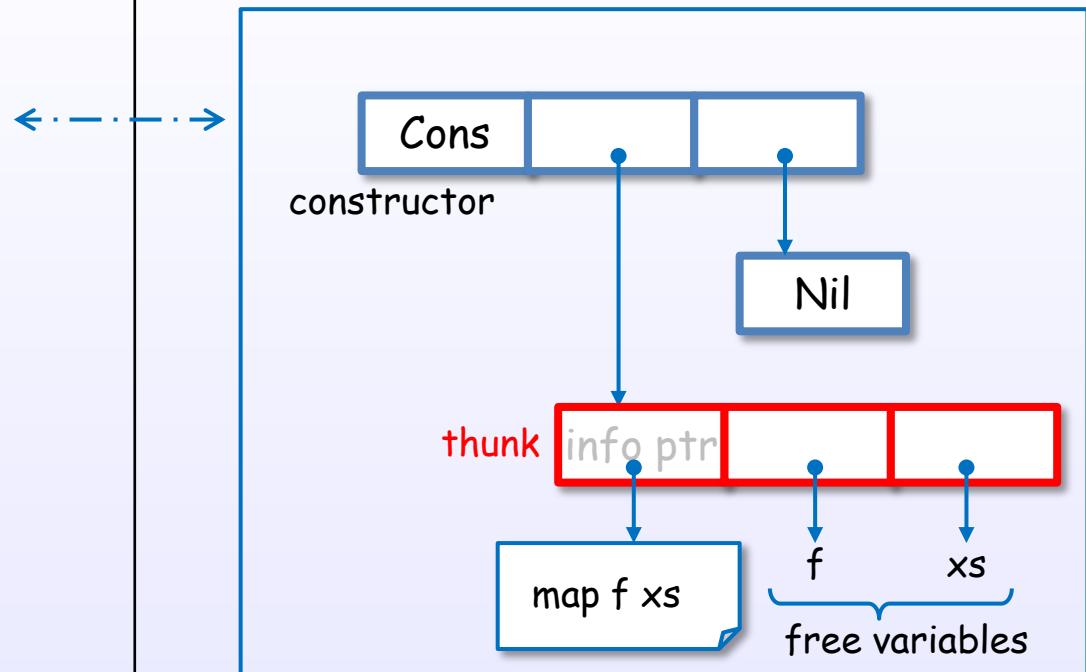
Haskell's constructors are lazy constructors.

Example of WHNF for a data value

Haskell code



GHC's internal representation



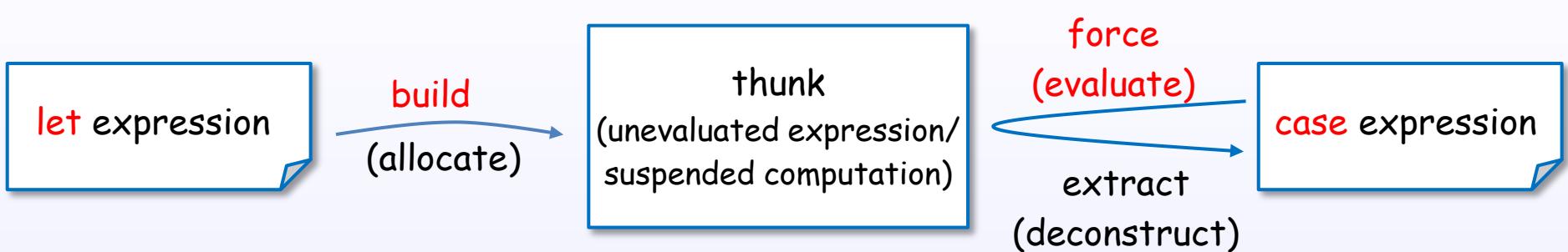
3. Internal representation of expressions

let, case expression

let, case expression

let and case expressions are special role in the evaluation

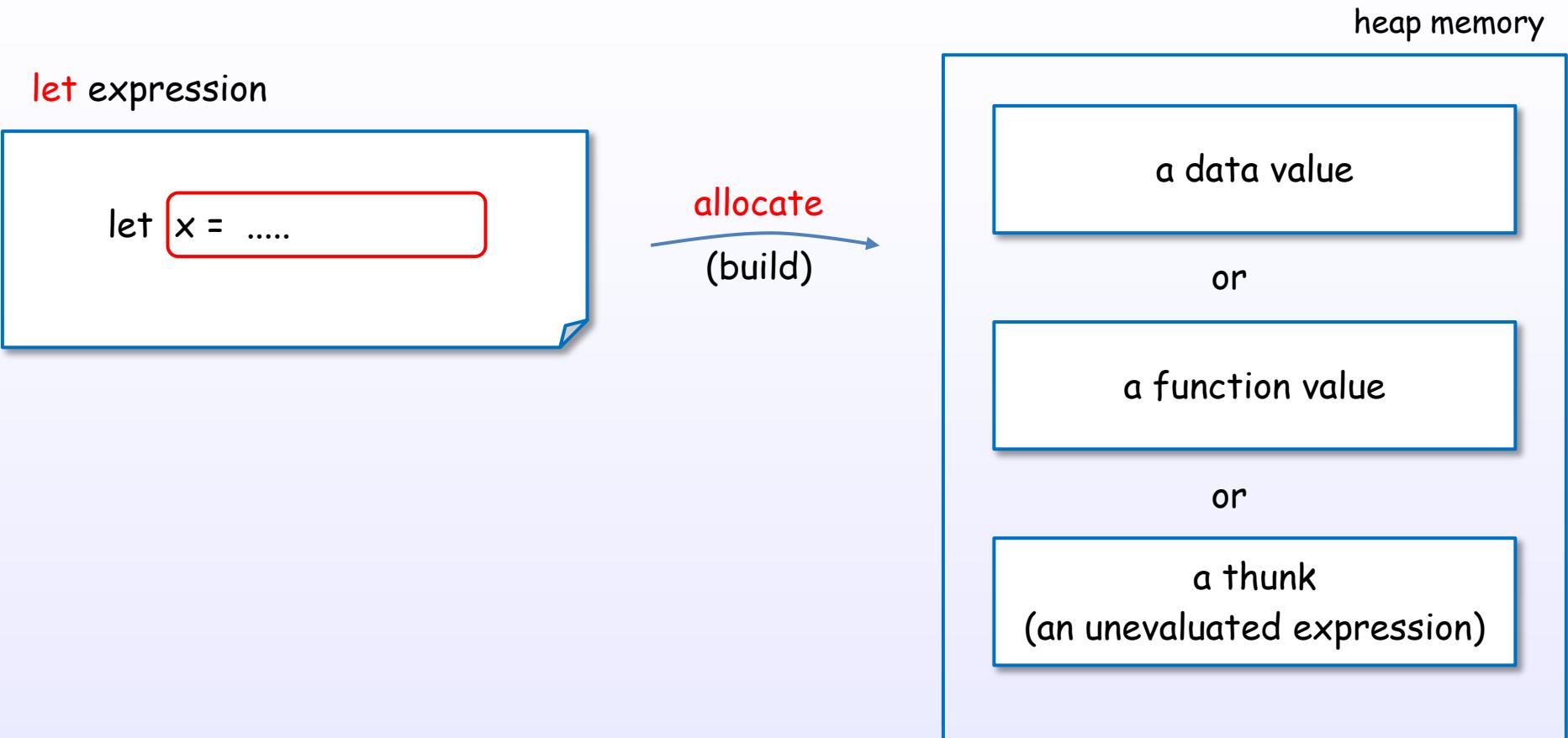
let/case expressions and thunk



A let expression may build a thunk.

A case expression forces and deconstructs the thunk.

A let expression may allocates a heap object



A let expression may allocates an object in the heap.
(If GHC optimize it, the let expression may not allocate.)

* At exactly, STG language's let expression rather than Haskell's let expression

Example of let expressions

Haskell code

```
let x = Just 5
```

allocate

```
let x =  $\lambda y \rightarrow y + z$ 
```

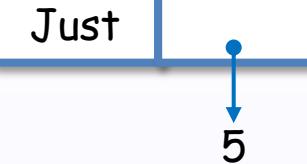
allocate

```
let x = take (n+1) xs
```

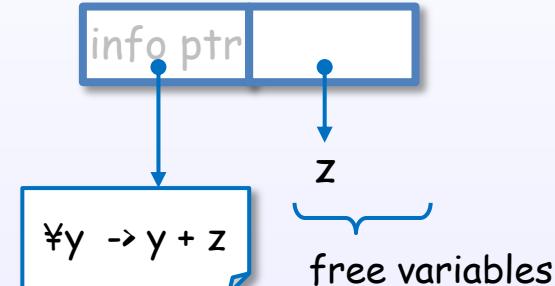
allocate
(build)

GHC's internal representation

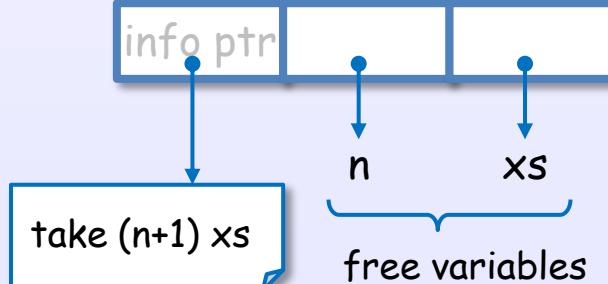
a data value



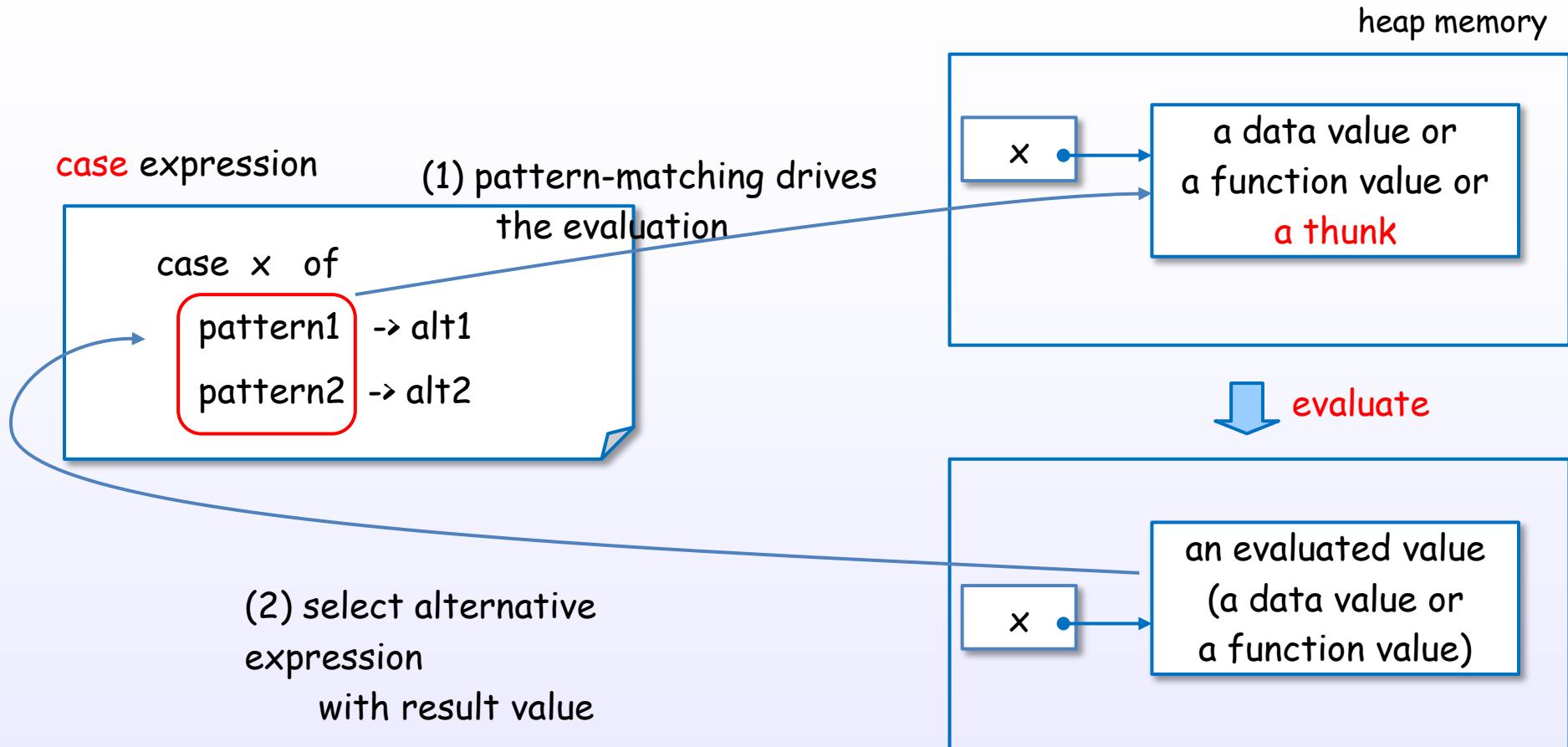
a function value



a thunk



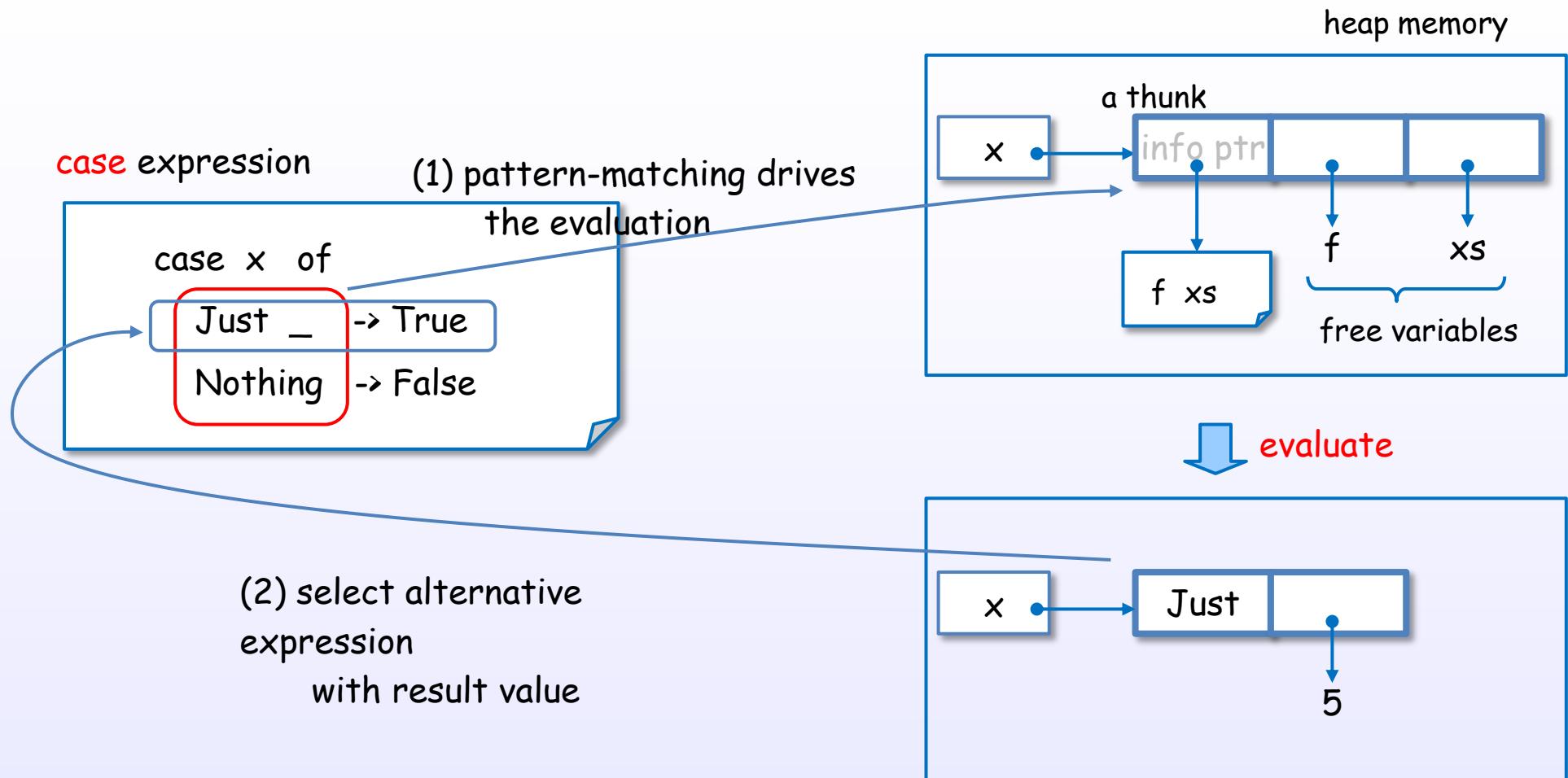
A case expression allocates a heap object



A case expression evaluates a subexpression
and optionally performs case analysis on its value.

