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Entropy, distance measure and similarity measure of fuzzy sets and their relations

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Abstract: The axiom definitions of entropy, distance measure and similarity measure of fuzzy sets are systematically given and basic relations between these measures are discussed. The concepts of σ- (resp. sub-σ-)entropy, σ- (resp. sub-σ-)distance measure and σ- (resp. sub-σ-)similarity measure are proposed, and some examples of them are listed. The relations between σ- (resp. sub-σ-)entropy, σ- (resp. sub-σ-)distance measure and σ- (resp. sub-σ-)similarity measure are obtained.

Keywords: Fuzzy set; entropy; distance measure; similarity measure.

1. Introduction

The entropy of a fuzzy set is a measure of fuzziness of the fuzzy set. De Luca and Termini [2] introduced the axiom construction of entropy of fuzzy sets and referred to Shannon's probability entropy; they gave an example of entropy of a fuzzy set for a finite universal set. Kaufmann [3] pointed out that an entropy of a fuzzy set can be gotten through the distance between the fuzzy set and its nearest non-fuzzy set. Yager [4] defined the entropies of a fuzzy set by distances between the fuzzy set and its complement. Loo [6] has proposed an entropy of a fuzzy set as $F(\sum_{i=1}^{n} c_i f_i(x_i))$ (the meaning of this formula will be explained in Section 2) and the entropies proposed by De Luca and Termini [2] and Kaufmann [3] are special cases of this entropy.

A distance measure of two fuzzy sets is a measure that describes the difference between fuzzy sets. A lot of researchers used it but no axiom definition for it has been seen. Related with the concept of distance measure, the similarity measure of two fuzzy sets introduced by Wang [7] and other people is a measure that indicates the similarity between fuzzy sets. It is easy to see that the distance measure and similarity measure are two dual concepts.

In this paper, (1) we systematically give the axiom definitions of entropy, distance measure and similarity measure of fuzzy sets and discuss some basic relations between these measures; (2) ee introduce the concepts σ- (resp. sub-σ-)entropy, σ- (resp. sub-σ-)distance measure and σ- (resp. sub-σ-)similarity measure of fuzzy sets which characterize 'additive' properties of entropy, distance measure and similarity measure and show that most of the entropies proposed in [2–7] are σ-entropies or sub-σ-entropies; (3) we study the relations between σ- (resp. sub-σ-)entropy, σ- (resp. sub-σ-) distance measure and σ- (resp. sub-σ-)similarity measure and the best result we obtain is that the σ-entropy and the sub-σ-entropy can be generated by distance measure and similarity measure.

Throughout this paper, $R^+ = [0, +\infty)$; X is the universal set; $\mathscr{F}(X)$ is the class of all fuzzy sets of X; $\mu_A(x)$ is the membership function of $A \in \mathscr{F}(X)$; $\mathscr{P}(X)$ is the class of all crisp sets of X; $[\frac{1}{2}]_X$ is the fuzzy set of X for which $\mu_{[\frac{1}{2}]_X}(x) = \frac{1}{2}$, $\forall x \in X$; $\mathscr{F}$ is a sub-class of $\mathscr{F}(X)$ with (1) $\mathscr{P}(X) \subset \mathscr{F}$, (2) $[\frac{1}{2}]_X \in \mathscr{F}$, (3) $A, B \in \mathscr{F} \Rightarrow A \cup B \in \mathscr{F}$, $A^c \in \mathscr{F}$, where $A^c \in \mathscr{F}(X)$ is the complement of $A \in \mathscr{F}(X)$, i.e., $\mu_{A^c}(x) = 1 - \mu_A(x)$, $\forall x \in X$.

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2. Axiom definition of entropy, distance measure and similarity measure of fuzzy sets

2.1. Axiom definition of entropy of fuzzy sets

Definition 2.1. A real function $e:\mathscr{F}\to R^+$ is called an entropy on $\mathscr{F}$ if e has the following properties:

(EP1) $e(D)=0,\ \forall D\in\mathscr{P}(X)$;

(EP2) $e([\frac{1}{2}]_X)=\max_{A\in\mathscr{F}} e(A)$;

(EP3) $\forall A,\ B\in\mathscr{F}$, if $\mu_B(x)\geqslant\mu_A(x)$ when $\mu_A(x)\geqslant\frac{1}{2}$ and $\mu_B(x)\leqslant\mu_A(x)$ when $\mu_A(x)\leqslant\frac{1}{2}$, then $e(A)\geqslant e(B)$;

(EP4) $e(A^c)=e(A),\ \forall A\in\mathscr{F}$.

Example 2.1 [2]. Let $X=\{x_1,x_2,\ldots,x_n\}$ and define

$$e_1(A)=-K\sum_{i=1}^{n} S(\mu_A(x_i))\quad \forall A\in\mathscr{F}(X).$$

Then e_1 is an entropy on $\mathscr{F}(X)$, where $S(x)=-x\ln x-(1-x)\ln(1-x),\ 0\leqslant x\leqslant 1$, and we take the convention that $0\ln 0=0$.

Example 2.2 [3]. Let $X=\{x_1,x_2,\ldots,x_n\}$. For any $A\in\mathscr{F}(X)$, take $\hat{A}\in\mathscr{P}(X)$ with

$$\mu_{\hat{A}}(x)=\begin{cases}1 & \text{when } \mu_A(x)>\frac{1}{2},\\ 0 & \text{when } \mu_A(x)\leqslant\frac{1}{2},\end{cases}$$

and define

$$e_2(A)=\Big(\sum_{i=1}^{n}|\mu_A(x_i)-\mu_{\hat{A}}(x_i)|^{\omega}\Big)^{1/\omega}\quad \forall A\in\mathscr{F}(X).$$

Then e_2 is an entropy on $\mathscr{F}(X)$, where $\omega\geqslant 1$.

Example 2.3 [4]. Let $X=\{x_1,x_2,\ldots,x_n\}$ and define

$$e_3(A)=1-\frac{D_p(A,A^c)}{n^{1/p}}\quad \forall A\in\mathscr{F}(X).$$

Then e_3 is an entropy on $\mathscr{F}(X)$, where D_p is defined as

$$D_p(A,B)=\Big(\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|^{p}\Big)^{1/p}\quad \forall A,\ B\in\mathscr{F}(X),\ p\in\{1,2,\ldots\}.$$

Example 2.4 [6]. Let $X=\{x_1,x_2,\ldots,x_n\}$ and define

$$e_4(A)=F\Big(\sum_{i=1}^{n} c_i f_i(\mu_A(x_i))\Big)\quad \forall A\in\mathscr{F}(X).$$

Then e_4 is an entropy on $\mathscr{F}(X)$, where $\forall i\in\{1,2,\ldots,n\}$, $c_i\in R^+$, f_i is a function from $[0,1]$ to $[0,1]$ such that

(1) $f_i(0)=f_i(1)=0$;

(2) $f_i(u)=f_i(1-u),\ \forall u\in[0,1]$;

(3) f_i is strictly increasing on $[0,\frac{1}{2}]$;

and F is a positive increasing function from R^+ to R^+ with $F(0)=0$.

