1. **a)** Let $\boldsymbol{F}(x,y) = 2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j}$

i. Show that $\boldsymbol{F}$ is a conservative vector field on the entire xy −plane.

ii. Find the potential function $\emptyset(x,y)$.

iii. Find $\int_{(2,-2)}^{(-1,0)} \boldsymbol{F}.\boldsymbol{dr}$ using (b)

i) $\frac{\partial}{\partial x} 3x^2y^2 = 6xy^2$

$\frac{\partial}{\partial y} 2xy^3 = 6xy^2$

ii) $\vec{F} = \vec{\nabla}\emptyset = \frac{\partial}{\partial x}\emptyset\,\hat{i} + \frac{\partial}{\partial y}\emptyset\,\hat{j}$

$\frac{\partial}{\partial x}\emptyset = 2xy^3 \qquad \frac{\partial}{\partial y}\emptyset = 3x^2y^2$

$\Rightarrow \emptyset = \int 2xy^3\,dx = x^2y^3 + h(y)$

$\Rightarrow \frac{\partial}{\partial y}\emptyset = 3x^2y^2 + h'(y)$

$h'(y) = 0 \Rightarrow h(y) = c$

$\emptyset = x^2y^3 + c$

iii) $\int_{(2,-2)}^{(-1,0)} F.dr = \emptyset(-1,0) - \emptyset(2,-2)$

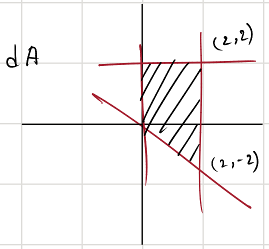
$= 0 - (-32)$

$= 32$

b) Using Green's theorem find the value of $\oint_c \boldsymbol{F}.\boldsymbol{dr}$

Where $\boldsymbol{F}(x,y) = (e^x - y^3)\boldsymbol{i} + (cosy + x^3)\boldsymbol{j}$ and C is the closed curve bounded by the rectangular region with boundary line $x = 0$, $x = 2, y = -x$ and $y = 2$.

$\oint_c F\,dr = \iint_R \frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\,dA$



$= \iint_R 3x^2 - (-3y^2)\,dA$

$= \int_0^2 \int_{-x}^2 3x^2 + 3y^2\,dy\,dx$

$= \int_0^2 3x^2 \cdot y\Big]_{-x}^{2} + y^3\Big]_{-x}^{2}\,dx$

$= \int_0^2 3x^2(2+x) + (2^3 + x^3)\,dx$

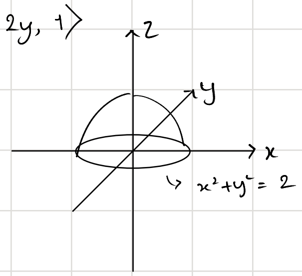
$= \int_0^2 6x^2 + 3x^3 + 8 + x^3\,dx$

$= \int_0^2 4x^3 + 6x^2 + 8\,dx$

$= \left[x^4 + 2x^3 + 8x\right]_0^2 = 48$

3. **a)** Find the flux of the vector field $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ across σ, where σ is the portion of the surface $\boldsymbol{z = 2 - x^2 - y^2}$ that lies above the xy − plane, and suppose that σ is oriented up.

$\nabla G = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle = \langle 2x, 2y, 1\rangle$



$\iint_\sigma \vec{F}\cdot\nabla G\,dA$

$= \iint_\sigma 2x^2 - 2y^2 + z\,dA$

$= \iint_\sigma 2x^2 - 2y^2 + 2 - x^2 - y^2\,dA$

$= \iint_\sigma x^2 - 3y^2 + 2\,dA$

$= \int_0^{\sqrt{2}} \int_0^{2\pi} (r^2\cos^2\theta - 3r^2\sin^2\theta + 2)\,r\,d\theta\,dr$

$= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3\cos^2\theta - 3r^3\sin^2\theta + 2r\,d\theta\,dr$

$= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3\,\frac{1}{2}(1 + \cos 2\theta) - \frac{3}{2}r^3(1 - \cos 2\theta) + 2r\,d\theta\,dr$

