1. (6 points) The position function of a moving particle is given by $s(t) = t^2 - 1$, where s is in meters and t is in seconds.

(a) Find the average velocity of the particle over the time interval [1 2].

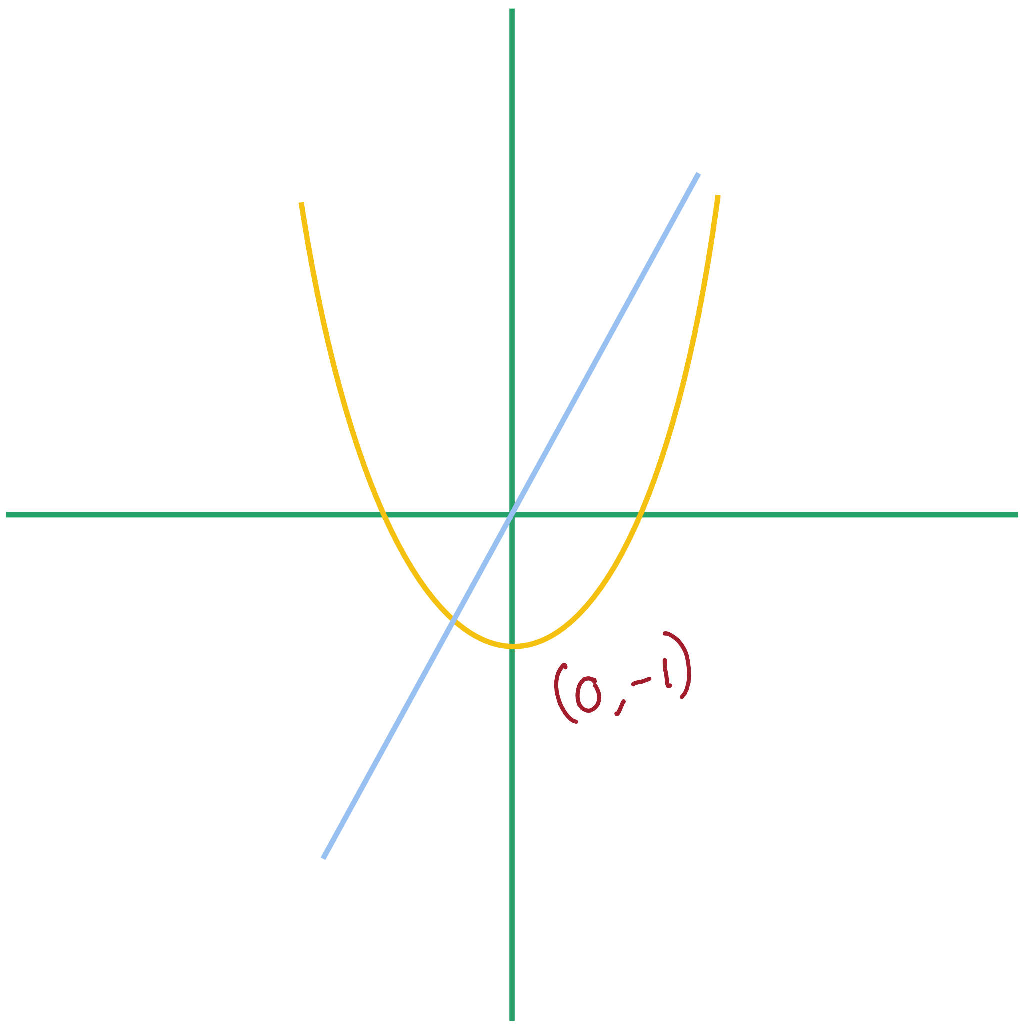
(b) Find the particle's instantaneous velocity at time $t = 1$ s.

