1. Let L_1 and L_2 be the lines [10]

L_1: $x = 5+2t$, $y = -2+6t$, $z = 3-2t$

L_2: $x = -2+t$, $y = 4+t$, $z = 3+2t$

i) Are the lines parallel?

ii) Do the lines intersect?

iii) Find the distance between lines?

iv) Find the intersection point of the line L_1 and the plane $x - y + 2z = 0$.

i) parallel vector of L_1, $\vec{v_1} = \langle 2, 6, -2 \rangle$

parallel vector of L_2, $\vec{v_2} = \langle 1, 1, 2 \rangle$

there is no scalar multiple k such that $\vec{v_1} = k\vec{v_2}$

therefore the lines are not parallel.

ii) let (x_0, y_0, z_0) be the point of intersection

$\therefore\ x_0 = 5+2t_1 = -2+t_2 \quad y_0 = -2+6t_1 = 4+t_2 \quad z_0 = 3-2t_1 = 3+2t_2$

$\Rightarrow 2t_1 - t_2 = -7 \quad \Rightarrow 6t_1 - t_2 = 6 \quad \Rightarrow 2t_1 + 2t_2 = 0$

$t_1 = \frac{13}{4} \quad t_2 = \frac{27}{2} \quad \Rightarrow t_1 + t_2 = 0$

but $\frac{13}{4} + \frac{27}{2} \neq 0$

therefore the lines don't intersect.

iii) Since the lines are not parallel and also do not intersect, they are skew.

now let $\vec{n}$ be the common normal at the two parallel planes that contain L_1 and L_2 respectively.

$$\therefore \vec{n} = \vec{v_1} \times \vec{v_2} = \langle 2, 6, -2 \rangle \times \langle 1, 1, 2 \rangle$$

$$= \langle 12+2, -(4+2), 2-6 \rangle$$

$$= \langle 14, -6, -4 \rangle$$

so the plane containing L_2 is

$$14(x+2) - 6(y-4) - 4(z-3) = 0$$

$$\Rightarrow 14x + 28 - 6y + 24 - 4z + 12 = 0$$

$$\Rightarrow 14x - 6y - 4z + 64 = 0$$

and $(5, -2, 3)$ is a point on L_1

$\therefore$ the distance between the lines

$$= \frac{|14 \cdot 5 - 6(-2) - 4 \cdot 3 + 64|}{\sqrt{14^2 + 6^2 + 4^2}}$$

$$= 8.509$$

iv) $x - y + 2z = 0$

$$\Rightarrow 5 + 2t - (-2+6t) + 2(3-2t) = 0$$

$$\Rightarrow 5 + 2t + 2 - 6t + 6 - 4t = 0$$

$$\Rightarrow 8t = 13$$

$$\Rightarrow t = \frac{13}{8}$$

$$x = 5 + \frac{13}{4} = \frac{33}{4}$$

$$y = -2 + 6 \times \frac{13}{8} = \frac{31}{4}$$

$$z = 3 - 2 \times \frac{13}{8} = -\frac{1}{4}$$

2. a) Find the vector component (orthogonal projection) of $u = \langle 4, -1, 0 \rangle$ along $w = \langle 2, 0, -1 \rangle$ and orthogonal to u. [4]

$\vec{w} = \langle 2, 0, -1 \rangle$

$$\hat{a} = \frac{1}{\sqrt{2^2+1^2}} \langle 2, 0, -1 \rangle$$

$$= \left\langle \frac{2}{\sqrt5}, 0, -\frac{1}{\sqrt5} \right\rangle$$

$$\vec{w} \cdot \hat{a} = \frac{8}{\sqrt5} - 0 + 0 = \frac{8}{\sqrt5}$$

orthogonal projection of $\vec{u}$ along $\vec{w}$

$$\vec{v_1} = \frac{8}{\sqrt5} \left\langle \frac{2}{\sqrt5}, 0, -\frac{1}{\sqrt5} \right\rangle$$

$$= \left\langle \frac{16}{5}, 0, -\frac{8}{5} \right\rangle$$

projection of $\vec{u}$ in the direction orthogonal to $\vec{u}$

$$|\vec{v_2}| = |\vec{u}| \cos 90^\circ = 0$$

therefore it's a null vector

b) Find the area of the triangle with vertices $A(0, 6, -3)$ $B(1, 0, 2)$ and $C(1, -4, 1)$.