* At exactly, STG language's case expression rather than Haskell's case expression

Example of a case expression



A case expression's pattern-matching says "I **need** the value".

pattern-matching in function definition

pattern-matching in **function definition**

```
f Just _ = True  
f Nothing = False
```

syntactic desugar

pattern-matching in **case expression**

```
f x = case x of  
    Just _ -> True  
    Nothing -> False
```

A function's pattern-matching is syntactic sugar of case expression.

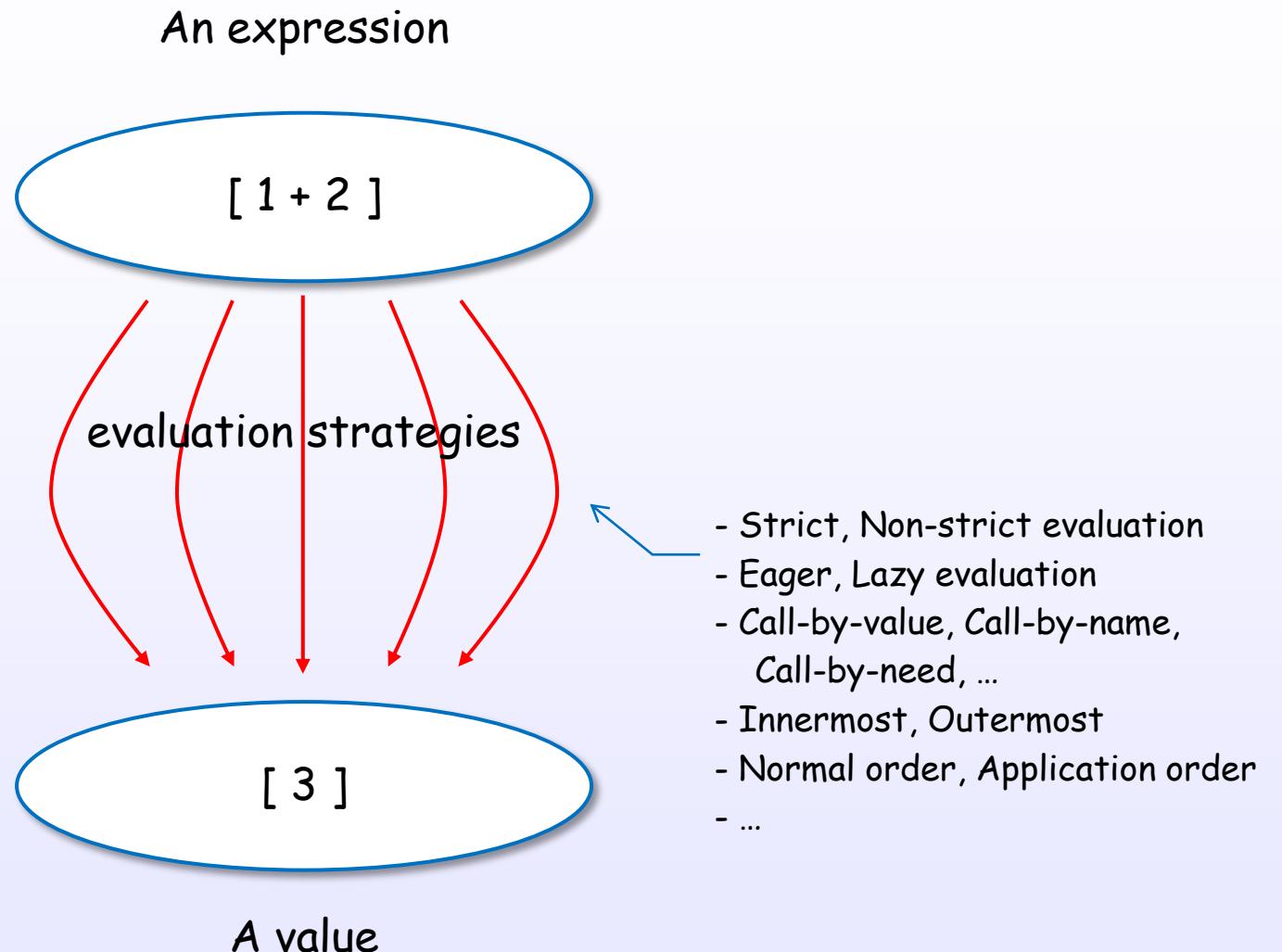
A function's pattern-matching also drives the evaluation.

4. Evaluation

4. Evaluation

Evaluation strategies

There are many evaluation approaches



Evaluation concept layer

Denotational semantics

Operational semantics
(Evaluation strategies / Reduction strategies)

Implementation techniques

Evaluation layer for GHC's Haskell

Denotational
semantics

Strict semantics

Non-strict semantics

Operational
semantics

Strict evaluation

Non-strict evaluation

Eager evaluation

Nondeterministic
evaluation

Lazy evaluation

...

Call-by-Value

Call-by-Name

Call-by-Need

...

Applicative order reduction

Normal order reduction

...

Rightmost reduction

Innermost reduction

Leftmost reduction

Outermost reduction

Implementation
techniques

Tree reduction

Lazy graph reduction

...

Evaluation layer for GHC's Haskell

Denotational
semantics

Strict semantics

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Non-strict evaluation

(Evaluation strategies/
Reduction strategies)

Eager evaluation

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Normal order reduction

Rightmost reduction

Innermost reduction

Leftmost reduction

Outermost reduction

Implementation
techniques

Tree reduction

Lazy graph reduction

Haskell 2010 specification

GHC's strategy

GHC's strategy

...

GHC's strategy

GHC's strategy

...

GHC's strategy

...

GHC's implementation

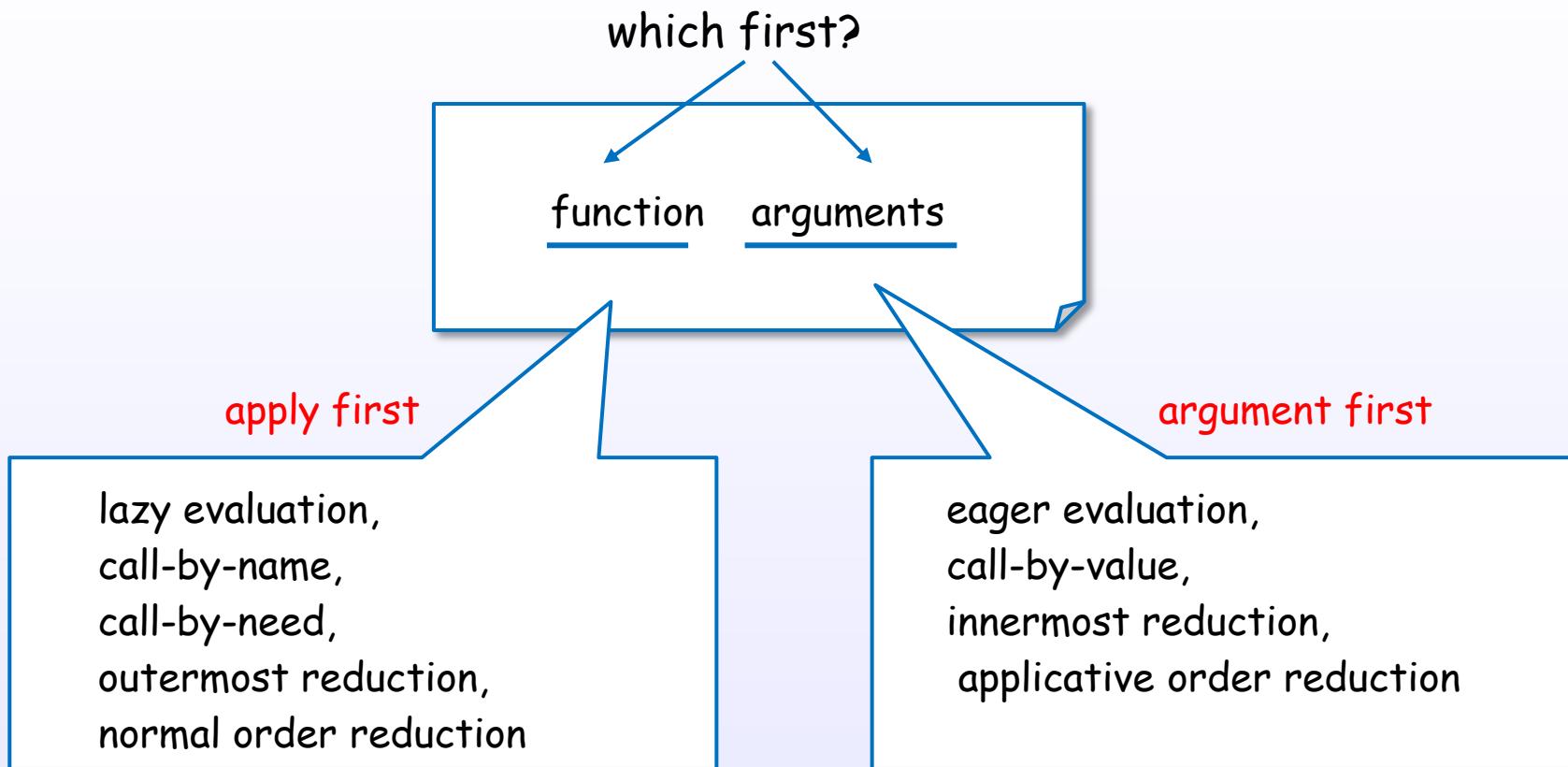
...

Evaluation strategies

Each evaluation strategy decides how to operate the evaluation about

ordering, region, trigger condition,
completion condition, re-evaluation, ...

One of the most importance is the order



Simple example of typical evaluations

call-by-value

default

C, Java, JavaScript,
Python, OCaml, Scheme, ...

square (1 + 2)

argument
evaluation
first



call-by-need

default

Haskell (GHC), ...

square (1 + 2)

apply
first



Simple example of typical evaluations

call-by-value

square (1 + 2)



square (3)



3 * 3



9

call-by-need

square (1 + 2)



(1 + 2) * (1 + 2)



(3) * (3)



9

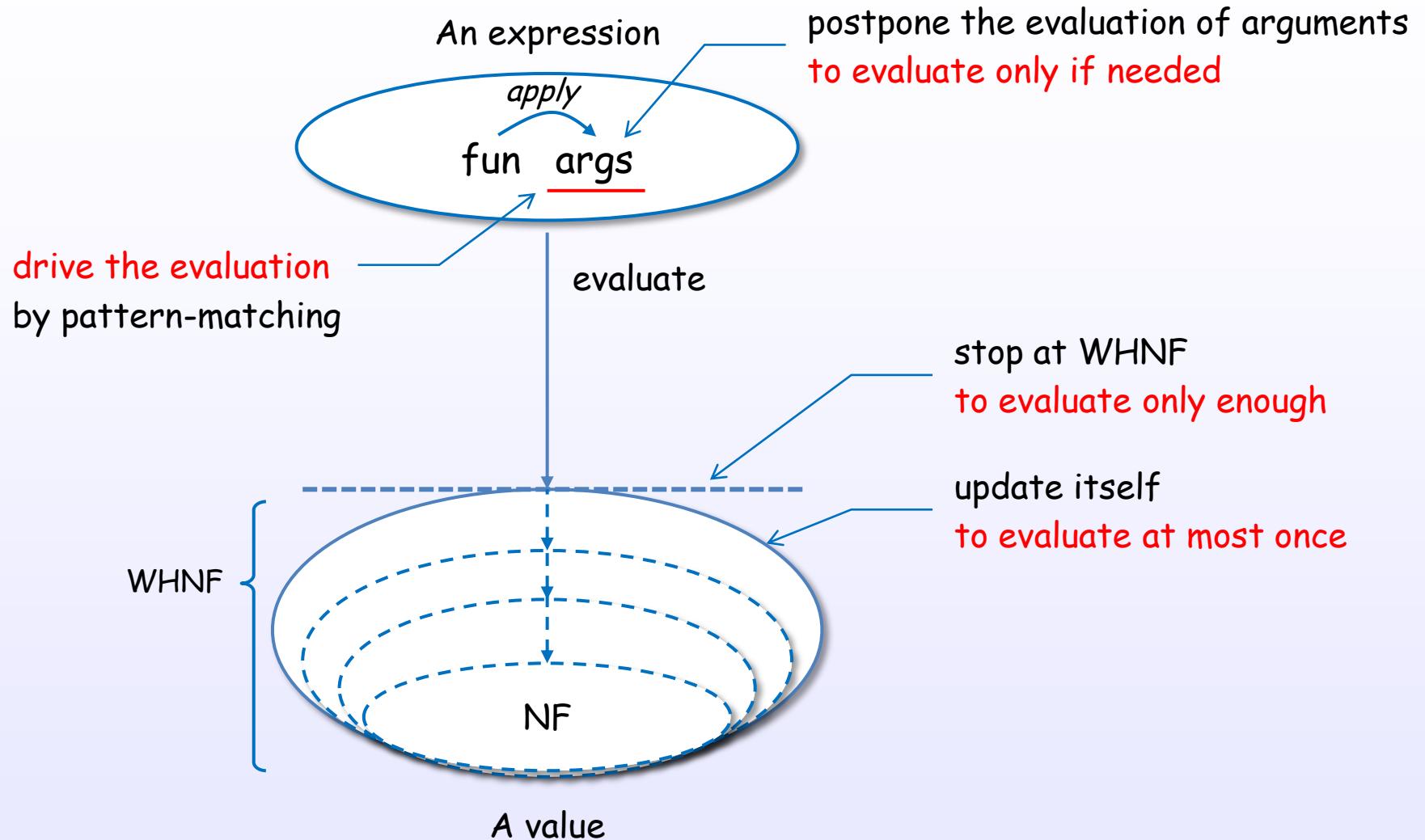
evaluation is
performed

evaluation is
delayed !

4. Evaluation

Evaluation in Haskell (GHC)

Key concepts of Haskell's lazy evaluation



Postpone the evaluation of arguments

Haskell code

```
tail (map abs [1, -2, 3])
```

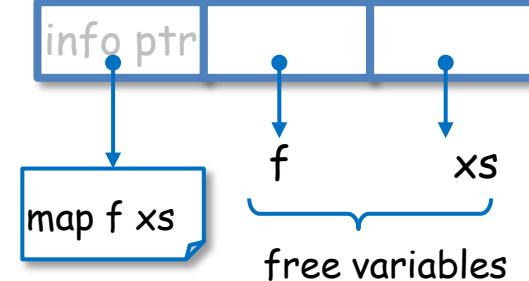
↓ 式確認！

internal translation

```
let thunk0 = map abs [1, -2, 3]  
in tail thunk0
```

postpone
(build)

a thunk



heap memory

postpone the evaluation by thunk build with let expression

4. Evaluation

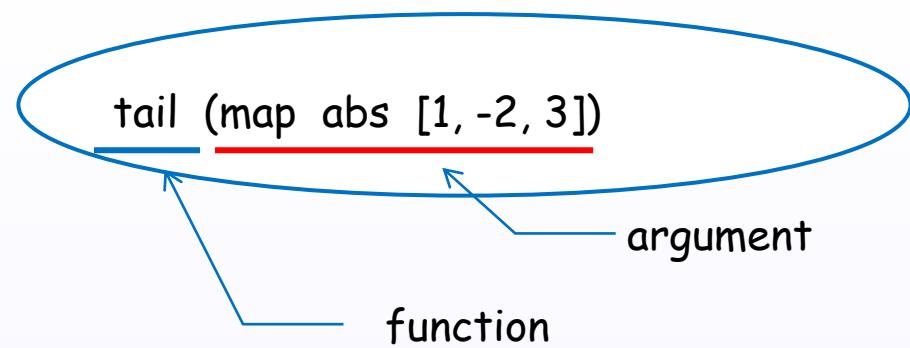
Examples of evaluation steps

(1) Example of GHC's evaluation

tail (map abs [1, -2, 3])

Let's evaluate. It's time to magic!