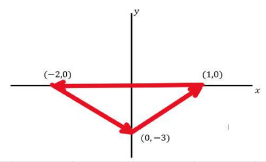
$= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta - \frac{3}{2} + \frac{3}{2}\cos 2\theta\right) + 2r\,d\theta\,dr$

$= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3(2\cos 2\theta - 1) + 2r\,d\theta\,dr$

$= \int_0^{\sqrt{2}} \left[r^3\left(2\cdot\frac{1}{2}\sin 2\theta - \theta\right) + 2r\cdot\theta\right]_0^{2\pi}\,dr$

$= \int_0^{\sqrt{2}} \left[r^3(-2\pi) + 2r\cdot 2\pi\right]\,dr = 2\pi\left[r^2 - \frac{1}{4}r^4\right]_0^{\sqrt{2}} = 2\pi$

2. **a)** Evaluate $\int_c \boldsymbol{xydx - xdy}$ along the curve shown in the figure



$\frac{y-0}{0+3} = \frac{x-1}{1-0}$

$\Rightarrow x = \frac{y}{3} + 1$

$\frac{y-0}{0+3} = \frac{x+2}{-2-0}$

$\Rightarrow x = -\frac{2}{3}y - 2$

$\int_c xy\,dx - x\,dy = \iint_R \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}xy\,dA$

$= \iint_R -1 - x\,dA$

$= -\int_{-3}^{0} \int_{-2/3\,y-2}^{\frac{1}{3}y+1} 1 + x\,dx\,dy$

$= -\int_{-3}^{0} \left(\frac{1}{3}y + 1 + \frac{2}{3}y + 2\right) + \frac{1}{2}x^2\Big]_{-2/3\,y-2}^{1/3\,y+1}\,dy$

$= -\int_{-3}^{0} y + 3 + \frac{1}{2}\left[(y/3 + 1)^2 - (-\frac{2}{3}y - 2)^2\right]dy$

$= -\int_{-3}^{0} y + 3 + \frac{1}{2}\left(\frac{y^2}{9} + \frac{2}{3}y + 1 - \frac{4}{9}y^2 - \frac{8}{3}y - 4\right)dy$

$= -\int_{-3}^{0} y + 3 + \frac{1}{2}\left(-\frac{1}{3}y^2 - 2y - 3\right)dy$

$= -\int_{-3}^{0} \cancel{y} + 3 - \frac{1}{6}y^2 - \cancel{y} - \frac{3}{2}\,dy$

$= -\int_{-3}^{0} -\frac{1}{6}y^2 + \frac{3}{2}\,dy$

$= \frac{1}{18}y^3\Big]_{-3}^{0} - \frac{3}{2}y\Big]_{-3}^{0}$

$= \frac{1}{18}\,27 - \frac{3}{2}(0+3)$

$= \frac{3}{2} - \frac{9}{2} = -3$

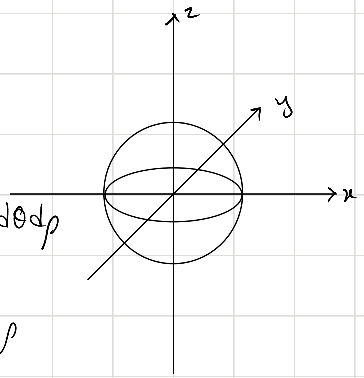
Or

Use the Divergence Theorem to find the outward flux of the vector field $\boldsymbol{F}(x,y,z) = 7x^3\boldsymbol{i} + 7y^3\boldsymbol{j} + 7z^3\boldsymbol{k}$ across the surface of the region that is enclosed by $\boldsymbol{z} = \sqrt{\boldsymbol{49 - x^2 - y^2}}$ and the plane $\boldsymbol{z = 0}$.

