1

a) Solve the following system by the elementary row operation.

$$2x - 2y + 4z + 6q = 0$$
$$6x + 2y - 4z + 2q = 4$$
$$-2x + 4y - 2z - 2q = -2$$

[4]

$$\left[\begin{array}{cccc|c} 2 & -2 & 4 & 6 & 0 \\ 6 & 2 & -4 & 2 & 4 \\ -2 & 4 & -2 & -2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & -2 & 4 & 6 & 0 \\ 0 & 8 & -16 & -16 & 4 \\ 0 & 2 & 2 & 4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 0 & 6 & 10 & -2 \\ 0 & 0 & -24 & -32 & 12 \\ 0 & 2 & 2 & 4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -1 \\ 0 & 0 & 6 & 8 & -3 \\ 0 & 1 & 1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -1 \\ 0 & 0 & 1 & 4/3 & -1/2 \\ 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$r_2 = -4r_1 + r_2$
$r_3 = r_1 + r_3$

$r_2 = r_2 - 4r_3$

$r_1 = r_1/2$
$r_2 = r_2/-4$
$r_3 = r_3/2$

$r_2 = r_2/6$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & 2/3 & -1/2 \\ 0 & 0 & 1 & 4/3 & -1/2 \end{array}\right] \leftarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 4/3 & -1/2 \\ 0 & 1 & 0 & 2/3 & -1/2 \end{array}\right] \leftarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 5 & -1 \\ 0 & 0 & 1 & 4/3 & -1/2 \\ 0 & 1 & 0 & 2/3 & -1/2 \end{array}\right]$$

swap r_2 r_3

$r_1 = r_1 - 3r_2$

$r_3 = r_3 - r_2$

$x + q = 1/2$

$y + 2/3\, q = -1/2$

$z + 4/3\, q = -1/2$

b) Solve the homogeneous system of linear equation.

$$3x - 6y + 3z - 3w = 0$$
$$-3y + 2z + w = 0$$
$$6x - 2y + z - w = 0$$

[3]

$$\left[\begin{array}{cccc|c} 3 & -6 & 3 & -3 & 0 \\ 0 & -3 & 2 & 1 & 0 \\ 6 & -2 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -2/3 & -1/3 & 0 \\ 6 & -2 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & -5/3 & 0 \\ 0 & 1 & -2/3 & -1/3 & 0 \\ 0 & 10 & -5 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & -5/3 & 0 \\ 0 & 1 & -2/3 & -1/3 & 0 \\ 0 & 0 & 5/3 & 25/3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & -5/3 & 0 \\ 0 & 1 & -2/3 & -1/3 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$r_1 = r_1/3$
$r_2 = r_2/-3$

$r_1 = r_1 + 2r_2$
$r_3 = -6r_1 + r_3$

$r_3 = -10r_2 + r_3$

$r_3 = r_3 \times \frac{3}{5}$

c) Find the $\boxed{\text{reduced row}}$ echelon form the matrix. [3]

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 0 \end{matrix} \rightarrow \begin{matrix} 2 & 4 & 1 \\ 0 & 1 & 1 \\ 3 & 2 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 1 \\ 3 & 2 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 1 \\ 0 & -4 & -3/2 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 5/2 \end{matrix}$$

swap $(1,2)$ $\quad r_1 = r_1/2 \quad r_3 = -3r_1 + r_3 \quad r_3 = 4r_2 + r_3$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \leftarrow \begin{matrix} 1 & 0 & -3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$$

$r_3 = r_3 \times \frac{2}{5}$

$x = 0$

$y + 3w = 0$

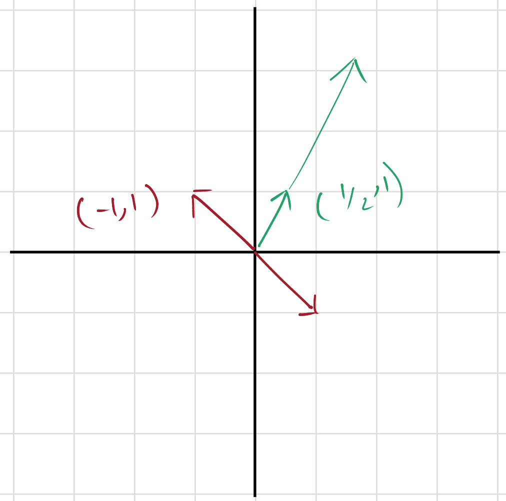
$z + 5w = 0$

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{matrix}$$

$r_1 = r_1 + r_3/3$

$r_2 = r_2 + \frac{2}{3} r_3$

2. a) Find the Eigenvalues and corresponding Eigenvector of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Also draw the Eigen-space in xy –plane. [4+1]



$AX - \lambda X = 0$

$\Rightarrow (A - \lambda I)X = 0$

$\det(A - \lambda I) = 0$

$\Rightarrow \det \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda = -1, 5$

for $\lambda = -1$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x + y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $y = p$

$x = -p$

$X = \begin{bmatrix} -p \\ p \end{bmatrix}$

$= p \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

for $\lambda = 5$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x - y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $y = q$

$x = q/2$

$X = \begin{bmatrix} q/2 \\ q \end{bmatrix}$

$= q \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

b) i) Find the inverse of $A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$. [4]

ii) If $X = A^{-1}A$, what is X? [1]

