1. **a)** Consider, $\boldsymbol{F(x,y)} = (2xy + y^2 + 1)\,\boldsymbol{i} + (x^2 + 2xy + 1)\,\boldsymbol{j}$

- **i)** Show that $\boldsymbol{F}$ is a conservative vector field on the entire $\boldsymbol{xy}$ −plane.
- **ii)** Find the potential function $\boldsymbol{\phi(x,y)}$.
- **iii)** Find $\int_{(0,0)}^{(2,3)} \boldsymbol{F.dr}$ using **ii)**

i) $\frac{\partial}{\partial y}(2xy + y^2 + 1) = 2x + 2y$

$\frac{\partial}{\partial x}(x^2 + 2xy + 1) = 2x + 2y$

ii) $\vec{F} = \vec{\nabla}\phi = \frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial}{\partial y}\phi\,\hat{j}$

$= (2xy + y^2 + 1)\,\hat{i} + (x^2 + 2xy + 1)\,\hat{j}$

$\frac{\partial}{\partial x}\phi = 2xy + y^2 + 1$

$\Rightarrow \phi = x^2y + xy^2 + x + h(y)$

$\Rightarrow \frac{\partial}{\partial y}\phi = x^2 + 2xy + h'(y)$

$h'(y) = 1 \Rightarrow h(y) = y$

$\phi = x^2y + xy^2 + x + y$

iii) $\int_{(0,0)}^{(2,3)} F.dr = \phi(2,3) - \phi(0,0)$

$= 35 - 0$

$= 35$

b) Using Green's theorem find the value of $\oint_C \boldsymbol{F.dr}$
Where $\boldsymbol{F}(x,y) = (\sin x\, e^{3x} - 5y^2)\boldsymbol{i} + (5y^3 + 2x)\boldsymbol{j}$ and C is the closed curve with parametric equations $\boldsymbol{x = 3cost}$, *and* $\boldsymbol{y = 3sint}$.

$\oint_C F.dr = \iint_R \frac{\partial}{\partial x}(5y^3 + 2x) - \frac{\partial}{\partial y}(\sin x\, e^{3x} - 5y^2)\, dA$

$= \iint_R 2 + 10y \, dA$

$= \int_0^3 \int_0^{2\pi} (2 + 10 \cdot r \sin t)\, r\, dt\, dr$

$= \int_0^3 r \left[2t - 10r\cos t\right]_0^{2\pi} dr$

$= \int_0^3 r \left[2 \cdot 2\pi - 10r(\cos 2\pi - \cos 0)\right] dr$

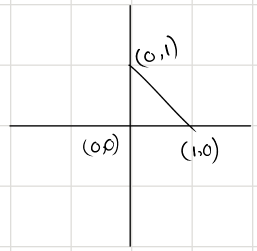
$= \int_0^3 4\pi r\, dr$

$= 2\pi \cdot r^2 \Big]_0^3$

$= 2\pi \cdot 3^2 = 18\pi$

2. **a)** Evaluate $\int_C (\boldsymbol{x+y})\boldsymbol{dx} + (\boldsymbol{-y})\boldsymbol{dy}$ along the traigle with vertices $(0,\ 0), (0,1)$ and $(1,0)$.

Grinch theorem



$\iint_R \frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x+y)\, dA$

$= \iint_R 0 - 1\, dA$

$= -\int_0^1 \int_0^{-x+1} dy\, dx$

$= -\int_0^1 -x + 1\, dx$

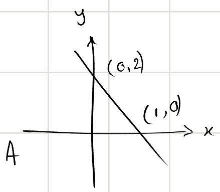
$= -\left[-\frac{1}{2}x^2 + x\right]_0^1$

$= -\frac{1}{2}$

$\frac{y-1}{1-0} = \frac{x-0}{0-1}$

$y = -x + 1$

b) Evaluate the surface integral $\iint_\sigma \boldsymbol{yz\, ds}$; σ is the part of the plane $\boldsymbol{2x + y + z = 2}$ that lies in the first octant.



$\iint_\sigma y(2 - 2x - y)\sqrt{(-2)^2 + (-1)^2 + 1}\, dA$

$= \iint_\sigma (2y - 2xy - y^2)\sqrt{4+1+1}\, dA$

$z = 2 - 2x - y$

$\frac{\partial z}{\partial x} = -2$

$\frac{\partial z}{\partial y} = -1$

$= \sqrt{6}\int_0^1 \int_0^{-2x+2} 2y - 2xy - y^2\, dy\, dx$

$= \sqrt{6}\int_0^1 \left[y^2 - xy^2 - \frac{1}{3}y^3\right]_0^{-2x+2} dx$

$= \sqrt{6}\int_0^1 (-2x+2)^2 - x(-2x+2)^2 - \frac{1}{3}(-2x+2)^3\, dx$

$= \sqrt{6}\int_0^1 4 - 8x + 4x^2 - x(4 - 8x + 4x^2) - \frac{1}{3}(8 - 24x + 24x^2 - 8x^3)\, dx$