The entropies mentioned in Example 2.1 and 2.2 are special cases of the one defined in Example 2.4 (see [9]).

Example 2.5. Let $X = [0, 1]$ and $\hat{\mathscr{F}} = \{A \in \mathscr{F}(X);\ \mu_A(x)$ is a measurable function with respect to Borel field $\mathscr{B}^1\}$. An entropy e_5 on $\hat{\mathscr{F}}$ can be defined as

$$e_5(A) = 1 - \left(\int_0^1 |\mu_A(x) - \mu_{A^c}(x)|^p \,\mathrm{d}x\right)^{1/p} \quad \forall A \in \hat{\mathscr{F}}$$

where $p \geqslant 1$.

2.2. Axiom definition of distance measure of fuzzy sets

Definition 2.2. A real function $d : \mathscr{F}^2 \to R^+$ is called a distance measure on $\mathscr{F}$ if d satisfies the following properties:

(DP1) $d(A, B) = d(B, A)$, $\forall A, B \in \mathscr{F}$;
(DP2) $d(A, A) = 0$, $\forall A \in \mathscr{F}$;
(DP3) $d(D, D^c) = \max_{A,B\in\mathscr{F}} d(A, B)$, $\forall D \in \mathscr{P}(X)$;
(DP4) $\forall A, B, C \in \mathscr{F}$, if $A \subset B \subset C$, then $d(A, B) \leqslant d(A, C)$ and $d(B, C) \leqslant d(A, C)$.

It is easy to see that (DP4) is equivalent to (DP4)′: $\forall A, B, C, D \in \mathscr{F}$, if $A \subset B \subset C \subset D$, then $d(B, C) \leqslant d(A, D)$.

Example 2.6. Following Example 2.3, D_p is a distance measure on $\mathscr{F}(X)$, $p = 1, 2, \ldots$.

Example 2.7. Following Example 2.5,

$$d_p(A, B) = \left(\int_0^{\prime} |\mu_A(x) - \mu_B(x)|^p\right)^{1/p} \quad \forall A, B \in \hat{\mathscr{F}}$$

is a distance measure on $\hat{\mathscr{F}}$, where $p \geqslant 1$.

2.3. Axiom definition of similarity measure of fuzzy sets

Definition 2.3. A real function $s : \mathscr{F}^2 \to R^+$ is called a similarity measure, if s has the following properties:

(SP1) $s(A, B) = s(B, A)$, $\forall A, B \in \mathscr{F}$;
(SP2) $s(D, D^c) = 0$, $\forall D \in \mathscr{P}(X)$;
(SP3) $s(C, C) = \max_{A,B\in\mathscr{F}} s(A, B)$, $\forall C \in \mathscr{F}$;
(SP4) $\forall A, B, C \in \mathscr{F}$, if $A \subset B \subset C$, then $s(A, B) \geqslant s(A, C)$ and $s(B, C) \geqslant s(A, C)$.

We can prove that (SP4) is equivalent to (SP4)′: $\forall A, B, C, D \in \mathscr{F}$, if $A \subset B \subset C \subset D$, then $s(B, C) \geqslant s(A, D)$.

Example 2.8. Following Example 2.5, for $p \geqslant 1$, define

$$s_p(A, B) = 1 - \left(\int_0^1 |\mu_A(x) - \mu_B(x)|^p\right)^{1/p} \quad \forall A, B \in \hat{\mathscr{F}}.$$

Then s_p is a similarity measure on $\hat{\mathscr{F}}$.

3. The relations between entropy, distance measure and similarity measure

3.1. Normal entropy, normal distance measure and normal similarity measure

Definition 3.1. If an entropy e on $\mathscr{F}$ satisfies $e([\frac{1}{2}]_X) = 1$, then we call e a normal entropy on $\mathscr{F}$.

Proposition 3.1. *If e is an entropy on $\mathscr{F}$, then $\hat{e}$ given by*

$$\hat{e}(A) = \frac{e(A)}{e([\frac{1}{2}]_X)} \quad \forall A \in \mathscr{F}$$

is a normal entropy on $\mathscr{F}$.

Definition 3.2. If a distance measure d on $\mathscr{F}$ satisfies $\max_{A,B \in \mathscr{F}} d(A, B) = 1$, then we call d a normal distance measure on $\mathscr{F}$.

Proposition 3.2. *If d is a distance measure on $\mathscr{F}$, then $\hat{d}$ given by*

$$\hat{d}(A, B) = \frac{d(A, B)}{\max_{C,D \in \mathscr{F}} d(C, D)} \quad \forall A, B \in \mathscr{F}$$

is a normal distance measure on $\mathscr{F}$.

Definition 3.3. If a similarity measure s on $\mathscr{F}$ satisfies $\max_{A,B \in \mathscr{F}} s(A, B) = 1$, then we call s a normal similarity measure on $\mathscr{F}$.

Proposition 3.3. *If s is a similarity measure on $\mathscr{F}$, then $\hat{s}$ given by*

$$\hat{s}(A, B) = \frac{s(A, B)}{\max_{C,D \in \mathscr{F}} s(C, D)} \quad \forall A, B \in \mathscr{F}$$

is a normal similarity measure on $\mathscr{F}$.

From Proposition 3.1, we know that the difference between an entropy and a normal entropy is only a multiplying constant, and similar conclusions are true for distance measure and similarity measure. For these reasons, from now on, we assume that the entropies, distance measures and similarity measures are normal.

3.2. The relations between entropy, distance measure and similarity measure

Proposition 3.4. *There exists a one-to-one correlation between all distance measures and all similarity measures, and a distance measure d and its corresponding similarity measure s satisfy $d + s = 1$.*

We call $s = 1 - d$ the similarity measure generated by distance measure d and denote it by $s\langle d \rangle$, and $d = 1 - s$ the distance measure generated by similarity measure s and denote it by $d\langle s \rangle$.

Proposition 3.5. *If s is a similarity measure on $\mathscr{F}$, define*

$$e(A) = s(A, A^c), \quad \forall A \in \mathscr{F}.$$

Then e is an entropy on $\mathscr{F}$.