(c) Draw the position function $s(t)$, and velocity function $v(t)$ in the same plot.

a) $\dfrac{s(2)-s(1)}{2-1} = \dfrac{3-0}{1} = 3$

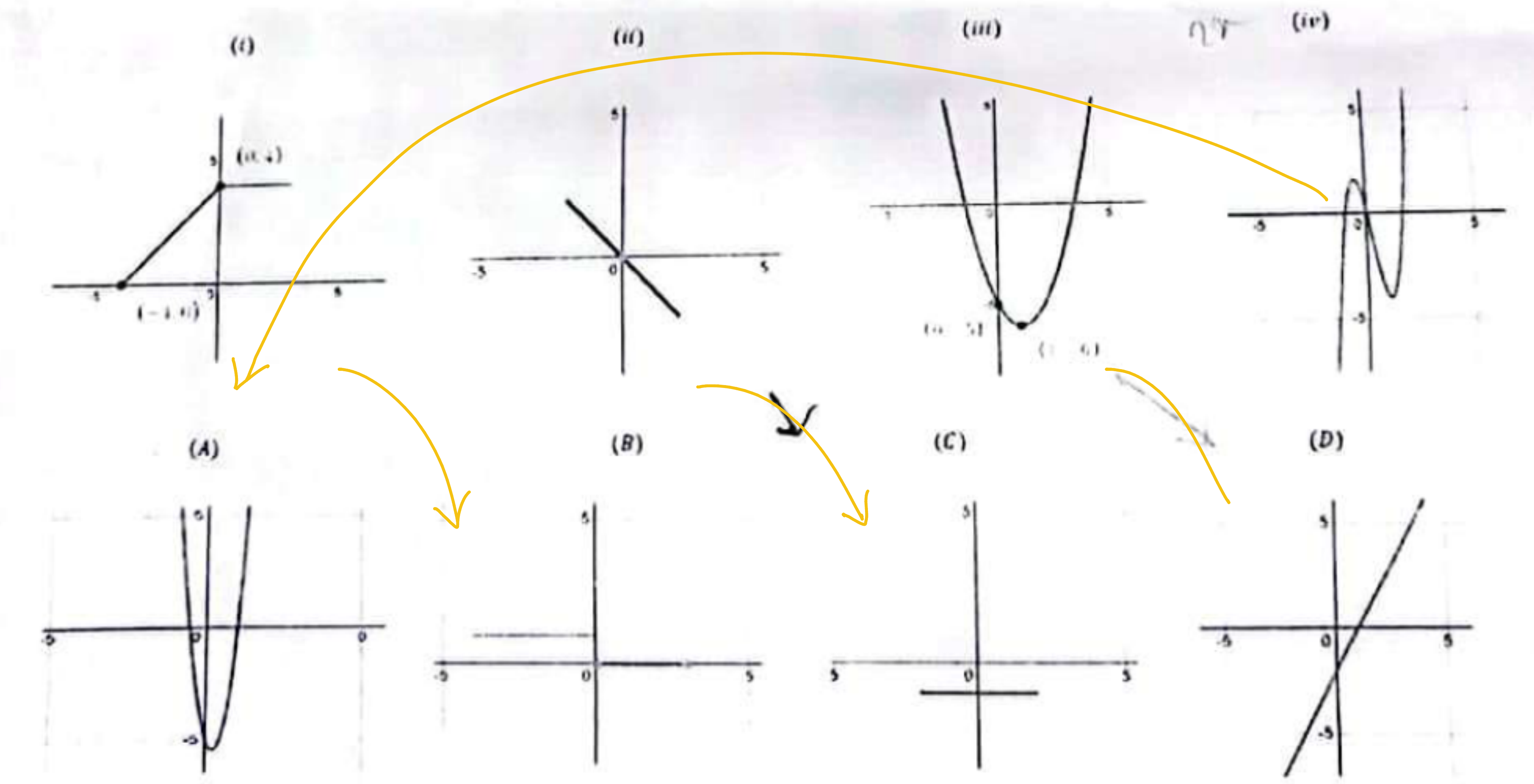
b) $\lim\limits_{t\to 1} \dfrac{s(t)-s(1)}{t-1} = \lim\limits_{t\to 1} \dfrac{t^2-1-\cancel{1+1}}{t-1} = \lim\limits_{t\to 1} t+1 = 2$

c)



2. (4 points) Match each of the following graphs($i - iv$) with its derivative graph($A - D$).

