Homework Assignment 2

(Submission deadline: April 16th 10:20, 2024, Beijing time, before the class starts. No submission after the deadline will be accepted.)

Notations: $\mathbb{N}$ denotes the set of natural numbers. For a set X, we write $x \in X$ when x is an element of X.

Question 1. Suppose Σ is a set of formulas that is inconsistent. Prove $\Sigma \vdash \varphi$ holds for any formula φ.

Hint: get $\perp \to (\neg\varphi \to \perp)$ and $(\neg\varphi \to \perp) \to \varphi$ from axioms, and then derive φ by Modus Ponens. $(\neg\varphi \to \perp) \to \varphi$ is easy to find, how about $\perp \to (\neg\varphi \to \perp)$? Recall $(\varphi \to \psi)$ means $(\neg\varphi \vee \psi)$.

Question 2. Prove the opposite of Deduction Lemma; that is, if $\Sigma \vdash (\varphi \to \psi)$ then $\Sigma \cup \{\varphi\} \vdash \psi$.

Hint: by assumption, there is a proof $\varphi_1, \ldots, \varphi_n$ of $(\varphi \to \psi)$ from Σ. How can you modify it into a proof of φ from $\Sigma \cup \{\varphi\}$?

Question 3. Suppose a set of formulas Σ_n is given for each $n \in \mathbb{N}$, and suppose $\Sigma_n \subseteq \Sigma_{n+1}$ holds for each n. Further assume Σ_n is consistent for each n. Under these assumptions, prove $\Sigma_\infty = \bigcup_{n\in\mathbb{N}} \Sigma_n$ (i.e., the union of Σ_n for all n) is also consistent.

Hint: prove by contradiction; that is, assume Σ_∞ is inconsistent and derive contradiction. In particular, if Σ_∞ is inconsistent, then you can show Σ_n is inconsistent for some n.

Question 4. Let Σ be a complete set of formulas (p13 of slides, Lecture 5). Prove $(\varphi_1 \vee \varphi_2) \in \Sigma$ if and only if $\varphi_1 \in \Sigma$ or $\varphi_2 \in \Sigma$, in the following steps:

1. Prove, for a given set Σ' of formulas and φ, if $\Sigma' \vdash \varphi$ and $\Sigma' \vdash \neg\varphi$, then Σ' is inconsistent.
2. Prove, for any φ, we have $\Sigma \vdash \varphi$ if and only if $\varphi \in \Sigma$.

3. Prove, for any φ_1, φ_2, we have $(\varphi_1 \vee \varphi_2) \in \Sigma$ if and only if $\varphi_1 \in \Sigma$ or $\varphi_2 \in \Sigma$.

Hint: For 1, find a suitable axiom and derive $\Sigma' \vdash \perp$. For 2, it is easy to show $\varphi \in \Sigma \Rightarrow \Sigma \vdash \varphi$; then prove the opposite implication by contradiction, using 1. For 3, assume $\varphi_1 \in \Sigma$ or $\varphi_2 \in \Sigma$ and show $\Sigma \vdash (\varphi_1 \vee \varphi_2)$, then use 2; this proves the "if" part of 3, and the "only if" part can be proved similarly.