Course Title :
Population Dynamics [Environmental Sciences]
Environmental Management and Policy III

(Advanced course of)

The Theory in Bio-Demography

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Matrix model

Oct. 10 and 17

Chapter 1

Dynamics of one variable

Description of dynamics

Ta Panta rhei (in Greek) "Everything flows"
by Heraclitus (500 B.C.)

1) Differential equation

$x(t)$: The amount of x at time t

(Time continuous)

$$\frac{dx}{dt} = f(x)$$

2) Difference equation

x_t : The amount of x at time t

(Time discrete)

$$x_{t+1} = f(x_t)$$

A rule of the dynamics

Example of x : Population size, Age, Number of planets, fixed deposit

If we identify the rule, we can forecast the future

Section 1 Rule $f(x)$

$$x_{t+1} = f(x_t)$$

- EX 1 invariant — No. of planets
- EX 2 constant increase — Age
- EX 3 constant multiplication — cell division, fixed deposit
- EX 4 more complicated — Population size or most cases

■ Show the rule by graph

Linear : $x_{t+1} = rx_t + b$

Hor. axis : x_t Ver. axis: x_{t+1}

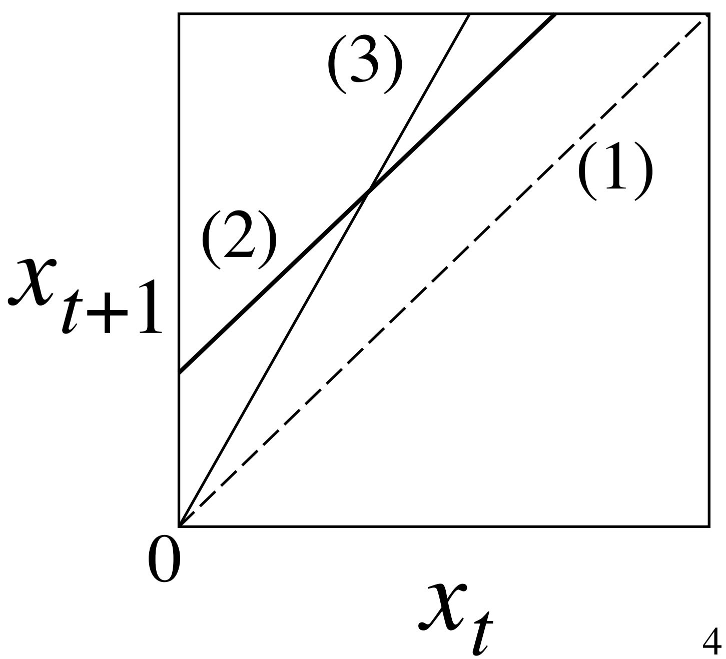
Equation of rules

$f(x_t)$

(1) x_t

(2) $x_t + b$

(3) ax_t



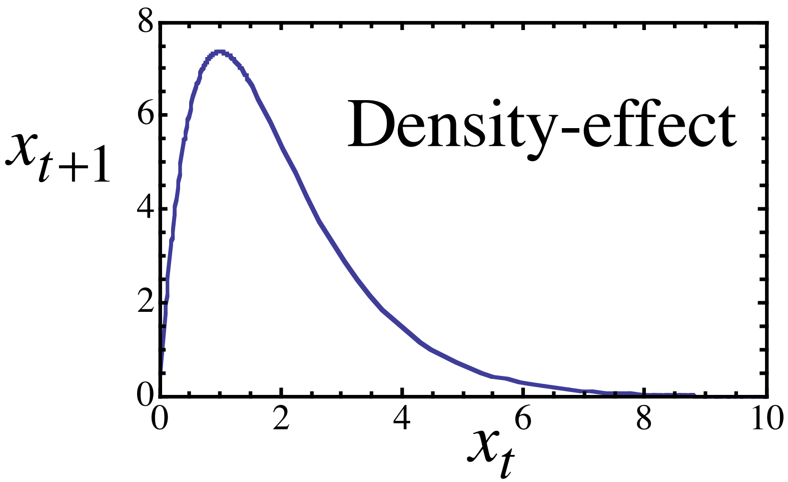
They are all linear functions.

- EX 1 invariant No. of plants
- EX 2 constant increase Age
- EX 3 constant multiplication cell division, fixed deposit
- EX 4 more complicated Population size or most cases

■ Show the rule by graph

An annual plant in sand dune
Draba verna

Rule $f(x_t)$: One-modal function



Complicated function,

Not linear

Section 2 Forecasting

Linear : $x_{t+1} = rx_t + b$

(1) Graph Hor. axis : t Vert. axis : x_t

(2) Solving analytically (Linear: possible, Most cases: impossible)

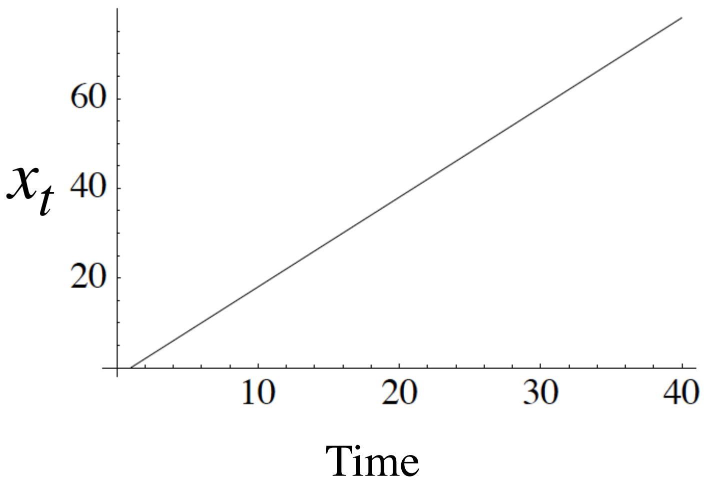
Impossible---> Numerical calculation

- EX 1 invariant $x_t = x_0$ $r = 1, b = 0$ x_0 : At time 0
- EX 2 constant increase $x_t = x_0 + bt$ $r = 1$, b : constant
- EX 3 constant multiplication