(2) How to postpone the evaluation of arguments?



(3) GHC translates internally the expression

tail (map abs [1, -2, 3])

internal translation

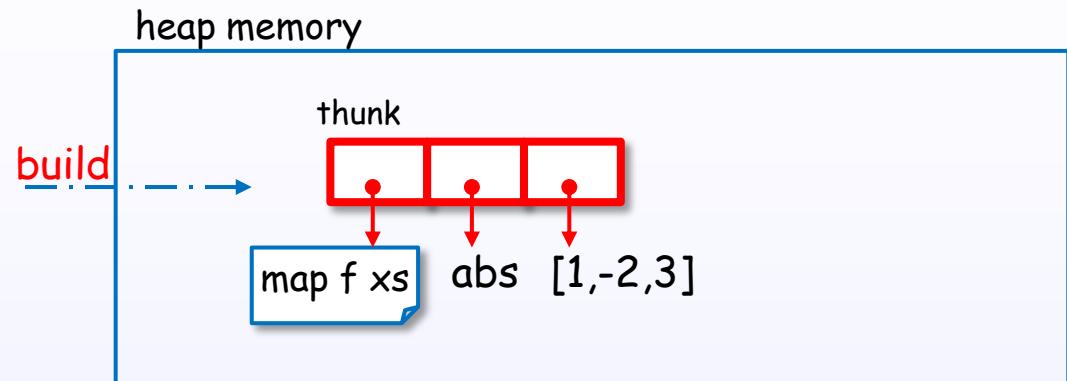
let thunk0 = map abs [1, -2, 3]
in tail thunk0

(4) a let expression builds a thunk

tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]
in tail thunk0



(5) function apply to argument

tail (map abs [1, -2, 3])

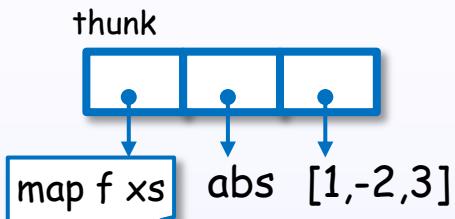
internal translation

let thunk0 = map abs [1, -2, 3]

in tail thunk0

apply

heap memory



(6) tail is defined here

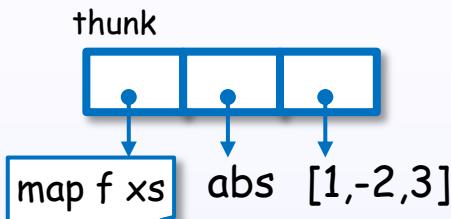
tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]
in tail thunk0

tail (_:xs) = xs

heap memory



(7) function is syntactic sugar

tail (map abs [1, -2, 3])

internal translation

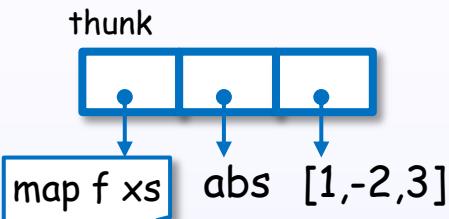
let thunk0 = map abs [1, -2, 3]
in tail thunk0

syntactic
desugar

tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

heap memory



(8) substitute function body (beta reduction)

tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]
in tail thunk0

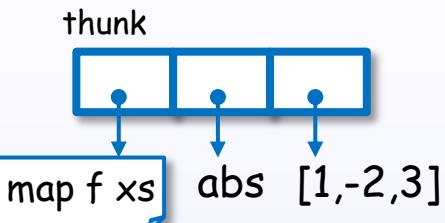
tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

case thunk0 of
(_:xs) -> xs

reduction

heap memory



(9) case pattern-matching drives the evaluation

tail (map abs [1, -2, 3])

internal translation

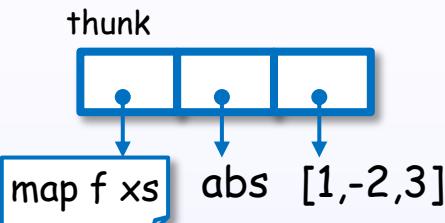
let thunk0 = map abs [1, -2, 3]
in tail thunk0

tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

case thunk0 of
(_:xs) -> xs

heap memory



evaluate

(10) but, stop at WHNF

tail (map abs [1, -2, 3])

internal translation

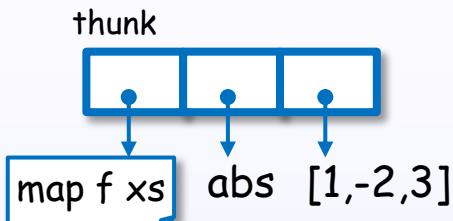
let thunk0 = map abs [1, -2, 3]
in tail thunk0

tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

case thunk0 of
(_:xs) -> xs

heap memory



evaluate

constructor

WHNF

Cons

thunk

abs x

1

thunk

map f xs

abs

[-2,3]

(11) bind variables to result

tail (map abs [1, -2, 3])

internal translation

let thunk0 = map abs [1, -2, 3]
in tail thunk0

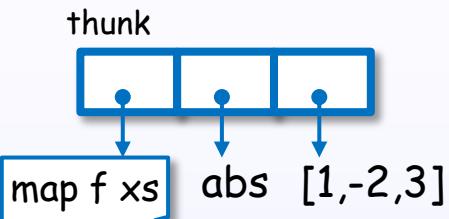
tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

case thunk0 of
(_:xs) -> xs

case (abs 1) : (map abs [-2, 3]) of
(_:xs) -> xs

heap memory



evaluate

constructor



thunk

abs x

1

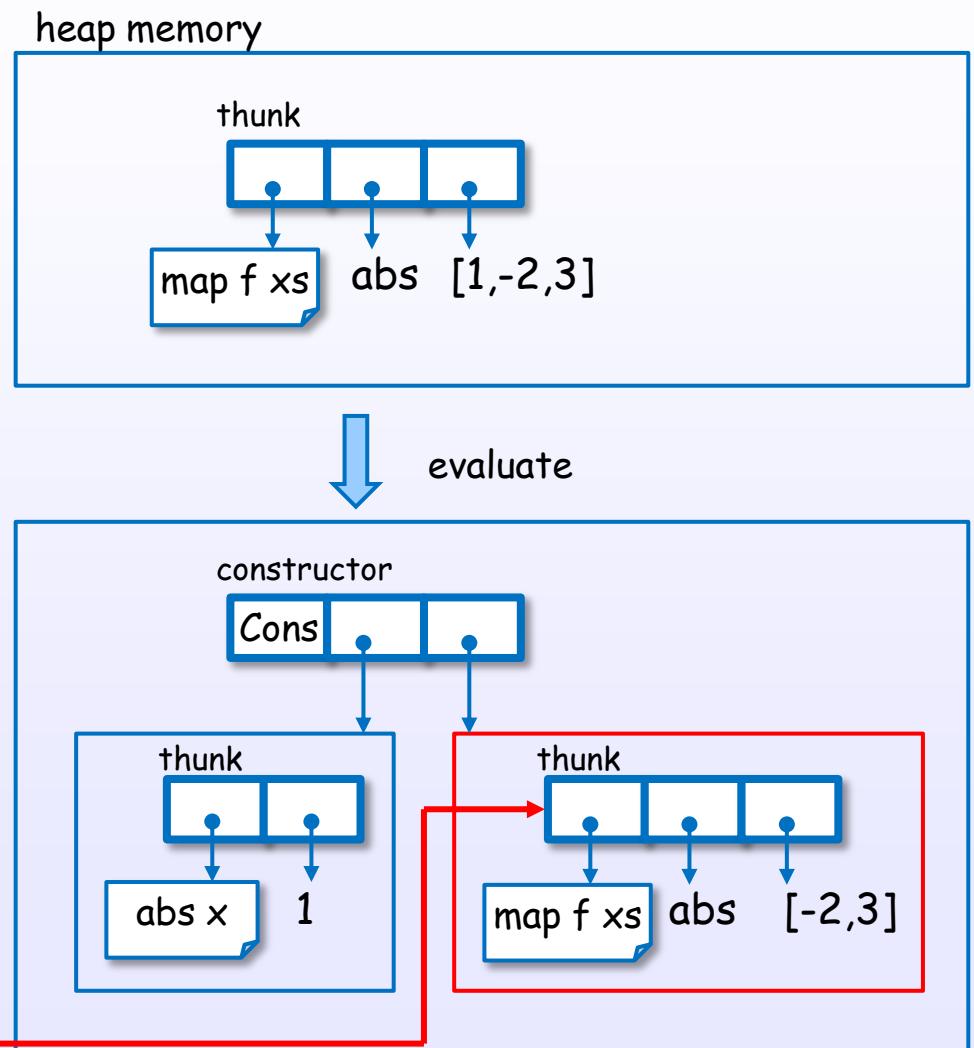
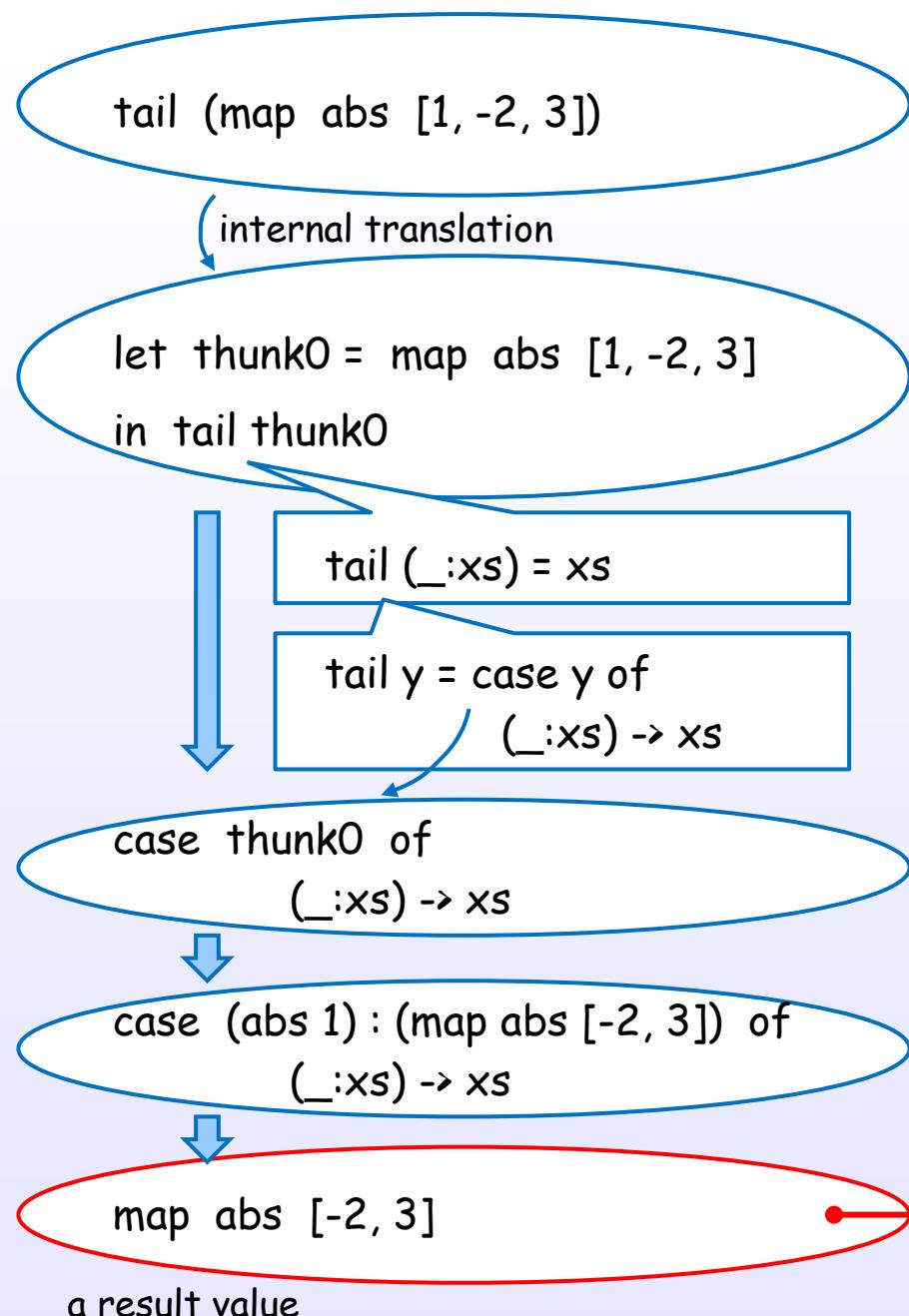
thunk

map f xs

abs

[-2,3]

(12) return the value



key points

tail (map abs [1, -2, 3])

let thunk0 = map abs [1, -2, 3]
in tail thunk0

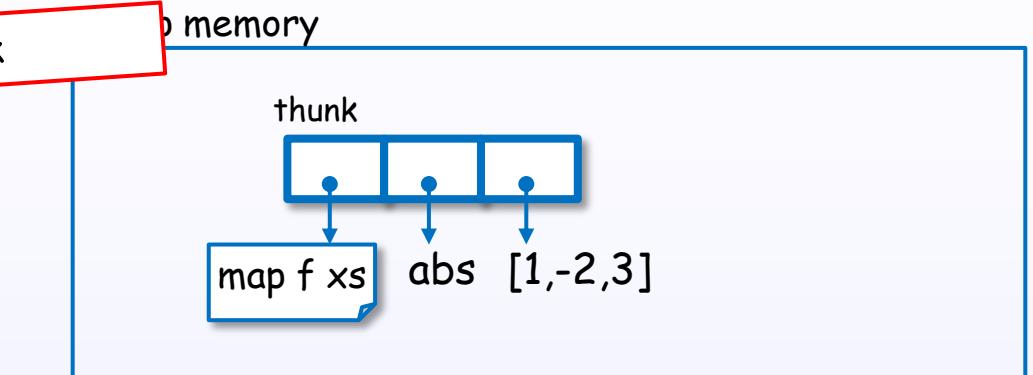
tail (_:xs) = xs

tail y = case y of
(_:xs) -> xs

case thunk0 of
(_:xs) -> xs

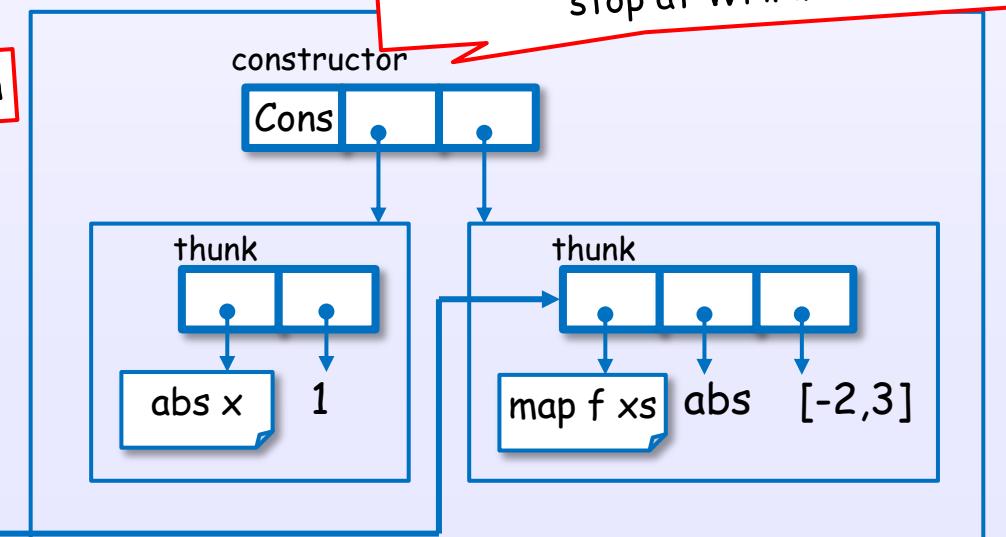
case (abs 1) : (map abs [-2, 3]) of
(_:xs) -> xs

