1. a) Identify and sketch the graph of the Conic.

$$16x^2 - y^2 - 32x - 6y = 57$$

$16x^2 - y^2 - 32x - 6y = 57$

$\Rightarrow 16(x^2 - 2x) - (y^2 + 6y) = 57$

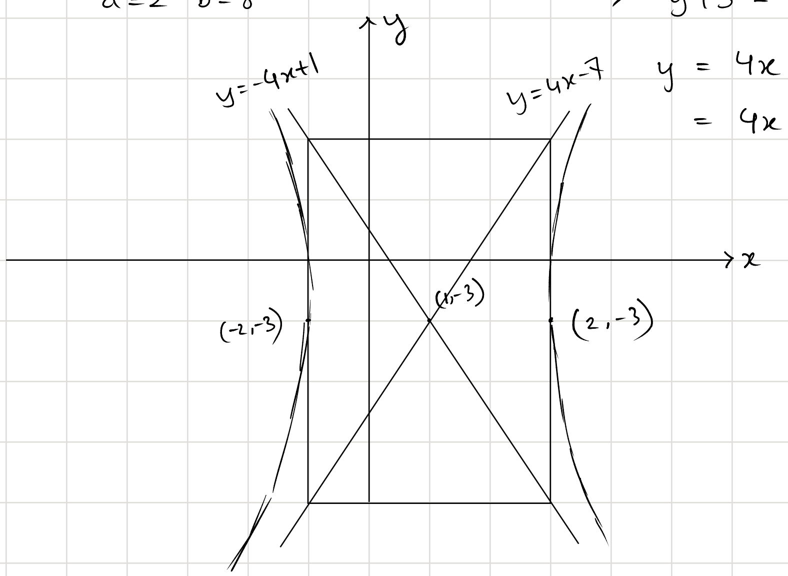
$\Rightarrow 16(x^2 - 2x + 1) - (y^2 + 2 \cdot y \cdot 3 + 3^2) = 57 + 16 - 9$

$\Rightarrow 16(x-1)^2 - (y+3)^2 = 64$

$\Rightarrow \dfrac{(x-1)^2}{2^2} - \dfrac{(y+3)^2}{8^2} = 1$ $\qquad \dfrac{y+3}{8} = \pm \dfrac{x-1}{2}$ [asymptotes]

$a = 2 \quad b = 8$ $\qquad \Rightarrow y + 3 = \pm 4(x-1)$

$y = 4x - 4 - 3 = 4x - 7$ $\qquad y = -4x + 4 - 3 = -4x + 1$



b) Rotate the coordinate axes to remove the xy-term, then identify the type of Conic.

$$x^2 - 10\sqrt{3}xy + 11y^2 + 64 = 0$$

$x^2 - 10\sqrt{3}\,xy + 11y^2 + 64 = 0$

$A = 1 \quad B = -10\sqrt{3} \quad C = 11 \quad D = 0 \quad E = 0 \quad F = 64$

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{10}{10\sqrt{3}}$

$\Rightarrow \tan 2\theta = \sqrt{3} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\sqrt{3} = 30°$

$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta = -4$

$C' = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta = 16$

$D' = D\cos\theta + E\sin\theta = 0$

$E' = -D\sin\theta + E\cos\theta = 0$

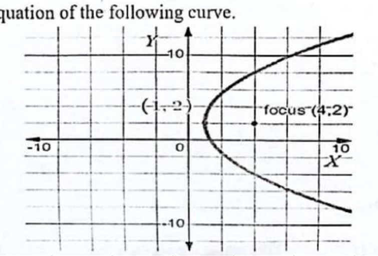
$F' = 64$

$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$

$\Rightarrow -4x'^2 + 16y'^2 + 64 = 0$

$\Rightarrow \dfrac{x'^2}{16} - \dfrac{y'^2}{4} = 1$ [hyperbola]

c) Write the equation of the following curve.