Figure 1:



3. (4 points) Consider $f(x) = x^3$ and $g(x) = x - 3$ are two functions.

(a) Find $y = f(g(x))$ and y' using the chain rule.

(b) Find the equation of the tangent line to the graph of the function at $x = 3$

(c) Draw the graph of the function $y = f(g(x))$ with the tangent line at $x = 3$.

a) $y = f(g(x))$

$y' = f'(g(x)) \times g'(x)$

$f(x) = x^3$ $\quad f'(g(x)) = f'(x-3) = 3(x-3)^2$

$\Rightarrow f'(x) = 3x^2$

$g(x) = x-3$ $\quad \therefore y' = 3(x-3)^2 \times 1 = 3(x-3)^2$

$g'(x) = 1$

b) $y = (x-3)^3$

$= (3-3)^3 = 0$

$y' = 3(3-3)^2$

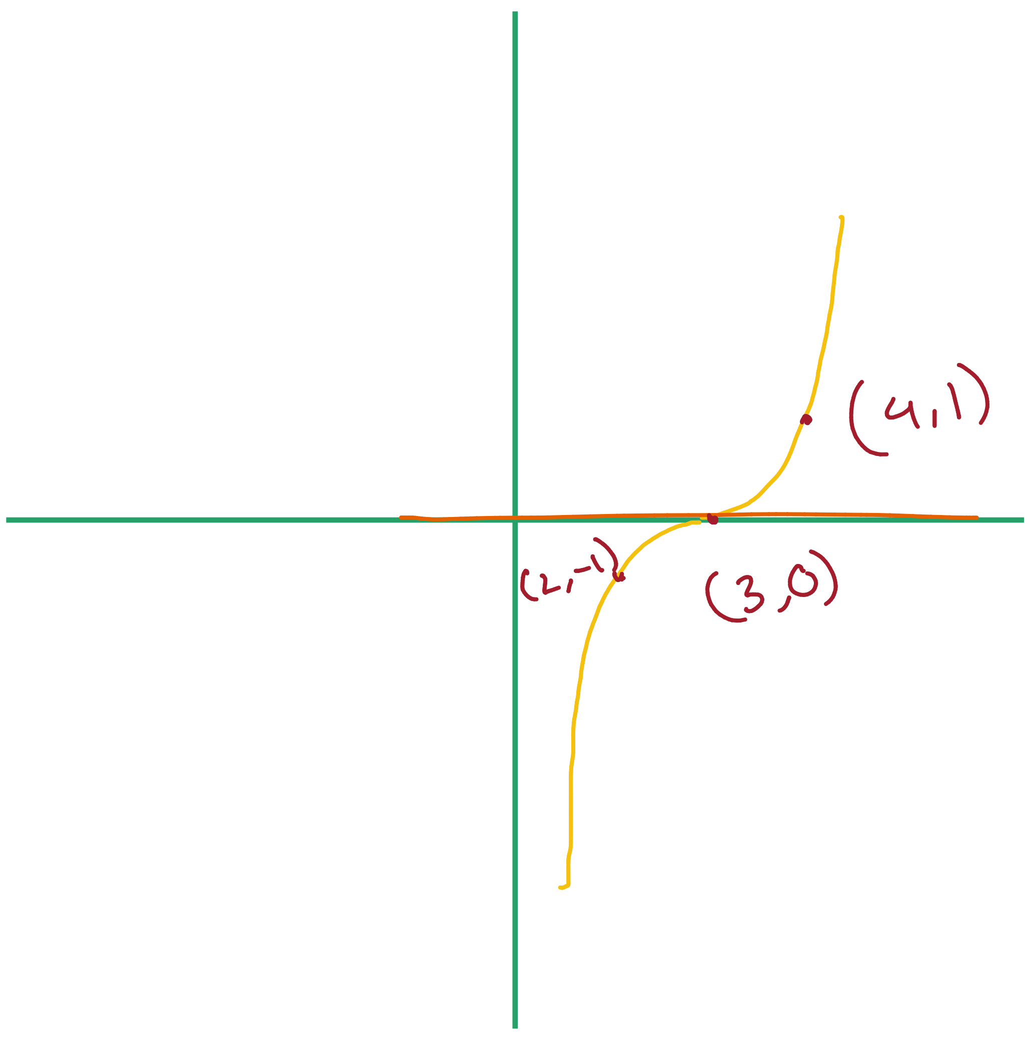
$= 0$

$(0,0)$ slope 0

$y - 0 = 0(x-0)$

$\Rightarrow y = 0$

c)



4. (4 points) Consider the function

$\int u^4 . du$

$$f(x) = \begin{cases} 2 & \text{if } 2 \le x \le 5 \\ 2x - 2 & \text{if } x < 2. \end{cases}$$

(a) Determine whether $f(x)$ is differentiable at $x = 2$. Is the function continuous at $x = 2$? If not, why?

(b) Sketch the graph of $f(x)$ and its derivative.

a) LHL $\quad$ RHL $\quad$ continuous at $x = 2$

$\lim_{x \to 2^-} 2x - 2$ $\quad \lim_{x \to 2^+} 2$

$= 2$ $\quad = 2$

L.H.D

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{2x-2-(2\cdot 2-2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{2x-4}{x-2}$$