map abs [-2, 3]



evaluate

stop at WHNF



a result value

4. Evaluation

Examples of evaluations

Example of repeat

repeat 1



1 : repeat 1



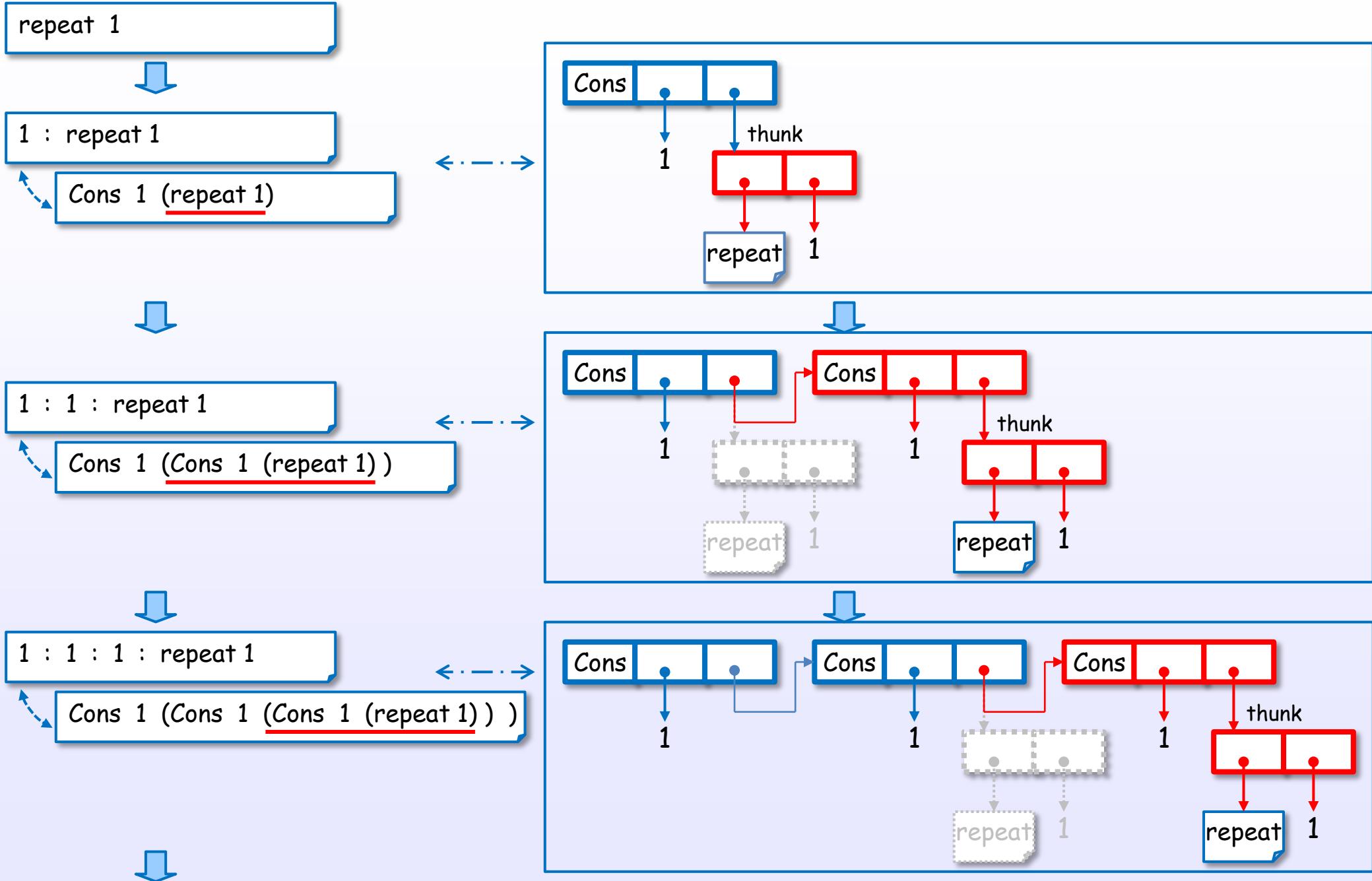
1 : 1 : repeat 1



1 : 1 : 1 : repeat 1



Example of repeat



References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of map

```
map f [1, 2, 3]
```



```
f 1 : map f [2, 3]
```



```
f 1 : f 2 : map f [3]
```



```
f 1 : f 2 : f 3
```



...

Example of map

map f [1, 2, 3]



f 1 : map f [2, 3]



Cons (f 1) (map f [2, 3])



f 1 : f 2 : map f [3]



Cons (f 1) (Cons (f 2) (map f [3]))



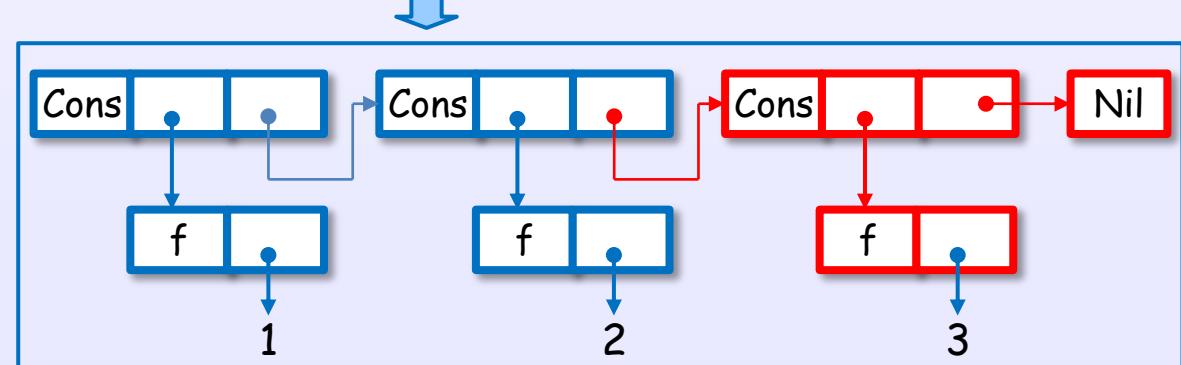
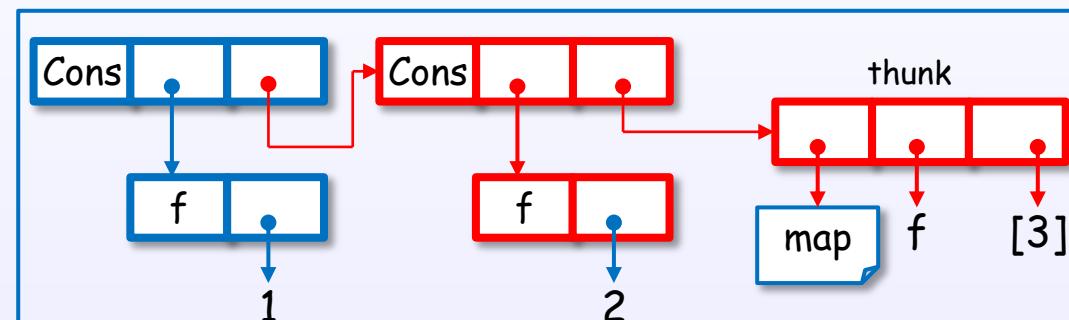
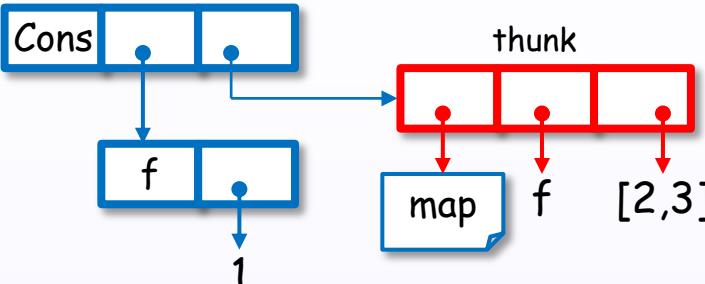
f 1 : f 2 : f 3



Cons (f 1) (Cons (f 2) (Cons (f 3) Nil))



...



References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of foldl (non-strict)

```
foldl (+) 0 [1 .. 100]
```



```
foldl (+) (0 + 1) [2 .. 100]
```



```
foldl (+) (((0 + 1) + 2) [3 .. 100]
```



```
foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]
```



...

References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of foldl (non-strict)

`foldl (+) 0 [1 .. 100]`



`foldl (+) (0 + 1) [2 .. 100]`

`let thunk1 = (0 + 1)
in foldl (+) thunk1 [2 .. 100]`



`foldl (+) ((0 + 1) + 2) [3 .. 100]`

`let thunk2 = (thunk1 + 2)
in foldl (+) thunk2 [3 .. 100]`



`foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]`

`let thunk3 = (thunk2 + 3)
in foldl (+) thunk3 [4 .. 100]`

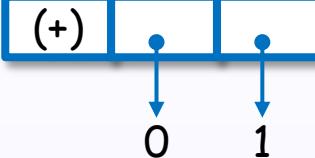


...

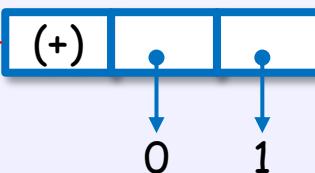
heap memory

*show only accumulation value

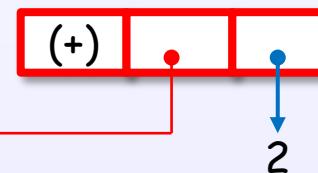
thunk1



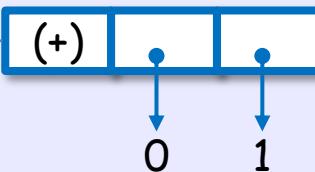
thunk1



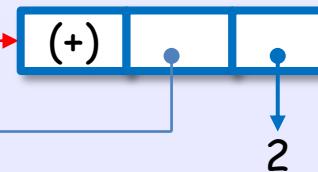
thunk2



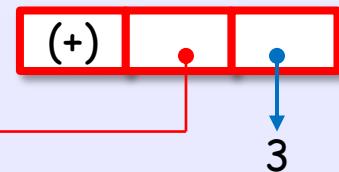
thunk1



thunk2



thunk3



increasing heap ...



References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of foldl' (strict)

```
foldl' (+) 0 [1 .. 100]
```



```
foldl' (+) (0 + 1) [2 .. 100]
```



```
foldl' (+) (1 + 2) [3 .. 100]
```



```
foldl' (+) (3 + 3) [4 .. 100]
```



...

References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of foldl' (strict)

`foldl' (+) 0 [1 .. 100]`



`foldl' (+) (0 + 1) [2 .. 100]`

```
let thunk1 = (0 + 1)
in thunk1 `seq`
  foldl' (+) thunk1 [2 .. 100]
```



`foldl' (+) (1 + 2) [3 .. 100]`

```
let thunk2 = (1 + 2)
in thunk2 `seq`
  foldl' (+) thunk2 [3 .. 100]
```

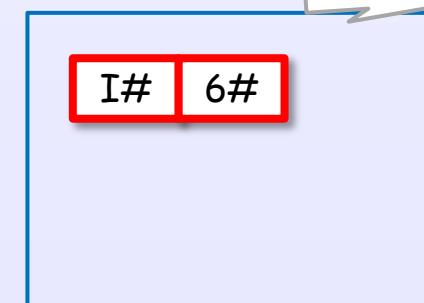
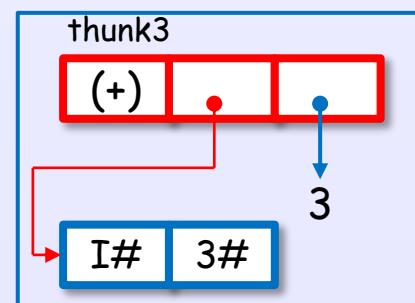
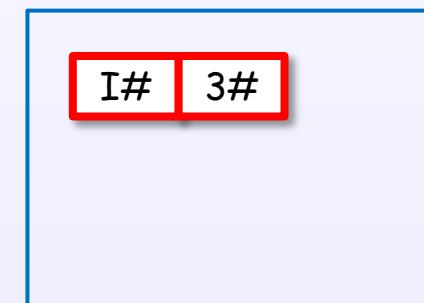
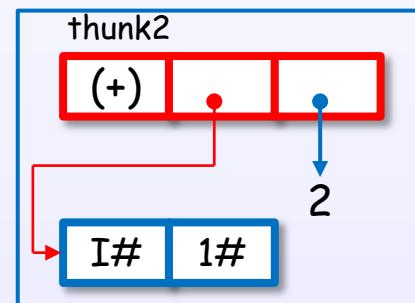
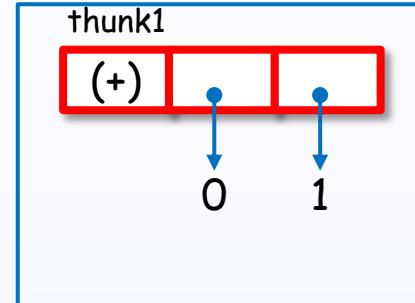


`foldl' (+) (3 + 3) [4 .. 100]`

```
let thunk3 = (3 + 3)
in thunk3 `seq`
  foldl' (+) thunk3 [4 .. 100]
```



heap memory



fixed heap size

...

