1. **a)** Consider, $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = e^x \boldsymbol{siny}\ \boldsymbol{i} + e^x \boldsymbol{cosy}\ \boldsymbol{j}$

i) Show that $\boldsymbol{F}$ is a conservative vector field on the entire $\boldsymbol{xy}$ −plane.
ii) Find the potential function $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y})$.
iii) Find $\int_{(0,0)}^{(1,\frac{\pi}{2})} \boldsymbol{F}.\boldsymbol{dr}$ using **ii)**

i) $\frac{\partial}{\partial x} e^x \cos y = e^x \cos y$

$\frac{\partial}{\partial y} e^x \sin y = e^x \cos y$

ii) $\vec{F} = \vec{\nabla} \phi$

$\Rightarrow e^x \sin y\, \hat{i} + e^x \cos y\, \hat{j} = \frac{\partial}{\partial x}\phi\, \hat{i} + \frac{\partial}{\partial y}\phi\, \hat{j}$

$\therefore \frac{\partial}{\partial x}\phi = e^x \sin y$ and $\frac{\partial}{\partial y}\phi = e^x \cos y$

$\Rightarrow \phi = \int e^x \sin y \, dx$

$= e^x \sin y + h(y)$

$\Rightarrow \frac{\partial}{\partial y}\phi = e^x \cos y + h'(y)$

$h'(y) = 0$

$\Rightarrow h(y) = c$

$\therefore \phi = e^x \sin y + c$

iii) $\int_{(0,0)}^{(1,\pi/2)} F.dr = \phi(1,\pi/2) - \phi(0,0)$

$= e \sin\frac{\pi}{2} - e^0 \sin 0$

$= e$

b) Using Green's theorem find the value of $\oint_C \boldsymbol{F}.\boldsymbol{dr}$
Where $\boldsymbol{F}(x,y) = (25e^{3x} - y^3)\boldsymbol{i} + (5y^3 + x^3)\boldsymbol{j}$ and C is the closed circle with parametric equations $\boldsymbol{x} = \boldsymbol{cost}, and\ \boldsymbol{y} = \boldsymbol{sint}$.

$\oint F.dr = \iint_R \frac{\partial}{\partial x}(5y^3 + x^3) - \frac{\partial}{\partial y}(25e^{3x} - y^3)\, dA$

$= \iint_R 3x^2 - (-3y^2)\, dA$

$= \int_0^1 \int_0^{2\pi} 3r^2 . r\, d\theta\, dr$

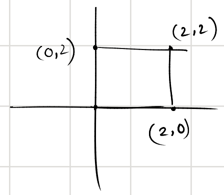
$= \int_0^1 3r^3 . 2\pi\, dr$

$= 6\pi \frac{1}{4} r^4 \Big]_0^1$

$= \frac{3}{2}\pi$

2. **a)** Evaluate $\int_C (x + y)dx + (-y - x)dy$ along the rectangle with vertices $(0,\ 0), (0, 2), (2,2)$ and $(2, 0)$.

$\int_C (x+y)\, dx + (-y-x)\, dy$



$= \iint_R \frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial y}(x+y)\, dA$

$= \iint_R -1 - 1\, dA$

$= \int_0^2 \int_0^2 -2\, dy dx$

$= \int_0^2 -2 \cdot 2\, dx$

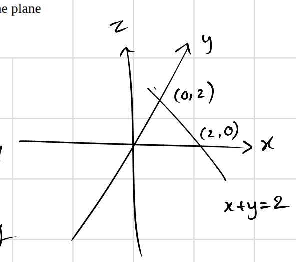
$= -2 \cdot 2 \cdot 2$

$= -8$

b) Evaluate the surface integral $\iint_\sigma 2xz\, ds$; σ is the part of the plane $x + y + z = 2$ that lies in the first octant.

$z = 2 - x - y$

$ds = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx dy$



$= \sqrt{(-1)^2 + (-1)^2 + 1}\ dx\, dy$

$= \sqrt{3}\ dx\, dx$

$\iint_\sigma 2xz\, ds = \int_0^2 \int_0^{-x+2} 2x(-x-y+2)\sqrt{3}\, dy dx$

$= \int_0^2 2x \left(-xy\Big]_0^{-x+2} - \frac{1}{2}y^2\Big]_0^{-x+2} + 2y\Big]_0^{-x+2}\right)\sqrt{3}\, dx$

$= \int_0^2 2x \left[-x(-x+2) - \frac{1}{2}(x^2 - 4x + 4) + 2(-x+2)\right]\sqrt{3}\, dx$

$= \int_0^2 \left[-2x^2(-x+2) - x(x^2 - 4x + 4) + 4x(-x+2)\right]\sqrt{3}\, dx$

$= \int_0^2 \left[2x^3 - 4x^2 - x^3 + \cancel{4x^2} - 4x - \cancel{4x^2} + 8x\right]\sqrt{3}\, dx$

$= \int_0^2 \left[x^3 - 4x^2 + 4x\right]\sqrt{3}\, dx$

$= \sqrt{3}\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2$

$= \sqrt{3} \cdot \frac{4}{3}$

$= \frac{4}{\sqrt{3}}$

3. **a)** Find the flux of the vector field $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + 2z\boldsymbol{k}$ across σ, where σ is the portion of the cone $\boldsymbol{z} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$ between the planes $\boldsymbol{z} = \boldsymbol{1}$ and $\boldsymbol{z} = \boldsymbol{4}$, oriented upward unit normal.