We call e defined in Proposition 3.5 the entropy generated by similarity measure s and denote it by $e\langle s \rangle$.

Proposition 3.6. *If d is a distance measure on $\mathscr{F}$, define*

$$e(A) = 1 - d(A, A^c) \quad \forall A \in \mathscr{F}$$

then e is an entropy on $\mathscr{F}$.

We call e mentioned in Proposition 3.6 the entropy generated by distance measure d and denote it by $e\langle d \rangle$.

4. σ-Entropy and sub-σ-entropy of fuzzy sets

The entropy of a fuzzy set A is a measure of fuzziness of A. If we divide X into two parts D and D^c, where $D, D^c \in \mathscr{P}(X)$, then the fuzzy set $A \cap D$ has fuzziness only on D and the fuzzy set $A \cup D^c$ has fuzziness only on D^c. It is very natural for us to realize that the fuzziness of A should be the 'sum' of the fuzziness of $A \cap D$ and the fuzziness of $A \cap D^c$. Particularly, it is reasonable to assume that the 'sum' is the sum between real numbers. Definition 4.1 gives the formal description of this idea.

Definition 4.1. Let e be an entropy on $\mathscr{F}$. If for any $A \in \mathscr{F}$, we have

$$e(A) = e(A \cap D) + e(A \cap D^c) \quad \forall D \in \mathscr{P}(X) \tag{4.1}$$

then we call e a σ-entropy on $\mathscr{F}$.

Example 4.1. The entropy e_1 (see Example 2.1) is a σ-entropy on $\mathscr{F}(X)$.

Example 4.2. When $\omega = 1$, the entropy e_2 (see Example 2.2) is a σ-entropy on $\mathscr{F}(X)$.

Example 4.3. When $p = 1$, the entropy e_3 (see Example 2.3) is a σ-entropy on $\mathscr{F}(X)$.

Example 4.4. When $F(x) = cx$ (c is a constant), the entropy e_4 (see Example 2.4) is a σ-entropy on $\mathscr{F}(X)$.

Example 4.5. When $p = 1$, the entropy e_5 (see Example 2.5) is a σ-entropy on $\hat{\mathscr{F}}$.

Note. The entropies defined in Section 2 may not be normal, but we still call them σ-entropies if they satisfy (4.1). In the following, we shall meet similar problems as for distance measures and similarity measures.

In the following, in order to define sub-σ-entropy, which indicates a 'weak' additive property of the entropy, we introduce the concept of F-function.

Definition 4.2. A real function $F: R^+ \to R^+$ is called an F-function, if F is increasing on R^+ and $F(0) = 0$.

Proposition 4.1. *If e is an entropy on $\mathscr{F}$ and F is an* F*-function, then $F(e)$ is an entropy on $\mathscr{F}$.*

Definition 4.3. Let e be an entropy on $\mathscr{F}$. We call e a sub-σ-entropy, if there exist an F-function F and σ-entropy $\bar{e}$ such that $e = F(\bar{e})$.

It is obvious that a σ-entropy is a sub-σ-entropy.

Example 4.6. The entropy e_2 is a sub-σ-entropy on $\mathscr{F}(X)$ for every $\omega \geqslant 1$. The corresponding F-function F is $x^{1/\omega}$ and the corresponding σ-entropy $\bar{e}$ is

$$\bar{e}(A) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_{\hat{A}}(x_i)|^{\omega} \quad \forall A \in \mathscr{F}(X).$$

Example 4.7. The entropy e_3 is a sub-σ-entropy for every $p = 1, 2, \ldots$. The corresponding F-function F is $1 - (1-x)^{1/p}$ and the corresponding σ-entropy $\bar{e}$ is

$$\bar{e}(A) = 1 - \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_{A^c}(x_i)|^{p} \quad \forall A \in \mathscr{F}(X).$$

Example 4.8. The entropy e_4 is a sub-σ-entropy on $\mathscr{F}(X)$. The corresponding F-function F is that mentioned in Example 2.4 and the corresponding σ-entropy $\bar{e}$ is

$$\bar{e}(A) = \sum_{i=1}^{n} c_i f_i(\mu_A(x_i)) \quad \forall A \in \mathscr{F}(X).$$

Example 4.9. The entropy e_5 is a sub-σ-entropy on $\hat{\mathscr{F}}$ for every $p \geq 1$. The corresponding F-function F is $1-(1-x)^{1/p}$ and the corresponding σ-entropy $\bar{e}$ is

$$\bar{e}(A) = 1 - \int_0^1 |\mu_A(x) - \mu_{A^c}(x)|^p \, dx \quad \forall A \in \hat{\mathscr{F}}.$$

At the end of this section, we give another description of σ-entropy.

Proposition 4.2. *An entropy e on $\mathscr{F}$ is a σ-entropy iff for any $A, B \in \mathscr{F}$, there holds*

$$e(A \cup B) + e(A \cap B) = e(A) + e(B). \tag{4.2}$$

Proof. *Necessity*: Let $A, B \in \mathscr{F}$ and D denote the crisp set $\{x \in X;\ \mu_A(x) \geq \mu_B(x)\}$. Since e is a σ-entropy, we have

$$e(A \cup B) = e((A \cup B) \cap D) + e((A \cup B) \cap D^c) = e(A \cap D) + e(B \cap D^c)$$

and

$$e(A \cap B) = e((A \cap B) \cap D) + e((A \cap B) \cap D^c) = e(B \cap D) + e(A \cap D^c).$$

Therefore,

$$e(A \cup B) + e(A \cap B) = (e(A \cap D) + e(A \cap D^c)) + (e(B \cap D) + e(B \cap D^c)) = e(A) + e(B).$$

Sufficiency: Suppose that for any $A, B \in \mathscr{F}$, (4.2) holds. Let B mentioned above be a crisp set D. Then we have

$$\begin{aligned} e(A) = e(A) + e(D) &= e(A \cap D) + e(A \cup D) = e(A \cap D) + e((A \cap D^c) \cup \mathrm{D}) \\ &= e(A \cap D) + (e(A \cap D^c) + e(D) - e((A \cap D^c) \cap D)) \\ &= e(A \cap D) + e(A \cap D^c) \end{aligned}$$

The proof is completed.

Note. Since $\mathscr{F}$ is closed under the operations union and complement, we can easily verify that $\mathscr{F}$ is closed under the operation intersection. Then, for any $A, B \in \mathscr{F}$, there holds $A \cap B \in \mathscr{F}$.