References : [D5], [D6], [D8], [D9], [D10], [H10]

Example of foldl (non-strict) and foldl' (strict)

foldl (+) (0 + 1) [2 .. 100]



foldl' (+) (0 + 1) [2 .. 100]

foldl (+) ((0 + 1) + 2) [3 .. 100]



foldl' (+) (1 + 2) [3 .. 100]



foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]



foldl' (+) (3 + 3) [4 .. 100]

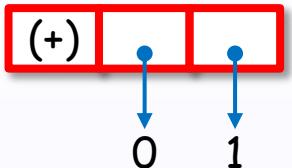


References : [D5], [D6], [D8], [D9], [D10], [H10]

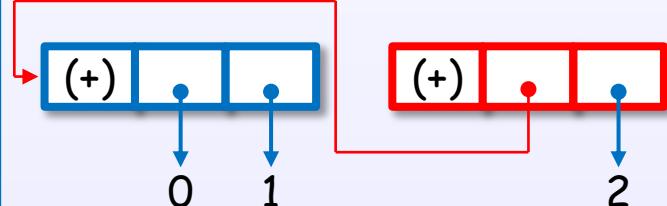
Example of foldl (non-strict) and foldl' (strict)

foldl (+) (0 + 1) [2 .. 100]

heap memory

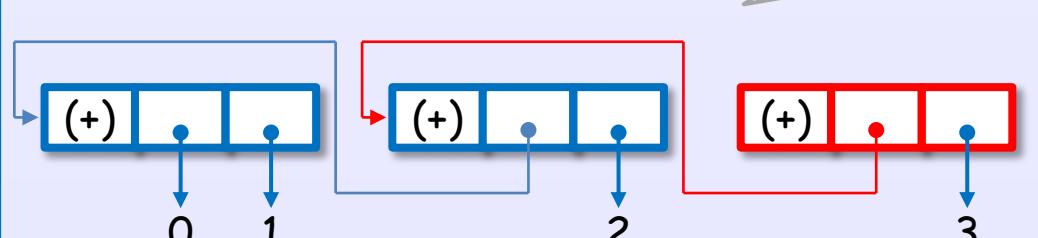


foldl (+) ((0 + 1) + 2) [3 .. 100]

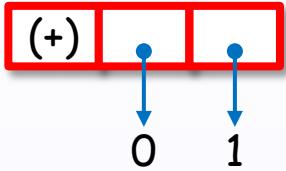


foldl (+) (((0 + 1) + 2) + 3) [4 .. 100]

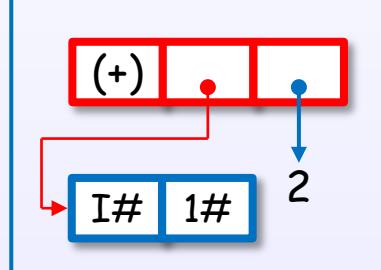
increasing heap ...



foldl' (+) (0 + 1) [2 .. 100]

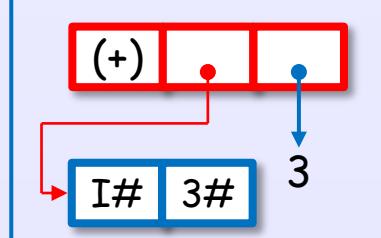


foldl' (+) (1 + 2) [3 .. 100]



foldl' (+) (3 + 3) [4 .. 100]

fixed heap size

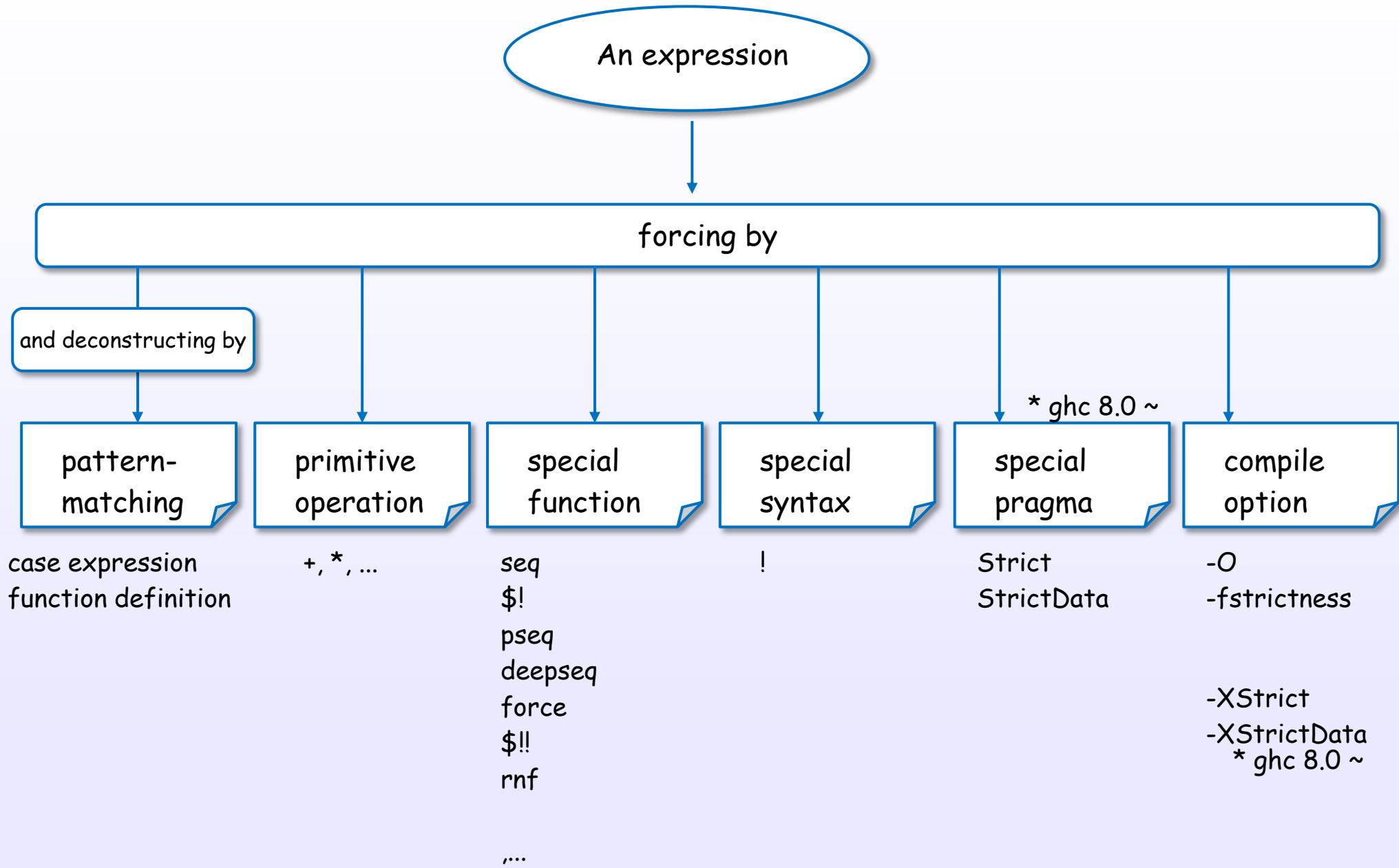


References : [D5], [D6], [D8], [D9], [D10], [H10]

4. Evaluation

Controlling the evaluation

How to drive evaluation



(1) Evaluation by pattern-matching

case expression

```
case ds of
  x:xs -> f x xs
  []      -> False
```

forcing
(drive the evaluation of the thunk)

function definition

```
f Just _ = True
f Nothing = False
```

forcing
(drive the evaluation of the thunk)

(1) Evaluation by pattern-matching

strict pattern

case expression

```
case ds of
  x:xs -> f x xs
  []      -> False
```

function definition

```
f Just _ = True
f Nothing = False
```

drive the evaluation

lazy pattern

let binding pattern

```
let (x:xs) = fun args
```

irrefutable patterns [H1] 3.17

```
f ~(Just _) = True
f ~(Nothing) = False
```

postpone the evaluation

There are two kinds of pattern-matching.

(2) Evaluation by primitive operation

primitive operation

$$f \ x \ y = x \underset{\text{---}}{+} y$$

forcing of x and y
(drive the evaluation of the thunk)

$+, *, ...$

primitive operations are defined like as

$$(+)(I\# a)(I\# b) = I\# (a+b)$$

pattern-matching

(3) Evaluation by special function

special function for WHNF

`seq x y`

`f $! x`

`pseq x y`

forcing
(drive the evaluation of the thunk)

special function for NF

`deepseq x y`

`f $!! x`

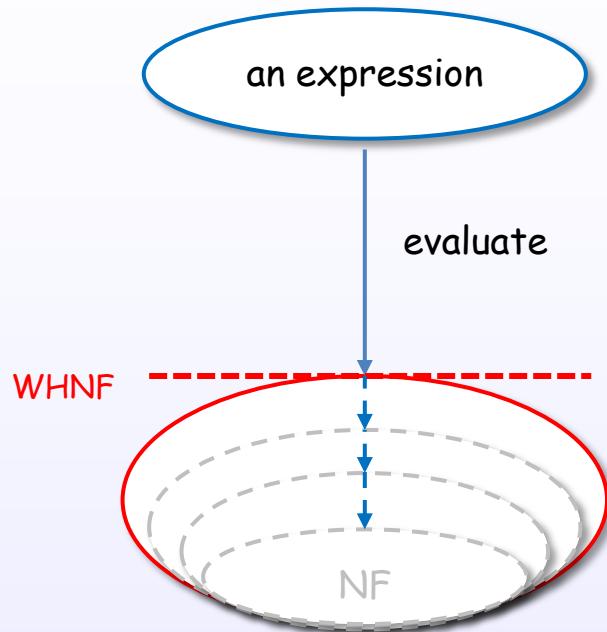
`force x`

`rnf x`

Please refer the document more detail. [xx]
hoogle or hackage

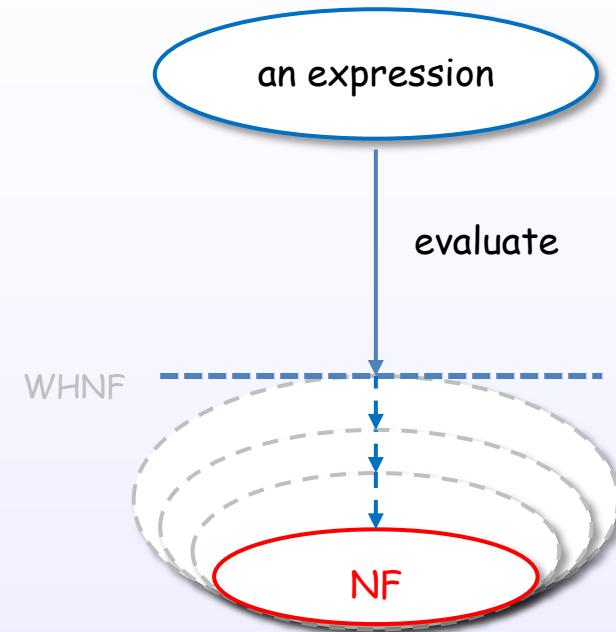
(3) Evaluation by special function

to WHNF



seq
\$!
pseq

to NF

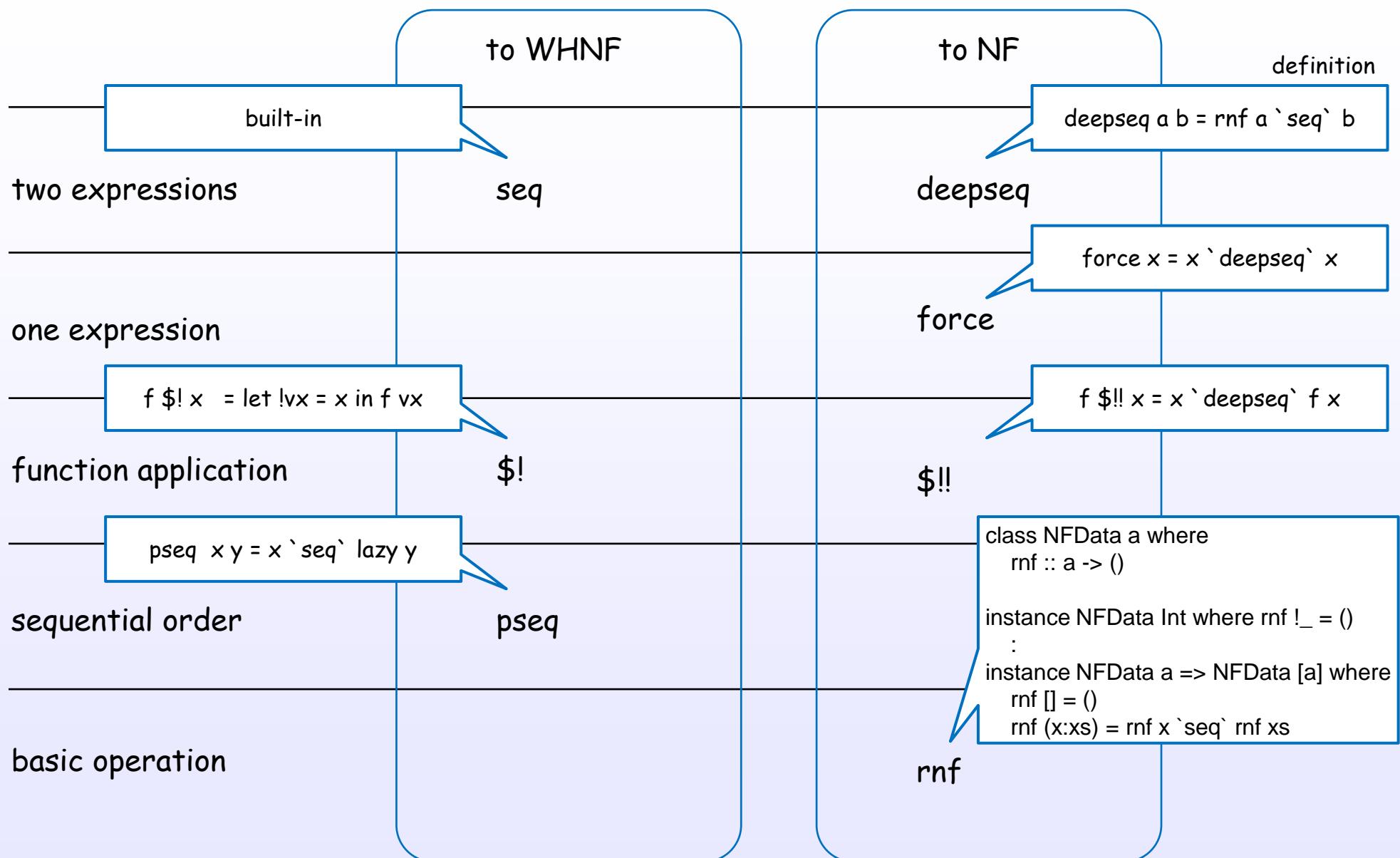


deepseq
force
\$!!
rnf

(3) Evaluation by special function

	to WHNF	to NF
two expressions	seq	deepseq
one expression		force
function application	\$!	\$!!
sequential order	pseq	
basic operation		rnf

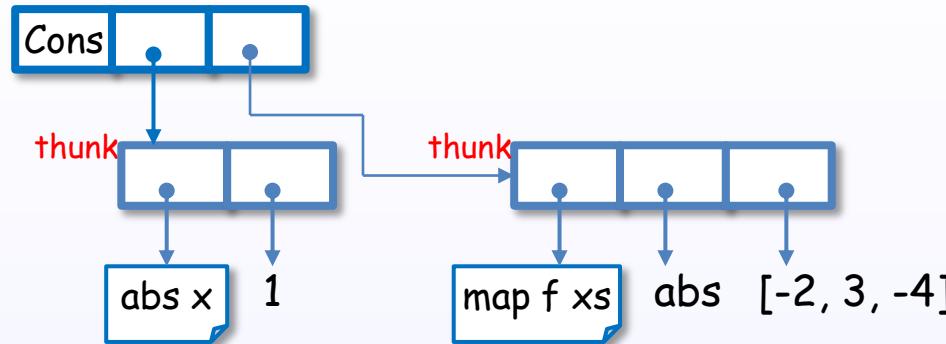
(3) Evaluation by special function



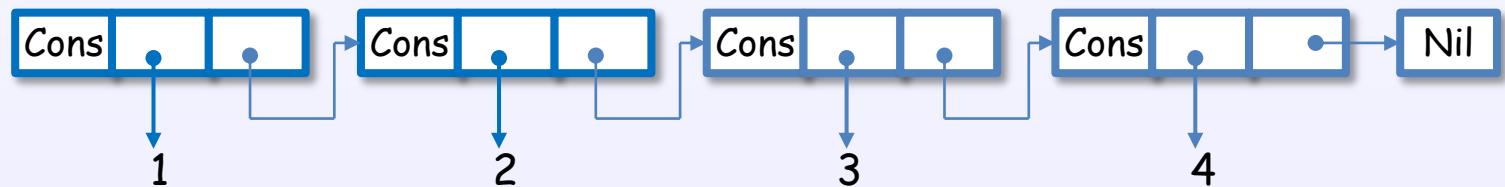
(3) Evaluation by special function

`a = map abs [1, -2, 3, -4]`

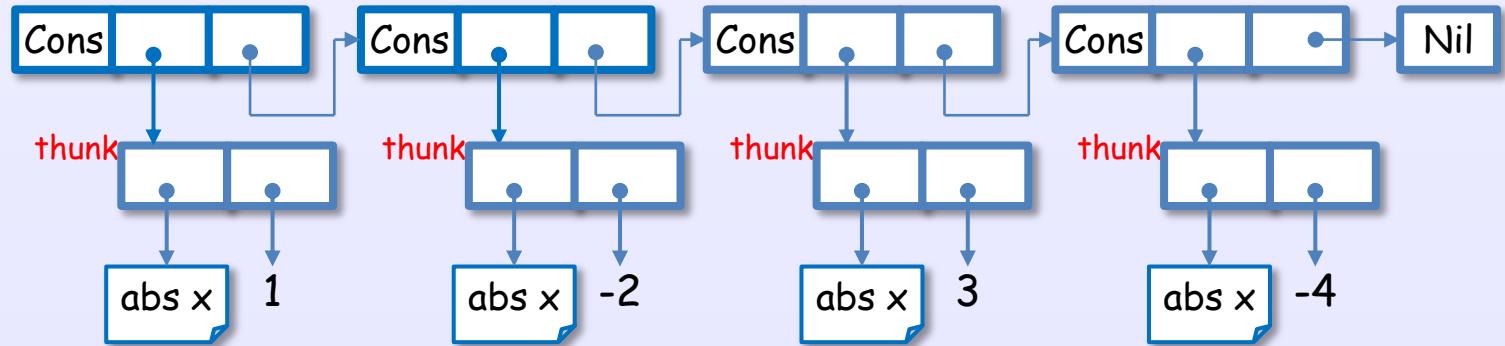
`seq a ()`



`deepseq a ()`



`length a`



(4) Evaluation by special syntax

Strictness annotation

Bang pattern[H2] 7.19

see also Strict pragma

```
{-# LANGUAGE BangPatterns #-}
```

```
f !xs = g xs
```

Strictness flag [H1] 4.2.1

see also StrictData and Strict pragma

```
data Pair = Pair !a !b
```

[slpj-book] Ch.22

Please refer the document more detail. [xx]

[Denotational]

(5) Evaluation by special pragma

special pragma

see also bang pattern and strictness flag

```
{-# LANGUAGE Strict #-}
```

```
let f xs = g xs in f ys  
data Pair = Pair a b
```

* ghc 8.0 ~

see also strictness flag

```
{-# LANGUAGE StrictData #-}
```

```
data Pair = Pair a b
```

Strict and StrictData pragmas are module level control.
These can use in ghc 8.0 or later.