$$= 2$$

R.H.D

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

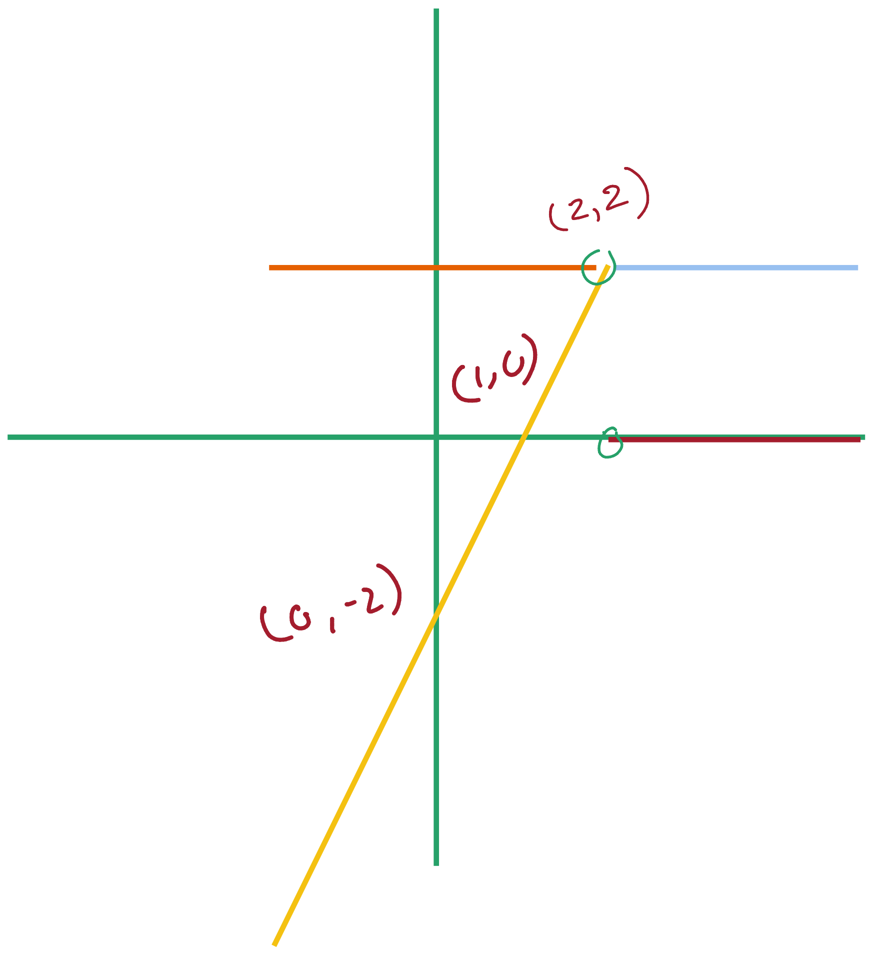
$$= \lim_{x\to 2^+} \frac{2-2}{x-2}$$

$$= 0$$

not continuous at $x=2$

therefore not diff at $x=2$

b)



5. (4 points) Consider the function over the given interval.

$f(x) = 2x + 3;\ [-1, x]$

(a) Use an area formula from geometry to find the area function $A(x)$ that gives the area between the graph of the function $f(x)$ and the interval $[-1, x]$.

(b) Confirm that $A'(x) = f(x)$.

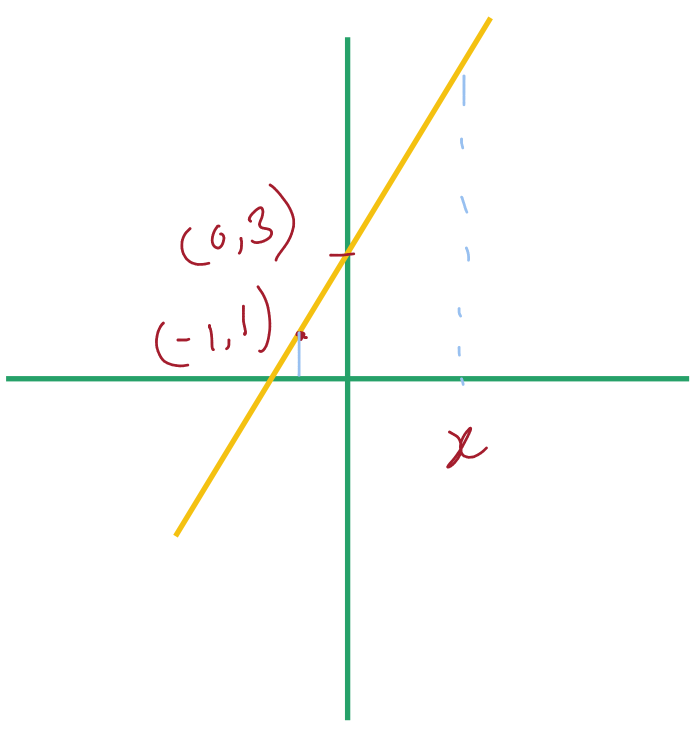
(c) Also, verify your answer by direct integration.