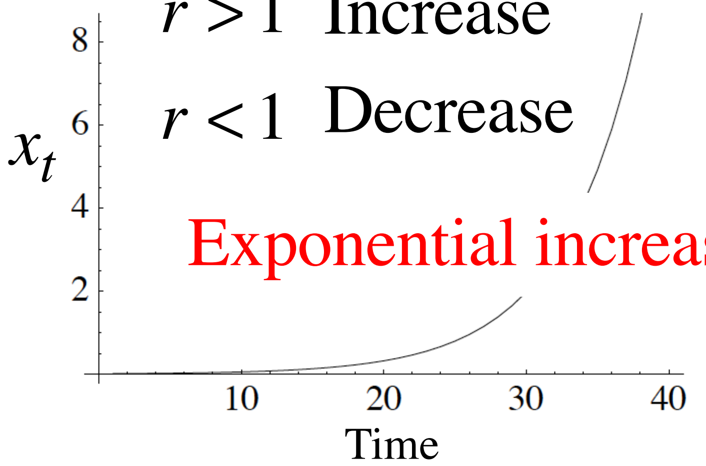
$x_t = x_0 r^t$ $b = 0$, r : magnitude

Linear example : 1 , 2 , 3

EX2



EX3



$r > 1$ Increase

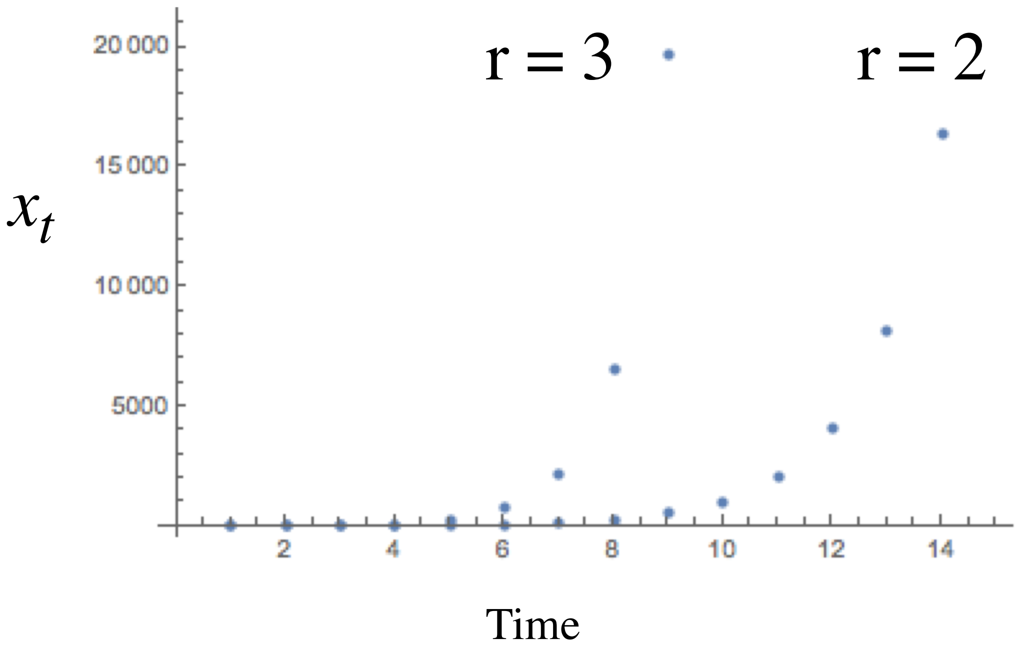
$r < 1$ Decrease

Exponential increase

Please remember：Semi-log graph

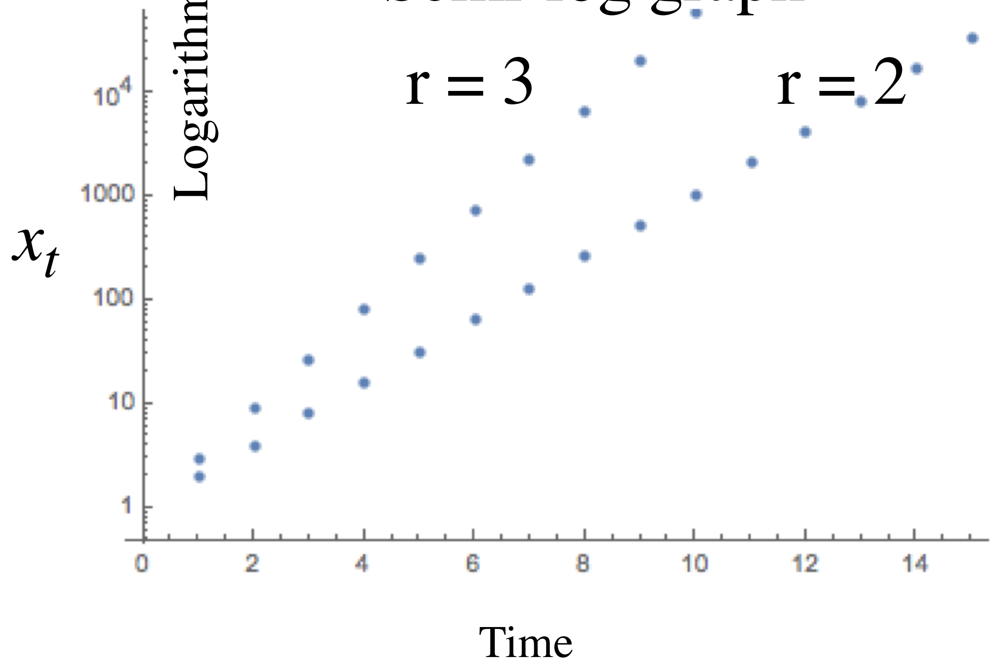
$x_t = x_0 r^t$

geometric series
exponential increase



Too rapid increase

Semi-log graph



Exponential increase --→
Linear increase

Reason: Exponential increase --→ Linear increase

$$x_t = x_0 r^t$$

$$\underline{\ln x_t} = \ln(x_0 r^t) = \ln(x_0) + \ln(r^t)$$

$$= \ln r \times \underline{t} + \ln x_0$$

Y = ln r X+ ln x_0

(Line with slope: ln r and intercept: ln x_0)