$$\operatorname{div} \cdot \vec{F} = 21x^2 + 21y^2 + 21z^2$$

$$21 \iiint_G x^2 + y^2 + z^2 \, dv$$

$$= 21 \int_0^7 \int_0^{2\pi} \int_0^{\pi/2} \rho^2 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$



$$= 21 \int_0^7 \int_0^{2\pi} \rho^4 \Big[-\cos\phi\Big]_0^{\pi/2} d\theta \, d\rho$$

$$= 21 \int_0^7 \int_0^{2\pi} \rho^4 \left(0 - (-1)\right) d\theta \, d\rho$$

$$= 21 \int_0^7 \rho^4 \cdot 2\pi \, d\rho$$

$$= 21 \cdot 2\pi \cdot \frac{1}{5} \rho^5 \Big]_0^7$$

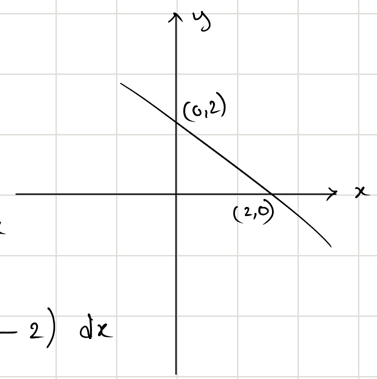
$$= \frac{1}{5} 705894\pi$$

$z = 2 - x - y$

b) Evaluate the surface integral $\iint_\sigma (\boldsymbol{2x - 2y}) \, \boldsymbol{ds}$; σ is the part of the plane $\boldsymbol{x + y + z = 2}$ that lies in the first octant.

$x + y = 2 \Rightarrow y = -x + 2$

$$\iint_\sigma 2x - 2y \sqrt{1^2 + 1^2 + 1} \, dA$$



$$= \int_0^2 \int_0^{-x+2} (2x - 2y) \sqrt{3} \, dy \, dx$$

$$= \int_0^2 2\sqrt{3} \left[x(-x+2) - \frac{1}{2}(-x+2)^2 \right] dx$$

$$= \int_0^2 2\sqrt{3} \left(-x^2 + 2x - \frac{1}{2}x^2 + 2x - 2 \right) dx$$

$$= 2\sqrt{3} \int_0^2 -\frac{3}{2}x^2 + 4x - 2 \, dx$$

$$= \sqrt{3} \int_0^2 -3x^2 + 8x - 4 \, dx$$

$$= \sqrt{3} \left[-x^3 + 4x^2 - 4x \right]_0^2$$

$$= 0$$

4.

a) Use cylindrical coordinate to evaluate $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \boldsymbol{y^2 dzdydx}$.

$$\int_0^3 \int_0^{2\pi} \int_0^{9-r^2} r^2 \sin^2\theta \cdot r \cdot dz \, d\theta \, dr$$

$$= \int_0^3 \int_0^{2\pi} r^3 \sin^2\theta \cdot (9 - r^2) \, d\theta \, dr$$

$$= \int_0^3 (9r^3 - r^5) \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) \, d\theta \, dr$$

$$= \int_0^3 (9r^3 - r^5) \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{2\pi} dr$$

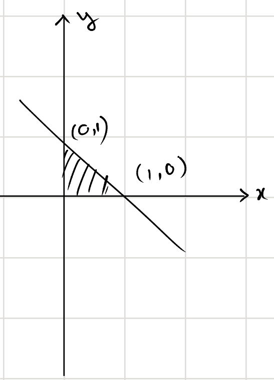
$$= \int_0^3 (9r^3 - r^5) \frac{1}{2} (2\pi - 0) \, dr$$

$$= \pi \left[\frac{9}{4} r^4 - \frac{1}{6} r^6 \right]_0^3$$

$$= \frac{1}{4} 243\pi$$

b) Use a triple integral to find the volume of the solid bounded by the surface $\boldsymbol{z = x^2 sinxy}$ and the enclosed region in $\boldsymbol{xy}$ –plane by the equations $\boldsymbol{x + y = 1, x = 0}$ *and* $\boldsymbol{y = 0}$.

$$\int_0^1 \int_0^{-x+1} \int_0^{x^2 \sin xy} dz \, dy \, dx$$



$$= \int_0^1 \int_0^{-x+1} x^2 \sin xy \, dy \, dx$$

$$= \int_0^1 x^2 \frac{1}{x} \left[-\cos xy \right]_0^{-x+1} dx$$

$$= \int_0^1 x \left[-\cos(-x^2 + x) + 1 \right] dx$$

$=$