(6) Evaluation by compile option

compile option

```
$ ghc -O
```

```
$ ghc -fstrictness
```

compile option * ghc 8.0 ~

```
$ ghc -XStrict
```

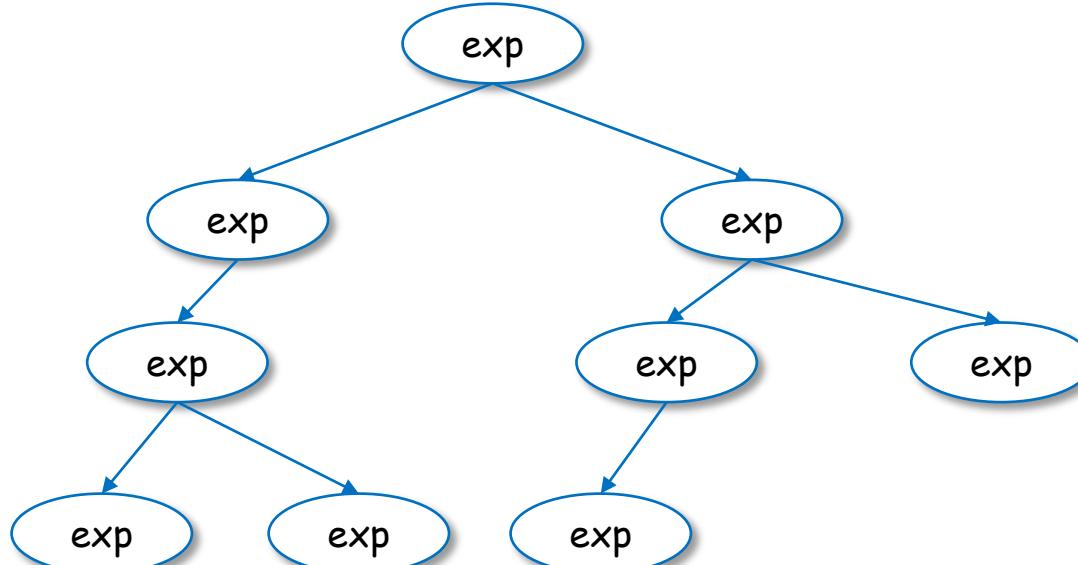
```
$ ghc -XStrictData
```

5. Implementation of evaluator

5. Implementation of evaluator

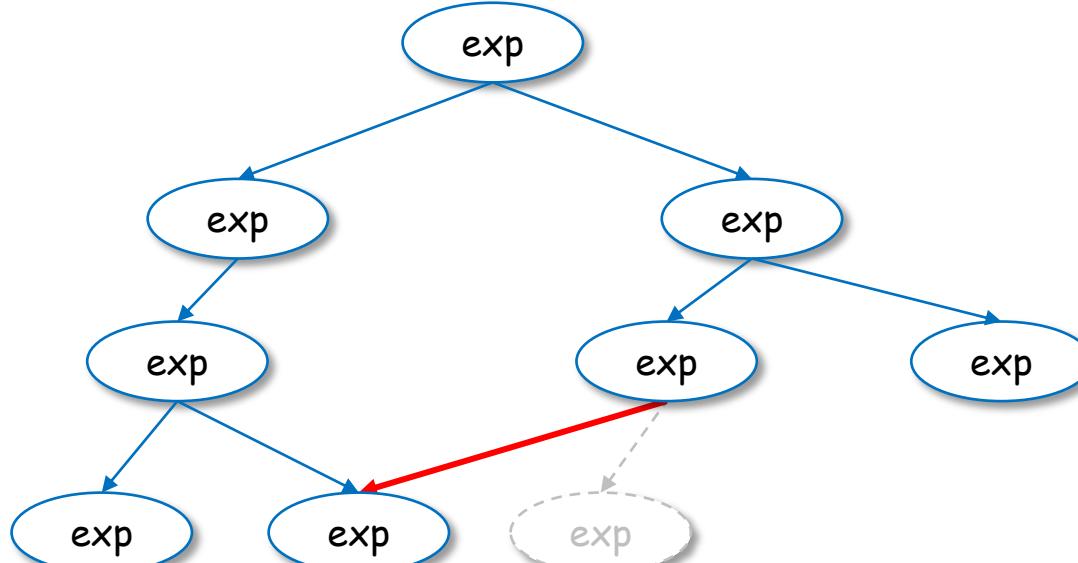
Lazy graph reduction

Tree



An expression can be represented with the form of Abstract Syntax **Tree** (AST). AST is reduced using stack (sequential access memory).

Graph



An expression can be also represented with the form of Graph.

Graph can share subexpressions to evaluate at once.

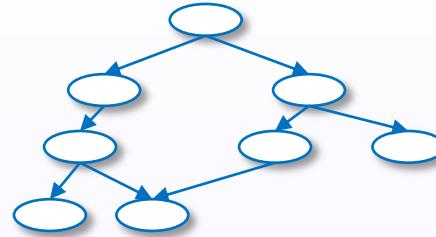
So, graph is reduced using heap (random access memory) rather than stack.

5. Implementation of evaluator

STG-machine

Abstract machine

Graph
(expression)



Evaluator
(abstract machine)

STG-machine

GHC uses abstract machine to reduce the expression.
It's called "STG-machine".

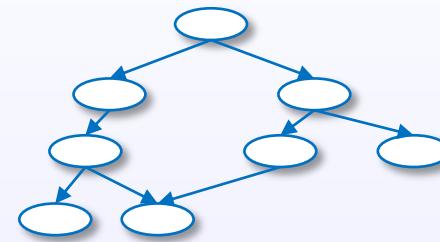
Concept layer

Haskell code

`take 5 [1..10]`

:

Graph
(internal representation
of expression)



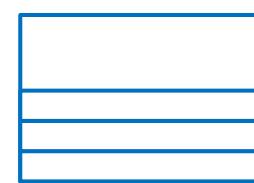
Evaluator (reducer, executer)
(abstract machine)

STG-machine

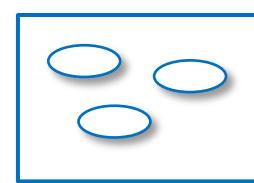
STG Registers

R1, ...

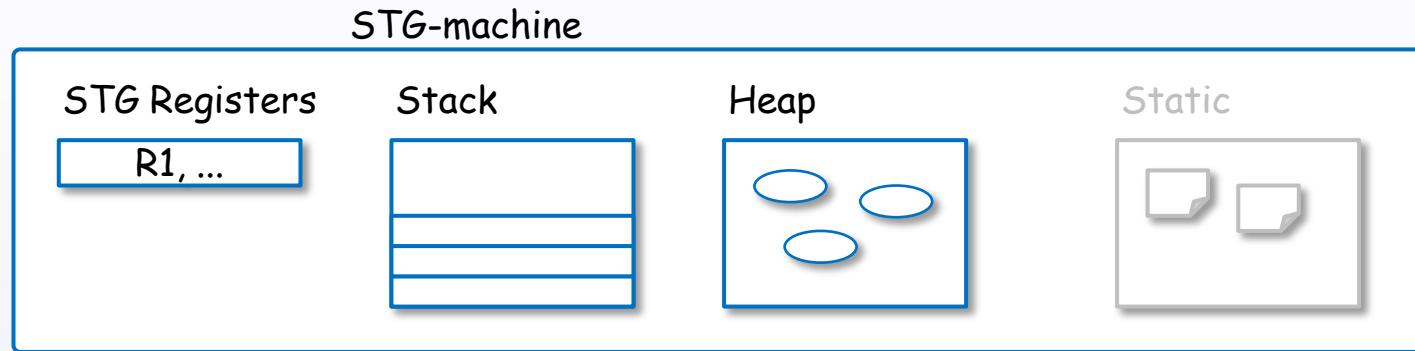
Stack



Heap



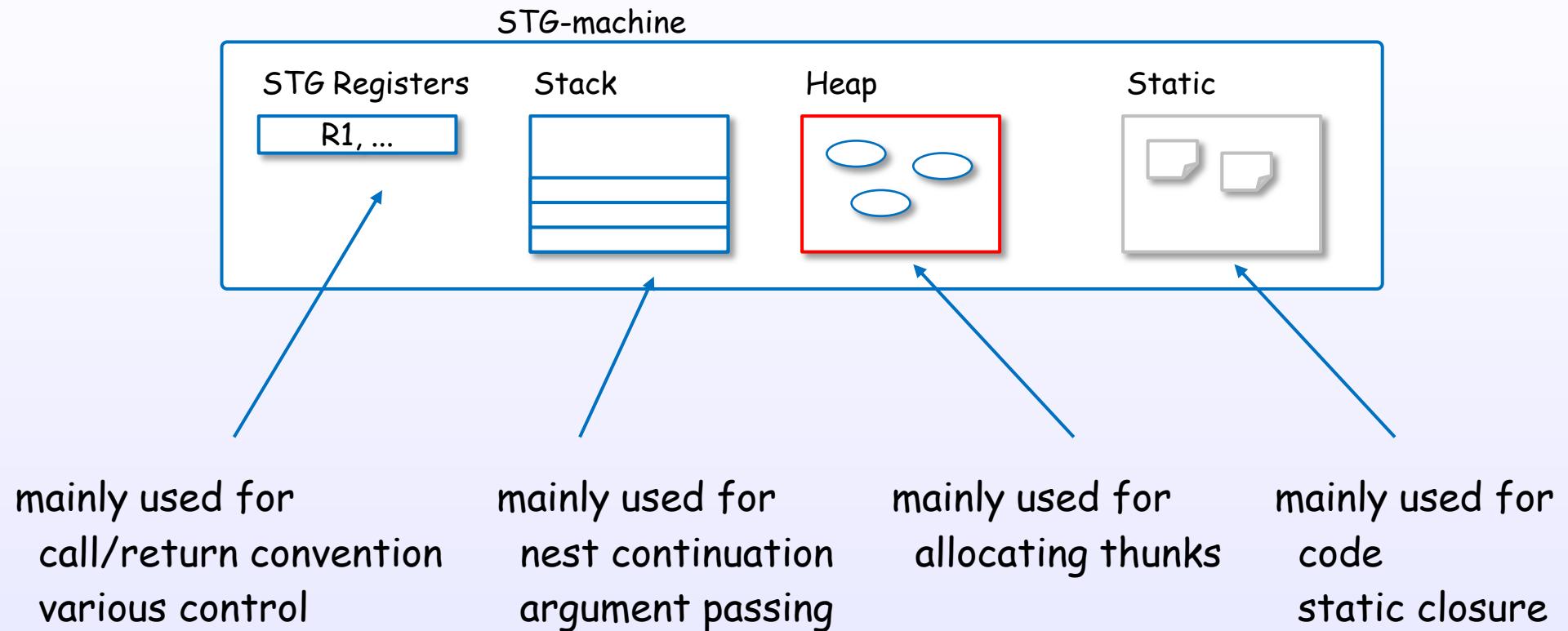
STG-machine



STG-machine is abstraction machine
which is defined by operational semantics.

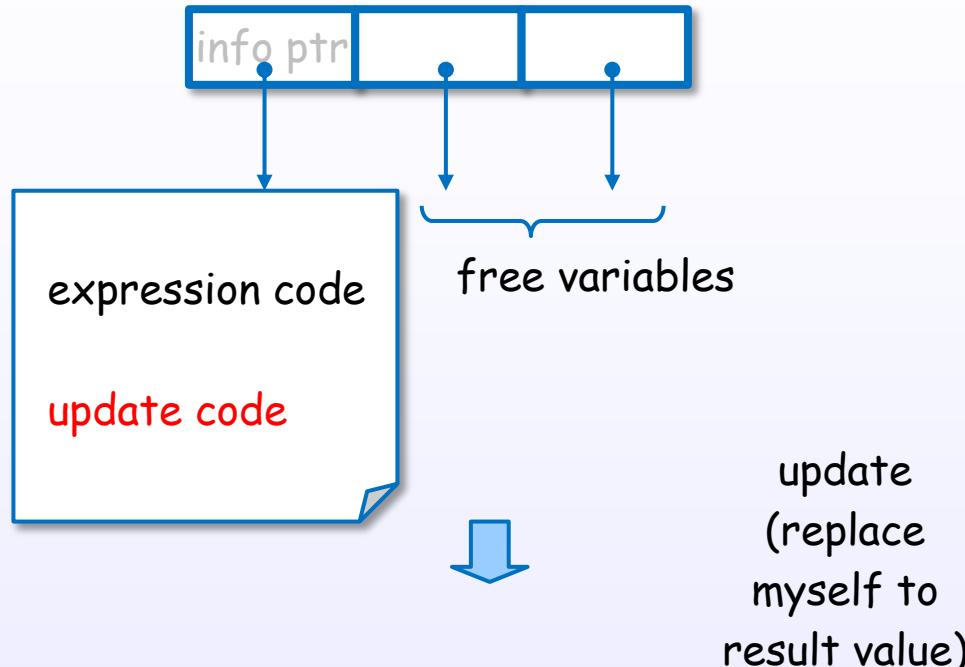
STG-machine efficiently performs lazy graph reduction.