parabola, vertex at $(1, 2)$, $p = 4 - 1 = 3$

$(y-2)^2 = 4 \cdot 3 \cdot (x-1)$

2. a) Consider, $F(x,y) = 2xe^y\,i + x^2e^y\,j$

i) Show that F is a conservative vector field on the entire xy-plane.

ii) Find the potential function $\phi(x,y)$.

iii) Find $\int_{(0,0)}^{(3,2)} F.dr$ using ii).

i) $F(x,y) = 2xe^y\hat{i} + x^2e^y\hat{j}$

$\dfrac{\partial}{\partial y} 2xe^y = 2xe^y$ $\qquad \vec{F}$ is conservative

$\dfrac{\partial}{\partial x} x^2e^y = 2xe^y$

ii) Since $\vec{F}$ is conservative there exits a potential function ϕ such that $\vec{F} = \vec{\nabla}\phi = \dfrac{\partial}{\partial x}\phi\,\hat{i} + \dfrac{\partial}{\partial y}\phi\,\hat{j}$

$\Rightarrow 2xe^y\hat{i} + x^2e^y\hat{j} = \dfrac{\partial}{\partial x}\phi\,\hat{i} + \dfrac{\partial}{\partial y}\phi\,\hat{j}$

$\dfrac{\partial}{\partial x}\phi = 2xe^y$ and $\dfrac{\partial}{\partial y}\phi = x^2e^y$

$\Rightarrow \phi = \int 2xe^y\,dx = x^2e^y + h(y)$

$\Rightarrow \dfrac{\partial}{\partial y}\phi = x^2e^y + h'(y)$

$h'(y) = 0$

$h(y) = c$

$\therefore \phi = x^2e^y + c$

iii) $\displaystyle\int_{(0,0)}^{(3,2)} F \cdot dr = \phi(3,2) - \phi(0,0)$

$= 3^2 \cdot e^2 - 0^2 \cdot e^0$

$= 9e^2$

b) Find the work done by the force field $F(x,y) = (e^x - y^3)i + (\cos y + x^3)j$ on the particle that travels once around the unit circle $x^2 + y^2 = 1$.

$\displaystyle\int_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy$

$\displaystyle= \iint_R \frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\,dA$

$\displaystyle= \iint_R 3x^2 + 3y^2\,dA$

$\displaystyle= \int_0^1\int_0^{2\pi} 3r^2\,r.d\theta\,dr$

$\displaystyle= \int_0^1 3r^3 \cdot 2\pi\,dr$

$= 6\pi\,\frac{1}{4}r^4\Big]_0^1$

$= \dfrac{6\pi}{4} = \dfrac{3}{2}\pi$

c) Determine the constant a so that the vector $V(x,y,z) = (x+3y)i + (y-2z)j + (x+az)k$ is divergence free.

$\vec{\nabla} \cdot \vec{V}(x,y,z) = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az)$

$= 1 + 1 + a$

$= 2 + a$

$\Rightarrow 2 + a = 0$

$\Rightarrow a = -2$

3. a) Evaluate the line integral along the curve C $\int_C (x+2y)dx + (x-y)dy$ where $C: x = 2\cos t$ $y = 4\sin t \left(0 \le t \le \frac{\pi}{4}\right)$.

$$dx = -2\sin t\, dt$$
$$dy = 4\cos t\, dt$$

$$\int_0^{\pi/4} (2\cos t + 8\sin t)(-2\sin t\, dt) + (2\cos t - 4\sin t)(4\cos t\, dt)$$

$$= \int_0^{\pi/4} \left(-4\cos t \sin t - 16\sin^2 t + 8\cos^2 t - 16\cos t \sin t\right) dt$$

$$= \int_0^{\pi/4} \left(-20\cos t \sin t - 16 + 16\cos^2 t + 8\cos^2 t\right) dt$$