Slope ln r is positive when r >1	Linear increase
negative whenr <1	Linear decrease

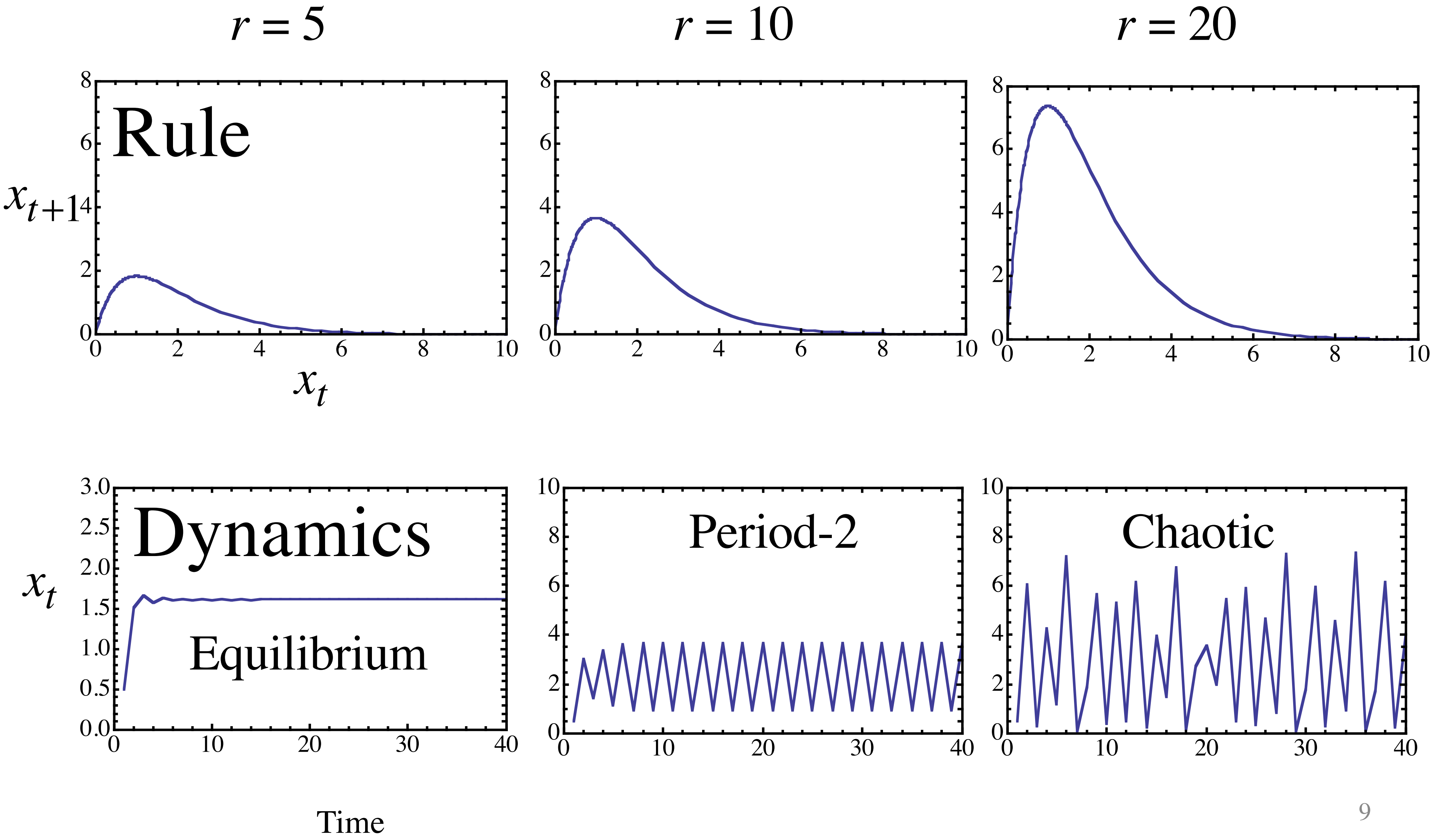
EX4 : Annual plant in sand dune

Rule: $f(x_t) = r\, x_t \exp(-x_t)$

r : fecundity

(Ricker–type density dependence)

What if fecundity increases gradually?



Exercise

Solve the following linear equations:

$$x_{t+1} = f(x_t)$$

x_0: initial value at time zero

- EX 1 invariant $f(x_t) = x_t$
- EX 2 constant increase $f(x_t) = x_t + b$
- EX 3 constant multiplication $f(x_t) = rx_t$
- general linear equation $f(x_t) = rx_t + b$

Note: linear difference equation can be solved analytically.

Summary

✧ The rules of dynamics have many varieties; from simple (linear) to complicated.

✧ We can obtain the solution analytically if simple, but cannot if complicated.

✧ Linear one-variable dynamics is easy to understand whether it increases or not. When $r > 1$, it increases. Otherwise, not.

Caution

However, one-variable model is not useful to treat practical problems we are faced because real world is a multiple–variable system and the variables interact with each other and change simultaneously.

Short tale: A village with child and adult population

Many children and adult individuals in a village. Each child stays at child population with probability 0.7 next year and grows up to adult with probability 0.1. Furthermore, the dying probability of adults is 0.4 and each adult reproduces 1.2 children on average.

Question: Will the village population increase or decrease in future?