5. σ-Distance measure and sub-σ-distance measure of fuzzy sets

Similar to the concepts of σ-entropy and sub-σ-entropy, in this section, we introduce the concepts of σ-distance measure and sub-σ-distance measure.

Definition 5.1. Let d be a distance measure on $\mathscr{F}$. We call d a σ-distance measure if for any $A, B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, there holds

$$d(A, B) = d(A \cap D, B \cap D) + d(A \cap D^c, B \cap D^c). \tag{5.1}$$

Example 5.1. When $p = 1$, the distance measure D_p (see Example 2.3 and 2.6) is a σ-distance measure on $\mathscr{F}(X)$.

Example 5.2. When $p = 1$, the distance measure d_p (see Example 2.7) is a σ-distance measure on $\hat{\mathscr{F}}$.

Proposition 5.1. *If d is a distance measure on $\mathscr{F}$ and F is an* F-*function, then* F(d) *is a distance measure on $\mathscr{F}$.*

Definition 5.2. A distance measure d on $\mathscr{F}$ is called a sub-σ-distance measure, if there exist an F-function F and a σ-distance measure $\bar{d}$ such that $d = F(\bar{d})$.

It is clear that a σ-distance measure on $\mathscr{F}$ is a sub-σ-distance measure on $\mathscr{F}$.

Example 5.3. The distance measure D_p is a sub-σ-distance measure on $\mathscr{F}(X)$ for every $p = 1, 2, \ldots$ The corresponding F-function F is $x^{1/p}$ and the corresponding σ-distance measure $\bar{d}$ is

$$\bar{d}(A, B) = \frac{1}{n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|^p \quad \forall A, B \in \mathscr{F}(X).$$

Example 5.4. The distance measure d_p $(p \geqslant 1)$ is a sub-σ-distance measure on $\hat{\mathscr{F}}$. The corresponding F-function F is $x^{1/p}$ and the corresponding σ-distance measure $\bar{d}$ is

$$\bar{d}(A, B) = \int_0^1 |\mu_A(x) - \mu_B(x)|^p \, \mathrm{d}x \quad \forall A, B \in \hat{\mathscr{F}}.$$

6. σ-Similarity measure and sub-σ-similarity measure of fuzzy sets

Analogous to σ-entropy (resp. sub-σ-entropy) and σ-distance measure (resp. sub-σ-distance measure, σ-similarity measure (resp. sub-σ-similarity measure) which we shall define also describes an additive property of similarity measure, although the definition for it seems unnatural.

Definition 6.1. Let s be a similarity measure on $\mathscr{F}$. We call s a σ-similarity measure on $\mathscr{F}$, if for any $A, B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, there holds

$$s(A, B) = s(A \cap D, B \cup D^c) + s(A \cap D^c, B \cup D). \tag{6.1}$$

Example 6.1. When $p = 1$, the similarity measure s_p on $\hat{\mathscr{F}}$ (see Example 2.8) is a σ-similarity measure on $\hat{\mathscr{F}}$.

Proposition 6.1. *A similarity measure on $\mathscr{F}$ is a σ-similarity measure on $\mathscr{F}$ iff for any $A, B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, there holds*

$$s(A, B) = s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c). \tag{6.2}$$

Proof. We only prove that (6.1) implies (6.2). The proof of the conclusion that (6.2) implies (6.1) is similar.

In (6.1), replacing A by $A \cup D$, B by $B \cap D^c$, we have

$$\begin{aligned} s(A \cup D, B \cap D^c) &= s((A \cup D) \cap D, (B \cap D^c) \cup D^c) + s((A \cup D) \cap D^c, (B \cap D^c) \cup D) \\ &= s(D, D^c) + s(A \cap D^c, B \cup D). \end{aligned}$$

Since s is a similarity measure, then $s(D, D^c) = 0$ and hence

$$s(A \cup D, B \cap D^c) = s(A \cap D^c, B \cup D)$$

and the conclusion follows.

Note. From the proof of Proposition 6.1, we know that (6.1) is equivalent to (6.2) only under the condition that $s(D, D^c) = 0$, $\forall D \in \mathscr{P}(X)$.

Proposition 6.2. *If s is a similarity measure on $\mathscr{F}$ and F is an* F*-function, then $F(s)$ is a similarity measure on $\mathscr{F}$.*

Definition 6.2. A similarity measure s on $\mathscr{F}$ is called a sub-σ-similarity measure if there exist an F-function F and a σ-similarity measure $\bar{s}$ such that $s = F(\bar{s})$.

Obviously, a σ-similarity measure on $\mathscr{F}$ is a sub-σ-similarity measure on $\mathscr{F}$.

Example 6.2. The similarity measure s_p is a sub-σ-similarity measure on $\hat{\mathscr{F}}$. The corresponding F-function F is $1-(1-x)^{1/p}$ and the corresponding σ-similarity measure $\bar{s}$ is

$$\bar{s}(A, B) = 1 - \int_0^1 |\mu_A(x) - \mu_B(x)|^p \, \mathrm{d}x \quad \forall A, B \in \hat{\mathscr{F}}.$$

7. The relations between σ-similarity measure and σ-distance measure

Theorem 7.1. *If s is a σ-similarity measure on $\mathscr{F}$, then $d\langle s\rangle$ is a σ-distance measure on $\mathscr{F}$.*

Proof. For any $A, B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, from (6.2), we have

$$\begin{aligned} d\langle s\rangle(A \cap D, B \cap D) &= 1 - s(A \cap D, B \cap D) \\ &= 1 - s((A \cap D) \cap D, (B \cap D) \cup D^c) - s((A \cap D) \cup D, (B \cap D) \cap D^c) \\ &= 1 - s(A \cap D, B \cup D^c) - s(D, \emptyset). \end{aligned}$$

Similarly,

$$d\langle s\rangle(A \cap D^c, B \cap D^c) = 1 - s(A \cup D, B \cap D^c) - s(\emptyset, D^c).$$

Therefore,

$$\begin{aligned} d\langle s\rangle(A \cap D, B \cap D) + d\langle s\rangle(A \cap D^c, B \cap D^c) = {} & 2 - (s(\emptyset, D^c) + s(D, \emptyset)) \\ & - (s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c)). \end{aligned}$$

From (6.2), we know

$$s(\emptyset, D^c) + s(D, \emptyset) = s(\emptyset, \emptyset) = 1$$

and

$$s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c) = s(A, B).$$

Thus,

$$d\langle s\rangle(A \cap D, B \cap D) + d\langle s\rangle(A \cap D^c, B \cap D^c) = 1 - s(A, B) = d\langle s\rangle(A, B)$$

and the conclusion follows.