b) $A(x) = x^2 + 3x + 2$

$$A'(x) = 2x + 3 + 0$$

$$= f(x)$$

a) $\int_{-1}^{x} 2x+3 \, dx$



$$\frac{1}{2} \times (2x+3+1) \times (x+1)$$

$$= (x+2)(x+1)$$

$$= x^2 + 2x + x + 2$$

$$= x^2 + 3x + 2$$

c) $\int_{-1}^{x} 2x+3 \, dx$

$$= \left[2\frac{x^2}{2} + 3x\right]_{-1}^{x}$$

$$= (x^2 + 3x - 1 + 3)$$

$$= x^2 + 3x + 2$$

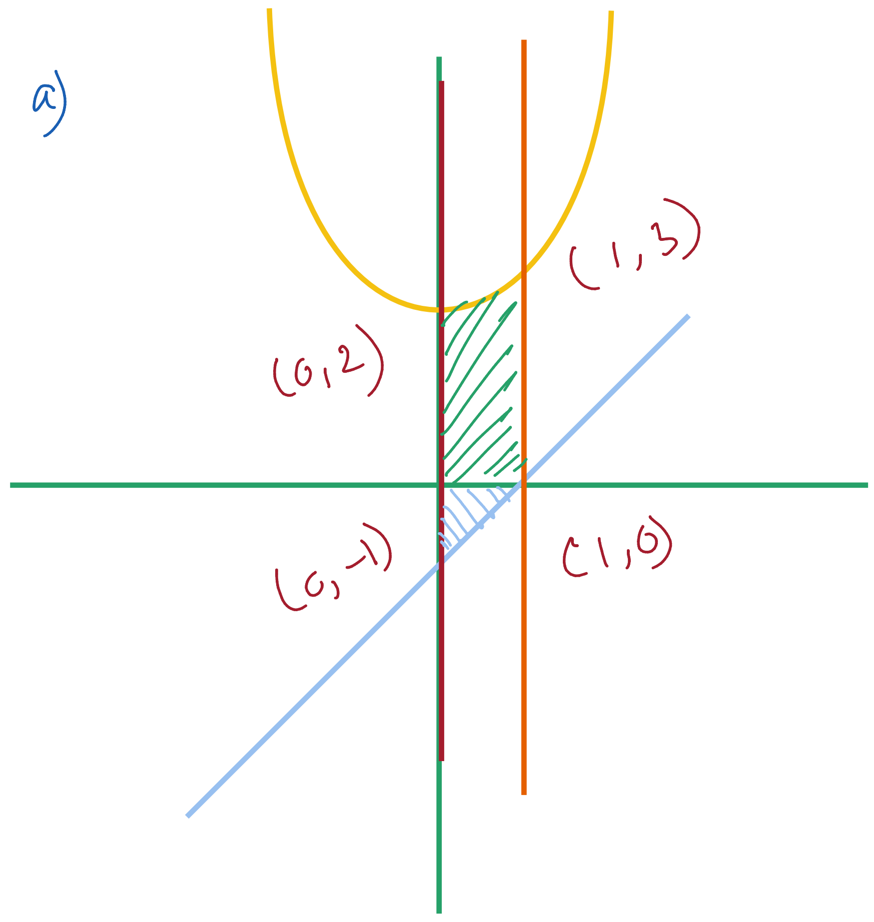
6. (6 points) (a) Draw the graphs of $y = x^2 + 2$, $y = x - 1$, $x = 0$ and $x = 1$.

(b) Find the area of the region enclosed by the graphs in part (a).

(c) Draw the graphs of $x = y^2$ and $y = x - 6$.

(d) Find the area of the region enclosed by the graphs in part (c).

a)