Chapter 2

Dynamics of two variables

Section 1 Rule $\boldsymbol{f}(\boldsymbol{x})$: vector

$$\vec{x}_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\vec{x}_{t+1} = \vec{f}(\vec{x}_t) \iff \begin{cases} x_{1,t+1} = f_1(x_{1,t}, x_{2,t}) \\ x_{2,t+1} = f_2(x_{1,t}, x_{2,t}) \end{cases}$$

（1）Two variables are independent (No interaction)

EX 1　Invariant　Diameters of the earth and the moon

E : Unity matrix

Earth
Moon

$$\begin{cases} x_{1,t+1} = x_{1,t} \\ x_{2,t+1} = x_{2,t} \end{cases} \Rightarrow \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Same as one-variable system

EX 2　Constant multiplication　Two fixed deposits with different interest rate

$$\begin{cases} x_{1,t+1} = (1+a_1)x_{1,t} \\ x_{2,t+1} = (1+a_2)x_{2,t} \end{cases} \Rightarrow \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1+a_1 & 0 \\ 0 & 1+a_2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Note: Diagonal matrix

（２）Two variables are dependent（With interaction)

EX３　Population sizes of Tokyo and others（Immigration to Tokyo with rate c）

Tokyo

Others

$$\begin{cases} x_{1,t+1} = x_{1,t} + c x_{2,t} \\ x_{2,t+1} = (1-c) x_{2,t} \end{cases} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1-c \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

EX４　Population size of Tokyo and others

（Immigration to Tokyo with rate c_1, and from Tokyo with rate c_2）

$$\begin{cases} x_{1,t+1} = (1-c_2) x_{1,t} + c_1 x_{2,t} \\ x_{2,t+1} = c_2 x_{1,t} + (1-c_1) x_{2,t} \end{cases} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1-c_2 & c_1 \\ c_2 & 1-c_1 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Not diagonal matrix.

The rule is not so simple in most cases, where we cannot describe the dynamics using matrix. We describe the dynamics using a matrix in simple linear cases.

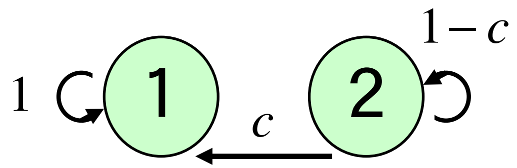
Flow chart of the dynamics.
Let's see the flow of individuals between states！

EX 3　Population sizes of Tokyo and others（Immigration to Tokyo with rate c）

Living in Tokyo

Living in other area

$$\begin{cases} x_{1,t+1} = x_{1,t} + c x_{2,t} \\ x_{2,t+1} = (1-c) x_{2,t} \end{cases} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1-c \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$



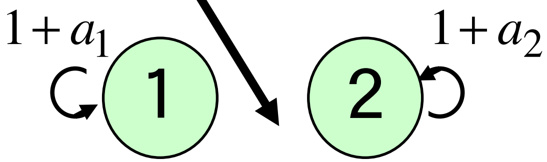
$$\begin{pmatrix} 1 & c \\ 0 & 1-c \end{pmatrix}$$

The parameters attached with arrow from state j to state i are assigned in (i, j) elements in the matrix.

Flow chart and matrix

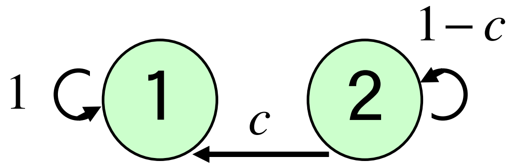
If independent, no flow between states.

EX 2

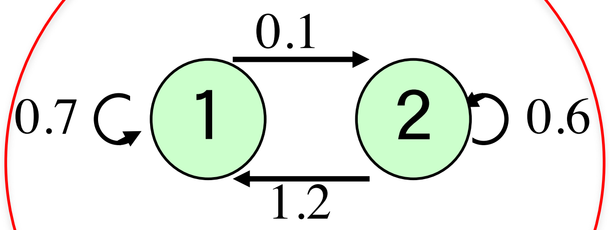


$$\begin{pmatrix} 1+a_1 & 0 \\ 0 & 1+a_2 \end{pmatrix}$$

EX 3



$$\begin{pmatrix} 1 & c \\ 0 & 1-c \end{pmatrix}$$



$$\begin{pmatrix} 0.7 & 1.2 \\ 0.1 & 0.6 \end{pmatrix}$$

Parameters from state j to state i are assigned in (i, j) elements.

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1+a_1 & 0 \\ 0 & 1+a_2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1-c \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 1.2 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Using the matrices, we can construct the linear difference equations as follows:

$$\vec{x}_{t+1} = \mathbf{A}\vec{x}_t$$

Simple examples of $\vec{x}_{t+1} = \vec{f}(\vec{x}_t)$

Ex. 5 A village with child and adult population

Each child stays at child population with probability 0.7 next year and grows up to adult with probability 0.1. Furthermore, the dying probability of adults is 0.4 and each adult reproduces 1.2 children on average.

$$\begin{cases} x_{1,t+1} = 0.7x_{1,t} + 1.2x_{2,t} \\ x_{2,t+1} = 0.1x_{1,t} + 0.6x_{2,t} \end{cases} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 1.2 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Real story : The probability that individuals with age *n* move to age $n+1$ depends on age *n*. The fecundity also depends on the age. ・・・・・・・・・・・・・・・・・・・・・・・・・・・・・

So complicated.

How can we solve the demography?

Section 2 Graphical description of two–variable dynamics

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \vec{x}_{t+1} = \mathbf{A}\vec{x}_t \;\Rightarrow\; \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Vector at time 0 $\begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Exercise

Obtain the vectors at time 1, 2, 3, 4 and 5 by yourself using the following equation!!

