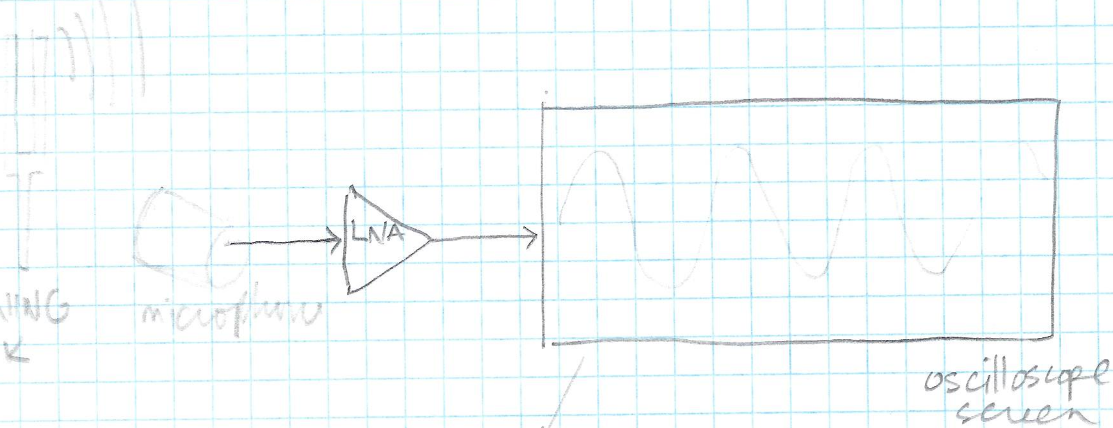
Fourier Transforms

- we distinguish between the time and frequency domain...
- the foundation of all Fourier Transforms is the recognition that many signals can be represented or decomposed into a series of sinusoids or "pure" tones
- What is a pure tone?



we see a sinusoidal variation

- even a tuning fork is not a pure tone, but close!

pure tone at $\Omega = \Omega_0$:

$$\left.\begin{array}{l} \sin(\Omega_0 t + \varphi) \\ \cos(\Omega_0 t + \varphi) \\ e^{j(\Omega_0 t + \varphi)} \end{array}\right\} \text{all qualify}$$

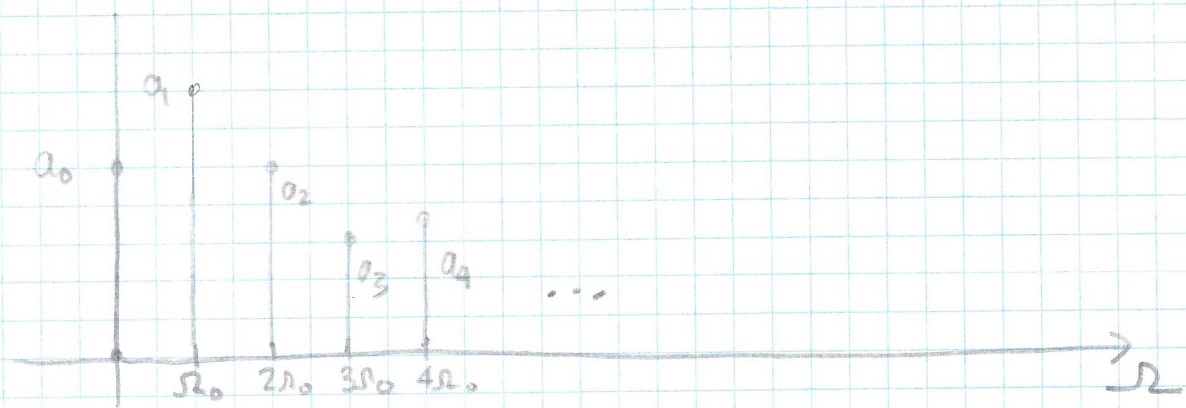
- and, indeed, we can represent signals based on Fourier Series using any of these (one or more)

