Q1 **(a)** Farid goes shopping and buys three different types of fruit. The first matrix below shows the number of kilograms of each fruit bought during two different weeks. The second matrix shows the price per kilogram, in cents of each fruit.

$$\begin{array}{ccc} & \text{bananas} & \text{apples} & \text{grapes} \\ \text{Week 1} & 1 & 2 & 0.5 \\ \text{Week 2} & 1.5 & 1 & 1 \end{array} \qquad \text{Kg}$$

Speed

$$\text{price/kg} \quad \text{time} \begin{pmatrix} 290 \\ 160 \\ 640 \end{pmatrix} \begin{array}{l} \text{bananas} \\ \text{apples} \\ \text{grapes} \end{array}$$

Given that, $F = \begin{pmatrix} 1 & 2 & 0.5 \\ 1.5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 290 \\ 160 \\ 640 \end{pmatrix}$. distance

1×290

(i) Find the value of F and interpret the result.
(ii) Using matrix multiplication find the total amount spent on fruits for two weeks. **[3+1]**

a) (i)

$$\begin{bmatrix} 290 + 320 + 320 \\ 435 + 160 + 640 \end{bmatrix} = \begin{bmatrix} 930 \\ 1235 \end{bmatrix}$$

total spent in first week 930 cents = 9.3 \$

second week 1235 cents = 12.35 \$

(ii)

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 930 \\ 1235 \end{bmatrix} = \begin{bmatrix} 930 + 1235 \end{bmatrix} = 2165$$

total spent 2165 cents = 21.65 \$

(b) Given that,

week 1
week 2

$$2x + 3y - z = 5$$
$$x - y + 2z = 0$$
$$-3x + y + z = 8$$

(i) Write the above system of linear equations in the form AX = B, where A, X and B are matrices.
(ii) Find the inverse of A and hence solve the above system of linear equations.

i)

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$$

ii)

$$\begin{array}{ccccccc} 2 & 3 & -1 & 5 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ -3 & 1 & 1 & 8 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{ccccccc} 0 & 5 & -5 & 5 & 1 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 7 & 8 & 0 & 3 & 1 \end{array} \rightarrow \begin{array}{ccccccc} 0 & 1 & -1 & 1 & 1/5 & -2/5 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 7 & 8 & 0 & 3 & 1 \end{array}$$

$r_1 = r_1 - 2r_2$

$r_3 = 3r_2 + r_3$

$r_1 = r_1/5$

$$\begin{array}{ccccccc} 0 & 1 & 0 & 3 & 7/25 & 1/25 & 1/5 \\ 1 & 0 & 0 & -1 & 3/25 & 4/25 & -1/5 \\ 0 & 0 & 1 & 2 & 2/25 & 11/25 & 1/5 \end{array} \leftarrow \begin{array}{ccccccc} 0 & 1 & -1 & 1 & 1/5 & -2/5 & 0 \\ 1 & 0 & 1 & 1 & 1/5 & 3/5 & 0 \\ 0 & 0 & 1 & 2 & 2/25 & 11/25 & 1/5 \end{array} \leftarrow \begin{array}{ccccccc} 0 & 1 & -1 & 1 & 1/5 & -2/5 & 0 \\ 1 & 0 & 1 & 1 & 1/5 & 3/5 & 0 \\ 0 & 0 & 5 & 10 & 2/5 & 11/5 & 1 \end{array}$$

$r_1 = r_1 + r_3$

$r_2 = r_2 - r_3$

$r_3 = r_3/5$

$r_2 = r_1 + r_2$

$r_3 = 2r_1 + r_3$

$$\begin{array}{ccccccc} 1 & 0 & 0 & -1 & 3/25 & 4/25 & -1/5 \\ 0 & 1 & 0 & 3 & 7/25 & 1/25 & 1/5 \\ 0 & 0 & 1 & 2 & 2/25 & 11/25 & 1/5 \end{array}$$

swap (1,2)

$x = -1$

$y = 3$

$z = 2$

Q2 (a) State (do not solve) how many solutions does the following set of equations have?

$$x + 3y - z = 5$$
$$x + y + 2z = -3$$

(b) Find the Eigenvalues and corresponding Eigenvector of the Matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$.

a)

$$\begin{matrix} 1 & 3 & -1 & 5 \\ 1 & 1 & 2 & -3 \end{matrix}$$

infinite solutions

b)

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow -(1-\lambda)(2+\lambda) = 0$$

$$\Rightarrow 2 - 2\lambda + \lambda - \lambda^2 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 2 = 0$$

$$\Rightarrow \lambda = 1, -2$$

for $\lambda = 1$

$$\begin{bmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 + 0 \\ x - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $x = p$

$y = \frac{1}{3} p$

$$\begin{bmatrix} p \\ p/3 \end{bmatrix} = p \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$

for $\lambda = -2$

$$\begin{bmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

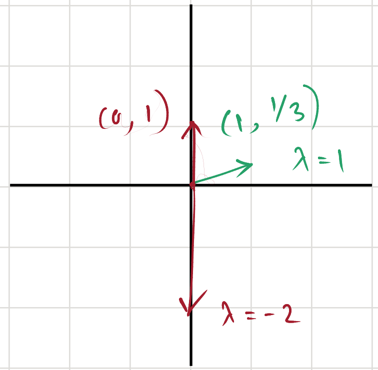
$$\Rightarrow \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3x \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = 0$

$y = p$

$$\begin{bmatrix} 0 \\ p \end{bmatrix} = p \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