$$\vec{x}_{t+1} = \mathbf{A}\vec{x}_t$$

t	0	1	2	3		∞
$x1$	2	5	14	41	⋰	α
$x2$	1	4	13	40	⋰	α

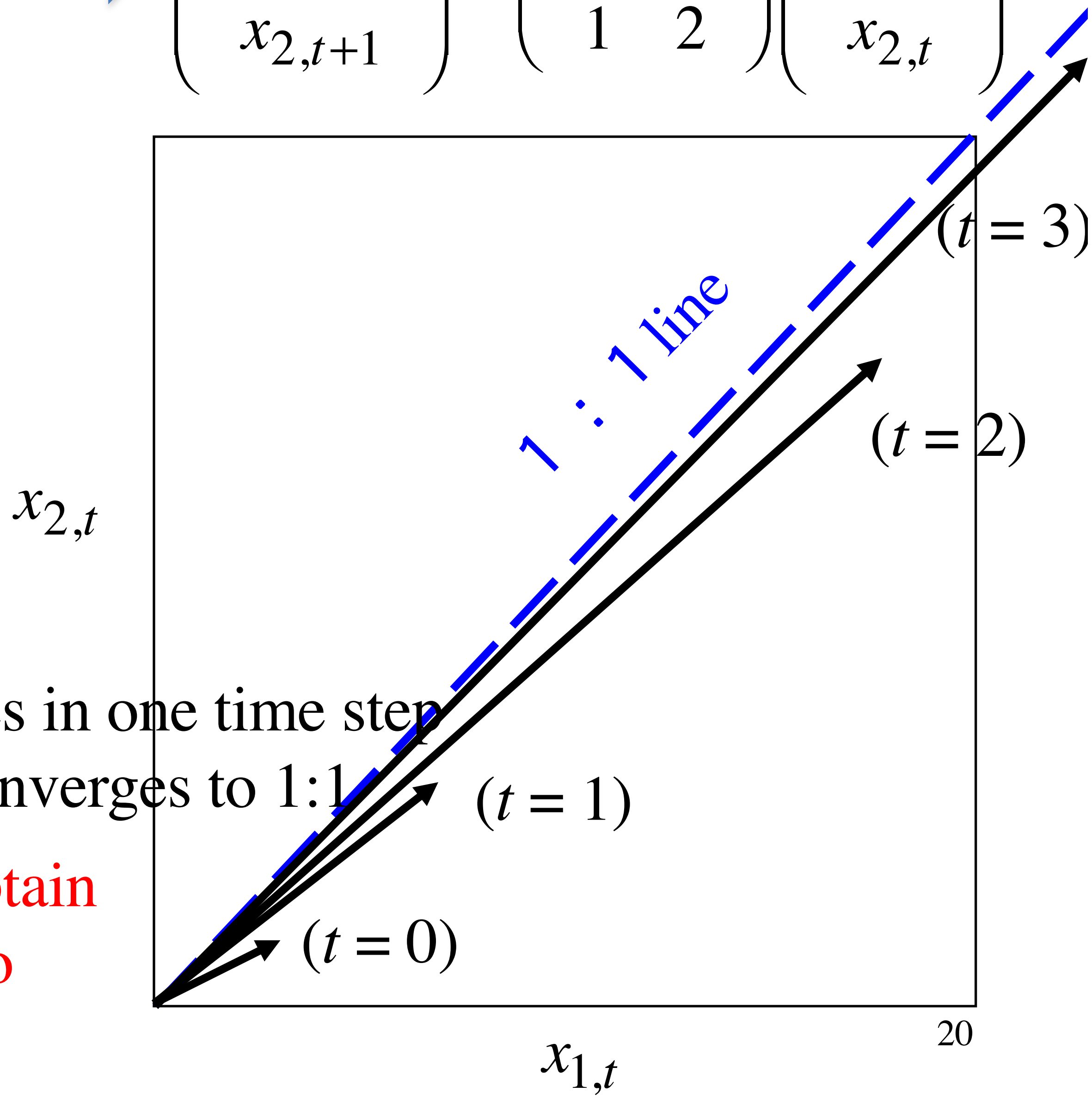
Do you feel any specific trend?

Section 2 Graphical description of two–variable dynamics

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \vec{x}_{t+1} = \mathbf{A}\vec{x}_t \Rightarrow \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

Vector at time 0 $\begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

t	0	1	2	3		∞
$x1$	2	5	14	41	∴	α
$x2$	1	4	13	40	∴	α



Getting bigger by about 3 times in one time step

The ratio between elements converges to 1:1

3 times? 1:1? How can we obtain the magnification and the ratio using math?

How to obtain the magnification and the ratio.

Suppose there is a vector **u**

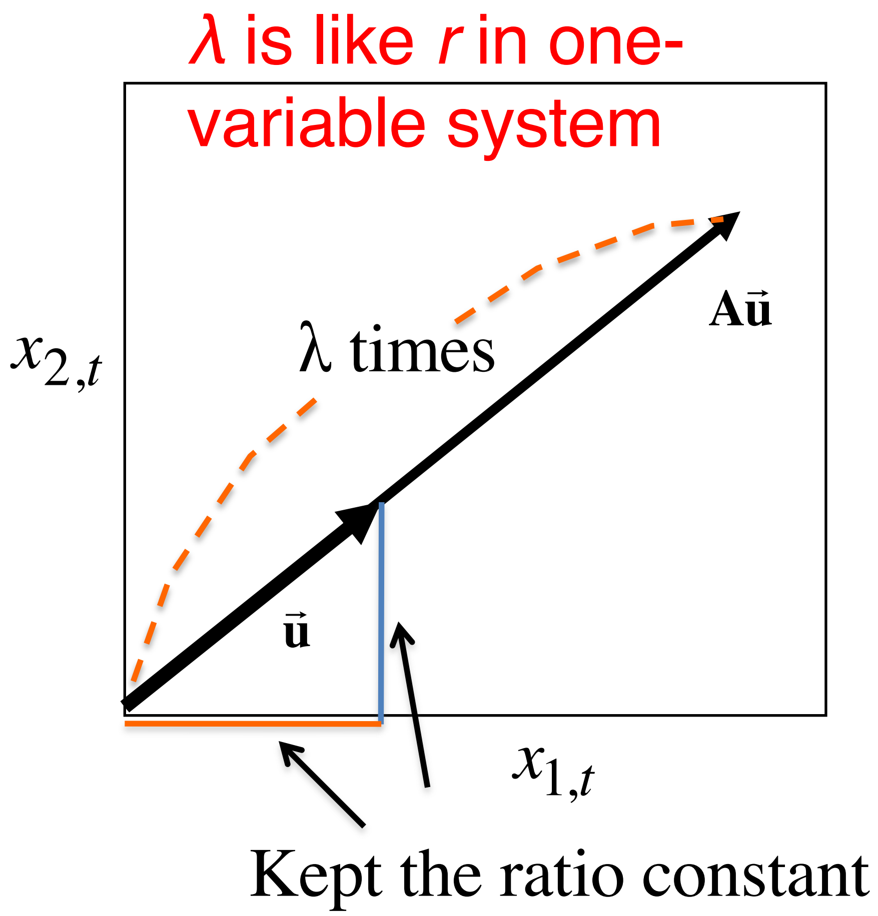
(1) Multiply matrix **A** $\quad \mathbf{A}\vec{\mathbf{u}}$