$\vec{AB} = \langle 1, -6, 5 \rangle$

$\vec{AC} = \langle 1, -10, 4 \rangle$

$$\vec{AB} \times \vec{AC} = \langle -24+50, -(4-5), -10+6 \rangle$$

$$= \langle 26, 1, -4 \rangle$$

$$\text{area of the triangle} = \frac{1}{2}\sqrt{26^2 + 1^2 + 4^2}$$

$$= \frac{3}{2}\sqrt{77}$$

c) Evaluate the double integral $\int_0^{\ln 3} \int_0^1 xye^{x^2 y}\,dx\,dy$

$$\int_0^{\ln 3} \int_0^1 xye^{x^2 y}\,dx\,dy$$

$$= \int_0^{\ln 3} \int_0^y \frac{1}{2} e^u \,du\,dy$$

$u = x^2 y$

$du = 2xy\,dx$

$x = 0 \quad u = 0$

$x = 1 \quad u = y$

$$= \int_0^{\ln 3} \frac{1}{2}(e^y - 1)\,dy$$

$$= \frac{1}{2}\left(e^y \Big|_0^{\ln 3} - y \Big|_0^{\ln 3} \right)$$

$$= \frac{1}{2}\left[(3-1) - (\ln 3 - 0) \right] = \frac{1}{2}(2 - \ln 3) = 1 - \frac{1}{2}\ln 3$$

3. **a)** Evaluate $\int_1^5 \int_x^{x^2} \int_0^{\ln z} xe^y \, dydzdx$

$$\int_0^{\ln z} xe^y \, dy = x\left(e^{\ln z} - e^0\right) = x(z-1)$$

$$\int_x^{x^2} x(z-1)\, dz = x\left[\frac{1}{2}z^2 - z\right]_x^{x^2} = x\left[\frac{1}{2}x^4 - x^2 - \left(\frac{1}{2}x^2 - x\right)\right]$$

$$= x\left(\frac{1}{2}x^4 - \frac{3}{2}x^2 + x\right)$$

$$= \frac{1}{2}x^5 - \frac{3}{2}x^3 + x^2$$

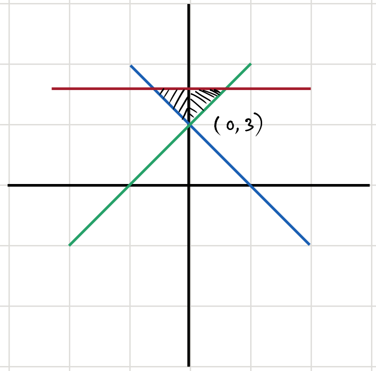
$$\int_1^5 \frac{1}{2}x^5 - \frac{3}{2}x^3 + x^2 \, dx$$

$$= \left[\frac{1}{12}x^6 - \frac{3}{8}x^4 + \frac{1}{3}x^3\right]_1^5$$

$$= \left(\frac{1}{12}5^6 - \frac{3}{8}5^4 + \frac{1}{3}5^3\right) - \left(\frac{1}{12} - \frac{3}{8} + \frac{1}{3}\right)$$

$$= \frac{3328}{3}$$

b) Use a double integral to find the area of region bounded by the lines $x + y = 3$, $y = x + 3$ and $y = 5$.



$$\int_3^5 \int_{3-y}^{y-3} dx\, dy$$

$$= \int_3^5 y - 3 - 3 + y \, dy$$

$$= \int_3^5 2y - 6 \, dy$$

$$= \left[y^2 - 6y\right]_3^5$$

$$= 5^2 - 30 - (3^2 - 18)$$

$$= -5 + 9$$

$$= 4$$