Similarly, we can prove:

Theorem 7.2. *If d is a σ-distance measure on $\mathscr{F}$, then $s\langle d\rangle$ is a σ-similarity measure on $\mathscr{F}$.*

8. The relations between σ-entropy and σ-similarity measure

Theorem 8.1. *If s is a σ-similarity measure on $\mathscr{F}$, then $e\langle s\rangle$ is a σ-entropy on $\mathscr{F}$.*

Proof. Let $A \in \mathscr{F}$ and $D \in \mathscr{P}(X)$. We have

$$\begin{aligned} e\langle s\rangle(A\cap D)+e\langle s\rangle(A\cap D^c) &= s(A\cap D,(A\cap D)^c)+s(A\cap D^c,(A\cap D^c)^c)\\ &= s(A\cap D, A^c\cup D^c)+s(A\cap D^c, A^c\cup D)\\ &= s(A, A^c) \quad \text{(from (6.1))}\\ &= e\langle s\rangle(A) \end{aligned}$$

and the proof is complete.

Next, we shall use a σ-entropy e on $\mathscr{F}$ to define a σ-similarity measure on $\mathscr{F}$. This is the main result of this paper. For the common entropy e on $\mathscr{F}$, it is impossible to construct a similarity measure by using the entropy e.

First of all, we give the following notations.

For $A, B \in \mathscr{F}(X)$, let $D_i \in \mathscr{P}(X)$ ($i = 1, 2, 3, 4$) as

$$D_1=\{x\in X;\ \mu_A(x)\geq\tfrac12,\ \mu_B(x)\geq\tfrac12\},\qquad D_2=\{x\in X;\ \mu_A(x)\geq\tfrac12,\ \mu_B(x)<\tfrac12\},$$
$$D_3=\{x\in X;\ \mu_A(x)<\tfrac12,\ \mu_B(x)\geq\tfrac12\},\qquad D_4=\{x\in X;\ \mu_A(x)<\tfrac12,\ \mu_B(x)<\tfrac12\}.$$

Then define $R_i \in \mathscr{F}(X)$ ($i = 1, 2, 3, 4, 5, 6, 7, 8$) as

$$\mu_{R_1}(x)=\begin{cases}\mu_A(x)\vee\mu_B(x), & x\in D_1,\\ \frac12, & x\notin D_1,\end{cases}\qquad \mu_{R_2}(x)=\begin{cases}\mu_A(x)\wedge\mu_B(x), & x\in D_1,\\ \frac12, & x\notin D_1,\end{cases}$$

$$\mu_{R_3}(x)=\begin{cases}\mu_A(x), & x\in D_2,\\ \frac12, & x\notin D_2,\end{cases}\qquad \mu_{R_4}(x)=\begin{cases}\mu_B(x), & x\in D_2,\\ \frac12, & x\notin D_2,\end{cases}$$

$$\mu_{R_5}(x)=\begin{cases}\mu_B(x), & x\in D_3,\\ \frac12, & x\notin D_3,\end{cases}\qquad \mu_{R_6}(x)=\begin{cases}\mu_A(x), & x\in D_3,\\ \frac12, & x\notin D_3,\end{cases}$$

$$\mu_{R_7}(x)=\begin{cases}\mu_A(x)\vee\mu_B(x), & x\in D_4,\\ \frac12, & x\notin D_4,\end{cases}\qquad \mu_{R_8}(x)=\begin{cases}\mu_A(x)\wedge\mu_B(x), & x\in D_4,\\ \frac12, & x\notin D_4.\end{cases}$$

Theorem 8.2. *Let e be a σ-entropy on $\mathscr{F}$. For $A, B \in \mathscr{F}$, define*

$$s(A,B)=\tfrac12(e(R_1)-e(R_2)+e(R_3)+e(R_4)+e(R_5)+e(R_6)-e(R_7)+e(R_8))-1. \tag{8.1}$$

Then s is a σ-similarity measure on $\mathscr{F}$.

Lemma 8.1. *If $T_1, T_2, \ldots, T_n \in \mathscr{P}(X)$, $\bigcup_{i=1}^n T_i = X$, $T_i \cap T_j = \emptyset$ when $i \neq j$ and e is a σ-entropy on $\mathscr{F}$, then $e(A) = \sum_{i=1}^n e(A \cap T_i)$, $\forall A \in \mathscr{F}$.*

Proof of Theorem 8.2. In the following steps (1)–(5), we prove that s defined by (8.1) satisfies (SP1), (SP2), (SP3) and (SP4) of Definition 2.3 and (6.1), (6.2). The conclusion that $0 \leq s(A, B) \leq 1$ $\forall A, B \in \mathscr{F}$ will be shown in (6).

(1) For any $A, B \in \mathscr{F}$, it is easy to see that $s(A, B) = s(B, A)$.

(2) For any $D \in \mathscr{P}(X)$, we prove that $s(D, D^c) = 0$. Since for D, D^c,

$$\mu_{R_1}(x) = \mu_{R_2}(x) = \mu_{R_7}(x) = \mu_{R_8}(x) = \tfrac{1}{2}, \quad \forall x \in X,$$

$$\mu_{R_3}(x) = \begin{cases} 1, & x \in D, \\ \frac{1}{2}, & x \in D^c, \end{cases} \qquad \mu_{R_4}(x) = \begin{cases} 0, & x \in D, \\ \frac{1}{2}, & x \in D^c, \end{cases}$$

$$\mu_{R_5}(x) = \begin{cases} 1, & x \in D^c, \\ \frac{1}{2}, & x \in D, \end{cases} \qquad \mu_{R_6}(x) = \begin{cases} 0, & x \in D^c, \\ \frac{1}{2}, & x \in D, \end{cases}$$

we have, for $i = 3, 4$,

$$e(R_i) = e(R_i \cap D) + e(R_i \cap D^c) = e(R_i \cap D^c) = e([\tfrac{1}{2}]_X \cap D^c),$$

and for $i = 5, 6$,

$$e(R_i) = e(R_i \cap D) + e(R_i \cap D^c) = e(R_i \cap D) = e([\tfrac{1}{2}]_X \cap D).$$

Thus,

$$\begin{aligned} s(D, D^c) &= \tfrac{1}{2}(e(R_1) - e(R_2) + e(R_3) + e(R_4) + e(R_5) + e(R_6) - e(R_7) + e(R_8)) - 1 \\ &= \tfrac{1}{2}\sum_{i=3}^{6} e(R_i) - 1 = e([\tfrac{1}{2}]_X \cap D^c) + e([\tfrac{1}{2}]_X \cap D) - 1 \\ &= e([\tfrac{1}{2}]_X) - 1 = 0. \end{aligned}$$