(i)
$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 4 & 1 & 1 & 0 & 1 \\ 0 & -7 & -1 & 0 & 1 & -2 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 4 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1/7 & 0 & -1/7 & 2/7 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 3/7 & 1 & 4/7 & -1/7 \\ 0 & 1 & 1/7 & 0 & -1/7 & 2/7 \\ 1 & 0 & 4/7 & 0 & 3/7 & 1/7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 7/3 & 4/3 & -1/3 \\ 0 & 1 & 1/7 & 0 & -1/7 & 2/7 \\ 1 & 0 & 4/7 & 0 & 3/7 & 1/7 \end{array}\right]$$

$r_1 = r_1 + r_3$

$r_2 = r_2 - 2r_3$

$r_2 = r_2 / -7$

$r_1 = r_1 - 4r_2$

$r_3 = r_3 - 3r_2$

$r_1 = r_1 \times \frac{7}{3}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 7/3 & 4/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1/3 \\ 1 & 0 & 0 & -4/3 & -1/3 & 1/3 \end{array}\right]$$

$r_2 = -\frac{1}{7} r_1 + r_2$

$r_3 = -\frac{4}{7} r_1 + r_3$

$$\leftarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/3 & -1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 7/3 & 4/3 & -1/3 \end{array}\right]$$

swap (1,3)

(ii)
$$X = A^{-1} \times A = I$$
$$\Rightarrow X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

Consider a Matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix}$

i) Find $p(A)$ for $p(x) = x^2 - 2x + 3$. [3]

ii) Verify that $(A^T)^{-1} = (A^{-1})^T$. [2]

iii) Find x, Such that $\operatorname{tr}(A) = x^2 + 2x$. [2]

iv) Find A^{-3}. [2]

v) Find AB, where $B = \begin{bmatrix} 1 & 5 \\ 1 & 9 \\ 7 & 4 \end{bmatrix}$. [1]

(i)
$$A^2 = \begin{bmatrix} 11 & -10 \\ -4 & 19 \end{bmatrix} \qquad -2A = \begin{bmatrix} -2 & -10 \\ -4 & 6 \end{bmatrix} \qquad 3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 11 & -10 \\ -4 & 19 \end{bmatrix} + \begin{bmatrix} -2 & -10 \\ -4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -20 \\ -8 & 28 \end{bmatrix}$$

(ii)
$$A = \begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 1 & 2 \\ 5 & -3 \end{bmatrix}$$
$$(A^T)^{-1} = \frac{1}{13}\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}$$
$$A^{-1} = \frac{1}{13}\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}$$
$$(A^{-1})^T = \frac{1}{13}\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}$$

equal

(iii) $-2 = x^2 + 2x$

$\Rightarrow x^2 + 2x + 2 = 0$

$\Rightarrow x = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \dfrac{\sqrt{-4}}{2}$

$= -1 \pm i$

(iv) $A^{-3} = (A \times A \times A)^{-1}$

$= \begin{bmatrix} -9 & 85 \\ 34 & -77 \end{bmatrix}^{-1}$

$= \begin{bmatrix} \frac{77}{2197} & \frac{85}{2197} \\ \frac{34}{2197} & \frac{9}{2197} \end{bmatrix}$

(v) A is 2×2

B is 3×2

$A \times B$ not possible.

(4) **a)** Solve $y'' - 8y' + 16y = e^{-4x} + \sin 4x - 2$.

$m^2 - 8m + 16 = 0$

$\Rightarrow m = 4$

$y_c = c_1 e^{4x} + c_2 x e^{4x}$

$y = \dfrac{1}{D^2 - 8D + 16} e^{-4x} = \dfrac{1}{16 + 32 + 16} e^{-4x} = \dfrac{1}{64} e^{-4x}$

$y = \dfrac{1}{D^2 - 8D + 16} \sin 4x = \dfrac{1}{-16 - 8D + 16} \sin 4x = -\dfrac{1}{8} \cdot \dfrac{1}{D} \sin 4x = -\dfrac{1}{8}\left(-\cos 4x \cdot \dfrac{1}{4}\right) = \dfrac{1}{32} \cos 4x$

$y = \dfrac{1}{D^2 - 8D + 16} (-2) = -2 \dfrac{1}{D^2 - 8D + 16} e^0 = -2 \dfrac{1}{16} = -\dfrac{1}{8}$

b) Solve the initial value problem

$$y'' - 6y' + 9y = 0 \; ; \; y(0) = -2 \, , \quad y'(0) = 1 \qquad [4]$$

$m^2 - 6m + 9 = 0$

$\Rightarrow m = 3$

$y = c_1 e^{3x} + c_2 x e^{3x}$

$y' = 3c_1 e^{3x} + 3c_2 x e^{3x} + 3c_2 e^{3x}$

$y(0) = -2$

$\Rightarrow c_1 = -2$

$y'(0) = 1$

$\Rightarrow 3c_1 + 3c_2 = 1$

$\Rightarrow 3c_2 = 1 + 6 = 7$

$\Rightarrow c_2 = 7/3$

$y = -2e^{3x} + \frac{7}{3} c_2 x e^{3x}$