$= \sqrt{6}\int_0^1 4 - 8x + 4x^2 - 4x + 8x^2 - 4x^3 - \frac{1}{3}(8 - 24x + 24x^2 - 8x^3)\, dx$

$= \frac{\sqrt{6}}{3}\int_0^1 3(-4x^3 + 12x^2 - 12x + 4) - (-8x^3 + 24x^2 - 24x + 8)\, dx$

$= \frac{\sqrt{6}}{3}\int_0^1 -12x^3 + 36x^2 - 36x + 12 + 8x^3 - 24x^2 + 24x - 8\, dx$

$= \frac{\sqrt{6}}{3}\int_0^1 -4x^3 + 12x^2 - 12x + 4\, dx$

$= \frac{\sqrt{6}}{3}\left[-x^4 + 4x^3 - 6x^2 + 4x\right]_0^1$

$= \frac{\sqrt{6}}{3} \cdot 1$

3. **a)** Find the flux of the vector field $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + 2z\boldsymbol{k}$ across σ, where σ is the portion of the cone $\boldsymbol{z} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$ between the planes $\boldsymbol{z} = \boldsymbol{2}$ and $\boldsymbol{z} = \boldsymbol{5}$, oriented upward unit normal.

$$\nabla G = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

$$\iint_R \vec{F} \cdot \nabla G \, dA = \iint_R -\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z \; dA$$

$$= \iint_R -\frac{x^2+y^2}{\sqrt{x^2+y^2}} + 2\sqrt{x^2+y^2} \, dA$$

$$= \iint_R \sqrt{x^2+y^2} \, dA \qquad x^2+y^2 = 2^2, \quad x^2+y^2 = 5^2$$

$$= \int_2^5 \int_0^{2\pi} r \cdot r \, d\theta \, dr$$

$$= 2\pi \int_2^5 r^2 \, dr = 78\pi$$

b) Use the Divergence Theorem to find the outward flux of the vector field $\boldsymbol{F}(x,y,z) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z^3\boldsymbol{k}$ across the surface of the region that is enclosed by $\boldsymbol{z} = \boldsymbol{36} - \boldsymbol{x}^2 - \boldsymbol{y}^2$ and the plane $\boldsymbol{z} = \boldsymbol{0}$.

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\int_0^6 \int_0^{2\pi} \int_0^{36-r^2} 3r^2 + 3z^2 \, dz \, r \, d\theta \, dr$$

$$= \int_0^6 \int_0^{2\pi} \left\{ 3r^2(36-r^2) + (36-r^2)^3 \right\} r \, d\theta \, dr$$

$$36^3 - 3 \cdot 36^2 r^2 + 3 \cdot 36 r^4 - r^6$$

$$108r^2 - 3r^4 + 36^3 - 3888r^2 + 108r^4 - r^6$$

$$\left(-r^6 + 105r^4 - 3780r^2 + 36^3\right) r$$

$$= \int_0^6 \int_0^{2\pi} \left(-r^7 + 105r^5 - 3780r^3 + 36^3 r\right) d\theta \, dr$$

$$= 2\pi \left[-\frac{1}{8}r^8 + \frac{105}{6}r^6 - \frac{3780}{4}r^4 + \frac{36^3}{2}r^2 \right]_0^6$$

$$= 2\pi \cdot 221616 = 443232\pi$$

4. **a)** Using triple integral find the volume of the solid bounded by the $\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 = \boldsymbol{4}$ and $\boldsymbol{xy} - \textit{plane}$.

$$\int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \rho^2 \left[-\cos\phi\right]_0^{\pi/2} d\theta \, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \rho^2 \, d\theta \, d\rho$$

$$= 2\pi \int_0^2 \rho^2 \, d\rho$$

$$= 2\pi \left. \frac{1}{3}\rho^3 \right]_0^2$$

$$= \frac{2}{3}\pi \cdot 2^3$$

$$= \frac{16}{3}\pi$$

b) Identify and Sketch the Conic $\boldsymbol{4x^2 - 25y^2 - 16x + 150y - 125 = 0}$. **[6]**

$$4(x^2 - 4x) - 25(y^2 - 6y) = 125$$

$$\Rightarrow 4(x^2 - 2 \cdot x \cdot 2 + 2^2) - 25(y^2 - 2 \cdot y \cdot 3 + 3^2) = 125 + 16 - 225$$

$$\Rightarrow 4(x-2)^2 - 25(y-3)^2 = -84$$

$$\Rightarrow -\frac{(x-2)^2}{21} + \frac{(y-3)^2}{\frac{84}{25}} = 1$$

$$a = \sqrt{21}$$

$$b = \frac{2\sqrt{21}}{5}$$

$$c = 3\frac{\sqrt{61}}{5}$$