1

a) The number of edges in a complete graph K_n is $\frac{n(n-1)}{2}$ and in a wheel graph W_{n-1} it is $2(n-1)$.

Here,

$$\begin{aligned} \frac{n(n-1)}{2} &= 2(n-1) \\ \frac{n}{2} &= 2 \\ n &= 4 \end{aligned}$$

2

a) $26^3 + 26^2 + 26 + 1 + 1$

b)

$$\begin{aligned} 4 \text{ front} + 3 \text{ back} &= \mathrm{C}(7,4) \times \mathrm{C}(9,3) \\ 5 \text{ front} + 2 \text{ back} &= \mathrm{C}(7,5) \times \mathrm{C}(9,2) \\ 6 \text{ front} + 1 \text{ back} &= \mathrm{C}(7,6) \times \mathrm{C}(9,1) \\ \text{Total} &= \text{sum} \end{aligned}$$

c) i. There are total 75 non-blue cards. So if only 7 cards are dealt, all of them can be non-blue.

ii.

$$\begin{aligned} k &= 4 \\ n &= 3 \\ N &= ? \\ \left\lceil \frac{N}{k} \right\rceil &= n \\ \left\lceil \frac{N}{4} \right\rceil &= 3 \\ N &= 9 \end{aligned}$$

3

Suppose,

$$P(n) \equiv \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

Basis case:

$$\begin{aligned} P(1) &\equiv \frac{1}{(3-1)(3+2)} = \frac{1}{6+4} \\ &\Rightarrow \frac{1}{2 \times 5} = \frac{1}{10} \\ &\Rightarrow \frac{1}{10} = \frac{1}{10} \end{aligned}$$

Inductive case:
Assuming $P(k)$ to be true, we need to prove $P(k+1)$ is true.

Now,

$$\begin{aligned}
P(k+1) \equiv \frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \ldots + \frac{1}{(3(k+1)-1)(3(k+1)+2)} \\
= \frac{k+1}{6(k+1)+4} \\
= \frac{k+1}{6k+10}
\end{aligned}$$

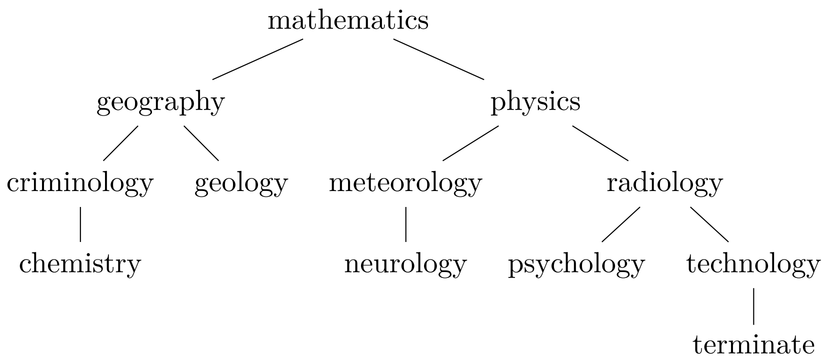
Let's manually add $\frac{1}{(3(k+1)-1)(3(k+1)+2)}$ to $P(k)$.

$$\begin{aligned}
&\frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3(k+1)-1)(3(k+1)+2)} \\
&= \frac{k}{6k+4} + \frac{1}{(3(k+1)-1)(3(k+1)+2)} \\
&= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)} \\
&= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \\
&= \frac{k(9k^2+21k+10)+6k+4}{(6k+4)(9k^2+21k+10)} \\
&= \frac{9k^3+21k^2+10k+6k+4}{(6k+4)(9k^2+21k+10)} \\
&= \frac{9k^3+21k^2+16k+4}{54k^3+126k^2+60k+36k^2+84k+40} \\
&= \frac{9k^3+21k^2+16k+4}{54k^3+126k^2+60k+36k^2+84k+40} \\
&= \frac{9k^3+21k^2+16k+4}{54k^3+162k^2+144k+40} \\
&= \frac{9k^3+9k^2+12k^2+12k+4k+4}{54k^3+90k^2+72k^2+120k+24k+40} \\
&= \frac{9k^2(k+1)+12k(k+1)+4(k+1)}{9k^2(6k+10)+12k(6k+10)+4(6k+10)} \\
&= \frac{(k+1)(9k^2+12k+4)}{(6k+10)(9k^2+12k+4)} \\
&= \frac{k+1}{6k+10}
\end{aligned}$$

So we find the same result by manually adding. Therefore, by mathematical induction it is proved that $P(n)$ is true.

4

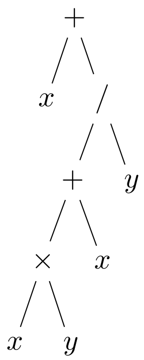
a) BST:



b) The height (h) of the tree is 4. There's a leaf (geology) at height 2. Therefore it's not a balanced tree.

The maximum limit of the number of leaves in a binary tree of height 4 is $2^4 = 16$.

c) Binary expression tree:



Expression in prefix notation: $(+ \quad x \quad (/ \quad (+ \quad (\times \quad x \quad y) \quad x) \quad y))$

Evaluating the expression where $x = 4$ and $y = 3$:

$$\begin{aligned}
&= (+ \quad (4 \quad (/ \quad (+ \quad (\times \quad 4 \quad 3) \quad 4) \quad 3))) \\
&= (+ \quad (4 \quad (/ \quad (+ \quad 12 \quad 4) \quad 3))) \\
&= (+ \quad (4 \quad (/ \quad 16 \quad 3))) \\
&= \left(+ \quad 4 \quad \frac{16}{3}\right) \\
&= \frac{28}{3}
\end{aligned}$$

5

a) There are 1 vertex with 1 children (h), 4 vertices with 2 children (e, g, d, i) and 3 vertices with 3 children (a, b, o). Therefore it is not a full m-ary tree as all the vertices should have a same number of children.

We can make it a binary tree by making c the child of h, f and r the child of p.

b) The number of leaves (l) in a full m-ary tree with i internal vertices and n vertices is $(m - 1) \cdot i + 1$.

Here,

$$
\begin{aligned}
(m-1)(136-109)+1 &= 109 \\
(m-1)27+1 &= 109 \\
27m-27+1 &= 109 \\
27m &= 109+26 \\
m &= \frac{135}{27} \\
m &= 5
\end{aligned}
$$

The number of edges in a 5-ary tree with 27 $(136-109)$ internal vertices is $27 \times 5 = 135$.