(3) For any $A \in \mathscr{F}$, we prove that $s(A, A) = 1$. Because for A, A, $R_1 = R_2$, $R_3 = R_4 = R_5 = R_6 = [\tfrac{1}{2}]_X$, and $R_7 = R_8$, we have

$$\begin{aligned} s(A, A) &= \tfrac{1}{2}(e(R_1) - e(R_2) + e(R_3) + e(R_4) + e(R_5) + e(R_6) - e(R_7) + e(R_8)) - 1 \\ &= \tfrac{1}{2}\sum_{i=3}^{6} e(R_i) - 1 = \tfrac{1}{2}\sum_{i=3}^{6} e([\tfrac{1}{2}]_X) - 1 = \tfrac{1}{2} \cdot 4 - 1 = 1. \end{aligned}$$

(4) For any $A, B, C \in \mathscr{F}$, we prove that if $A \subset B \subset C$, then $s(A, B) \geq s(A, C)$ and $s(B, C) \geq s(A, C)$. We only verify that $s(A, B) \geq s(A, C)$. The proof of the conclusion that $s(B, C) \geq s(A, C)$ is similar. Let

$$D_1 = \{x \in X; \mu_A(x) \geq \tfrac{1}{2}, \mu_B(x) \geq \tfrac{1}{2}\}, \qquad D_2 = \{x \in X; \mu_A(x) \geq \tfrac{1}{2}, \mu_B(x) < \tfrac{1}{2}\} = \emptyset,$$

$$D_3 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_B(x) \geq \tfrac{1}{2}\}, \qquad D_4 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_B(x) < \tfrac{1}{2}\},$$

$$\hat{D}_1 = \{x \in X; \mu_A(x) \geq \tfrac{1}{2}, \mu_C(x) \geq \tfrac{1}{2}\}, \qquad \hat{D}_2 = \{x \in X; \mu_A(x) \geq \tfrac{1}{2}, \mu_C(x) < \tfrac{1}{2}\} = \emptyset,$$

$$\hat{D}_3 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_C(x) \geq \tfrac{1}{2}\}, \qquad \hat{D}_4 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_C(x) < \tfrac{1}{2}\}.$$

It is easy to see that $D_1 = \hat{D}_1$, $D_3 \subset \hat{D}_3$, $D_4 \supset \hat{D}_4$ and $D_1 \cup D_3 \cup D_4 = \hat{D}_1 \cup \hat{D}_3 \cup \hat{D}_4 = X$.

Note that $D_1 = \hat{D}_1 = F_1$, $D_3 = F_2$, $\hat{D}_3 - D_3 = F_3$, $\hat{D}_4 = F_4$. For A, B, we have

$$\mu_{R_1}(x) = \begin{cases} \mu_B(x), & x \in F_1, \\ \frac{1}{2}, & x \notin F_1, \end{cases} \qquad \mu_{R_2}(x) = \begin{cases} \mu_A(x), & x \in F_1, \\ \frac{1}{2}, & x \notin F_1, \end{cases}$$

$$\mu_{R_3}(x) = \mu_{R_4}(x) = \tfrac{1}{2}, \quad x \in X,$$

$$\mu_{R_5}(x) = \begin{cases} \mu_B(x), & x \in F_2, \\ \frac{1}{2}, & x \notin F_2, \end{cases} \qquad \mu_{R_6}(x) = \begin{cases} \mu_A(x), & x \in F_2, \\ \frac{1}{2}, & x \notin F_2, \end{cases}$$

$$\mu_{R_7}(x) = \begin{cases} \mu_B(x), & x \in F_3 \cup F_4, \\ \frac{1}{2}, & x \notin F_3 \cup F_4, \end{cases} \qquad \mu_{R_8}(x) = \begin{cases} \mu_A(x), & x \in F_3 \cup F_4, \\ \frac{1}{2}, & x \notin F_3 \cup F_4. \end{cases}$$

For A, C, to avoid confusion of notations, we let $\hat{R}_i$ stand for R_i ($i = 1, 2, 3, 4, 5, 6, 7, 8$) and obtain

$$\mu_{\hat{R}_1}(x) = \begin{cases} \mu_C(x), & x \in F_1, \\ \frac{1}{2}, & x \notin F_1, \end{cases} \qquad \mu_{\hat{R}_2}(x) = \begin{cases} \mu_A(x), & x \in F_1, \\ \frac{1}{2}, & x \notin F_1, \end{cases}$$

$$\mu_{\hat{R}_3}(x) = \mu_{\hat{R}_4}(x) = \tfrac{1}{2}, \quad x \in X,$$

$$\mu_{\hat{R}_5}(x) = \begin{cases} \mu_C(x), & x \in F_2 \cup F_3, \\ \frac{1}{2}, & x \notin F_2 \cup F_3, \end{cases} \qquad \mu_{\hat{R}_6}(x) = \begin{cases} \mu_A(x), & x \in F_2 \cup F_3, \\ \frac{1}{2}, & x \notin F_2 \cup F_3, \end{cases}$$

$$\mu_{\hat{R}_7}(x) = \begin{cases} \mu_C(x), & x \in F_4, \\ \frac{1}{2}, & x \notin F_4, \end{cases} \qquad \mu_{\hat{R}_8}(x) = \begin{cases} \mu_A(x), & x \in F_4, \\ \frac{1}{2}, & x \notin F_4. \end{cases}$$

It is clear that

$$e(R_1) \geqslant e(\hat{R}_1), \qquad e(R_2) = e(\hat{R}_2), \qquad e(R_3) = e(R_4) = e(\hat{R}_3) = e(\hat{R}_4) = 1.$$

In the following, we prove $(*)$ $e(R_5) - e(R_7) \geqslant e(\hat{R}_5) - e(\hat{R}_7)$ and $(**)$ $e(R_6) + e(R_8) = e(\hat{R}_6) + e(\hat{R}_8)$ respectively. If $(*)$ and $(**)$ are proved, we can easily obtain that $s(A, B) \geqslant s(A, C)$. First, let us prove $(*)$. Because e is a σ-entropy on $\mathscr{F}$, we have

$$e(R_5) = e([\tfrac{1}{2}]_X \cap F_1) + e(B \cap F_2) + e([\tfrac{1}{2}]_X \cap F_3) + e([\tfrac{1}{2}]_X \cap F_4)$$

and

$$e(R_7) = e([\tfrac{1}{2}]_X \cap F_1) + e([\tfrac{1}{2}]_X \cap F_2) + e(B \cup F_3) + e(B \cap F_4).$$