b) $\int_0^1 x^2+2 \; dx$

$= \left[\frac{x^3}{3} + 2x\right]_0^1$

$= \frac{1}{3} + 2$

$= \frac{7}{3}$

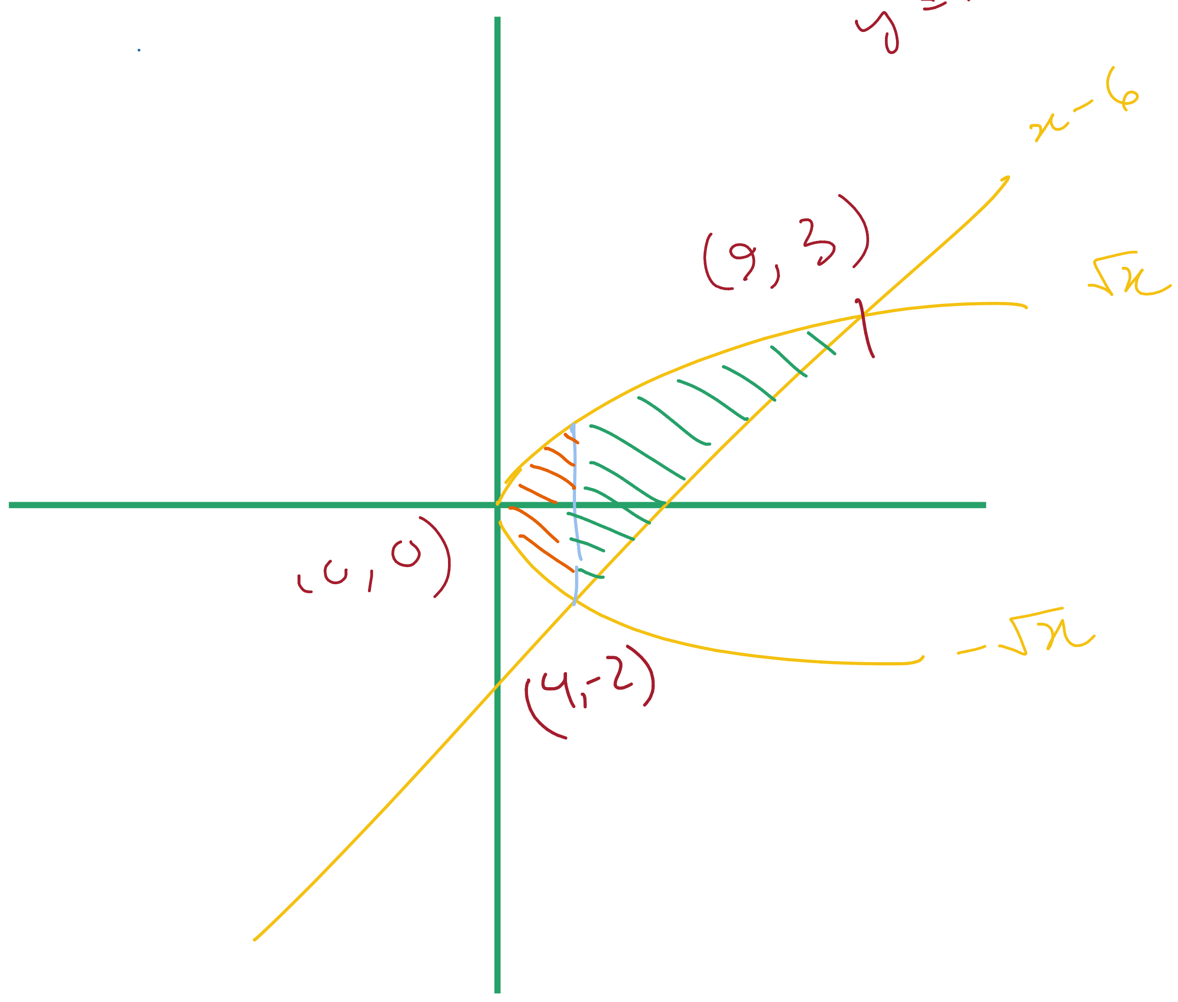
$\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$\frac{7}{3} + \frac{1}{2} = \frac{10}{6} = \frac{5}{3}$

$x = y^2$

$y = x-6$

c)



$x - 6 = \pm\sqrt{x}$

$\Rightarrow x^2 - 12x + 36 = x$

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow x^2 - 9x - 4x + 36 = 0$

$\Rightarrow x(x-9) - 4(x-9) = 0$

$\Rightarrow (x-4)(x-9) = 0$

$\Rightarrow x = 4, 9$

d) $\int_0^4 \sqrt{x} - (-\sqrt{x}) \; dx$

$= 2\int_0^4 \sqrt{x} \; dx$

$= 2 \cdot \frac{2}{3} \left[x^{3/2}\right]_0^4$

$= \frac{4}{3} \times 8$

$= \frac{32}{3}$

$\int_4^9 \sqrt{x} - x + 6 \; dx$

$= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 6x\right]_4^9$

$= \frac{2}{3} 27 - \frac{81}{2} + 54$

$- \left(\frac{2}{3} 8 - 8 + 24\right)$

$= \frac{61}{6}$

$\frac{32}{3} + \frac{61}{6} = \frac{125}{6}$

1. (8 points) Evaluate the following integrals (answer any 4):