$\nabla G = -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}$

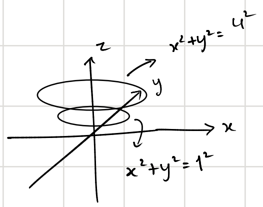
$= -\frac{2x}{2\sqrt{x^2+y^2}}\hat{i} - \frac{2y}{2\sqrt{x^2+y^2}}\hat{j} + \hat{k} = -\frac{x}{\sqrt{x^2+y^2}}\hat{i} - \frac{y}{\sqrt{x^2+y^2}}\hat{j} + \hat{k}$

$\iint_R \vec{F} \cdot \nabla G\, dA = \iint_R -\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z\, dA$

$= \iint_R -\frac{(x^2+y^2)}{\sqrt{x^2+y^2}} + 2\sqrt{x^2+y^2}\, dA$

$= \iint_R -\sqrt{x^2+y^2} + 2\sqrt{x^2+y^2}\, dA$

$= \iint_R \sqrt{x^2+y^2}\, dA$



$= \int_1^4 \int_0^{2\pi} r . r d\theta dr$

$= \int_1^4 r^2 . 2\pi\, dr$

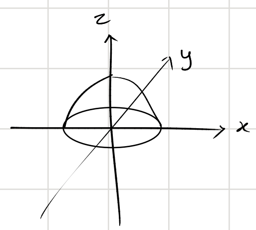
$= 2\pi \cdot \frac{1}{3} r^3\Big]_1^4 = 2\pi \cdot \frac{1}{3}(4^3 - 1^3) = 42\pi$

Or

Use the Divergence Theorem to find the outward flux of the vector field $\boldsymbol{F}(x,y,z) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z^3\boldsymbol{k}$ across the surface of the region that is enclosed by $\boldsymbol{z} = \boldsymbol{25} - \boldsymbol{x}^2 - \boldsymbol{y}^2$ and the plane $\boldsymbol{z} = \boldsymbol{0}$.

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3$

$= 3x^2 + 3y^2 + 3z^2$



$\iiint_G \operatorname{div}\vec{F}\, dv$

$= \int_0^5 \int_0^{2\pi} \int_0^{25-r^2} (3r^2 + 3z^2)\, dz\, r d\theta dr$

$= \int_0^5 \int_0^{2\pi} \left\{3r^2 . (25 - r^2) + (25 - r^2)^3\right\} r\, d\theta dr$

$= \int_0^5 \int_0^{2\pi} (75r^2 - 3r^4 + 15625 - 1875r^2 + 75r^4 - r^6)\, r\, d\theta dr$

$= \int_0^5 \int_0^{2\pi} (-r^6 + 72r^4 - 1800r^2 + 15625)\, r\ d\theta\ dr$

$= \int_0^5 (-r^7 + 72r^5 - 1800r^3 + 15625r)\ 2\pi\ dr$

$= \left[-\frac{1}{8}r^8 + \frac{72}{6}r^6 - \frac{1800}{4}r^4 + \frac{15625}{2}r^2\right]_0^5 \cdot 2\pi$

$= \frac{1}{2}\ 421875\,\pi$

4. **a)** Use cylindrical coordinate to evaluate

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{0}^{16-x^2-y^2}(x^2+y^2)\,dz\,dy\,dx.$$

$$\int_0^{2\pi}\int_0^4\int_0^{16-r^2} r^2\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 r^3\left(16-r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 16r^3 - r^5\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{16}{4}r^4 - \frac{1}{6}r^6\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\frac{1024}{3}\,d\theta$$

$$= \frac{1}{3}\,2048\pi$$

b) Find the volume of the sphere by using spherical coordinate system where the radius of sphere is 3.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\rho^2\sin\phi\cdot d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\sin\phi\left.\frac{1}{3}\rho^3\right]_0^3 d\phi\,d\theta$$

$$= \int_0^{2\pi} 9\left[-\cos\phi\right]_0^{\pi} d\theta$$

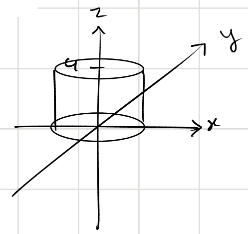
$$= \int_0^{2\pi} 9\left[1-(-1)\right] d\theta$$

$$= 18\cdot 2\pi$$

$$= 36\pi$$

Or

Using triple integral find the volume of the solid bounded by the $x^2+y^2=25$, $xy-plane$ and $z=4$.



$$\int_0^4\int_0^5\int_0^{2\pi} r\,d\theta\,dr\,dz$$

$$= \int_0^4\int_0^5 r\cdot 2\pi\cdot dr\,dz$$

$$= \int_0^4 2\pi\left.\frac{1}{2}r^2\right|_0^5 dz$$

$$= \pi\cdot 5^2\cdot 4$$

$$= 100\pi$$