(2) Multiply the magnification λ

$\lambda\vec{\mathbf{u}}$

The vector **u** exists such that these vector (1) and (2) match?

$$\lambda\vec{\mathbf{u}} = \mathbf{A}\vec{\mathbf{u}} \qquad \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



(To be continued)

$$\lambda\vec{\mathbf{u}} = \mathbf{A}\vec{\mathbf{u}}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow \quad \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix expression

$$(\lambda\mathbf{E} - \mathbf{A})\vec{\mathbf{u}} = \vec{0} \qquad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(To be continued)

$$(\lambda \mathbf{E} - \mathbf{A})\vec{\mathbf{u}} = \vec{0} \quad \Rightarrow \quad \mathbf{P}\vec{\mathbf{u}} = \vec{0}$$

<u>Theorem</u> If $\vec{\mathbf{u}}$ is non-zero, $\mathbf{P}\vec{\mathbf{u}} = \vec{0}$ has solutions if and only if

$$\det \mathbf{P} = 0 \quad \text{(determinant of matrix } \mathbf{P}\text{)}$$

Check the website.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{P} = \begin{pmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{pmatrix}$$

$$\det \begin{pmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{pmatrix} = (\lambda - 2)(\lambda - 2) - (-1)(-1) = 0$$

Eigenvalue equation

(To be continued)

$$\det\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-2)(\lambda-2)-(-1)(-1)=0$$

$$\Rightarrow \quad \lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)=0 \quad \Rightarrow \quad \lambda_1=3,\ \lambda_2=1$$

---

(1) When $\lambda_1=3$ $\quad \mathbf{P}=\begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix}$

$$\begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} u_1-u_2=0 \\ -u_1+u_2=0 \end{matrix} \quad \Rightarrow \quad \frac{u_1}{u_2}=\frac{1}{1}$$

Two eqs. are not independent

$$\vec{\mathbf{u}}_1=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

(2) When $\lambda_2=1$ $\quad \mathbf{P}=\begin{pmatrix} 1-2 & -1 \\ -1 & 1-2 \end{pmatrix}$ $\quad \vec{\mathbf{u}}_2=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} \beta \\ -\beta \end{pmatrix}$

$(Fine)$

Summary

Question: The results from multiplication (1) with matrix **A** and (2) with a scalar λ. *Does the vector* **u** exist such that these two results match?

$$\lambda\vec{\mathbf{u}} = \mathbf{A}\vec{\mathbf{u}} \qquad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The reason why the magnification is 3.

The reason why the ratio is equal to 1:1

$$\lambda_1 = 3 \quad \vec{\mathbf{u}}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

$$\lambda_2 = 1 \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \beta \\ -\beta \end{pmatrix}$$

They are called "eigenvalue" and corresponding "(right) eigenvector".

Factor arbitraliness $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

Normally used in matrix population model $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$, The sum equals one.

Exercise

Obtain the eigenvalues (λ) and the corresponding right eigenvectors ($\mathbf{u}$) of the following matrices.

(See p.24)

(a) $\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0.7 & 1.2 \\ 0.1 & 0.6 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 \\ 4 & -5 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 3 \\ 0 & 3 & 1 \end{pmatrix}$

Village example

Answer

(a) $\lambda = 7, -2$

$$\vec{\mathbf{u}} = \begin{pmatrix} 4\alpha \\ 5\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ -\beta \end{pmatrix}$$

(b) $\lambda = 1, \ 0.3$

$$\vec{\mathbf{u}} = \begin{pmatrix} 0.97\alpha \\ 0.24\alpha \end{pmatrix}, \begin{pmatrix} 0.95\beta \\ -0.32\beta \end{pmatrix}$$

(c) $\lambda = -1, -3$

$$\vec{\mathbf{u}} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \beta \\ 2\beta \end{pmatrix}$$

(d) $\lambda = 6, \ 1, \ -1$

$$\vec{\mathbf{u}} = \begin{pmatrix} \alpha \\ 5\alpha \\ 3\alpha \end{pmatrix}, \begin{pmatrix} -3\beta \\ 0 \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ -2\gamma \\ 3\gamma \end{pmatrix}$$

Section 3 General description of multi–variable dynamics

Matrix A is like r in one-variable system $x_t = x_0 r^t$

$$\vec{x}_{t+1} = \mathbf{A}\vec{x}_t \quad \Rightarrow \quad \text{Solution} \quad \vec{x}_t = \mathbf{A}^t \vec{x}_0$$

How to obtain the solution?

Population matrix of red sea-turtle

7-variable system

	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 7
Stage 1	0	0	0	0	127	4	80
Stage 2	0.675	0.737	0	0	0	0	0
Stage 3	0	0.049	0.661	0	0	0	0
Stage 4	0	0	0.015	0.691	0	0	0
Stage 5	0	0	0	0.052	0	0	0
Stage 6	0	0	0	0	0.809	0	0
Stage 7	0	0	0	0	0	0.809	0.808

What Nishimura-sensei showed

Again

$$\mathbf{x}(t) = \underbrace{\mathbf{A}^t}\ \mathbf{x}(0)$$

$$= \overbrace{\mathbf{U}\boldsymbol{\Lambda}^t\mathbf{U}^{-1}}\,\mathbf{x}(0) = \underbrace{\mathbf{U}\boldsymbol{\Lambda}^t}\ \underbrace{\mathbf{U}^{-1}\mathbf{x}(0)}$$

$$\overbrace{\mathbf{U}\boldsymbol{\Lambda}^t} = \underbrace{\begin{pmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_m \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m^t \end{pmatrix}}$$

$$\overbrace{\mathbf{U}^{-1}\mathbf{x}(0)} = \underset{m\times m}{\mathbf{U}^{-1}}\ \underset{m\times 1}{\mathbf{x}(0)} = \underbrace{\underset{m\times 1}{\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix}}}$$

$$= \overbrace{\begin{pmatrix}\lambda_1^t\mathbf{u}_1 & \lambda_2^t\mathbf{u}_2 & \cdots & \lambda_m^t\mathbf{u}_m\end{pmatrix}}\ \overbrace{\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix}} = g_1\lambda_1^t\mathbf{u}_1 + g_2\lambda_2^t\mathbf{u}_2 + \cdots + g_m\lambda_m^t\mathbf{u}_m$$

General formula of n by n matrix

Again

$*$ If there exists n linearly-independent right eigenvectors, $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \cdots$, of matrix **A**, then the solution of difference equation is

$$(*) \qquad \vec{\boldsymbol{x}}_t = \sum_{i=1}^{n} g_i (\lambda_i)^t \vec{\boldsymbol{u}}_i$$

where g_i satisfies $\vec{\boldsymbol{x}}_0 = \sum_{i=1}^{n} g_i \vec{\boldsymbol{u}}_i$

The key to solve: the eigenvalues and the eigenvectors.