(a) $\int 3x^4\sqrt{5+x^5}\,dx$ (d) $\int \sin^4 x\cos x\,dx$ (g) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$
(b) $\int x\sqrt{x-3}\,dx$ (e) $\int \frac{1}{1+4x^2}\,dx$
(c) $\int_1^e x^2\ln x\,dx$ (f) $\int e^x\sin(3x)\,dx$

a) $\int 3x^4\sqrt{5+x^5}\,dx$ $u = 5+x^5$
$du = 5x^4 dx$

$= \frac{1}{5}\int 3\sqrt{u}\,du$

$= \frac{3}{5}\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}\left(5+x^5\right)^{3/2} + C$

b) $\int x\sqrt{x-3}\,dx$

$u = x-3$ $x = u+3$
$du = dx$

$\int x\sqrt{u}\,du$

$= \int (u+3)\sqrt{u}\,du$

$= \int u^{\frac{3}{2}} + 3u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} + 2u^{3/2} + C$

$= \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$

c) $\int x^2\ln x\,dx$

$= \ln x\,\frac{x^3}{3} - \int \frac{1}{x}\frac{x^3}{3}\,dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{3}\frac{x^3}{3}$

$= \frac{1}{3}\left[x^3\ln x - x^3\right]_1^e$

d) $u = \sin x$

$\Rightarrow du = \cos x\,dx$

$\int u^4\,du$

$= \frac{1}{5}u^5 = \frac{1}{5}\sin^5 x\,dx$

e) $u = 2x$

$du = 2\,dx$

$\int \frac{1}{1+u^2}\frac{du}{2}$

$= \frac{1}{2}\tan^{-1}u = \frac{1}{2}\tan^{-1}2x + C$

g) $u = e^x - e^{-x}$

$\Rightarrow du = \left(e^x + e^{-x}\right)dx$

$\int \frac{e^x+e^{-x}}{u}\,\frac{du}{e^x+e^{-x}}$

$= \ln u = \ln|e^x - e^{-x}| + C$

f) $\int e^x\sin 3x\,dx =$

$= \sin 3x\,e^x - \int 3\cos 3x\,e^x\,dx$

$= e^x\sin 3x - 3\left[\cos 3x\,e^x + \int 3\sin 3x\,e^x\,dx\right]$

$= e^x\sin 3x - 3e^x\cos 3x - 9\int e^x\sin 3x\,dx$

$I = e^x\sin 3x - 3e^x\cos 3x - 9I$

$\Rightarrow I = \frac{1}{10}e^x\sin 3x - \frac{3}{10}e^x\cos 3x$

8. (4 points) Consider the function

$$f(x) = \begin{cases} |x+5| & \text{if } x \le 0 \\ -x+5 & \text{if } x > 0. \end{cases}$$

Use $f(x)$ to evaluate the following integrals:

(a) $\int_6^{10} f(x)\,dx$ (b) $\int_{-5}^{4} f(x)\,dx$

a) $\int_6^{10} -x+5\,dx$

$= \left[-\frac{x^2}{2} + 5x\right]_6^{10}$

DIY

Do It Yourself

b) $\int_{-5}^{0} -(x+5)\,dx$

$= -\left[\frac{x^2}{2} + 5x\right]_{-5}^{0}$

$= -\ 0 - \left(\frac{25}{2} - 25\right)$

$= 25/2$

$\int_0^4 -x+5\,dx$

$= \left[-\frac{x^2}{2} + 5x\right]_0^4$

$= -8 + 20 - 0$

$= 12$

$\frac{25}{2} + 12 = \frac{49}{2}$