Therefore,

$$e(R_5) - e(R_7) = e(B \cap F_2) + e([\tfrac{1}{2}]_X \cap F_3) + e([\tfrac{1}{2}]_X \cap F_4) - e([\tfrac{1}{2}]_X \cap F_2) - e(B \cap F_3) - e(B \cap F_4).$$

Similarly, we have

$$e(\hat{R}_5) - e(\hat{R}_7) = e(C \cap F_2) + e(C \cap F_3) + e([\tfrac{1}{2}]_X \cap F_4) - e([\tfrac{1}{2}]_X \cap F_2) - e([\tfrac{1}{2}]_X \cap F_3) - e(C \cap F_4).$$

Thus,

$$\begin{aligned}(e(R_5) - e(R_7)) - (e(\hat{R}_5) - e(\hat{R}_7)) = {} & (e(B \cap F_2) - e(C \cap F_2)) + (e([\tfrac{1}{2}]_X \cap F_3) - e(B \cap F_3)) \\ & + (e([\tfrac{1}{2}]_X \cap F_3) - e(C \cap F_3)) + (e(C \cap F_4) - e(B \cap F_4)).\end{aligned}$$

From the following results

$$e(C \cap F_2) \leqslant e(B \cap F_2), \quad e(B \cap F_3) \leqslant e([\tfrac{1}{2}]_X \cap F_3), \quad e(C \cap F_3) \leqslant e([\tfrac{1}{2}]_X \cap F_3), \quad e(B \cap F_4) \leqslant e(C \cap F_4),$$

we obtain

$$e(R_5) - e(R_7) \geqslant e(\hat{R}_5) - e(\hat{R}_7).$$

Second, we prove $(**)$. Since e is a σ-entropy on $\mathscr{F}$, then

$$e(R_6) = e(A \cap F_2) + e([\tfrac{1}{2}]_X \cap F_1) + e([\tfrac{1}{2}]_X \cap F_3) + e([\tfrac{1}{2}]_X \cap F_4)$$

and

$$e(\hat{R}_6) = e(A \cap F_2) + e([\tfrac{1}{2}]_X \cap F_1) + e(A \cap F_3) + e([\tfrac{1}{2}]_X \cap F_4).$$

Thus,

$$e(R_6) - e(\hat{R}_6) = e([\tfrac{1}{2}]_X \cap F_3) - e(A \cap F_3).$$

Similarly, we can obtain that

$$e(R_8) - e(\hat{R}_8) = e(A \cap F_3) - e([\tfrac{1}{2}]_X \cap F_3).$$

Therefore,

$$e(R_6) - e(\hat{R}_6) + e(R_8) - e(\hat{R}_8) = 0.$$

That is

$$e(R_6) + e(R_8) = e(\hat{R}_6) + e(\hat{R}_8).$$

(5) For any A, $B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, we verify (6.1) and (6.2). We only need to prove one of them, because they are equivalent in the condition that $s(D, D^c) = 0$, $\forall D \in \mathscr{P}(X)$, which is showed in (2).

Now we prove (6.2), i.e., for any A, $B \in \mathscr{F}$, $D \in \mathscr{F}(X)$, there holds

$$s(A, B) = s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c).$$

For A, $B \in \mathscr{F}$ and $D \in \mathscr{P}(X)$, write

$$D_1 = \{x \in X; \mu_A(x) \geqslant \tfrac{1}{2}, \mu_B(x) \geqslant \tfrac{1}{2}\}, \qquad D_2 = \{x \in X; \mu_A(x) \geqslant \tfrac{1}{2}, \mu_B(x) < \tfrac{1}{2}\},$$

$$D_3 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_B(x) \geqslant \tfrac{1}{2}\}, \qquad D_4 = \{x \in X; \mu_A(x) < \tfrac{1}{2}, \mu_B(x) < \tfrac{1}{2}\},$$

$$F_i = D_i \cap D, \quad \hat{F}_i = D_i \cap D^c, \qquad i = 1, 2, 3, 4.$$

For $A \cap D$, $B \cup D^c$, we have

$$\mu_{R_1}(x) = \begin{cases} \mu_A(x) \vee \mu_B(x), & x \in F_1, \\ \tfrac{1}{2}, & x \notin F_1, \end{cases} \qquad \mu_{R_2}(x) = \begin{cases} \mu_A(x) \wedge \mu_B(x), & x \in F_1, \\ \tfrac{1}{2}, & x \notin F_1, \end{cases}$$

$$\mu_{R_3}(x) = \begin{cases} \mu_A(x), & x \in F_2, \\ \tfrac{1}{2}, & x \notin F_2, \end{cases} \qquad \mu_{R_4}(x) = \begin{cases} \mu_B(x), & x \in F_2, \\ \tfrac{1}{2}, & x \notin F_2, \end{cases}$$

$$\mu_{R_5}(x) = \begin{cases} \mu_B(x), & x \in F_3, \\ 1, & x \in D^c, \\ \tfrac{1}{2}, & x \in F_1 \cup F_2 \cup F_4, \end{cases} \qquad \mu_{R_6}(x) = \begin{cases} \mu_A(x), & x \in F_3, \\ 0, & x \in D^c, \\ \tfrac{1}{2}, & x \in F_1 \cup F_2 \cup F_4, \end{cases}$$

$$\mu_{R_7}(x) = \begin{cases} \mu_A(x) \vee \mu_B(x), & x \in F_4, \\ \tfrac{1}{2}, & x \notin F_4, \end{cases} \qquad \mu_{R_8}(x) = \begin{cases} \mu_A(x) \wedge \mu_B(x), & x \in F_4, \\ \tfrac{1}{2}, & x \notin F_4. \end{cases}$$

From Lemma 8.1, we have

$$\begin{aligned} s(A \cap D, B \cup D^c) = \tfrac{1}{2}(&e((A \cup B) \cap F_1) - e((A \cap B) \cap F_1) \\ &+ e(A \cap F_2) + e(B \cap F_2) + e(A \cap F_3) + e(B \cap F_3) \\ &- e((A \cup B) \cap F_4) + e((A \cap B) \cap F_4)) + e([\tfrac{1}{2}]_X \cap F_1) + e([\tfrac{1}{2}]_X \cap F_4)). \end{aligned} \tag{i}$$