λ_i : eigenvalues that satisfies $\lambda_i \vec{\mathbf{u}}_i = \mathbf{A} \vec{\mathbf{u}}_i$ $\quad$ $\mathbf{E}$: Identity matrix

λ_i can be obtained by $\det(\lambda_i \mathbf{E} - \mathbf{A}) = 0$

Short tale: A village with child and adult population

Each child stays at child population with probability 0.7 next year and grows up to adult with probability 0.1. Furthermore, the dying probability of adults is 0.4 and each adult reproduces 1.2 children on average.

$$\begin{cases} x_{1,t+1} = 0.7x_{1,t} + 1.2x_{2,t} \\ x_{2,t+1} = 0.1x_{1,t} + 0.6x_{2,t} \end{cases} \qquad \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 1.2 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

$$(\lambda \mathbf{E} - \mathbf{A})\vec{\mathbf{u}} = \vec{0} \quad \Rightarrow \quad \det \begin{pmatrix} \lambda - 0.7 & -1.2 \\ -0.1 & \lambda - 0.6 \end{pmatrix} = 0$$

$$\lambda = 1,\ \ 0.3$$

$$\vec{\mathbf{u}} = \begin{pmatrix} 0.97\alpha \\ 0.24\alpha \end{pmatrix}, \begin{pmatrix} 0.95\beta \\ -0.32\beta \end{pmatrix}$$

(to be continued)

Solution formula $\quad \vec{\boldsymbol{x}}_t = \sum_{i=1}^{n} g_i(\lambda_i)^t \vec{\boldsymbol{u}}_i$

$$\vec{\boldsymbol{x}}_t = \sum_{i=1}^{2} g_i(\lambda_i)^t \vec{\boldsymbol{u}}_i = g_1(\lambda_1)^t \vec{\boldsymbol{u}}_1 + o_2(\lambda_2)^t \vec{\boldsymbol{u}}_2$$

$$= \sum_{i=1}^{2} g_i(\lambda_i)^t \vec{\boldsymbol{u}}_i = g_1(1)^t \begin{pmatrix} 0.97 \\ 0.24 \end{pmatrix} + g_2(0.3)^t \begin{pmatrix} 0.95 \\ -0.32 \end{pmatrix}$$

$$\approx (nearly\ equal) g_1(1)^t \begin{pmatrix} 0.97 \\ 0.24 \end{pmatrix}$$

Constant! $\quad = \begin{pmatrix} 0.97 g_1 \\ 0.24 g_1 \end{pmatrix}$

Time *t* disappears from the equation.
Does not depend on time *t*.

The village population neither increases nor decreases.

Summary

✧ In linear two-variable dynamics, we can understand the rule of the dynamics is understandable intuitively using the flow chart. The solution of the system is analytical.

✧The keys to solve linear systems are the eigenvalues and the eigenvectors.

✧The village population size in my short tale neither increases nor decreases as the time elapsed.

✧The multiple-variable linear dynamics can be applied to more general subjects (not limited in population dynamics).

Today's quiz as Report 1 (2020. 10. 10)

Obtain the solution of $\mathbf{x}(t)$ in the following difference equation.

$$\vec{x}_{t+1} = \mathbf{A}\vec{x}_t \qquad \text{Initial vector: } \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(a) $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Hint: obtain $\lambda_i, \vec{\boldsymbol{u}}_i$ first and obtain g_i so that the initial vector is satisfied when $t = 0$.

- ✓ **The due date and time : Oct. 30**
- ✓ **Filename: Your name and student number**
- ✓ **Send to: takada@ees.hokudai.ac.jp**

ベクトル・行列公式集

公式１：横ベクトルと縦ベクトルの積

$$\mathbf{a}\ \mathbf{b} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$$

結果はスカラー

ベクトルa, bの内積

公式２：行列と縦ベクトルの積

公式１より

$$\begin{array}{c} \mathbf{a}_1 \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \end{array} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \mathbf{b} = \begin{pmatrix} \mathbf{a}_1 \mathbf{b} \\ \mathbf{a}_2 \mathbf{b} \\ \vdots \\ \mathbf{a}_n \mathbf{b} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} a_{1k} b_k \\ \sum_{k=1}^{n} a_{2k} b_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk} b_k \end{pmatrix}$$

結果は縦ベクトル

ベクトル・行列公式集(続き)

公式３：横ベクトルと行列の積

結果はベクトル

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{pmatrix} = \mathbf{a}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}\mathbf{b}_1 & \mathbf{a}\mathbf{b}_2 & \cdots \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} a_k b_{k1} & \sum_{k=1}^{n} a_k b_{k2} & \cdots \end{pmatrix}$$

$\mathbf{b}_1$

公式２、３から行列同士の積の公式もできる

公式４：行列と対角行列の積

公式１より

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{pmatrix}$$

v1などが縦ベクトルに変わっても

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ v_{n1} & \cdots & \cdots & v_{nn} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{pmatrix}$$

結果は行列

解を知るには準備が必要：行列の対角化（ちょっと数学）

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

固有値、$\lambda_1, \lambda_2, \cdots, \lambda_n$ ，

固有ベクトル、$\mathbf{u}_1 = \begin{pmatrix} u_{11} \\ \vdots \\ u_{1n} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} u_{21} \\ \vdots \\ u_{1n} \end{pmatrix}, \cdots, \mathbf{u}_n = \begin{pmatrix} u_{n1} \\ \vdots \\ u_{nn} \end{pmatrix}$