STG-machine

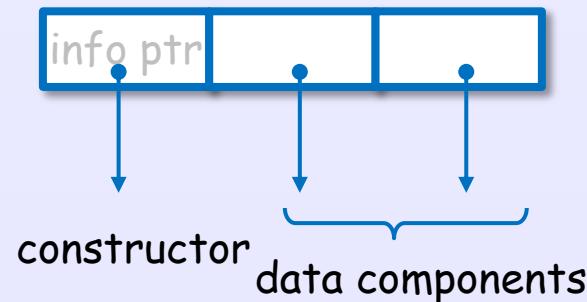


Self-updating model

a thunk



a data value

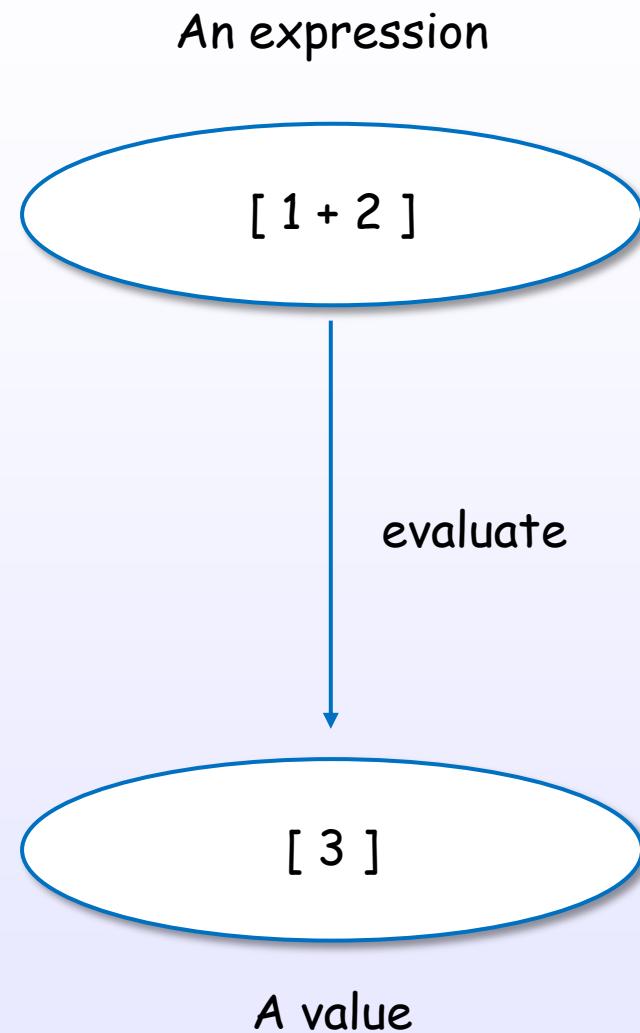


6. Semantics

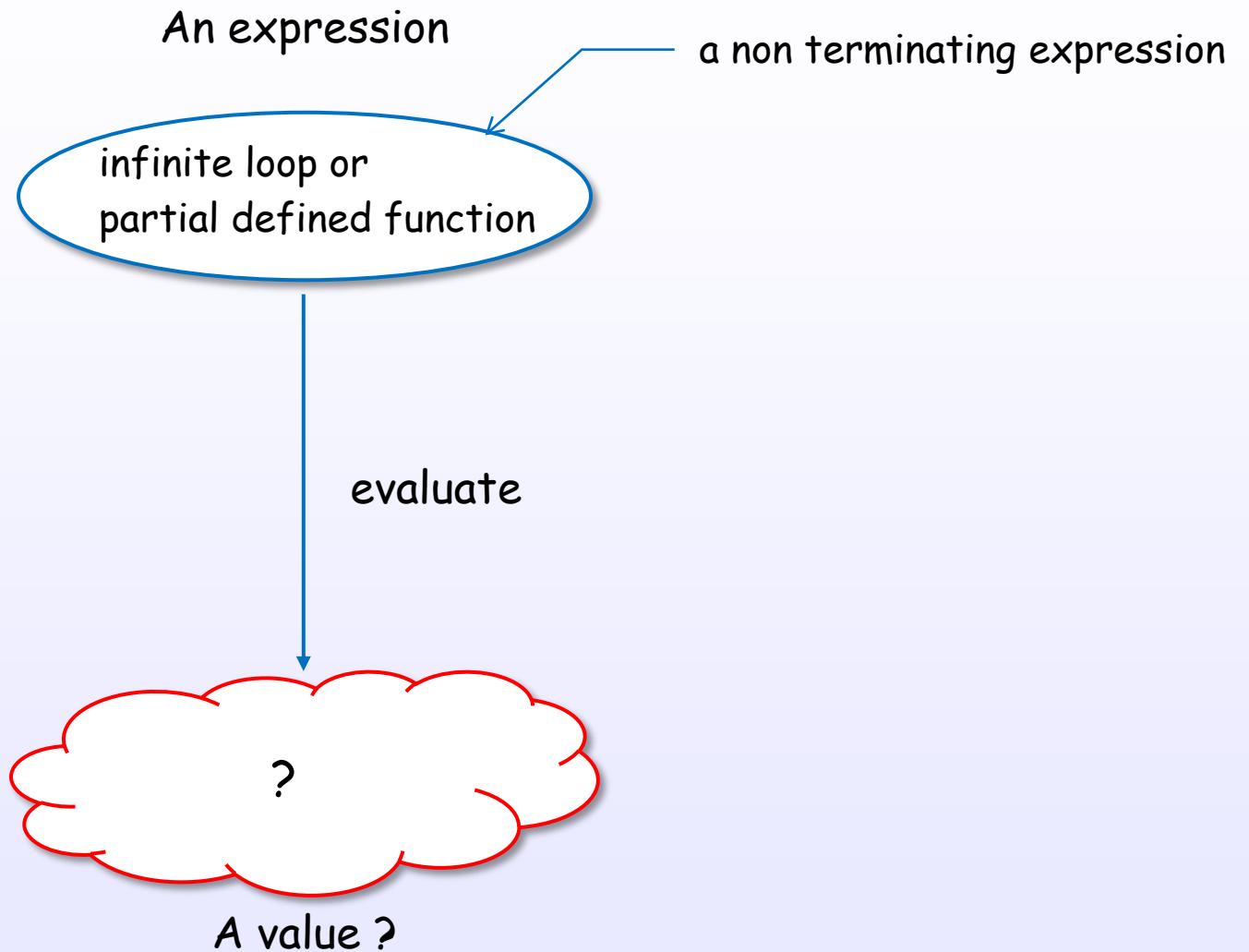
6. Semantics

Bottom

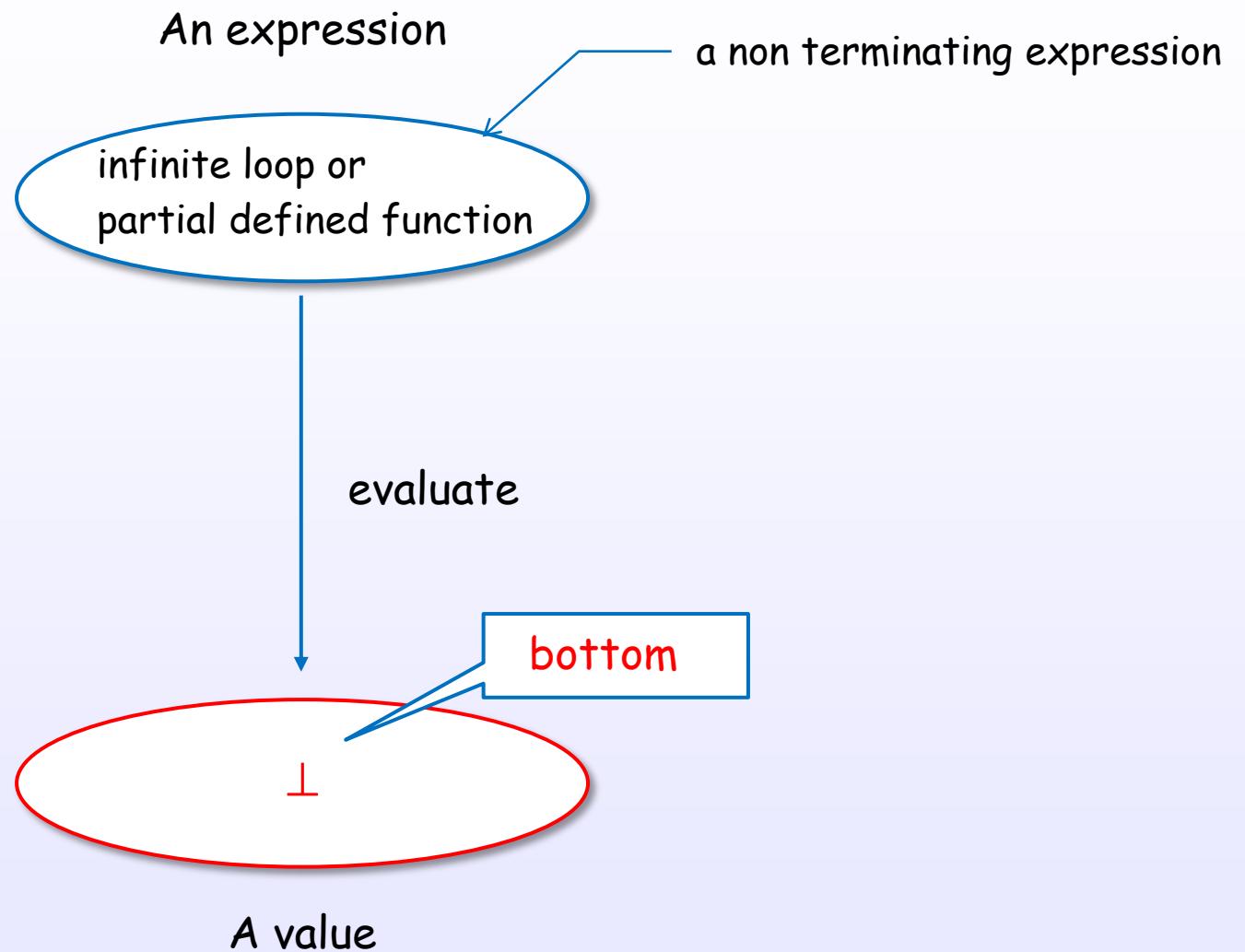
Well formed expression should have a value



What is a value in this case?

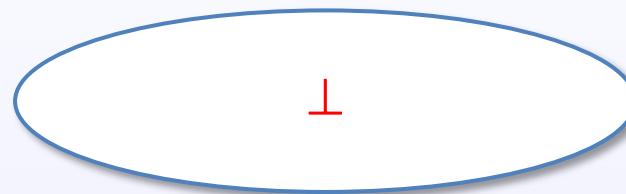


A value “bottom” is introduced



Bottom

A value



Bottom (\perp) is “an undefined value”.

Bottom (\perp) is “an non-terminating value”.

“undefined” represents bottom

Haskell code

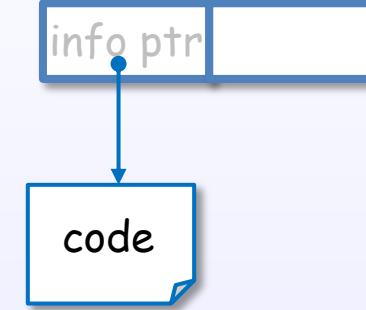
undefined :: a

Expression

\perp

GHC's internal representation

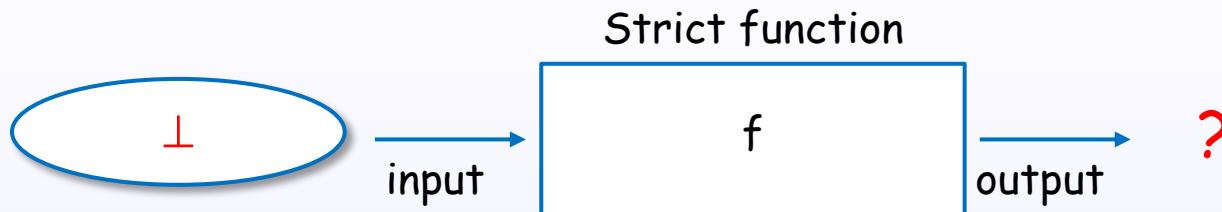
undefined :: a



6. Semantics

Non-strict Semantics

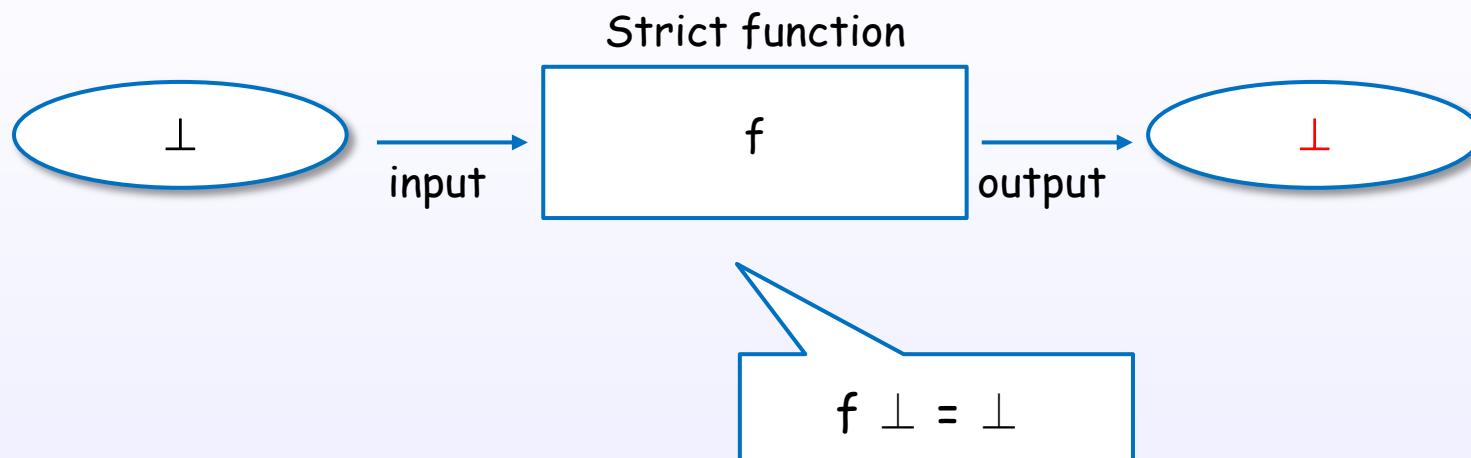
Strictness



Strictness is property of the function.

"given a non-terminating arguments, f will terminate?"

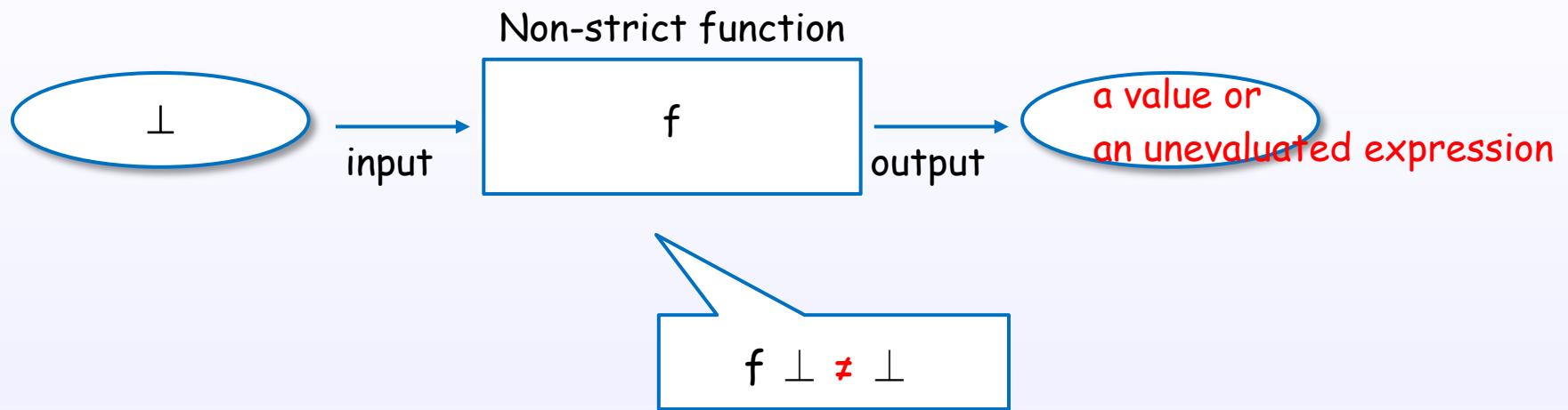
Strict function



Strict function's output is bottom if input is bottom.

given a non-terminating arguments, f will **not** terminate.