- in general, given a periodic signal, $r(t)$, we have the expansion

$$r(t) = a_0 + 2\sum_{k=1}^{\infty} b_k \cos(k\Omega_0 t) - 2\sum_{k=1}^{\infty} c_k \sin(k\Omega_0 t)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

→ other similar formulas exist



→ note that the spectrum is <u>discrete</u>

⇒ we associate discrete spectra with <u>periodicity</u> in the time domain

- for aperiodic signals in time, we have the Fourier Transform:

$$R(j\Omega) = \int_{-\infty}^{\infty} r(t)\, e^{-j\Omega t}\, dt$$

$$r(t) = \int_{-\infty}^{\infty} R(j\Omega)\, e^{j\Omega t}\, d\Omega$$

The DTFT (the discrete-time Fourier Transform)

- a mathematical tool for investigating the spectral properties of discrete-time signals:

$$(1) \quad X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

which exists if $x(n)$ is absolutely summable, i.e., if $\sum_{-\infty}^{\infty} |x(n)| < \infty$.

- its inverse, the IDTFT, is given by

$$(2) \quad x(n) \triangleq \mathcal{F}^{-1}\left[X(e^{j\omega})\right] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- note that (1) is CONTINUOUS in the (digital) frequency variable, ω, even though $x(n)$ is DISCRETE in its independent variable, n.

Some Properties

(1) PERIODICITY

$X(e^{j\omega})$ is periodic in ω with period 2π, that is,

$$X(e^{j\omega}) = X\left[e^{j(\omega + 2\pi k)}\right], \quad k \in \mathbb{Z}$$

$\Rightarrow$ we only need one period of $X(e^{j\omega})$ for most purposes (since the rest of the function is redundant)

Typically, we use $\omega \in [0, 2\pi]$ or $\omega \in [-\pi, \pi]$

(2) SYMMETRY

for real-valued $x(n)$, $X(e^{j\omega})$ exhibits conjugate symmetry, i.e.,

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

that is,

$$Re[X(e^{-j\omega})] = Re[X(e^{j\omega})] \quad \text{(even symmetry)}$$
$$Im[X(e^{-j\omega})] = -Im[X(e^{j\omega})] \quad \text{(odd symmetry)}$$

useful in plotting
$$\left\{ \begin{array}{ll} |X(e^{-j\omega})| = |X(e^{j\omega})| & \text{(even symmetry)} \\ \angle X(e^{-j\omega}) = -\angle X(e^{j\omega}) & \text{(odd symmetry)} \end{array} \right.$$

$\Rightarrow$ when plotting, we only need to view half of the period (for real signals), typically, $\omega \in [0, \pi]$.

other properties: linearity, time/frequency shifting, conjugation, convolution, etc., please see your text.

- If $x(n)$ is of finite duration, then we can compute (1) numerically.

- also, we can compute a <u>sampled-version</u> of $X(e^{j\omega})$, at equi-spaced frequencies

  – recall that if $x(n)$ is periodic, then it is equivalent to a sum of discrete sinusoids — the DTFT becomes discrete, i.e., the DTFT and the <u>Discrete Fourier Series</u> representation of the signal become equivalent.

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- and this is, ultimately what we'd like to do → have an exact representation of a discrete-time signal (not merely a sampled version of (1)) that is discrete so that it can be stored in a computer

- and we can accomplish this if we restrict ourselves to finite duration sequences, $x(n)$...

Here are some common DTFT pairs

Signal	Sequence $x(n)$	DTFT $X(e^{j\omega}),\ \omega \in [-\pi, \pi]$
unit impulse	$\delta(n)$	1
constant	1	$2\pi\,\delta(\omega)$
unit step	$u(n)$	$\frac{1}{1+e^{-j\omega}} + \pi\,\delta(\omega)$
complex exponential	$e^{j\omega_0 n}$	$2\pi\,\delta(\omega-\omega_0)$
cosine	$\cos(\omega_0 n)$	$\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$
sine	$\sin(\omega_0 n)$	$j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$

→ note how the sinusoids have DISCRETE Fourier Transforms? This will become important

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The DFS (Discrete Fourier Series)

- a signal $\tilde{x}(n)$, is PERIODIC if there exists a constant $N \neq 0$ such that

$$(3) \qquad \tilde{x}(n) = \tilde{x}(n + kN)$$

for any $n, k \in \mathbb{Z}$.