固有値の式

$$\mathbf{A} \cdot \mathbf{u}_1 = \lambda_1 \mathbf{u}_1$$
$$\mathbf{A} \cdot \mathbf{u}_2 = \lambda_2 \mathbf{u}_2$$
$$\vdots$$
$$\mathbf{A} \cdot \mathbf{u}_n = \lambda_n \mathbf{u}_n$$

縦ベクトル＝縦ベクトル

n本を横に
並べると

$$\left(\mathbf{A} \cdot \mathbf{u}_1, \mathbf{A} \cdot \mathbf{u}_2, \cdots, \mathbf{A} \cdot \mathbf{u}_n\right) = \left(\lambda_1 \mathbf{u}_1, \lambda_2 \mathbf{u}_2, \cdots, \lambda_n \mathbf{u}_n\right)$$

n 行 n 列　＝　n 行 n 列

行列の対角化

$$\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n) = \begin{pmatrix} u_{11} & \cdots & u_{n1} \\ \vdots & \ddots & \vdots \\ u_{1n} & \cdots & u_{nn} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\mathbf{A}\cdot\mathbf{u}_1 = \lambda_1\mathbf{u}_1$$

$$\mathbf{A}\cdot\mathbf{u}_2 = \lambda_2\mathbf{u}_2$$

$$\vdots$$

$$\mathbf{A}\cdot\mathbf{u}_n = \lambda_n\mathbf{u}_n$$

$$\Rightarrow \underbrace{\left(\mathbf{A}\cdot\mathbf{u}_1, \mathbf{A}\cdot\mathbf{u}_2, \cdots, \mathbf{A}\cdot\mathbf{u}_n\right)}_{} = \underbrace{\left(\lambda_1\mathbf{u}_1, \lambda_2\mathbf{u}_2, \cdots, \lambda_n\mathbf{u}_n\right)}_{}$$

n 行 n 列　＝　n 行 n 列　　公式 4 を使う

$$\mathbf{A}\cdot\left(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\right) = \left(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\right)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

n 行 n 列　＝　n 行 n 列

全部をまとめて
行列を使うと　　$\mathbf{A}\cdot\mathbf{U} = \mathbf{U}\cdot\Lambda$
簡単になりました

行列の対角化

$$\mathbf{A}\cdot\mathbf{U}=\mathbf{U}\cdot\boldsymbol{\Lambda}$$

右からUの逆行列をかけると

$$\mathbf{A}=\mathbf{U}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1}$$

$$\mathbf{A}^t=(\mathbf{U}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1})^t$$

行列Aの　t　乗を求めてみよう

$$=\overbrace{(\mathbf{U}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1})(\mathbf{U}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1})\cdots(\mathbf{U}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1})}^{t}$$

$$=\mathbf{U}\cdot\overbrace{(\boldsymbol{\Lambda}\cdot\underbrace{\mathbf{U}^{-1}\cdot\mathbf{U}}_{\mathbf{E}})(\boldsymbol{\Lambda}\cdot\underbrace{\mathbf{U}^{-1}\cdot\mathbf{U}}_{\mathbf{E}})\cdots(\boldsymbol{\Lambda}\cdot\underbrace{\mathbf{U}^{-1}\cdot\mathbf{U}}_{\mathbf{E}})}^{t-1}\cdot\boldsymbol{\Lambda}\cdot\mathbf{U}^{-1}$$

$$=\mathbf{U}\cdot\boldsymbol{\Lambda}^t\cdot\mathbf{U}^{-1}$$

$$=\mathbf{U}\cdot\begin{pmatrix}\boldsymbol{\lambda}_1^t & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \boldsymbol{\lambda}_n^t\end{pmatrix}\cdot\mathbf{U}^{-1}$$

$$\mathbf{E}=\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}$$

$$\mathbf{U}^{-1}\cdot\mathbf{U}=\mathbf{U}\cdot\mathbf{U}^{-1}=\mathbf{E}$$

連立差分方程式：行列モデル

$$\mathbf{x}(t) = \mathbf{A}^t \cdot \mathbf{x}(0)$$

$$\mathbf{x}(t) = \overbrace{\mathbf{U} \cdot \Lambda^t \cdot \mathbf{U}^{-1}}^{\mathbf{A}^t} \cdot \mathbf{x}(0) = \boxed{\mathbf{U} \cdot \Lambda^t} \cdot \mathbf{U}^{-1} \cdot \mathbf{x}(0)$$

$$\mathbf{U} \cdot \Lambda^t = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\lambda}_n^t \end{pmatrix}$$

公式４を使う

$$= \left(\boldsymbol{\lambda}_1^t \mathbf{u}_1,\ \boldsymbol{\lambda}_2^t \mathbf{u}_2,\ \cdots\ ,\ \boldsymbol{\lambda}_n^t \mathbf{u}_n \right)$$

連立差分方程式：行列モデル

$$\mathbf{x}(t) = \mathbf{A}^t \cdot \mathbf{x}(0)$$

$$= \mathbf{U} \cdot \Lambda^t \cdot \mathbf{U}^{-1} \cdot \mathbf{x}(0) = \boxed{\mathbf{U} \cdot \Lambda^t} \cdot \boxed{\mathbf{U}^{-1} \cdot \mathbf{x}(0)}$$

$$\mathbf{U} \cdot \Lambda^t = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\lambda}_n^t \end{pmatrix},$$

$$= \left(\boldsymbol{\lambda}_1^t \mathbf{u}_1,\ \boldsymbol{\lambda}_2^t \mathbf{u}_2,\ \cdots\ ,\ \boldsymbol{\lambda}_n^t \mathbf{u}_n \right)$$

$$\mathbf{U}^{-1} \cdot \mathbf{x}(0) = \left(\quad \right) \left(\quad \right) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

公式１を複数回使う

$$\mathbf{x}(t) = c_1 \boldsymbol{\lambda}_1^t \mathbf{u}_1 + c_2 \boldsymbol{\lambda}_2^t \mathbf{u}_2 + \cdots + c_n \boldsymbol{\lambda}_n^t \mathbf{u}_n$$

$$= \sum_{i=1}^{n} c_i (\boldsymbol{\lambda}_i)^t \vec{\mathbf{u}}_i$$

$$\mathbf{x}(0) = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n$$