(c) Solve the following system by **Gauss-Jordan elimination method.**

$$x - y + 2z + 3p + t = 3$$
$$3x + y - 2z + p - 4t = 0$$
$$-x + 2y - z - p + 2t = -1$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 1 & 3 \\ 3 & 1 & -2 & 1 & -4 & 0 \\ -1 & 2 & -1 & -1 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 1 & 3 \\ 0 & 4 & -8 & -8 & -7 & -9 \\ 0 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 5 & 4 & 5 \\ 0 & 0 & -12 & -16 & -19 & -17 \\ 0 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 5 & 4 & 5 \\ 0 & 0 & 1 & 4/3 & 19/12 & 17/12 \\ 0 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 1 & 4/3 & 19/12 & 17/12 \\ 0 & 1 & 0 & 2/3 & 17/12 & 7/12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -3/4 & 3/4 \\ 0 & 1 & 0 & 2/3 & 17/12 & 7/12 \\ 0 & 0 & 1 & 4/3 & 19/12 & 17/12 \end{bmatrix}$$

$r_2 = -3r_1 + r_2$, $r_3 = r_1 + r_2$

$r_1 = r_1 + r_3$, $r_2 = r_2 - 4r_3$

$r_2 = r_2 / -12$

$r_1 = r_1 - 3r_2$, $r_3 = r_3 - r_2$

swap $(2,3)$

$x + p - \frac{3}{4}t = \frac{3}{4}$

$\Rightarrow x = \frac{3}{4} - p + \frac{3}{4}t$

$y + \frac{2}{3}p + \frac{17}{12}t = \frac{7}{12}$

$\Rightarrow y = \frac{7}{12} - \frac{2}{3}p - \frac{17}{12}t$

$z + \frac{4}{3}p + \frac{19}{12}t = \frac{17}{12}$

$\Rightarrow z = \frac{17}{12} - \frac{4}{3}p - \frac{19}{12}t$

Q3 (a) The roots of the characteristics equation of a differential equation are -2, -2, and $1 \pm i\sqrt{2}$. Find the general solution of the differential equation.

(b) Solve the following differential equations.

(i) $2xy\,dx + (x^2 - 1)dy = 0$

(ii) $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

a) $y = c_1 e^{-2x} + c_2 x e^{-2x} + e^{x}\left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$

b) i) $2xy\,dx + (x^2-1)\,dy = 0$

$M \qquad N$

$M_y = 2x \qquad N_x = 2x \qquad$ exact ✓

$F_x = 2xy$

$\Rightarrow F = \int 2xy\,dx = x^2y + g(y) = x^2y - y$

$\Rightarrow F_y = x^2 + g'(y)$

$\Rightarrow x^2 - 1 = x^2 + g'(y)$

$\Rightarrow g'(y) = -1$

$\Rightarrow g(y) = -y$

ii) $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$

$M \qquad N$

$M_y = 2y \qquad N_x = 2x - y \qquad$ exact ✗

$y = vx \qquad dy = v\,dx + x\,dv$

$(x^2 + v^2x^2)\,dx + (x^2 - vx^2)(v\,dx + x\,dv) = 0$

$\Rightarrow (1 + v^2)\,dx + (1 - v)(v\,dx + x\,dv) = 0$

$\Rightarrow (1 + v^2)\,dx + v\,dx - v^2\,dx + x\,dv - vx\,dv = 0$

$\Rightarrow (1 + v^2 + v - v^2)\,dx + (x - vx)\,dv = 0$

$\Rightarrow (1 + v)\,dx = x(v - 1)\,dv$

$\Rightarrow \frac{v-1}{v+1}\,dv = \frac{1}{x}\,dx$

Q4 (a) Suppose a string is stretched and then released. The motion of the spring is affected by an external force $f(t) = e^{-3t}$. The position x of the mass at any time t starts from the equilibrium position is given by the differential equation.

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = e^{-3t}$$

Find the position x of the mass at any time t.

a) $m^2 + 4m + 4 = 0$

$\Rightarrow m = -2, -2$

$x_c = c_1 e^{-2t} + c_2 t e^{-2}$

$x_p = \dfrac{1}{D^2+4D+4} e^{-3t} = \dfrac{1}{9-12+4} e^{-3t} = e^{-3t}$

(b) Find the solution of the initial value problem

$4y'' - 8y' + 3y = 0 \qquad y(0) = 2 \qquad y'(0) = \frac{1}{2}$

$4m^2 - 8m + 3 = 0$

$m = 3/2, 1/2$

$y = c_1 e^{3t/2} + c_2 e^{t/2}$ $\qquad y' = \frac{3}{2} c_1 e^{3t/2} + \frac{1}{2} c_2 e^{t/2}$

$y(0) = 2$ $\qquad y'(0) = 1/2$

$\Rightarrow c_1 + c_2 = 2$ $\qquad \Rightarrow \frac{3}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2}$

$\Rightarrow 3c_1 + c_2 = 1$

$c_1, c_2 = -\frac{1}{2}, \frac{5}{2}$

$y = -\frac{1}{2} e^{3t/2} + \frac{5}{2} e^{t/2}$

$\Rightarrow v - 2\ln(v+1) + 1 = \ln x + c$

$\Rightarrow y/x - 2\ln(y/x + 1) + 1 = \ln x + c$

$\Rightarrow y/x - 2\ln(x+y) + 2\ln x + 1 = \ln x + c$

$\Rightarrow y/x - 2\ln(x+y) + \ln x + 1 = c$

$\Rightarrow y - 2x\ln(x+y) + x\ln x + x = cx$

$\int \frac{v-1}{v+1} dv$

$= \int \frac{v}{v+1} dv - \int \frac{1}{v+1} dv$

$= \int \frac{u-1}{u} du - \ln(v+1)$

$= \int 1 - \frac{1}{u} du - \ln(v+1)$

$= u - \ln u - \ln(v+1)$

$= v+1 - \ln(v+1) - \ln(v+1)$

$= v - 2\ln(v+1) + 1$