- FACT: a periodic signal can be synthesized as a linear combination of complex exponentials (or sinusoids) whose frequencies are multiples or harmonics of the fundamental frequency.

In the above definition, assuming N is the smallest integer such that (3) holds, then the fundamental frequency is

$$\omega_0 = \frac{2\pi}{N}$$

and the harmonics are the frequencies

$$\left\{ \omega_0 k = \left(\frac{2\pi}{N}\right)k \ , \quad k = 0, 1, 2, \ldots, N-1 \right\}$$

- Therefore, we can express $\tilde{x}(n)$ as

$$(4) \qquad \tilde{x}(n) = \left(\frac{1}{N}\right) \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\left(\frac{2\pi k}{N}\right)n} \ , \quad n = 0, \pm 1, \ldots$$

where $\tilde{X}(k)$ are called the DFS coefficients, and are given by

$$(5) \qquad \tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\left(\frac{2\pi k}{N}\right)n}$$

and note that $\tilde{X}(k)$ is itself a (complex-valued) periodic sequence with fundamental period equal to N, that is,

$$\tilde{X}(k+N) = \tilde{X}(k)$$

- (4) and (5) are called the DFS Representation for periodic sequences.

- Using

$$(6) \quad W_N \triangleq e^{-j\frac{2\pi}{N}}$$

we can express the pair as

$$(4)' \quad \tilde{X}(k) \triangleq DFS[\tilde{x}(n)] = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{nk}$$

$$(5)' \quad \tilde{x}(n) \triangleq IDFS[\tilde{X}(k)] = \left(\frac{1}{N}\right) \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-nk}$$

→ (4)' is sometimes called the "analysis equation"

→ (5)' is sometimes called the "synthesis equation"

NEXT: periodic extension of finite-duration sequence, $x(n)$

- a nice Matlab function from your textbook

```
function [Xk] = dfs(xn, N)
% computes DFS Coefficients
% Xk = DFS coeff. array over 0<= k <= N-1
% xn = one period of periodic signal over 0<=n<=N-1
% N = fundamental period of xn

n = [0:1:N-1];          % row vector for n
k = [0:1:N-1];          %  "     "     "  k
WN = exp(-j*2*pi/N);    % Wn factor
nk = n'*k;              % creates N x N matrix of nk values
WNnk = WN .^ nk;        % DFS Matrix
Xk = xn * WNnk;         % row vector for DFS coefficients

% end function [Xk] = dfs(xn, N)
```

Note that we can write

$$\underline{\tilde{X}} = W_N \, \underline{\tilde{x}}$$

$$\underline{\tilde{x}} = \left(\frac{1}{N}\right) W_N^{*} \, \underline{\tilde{X}}$$

where the <u>DFS Matrix</u>, W_N, is given by

$$W_N \triangleq \left[W_N^{kn}, \; 0 \le k, n \le N-1 \right] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & \cdots & W_N^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

→

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The DFS gives a means of calculating the DTFT — if $x(n)$ is a finite duration sequence then technically its spectrum is continuous, however, we can compute the DTFT at sample values $\omega = \left(\frac{2\pi}{N}\right)k$, by taking the periodic extension of $x(n)$.

$$\tilde{X}(k) = X(e^{j\omega})\Big|_{\omega = \left(\frac{2\pi}{N}\right)k}$$

DTFT of finite duration signal $x(n)$, $0 \le n \le N-1$

DFS of $\tilde{x}(n)$, the periodic extension of $x(n)$

- assign 5.6 FFT reading to review FFT algorithm