Non-strict function

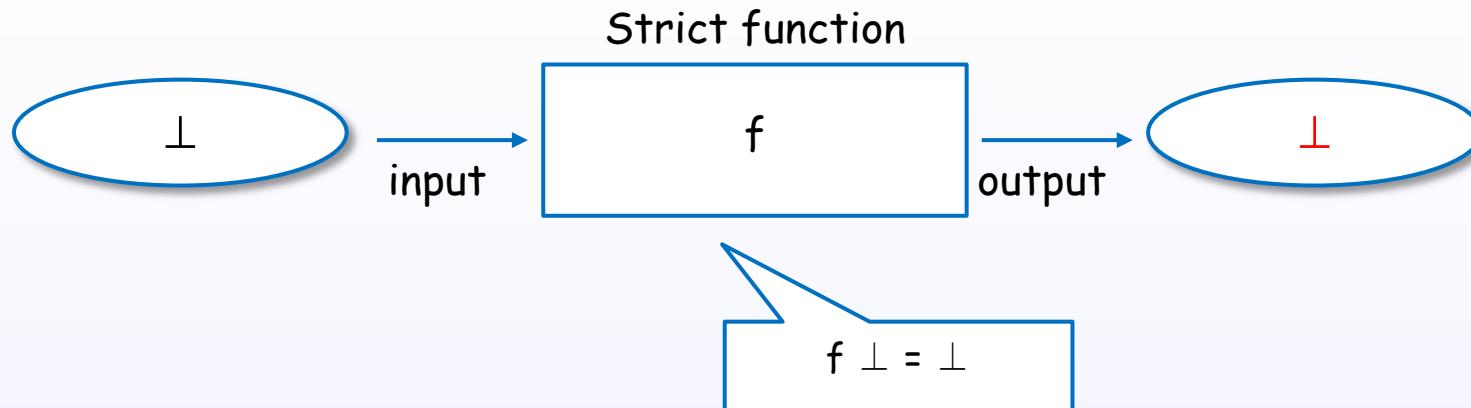


Non-strict function's output is **not** bottom if input is bottom.

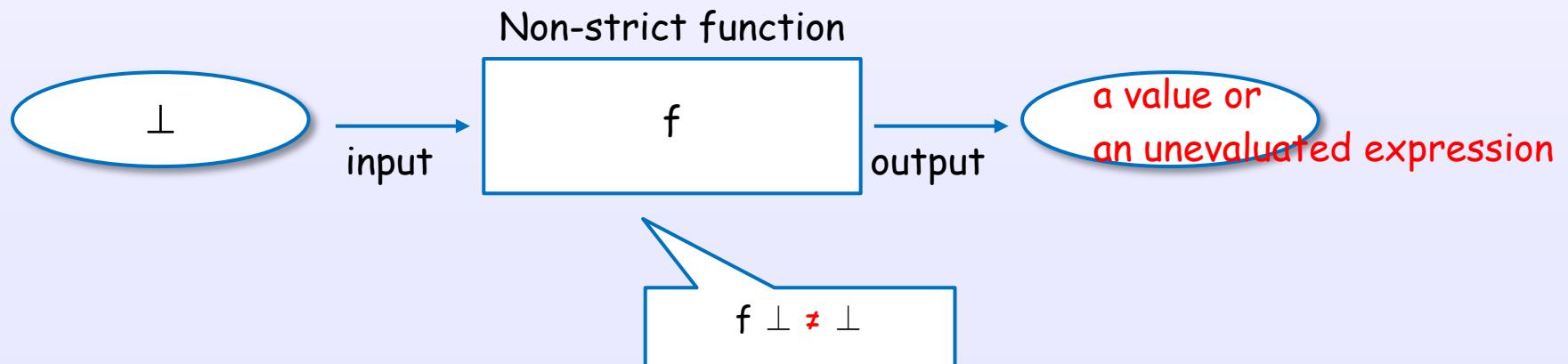
given a non-terminating arguments, f will terminate.

Strictness and Non-strictness

Strict



Non-strict

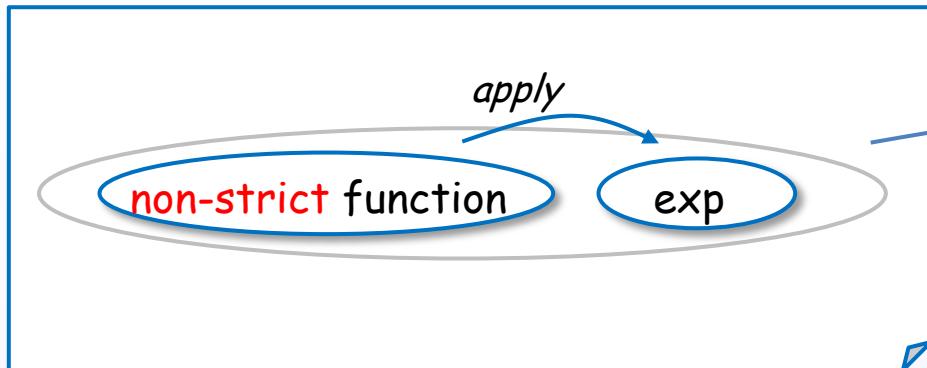


6. Semantics

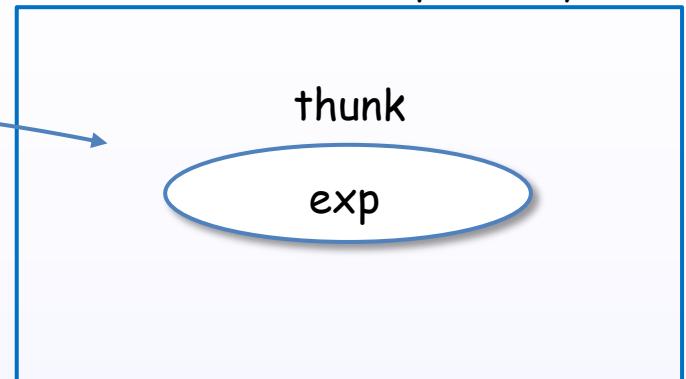
Strict analysis

Strict function's application doesn't build a thunk

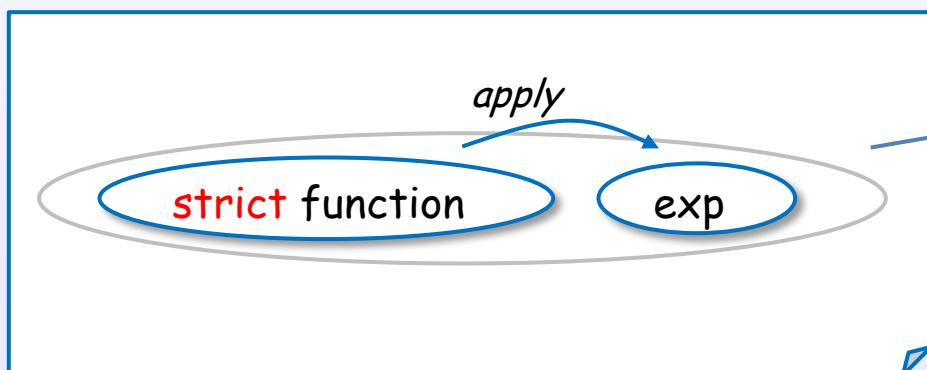
non-strict function



heap memory



strict function



heap memory



If GHC knows that a function is strict, arguments are evaluated before application.

GHC finds strict functions by "strictness analysis".

Strict analysis

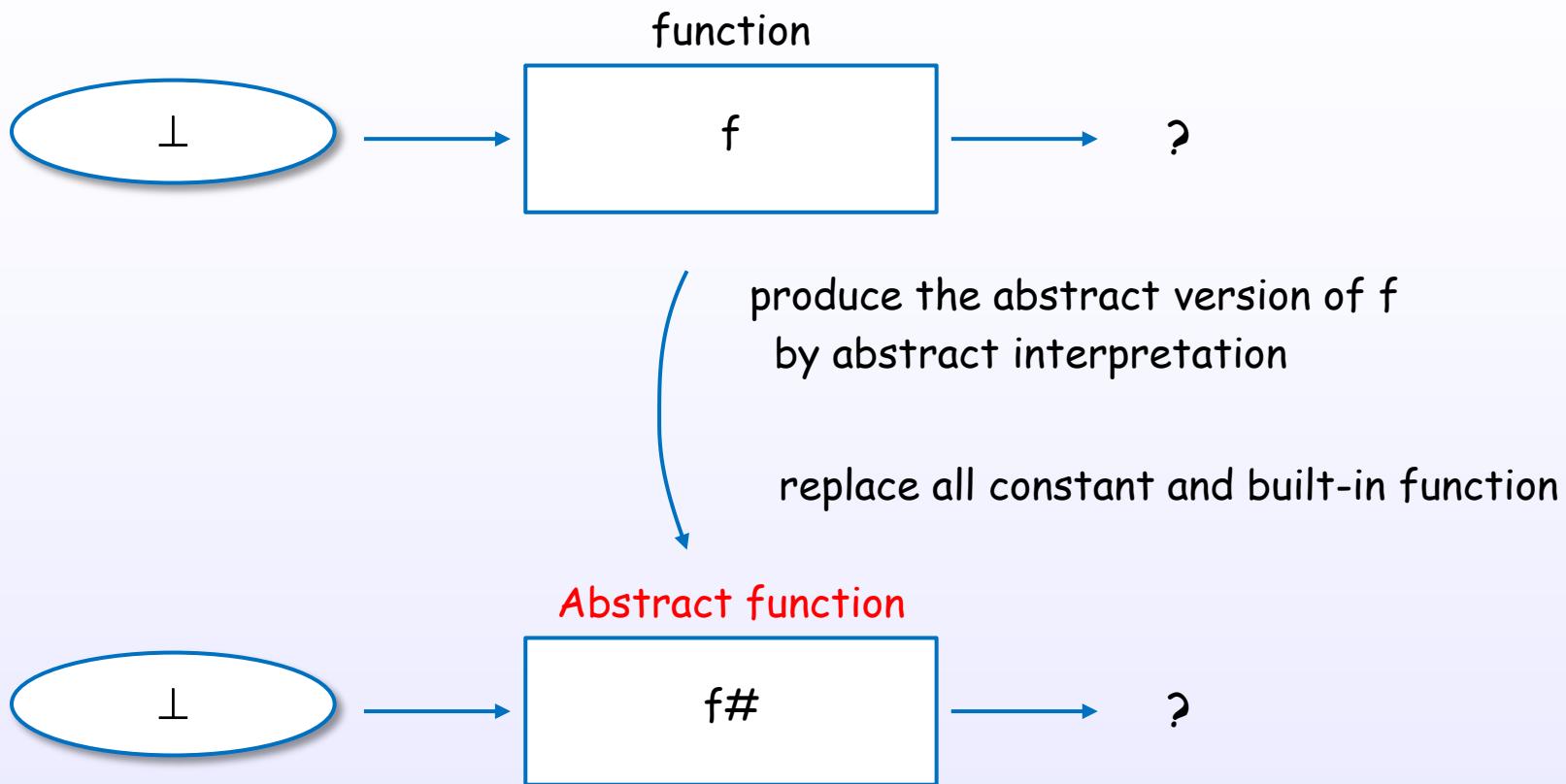
Strict analysis can determine which arguments a function is sure to evaluate.

"Does this function always need the value of its argument?"

given a function f, does $(f \ _| _) = \ _| _$?

-ddump-stranal
-ddump-strsigs
[user_guide] -fstrictness

Strict analysis using abstract function



$f\#$ gives a short-cut result.

Abstract function can decide output without going via full evaluation.

6. Semantics

Sequential order

Evaluation order

seq function

`seq a b`

specification

$\text{seq } a \ b = \perp, \quad \text{if } a = \perp$
 $= b, \quad \text{otherwise}$

"seq" function specifies strictness a and b.

But, "seq" don't guarantee the order a and b.

"b" may complete before "a".

If you need to control the order of "a" and "b",
you should use "pseq" rather than "seq".

seq and pseq

specification

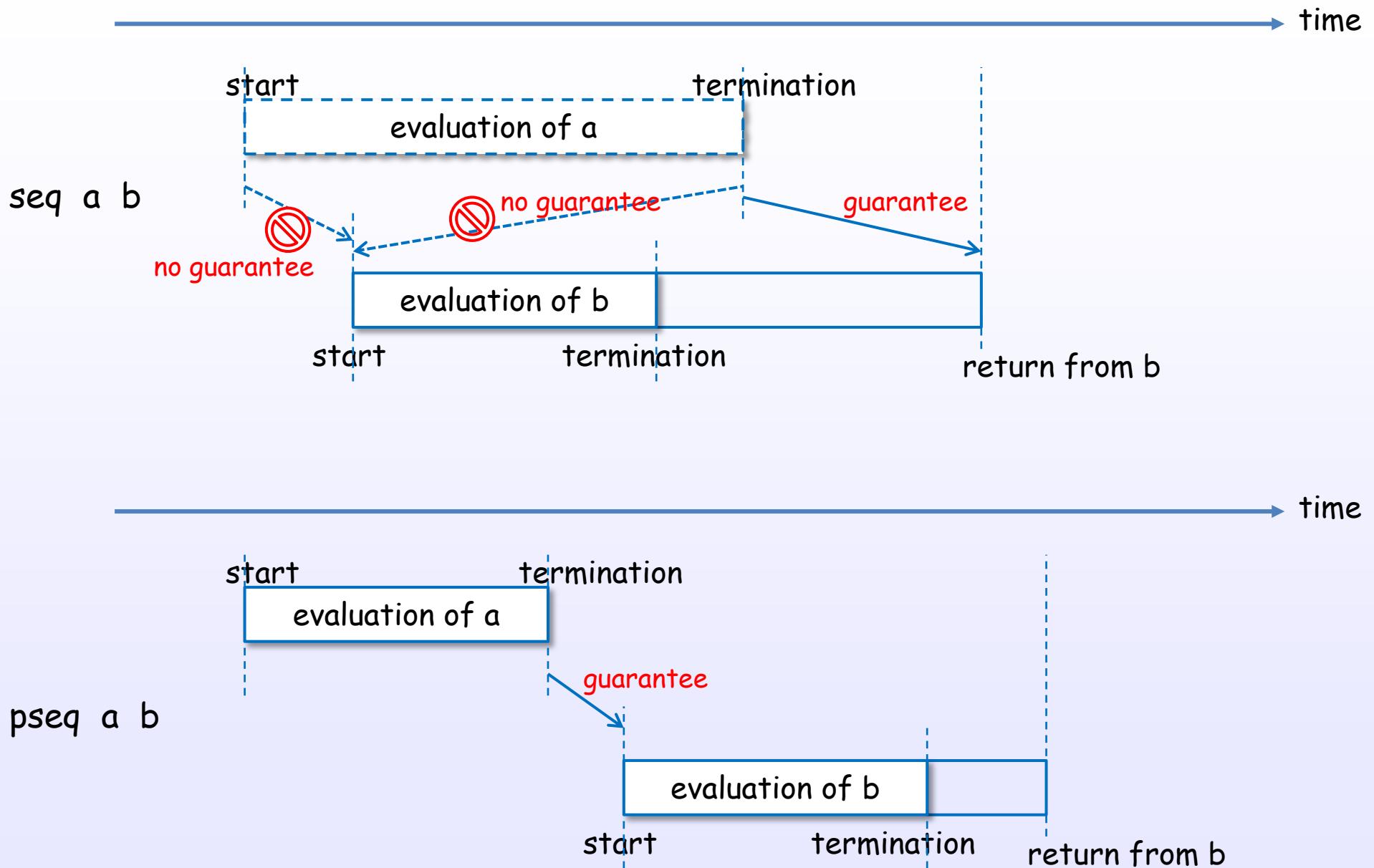
$$\begin{aligned} \text{seq } a \ b &= \perp, && \text{if } a = \perp \\ &= b, && \text{otherwise} \end{aligned}$$
$$\begin{aligned} \text{seq } \perp \ b &= \perp && a \text{ is strict} \\ \text{seq } a \ \perp &= \perp && b \text{ is strict} \end{aligned}$$

specification

$$\begin{aligned} \text{pseq } a \ b &= \perp, && \text{if } a = \perp \\ &= b, && \text{otherwise} \end{aligned}$$
$$\begin{aligned} \text{pseq } a \ \perp &= \perp && a \text{ is strict} \\ \text{pseq } \perp \ b &\neq \perp && b \text{ is non-strict} \end{aligned}$$

"pseq" guarantees the order "a" and "b".

Evaluation order of seq and pseq



7. Appendix

7. Appendix

References

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[W5] Haskell/Graph reduction

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Lazy,... zzz

to be as lazy as possible...