$$= \int_0^{\pi/4} \left(24\cos^2 t - 20\cos t \sin t - 16\right) dt$$

$$= \int_0^{\pi/4} \{12(1+\cos 2t) - 20\cos t \sin t - 16\} dt$$

$$= \int_0^{\pi/4} \left(12 + 12\cos 2t - 20\cos t \sin t - 16\right) dt$$

$$= 12t\Big]_0^{\pi/4} + 12 \cdot \frac{1}{2} \sin 2t\Big]_0^{\pi/4} - \underbrace{\int_0^{\pi/4} 20\cos t \sin t\, dt} - 16t\Big]_0^{\pi/4} \}$$

$$u = \sin t$$
$$du = \cos t\, dt$$
$$\int_0^{\pi/4} 20u\, du$$
$$10u^2\Big]_0^{\pi/4}$$
$$= 10\sin^2 t\Big]_0^{\pi/4}$$
$$= 5$$

$$= 3\pi + 6 - 5 - 4\pi$$

$$= -\pi + 1$$

b) Use the Divergence Theorem to find the outward flux of the vector field $F(x,y,z) = x^3 i + y^3 j + zk$ across the surface of the region that is enclosed by $x^2 + y^2 = 16$ and the plane $z = 0$ and $z = 3$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} y^3 + \frac{\partial}{\partial z} z$$

$$= 3x^2 + 3y^2 + 1$$

$$\int_0^3 \int_0^4 \int_0^{2\pi} (3r^2 + 1)\, r\, d\theta\, dr\, dz$$

$$= \int_0^3 \int_0^4 (3r^3 + r)\, 2\pi\, dr\, dz$$

$$= \int_0^3 2\pi \left[\frac{3}{4} r^4 + \frac{1}{2} r^2\right]_0^4 dz$$

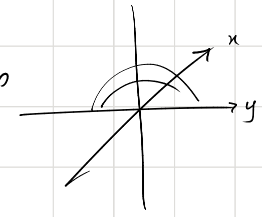
$$= 2\pi \cdot 200 \cdot 3$$

$$= 1200\pi$$

4. a) Use spherical coordinate systems to evaluate:

$$\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$

$$\int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$



$$= \int_0^3 \int_0^{2\pi} \rho^3 \left[-\cos\phi\right]_0^{\pi} d\theta\, d\rho$$

$$= \int_0^3 \int_0^{2\pi} \rho^3 \left[1 - (-1)\right] d\theta\, d\rho$$

$$= \int_0^3 2\rho^3 \cdot 2\pi\, d\rho$$

$$= 4\pi \left.\frac{1}{4}\rho^4\right]_0^3$$

$$= \pi \cdot 3^4$$

$$= 81\pi$$

b) Find the flux of the vector field $F(x,y,z) = xi + yj + 3zk$ across σ, where σ is the portion of the surface $z = 9 - x^2 - y^2$ that lies above the xy – plane and suppose that σ is oriented up.

$$x^2 + y^2 = 3^2$$

$$\nabla G = -\frac{\partial}{\partial x} z\, \hat{i} - \frac{\partial}{\partial y} z\, \hat{j} + \hat{k} = 2x\,\hat{i} + 2y\,\hat{j} + \hat{k}$$

$$\iint_\sigma \vec{F} \cdot \nabla G\, dA$$

$$= \iint_\sigma \left(x\,\hat{i} + y\,\hat{j} + 3z\,\hat{k}\right) \cdot \left(2x\,\hat{i} + 2y\,\hat{j} + \hat{k}\right) dA$$

$$= \iint_\sigma \left(2x^2 + 2y^2 + 3z\right) dA$$

$$= \iint_\sigma \left(2x^2 + 2y^2 + 27 - 3x^2 - 3y^2\right) dA$$

$$= \iint_\sigma \left(-x^2 - y^2 + 27\right) dA$$

$$= \int_0^3 \int_0^{2\pi} (-r^2 + 27)\, r\, d\theta\, dr$$

$$= \int_0^3 (27r - r^3) \cdot 2\pi\, dr$$

$$= 2\pi \left[\frac{27}{2} r^2 - \frac{1}{4} r^4\right]_0^3$$

$$= 2\pi \cdot \frac{405}{4} = \frac{1}{2} 405\pi$$