Similarly, we can obtain that

$$\begin{aligned} s(A \cup D, B \cap D^c) = \tfrac{1}{2}(&e((A \cup B) \cap \hat{F}_1) - e((A \cap B) \cap \hat{F}_1) \\ &+ e(A \cap \hat{F}_2) + e(B \cap \hat{F}_2) + e(A \cap \hat{F}_3) + e(B \cap \hat{F}_3) \\ &- e((A \cup B) \cap \hat{F}_4) + e((A \cap B) \cap \hat{F}_4)) + e([\tfrac{1}{2}]_X \cap \hat{F}_1) + e([\tfrac{1}{2}]_X \cap \hat{F}_4)). \end{aligned} \tag{ii}$$

By using Lemma 8.1, we can obtain that for any $A \in \mathscr{F}$ and T_1, $T_2 \in \mathscr{P}(X)$, $T_1 \cap T_2 = \emptyset$ (T_1, T_2 may not satisfy that $T_1 \cup T_2 = X$), there holds

$$e(A \cap (T_1 \cup T_2)) = e(A \cap T_1) + e(A \cap T_2).$$

From (i), (ii) and the above conclusion, we have

$$\begin{aligned} s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c) = \tfrac{1}{2}(&e((A \cup B) \cap D_1) - e((A \cap B) \cap D_1) \\ &+ e(A \cap D_2) + e(B \cap D_2) + e(A \cap D_3) + e(B \cap D_3) \\ &- e((A \cup B) \cap D_4) + e((A \cap B) \cap D_4)) \\ &+ e([\tfrac{1}{2}]_X \cap D_1) + e([\tfrac{1}{2}]_X \cap D_4)). \end{aligned} \tag{iii}$$

Observing (iii), we find that the value of $s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c)$ is completely determined by A, B. If we replace D in (iii) by X and D respectively, we have

$$s(A, B) + s(X, \emptyset) = s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c).$$

From (2), we know that $s(X, \emptyset) = 0$, and therefore

$$s(A, B) = s(A \cap D, B \cup D^c) + s(A \cup D, B \cap D^c).$$

(6) We verify that for any $A, B \in \mathscr{F}$, it is true that $0 \leqslant s(A, B) \leqslant 1$. First, let us prove that for any $A, B \in \mathscr{F}$,

$$s(A, B) = s(A \cap B, A \cup B).$$

Let $D = \{x \in X; \mu_A(x) \geqslant \mu_B(x)\} \subset \mathscr{P}(X)$. From (6.1) and (6.2) which are proved in (5), we have

$$\begin{aligned} s(A \cap B, A \cup B) &= s((A \cap B) \cap D, (A \cup B) \cup D^c) + s((A \cap B) \cap D^c, (A \cup B) \cup D) \quad \text{from (6.1)} \\ &= s(B \cap D, A \cup D^c) + s(A \cap D^c, B \cup D) \\ &= s(B \cap D, A \cup D^c) + s(B \cup D, A \cap D^c) \quad \text{from (1)} \\ &= s(A, B) \quad \text{from (6.2)}. \end{aligned}$$

Second, we prove that for any $A, B \in \mathscr{F}$, $0 \leqslant s(A, B) \leqslant 1$. For $A, B \in \mathscr{F}$, since $A \cap B \subset A \cup B \subset A \cup B$, we have from (4) that $s(A \cap B, A \cup B) \leqslant s(A \cup B, A \cup B)$, which by (3) equals 1. Therefore, $s(A, B) \leqslant 1$. Furthermore, for $A, B \in \mathscr{F}$, since $\emptyset \subset A \cap B \subset A \cup B \subset X$, it follows from (4) that $s(A \cap B, A \cup B) \geqslant s(\emptyset, X)$ which by (2) equals 0. That is $s(A, B) \geqslant 0$.

Now (1), (2), (3), (4), (5) and (6) ensure that s defined by (8.1) is a σ-similarity measure on $\mathscr{F}$.

For a σ-entropy e, we call the similarity measure s defined by (8.1) the similarity measure generated by e and denote it by $s\langle e \rangle$.

Theorem 8.3 *If e is a σ-entropy on $\mathscr{F}$, then*

$$\hat{e} = e$$

where $\hat{e} = e\langle s\langle e \rangle \rangle$.

Proof. Let $A \in \mathscr{F}$, then $\hat{e}(A) = s\langle e \rangle(A, A^c)$. For A, A^c, note

$$D = \{x \in X; \mu_A(x) \geqslant \tfrac{1}{2}\} \in \mathscr{P}(X).$$

We have

$$\mu_{R_1}(x) = \mu_{R_2}(x) = \mu_{R_7}(x) = \mu_{R_8}(x) = \tfrac{1}{2}, \quad \forall x \in X,$$

$$\mu_{R_3}(x) = \begin{cases} \mu_A(x), & x \in D, \\ \frac{1}{2}, & x \in D^c, \end{cases} \qquad \mu_{R_4}(x) = \begin{cases} \mu_{A^c}(x), & x \in D, \\ \frac{1}{2}, & x \in D^c, \end{cases}$$

$$\mu_{R_5}(x) = \begin{cases} \mu_{A^c}(x), & x \in D^c, \\ \frac{1}{2}, & x \in D, \end{cases} \qquad \mu_{R_6}(x) = \begin{cases} \mu_A(x), & x \in D^c, \\ \frac{1}{2}, & x \in D. \end{cases}$$

Therefore,

$$\begin{aligned} s\langle e \rangle(A, A^c) &= \tfrac{1}{2}(e(R_1) - e(R_2) + e(R_3) + e(R_4) + e(R_5) + e(R_6) - e(R_7) + e(R_8)) - 1 \\ &= \tfrac{1}{2}\sum_{i=3}^{6} e(R_i) - 1 \\ &= \tfrac{1}{2}((e(A \cap D) + e([\tfrac{1}{2}]_X \cap D^c)) + (e(A^c \cap D) + e([\tfrac{1}{2}]_X \cap D^c)) \\ &\quad + (e(A^c \cap D^c) + e([\tfrac{1}{2}]_X \cap D)) + (e(A \cap D^c) + e([\tfrac{1}{2}]_X \cap D))) - 1 \\ &= \tfrac{1}{2}(e(A) + e(A^c)) = \tfrac{1}{2}(e(A) + e(A)) = e(A), \end{aligned}$$

and the conclusion follows.

In this paper, the following problem can not be solved.
If s is a σ-similarity measure on $\mathscr{F}$, is it true that $s\langle e\langle s\rangle\rangle = s$?

9. The relations between sub-σ-entropy, sub-σ-distance measure and sub-σ-similarity measure

Sub-σ-entropy, sub-σ-distance measure and sub-σ-similarity measure can be obtained from σ-entropy, σ-distance measure and σ-similarity measure respectively through on F-function. The relations between them can be established through the connections between σ-entropy, σ-distance measure and σ-similarity measure.

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