ORIE 3800: Project

Out: April 12 | Instructor: Matthias Poloczek | Due: May 10

- Please submit your final report in the dropbox by **Wednesday noon, May 10 2017**. You do not need to hand in a printout of your code. In addition to handing in a copy of your report, please email your project report, your code, and any other material you want to submit to me (`poloczek@cornell.edu`) and Chamsi (`ch822@cornell.edu`). The subject of your email should be "ORIE 3800 Project." Your email must contain the names and netids of all team members.
- The project report must be **at most 3 pages**, excluding plots and tables. The format should be 11pt font, single spacing, one inch margin on each side. The report should detail your approach/solutions etc. Shorter concise reports are better, as long as they describe your approach effectively. Your report will not be graded based on the number of pages (as long as it obeys the above page limit).
- The late policy is the same as for homeworks.
- You are strongly encouraged to write your code in `Python`. If you want to write your code in `MATLAB`, then this is also allowed.
- You can have at most 3 people per team. Please send an email (one per team) to Chamsi and me by **noon on April 14 2017** with the names and netids of your team members. In your final report, please list the contribution of each member of the team (contributions can be overlapping). Please feel free to use the search function of `Piazza` for teammates or reach out to us before Fridays lecture.
- In addition to handing in your final results on Wednesday May 10, submit your interim results through email to `ch822@cornell.edu` as follows (all deadlines are at noon):
  - **April 14**: Each team emails the list of members in their team.
  - **April 21**: Submit your answers to problem 1 (optimal static strategy).

Project Description

We have a collection of k alternative treatments for a disease. Each treatment x is successful with probability $\theta_x \in [0,1]$, and unsuccessful with probability $1-\theta_x$. Thus, the state of the world is given by $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ and the state space is $\Omega = [0,1]^k$.

The value θ_x itself is unknown; our prior belief is

$$\theta_x \sim \mathrm{Beta}(\alpha_x, \beta_x),$$

where $(\alpha_x, \beta_x : x = 1, \dots, k)$ are given below. Note that our prior on $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ is independent across $x = 1, 2, \dots, k$.

In order to gather more information about the treatments, we are allowed to conduct N experiments: in each experiment we choose one alternative treatment and observe whether it is successful (represented by observing "1") or not (observing "0"). Each such test has a cost of zero, but their total number (the budget) is limited. Observe that, without loss of generality, an optimal strategy will conduct N tests. Let $x_n \in \{1, 2, \ldots, k\}$ be the treatment that we choose to try with our nth opportunity (with $n \in \{1, 2, \ldots, N\}$). Then the observation Y_n satisfies

$$P(Y_n = 1|x_n, \theta) = \theta_{x_n},$$
$$P(Y_n = 0|x_n, \theta) = 1 - \theta_{x_n}.$$

When we are done testing, we will choose a treatment. If we pick treatment $a \in \{1, \ldots, k\}$, we obtain a reward of $r(\theta, a) = \theta_a$. Our goal is to find a treatment of maximum expected reward.

Note that there are two types of decisions:

- We select the N treatments that are tested to gather more information about θ, and
- we pick the final recommendation a in order to maximize the expected reward.

We study the following two approaches of finding an optimal strategy to maximize the expected reward.

1. **Computation of an optimal static strategy.**

   Suppose you are required to choose $x_1, \ldots, x_N$ *before* seeing any of the observations $Y_1, \ldots, Y_N$. Such a sampling plan is called a "static strategy", since it does not take the observations into account.

   (a) Explain briefly why an optimal static strategy can be described in terms of a "plan" $n_1, \ldots, n_k$, where n_x is the number of times you sample alternative x. That is, explain why the ordering according to which the treatments are tested is not relevant for a static strategy. Recall that $\sum_{i=1}^{k} n_i = N$ holds.

   (b) Write code for calculating an optimal $(n_1, \ldots, n_k)$ and the corresponding expected value EV^* achieved by an optimal static strategy given a set of problem parameters. These are N, k, and $\forall i \in \{1, \ldots, k\} : \ (\alpha_i, \beta_i)$.

   (c) Plot EV^* versus N over the range $N = 0, 1, 2, \ldots, 25$ with $k = 5$, $\alpha_1 = \cdots = \alpha_k = 1$, $\beta_1 = \cdots = \beta_k = 1$. In a table, report an optimal $(n_1, \ldots, n_k)$ for each value of N.

   (d) Plot EV^* versus N over the range $N = 0, 1, 2, \ldots, 25$ with $k = 5$, $\alpha_1 = \beta_1 = 10$, $\alpha_2 = \cdots = \alpha_k = 1$, $\beta_2 = \cdots = \beta_k = 1$. In a table, report an optimal $(n_1, \ldots, n_k)$ for each value of N.

2. **Computation of an optimal sequential strategy.**

   Now we consider the case that the treatments are tested one after another and we may observe the outcome of each test before deciding which treatment to test next. That is, when selecting the nth treatment x_n to be tested, we may take the observations $Y_1, Y_2, \ldots, Y_{n-1}$ into account (and the respective choices $x_1, x_2, \ldots, x_{n-1}$).

   Write a code that takes the problem parameters as input and computes the *sequence* of N treatments to be tested under an optimal sequential strategy and the expected value of such a strategy. Note that now the ordering of the tests may be important.

   Let $(\alpha_x, \beta_x) = (1, 1)$ for all $x \in \{1, 2, \ldots, k\}$ and answer the following questions.

(a) Let $N = 2$ and $k = 4$. Give a table with the sequence of treatments to be tested of the optimal sequential strategy.

(b) Plot the expected value of an optimal sequential strategy as function of $N = 1, 2, \ldots, 10$ for each of the following values of k: $k = 2$, $k = 3$, $k = 4$, and $k = 5$.

(c) Plot the expected value of an optimal sequential strategy minus the expected value of an optimal static strategy as function of $N = 1, 2, \ldots, 10$ for each of the following values of k: $k = 2$, $k = 3$, $k = 4$, and $k = 5$.