Differential Forms Notes

Remy Goldschmidt

November 3, 2014

Chapter 1

Introduction

1.1 1-forms

- A 0-form is just a scalar function $P : \mathbb{R}^n \to \mathbb{R}$
- A 1-form is a linear combination of functions $(P_1, P_2, \dots P_n : \mathbb{R}^n \to \mathbb{R})$ and differential elements $(\mathrm{d}x_1, \mathrm{d}x_2, \dots \mathrm{d}x_n)$
- These are familiar from calculus: they are seen in, e.g.: $\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$
- For some vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ we define a 1-form $\widetilde{F} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$
- This generalizes in the obvious way to $\mathbb{R}^n$
- In $\mathbb{R}^n$ there are n basis 1-forms, i.e.: $\mathrm{d}x_1, \mathrm{d}x_2, \dots \mathrm{d}x_n$
- The degree of a form α is denoted by $|\alpha|$
- For example: $|\mathrm{d}x_1| = 1$, $|P| = 0$

1.2 Exterior products

- The exterior (or wedge) product of two forms α and β is denoted $\alpha \wedge \beta$
- $(\wedge) : \mathrm{Form}^p \to \mathrm{Form}^q \to \mathrm{Form}^{p+q}$
- The wedge product of two basis forms is a new basis form
- For example: $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ is a basis 2-form
- The wedge product is anticommutative on 1-forms: $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\,\mathrm{d}x_2 \wedge \mathrm{d}x_1$
- This generalizes to p and q-forms: $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}(\beta \wedge \alpha)$
- The wedge product is linear and associative. The wedge product on 0-forms is just scalar multiplication.
- $\mathrm{d}V = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$ in $\mathbb{R}^n$ (this is the volume form)
- The product of a 0-form and the volume form is known as a pseudoscalar, because it behaves like a scalar except its sign will flip under parity inversions

1.3 Interior products

- The interior product of a vector $\vec{r} : \mathbb{R}^n$ and a p-form α is denoted by $\iota_{\vec{r}}\,\alpha$, a $(p-1)$-form
- For a basis vector $\vec{e}_i$ and a basis form $\mathrm{d}x_j$, $\iota_{\vec{e}_i}\,\mathrm{d}x_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- The interior product is linear in both arguments:
  $a, b : \mathbb{R};\ \vec{r}, \vec{m}, \vec{n} : \mathbb{R}^n;\ \alpha, \beta : \mathrm{Form}^p$
  $\vec{s} = a(\vec{m} + \vec{n});\ \gamma = b(\alpha + \beta)$
  $\iota_{\vec{s}}\alpha = a(\iota_{\vec{m}}\alpha + \iota_{\vec{n}}\alpha)$
  $\iota_{\vec{r}}\gamma = b(\iota_{\vec{r}}\alpha + \iota_{\vec{r}}\beta)$
- The interior product is antisymmetric: $\iota_{\vec{m}}\iota_{\vec{n}}\alpha = -\iota_{\vec{n}}\iota_{\vec{m}}\alpha$
- The interior product is nilpotent: $\iota_{\vec{m}}\iota_{\vec{m}}\alpha = 0$
- The interior product over wedge products: $\iota_{\vec{m}}(\alpha \wedge \beta) = (\iota_{\vec{m}}\alpha) \wedge \beta + (-1)^{|\alpha|}(\alpha \wedge (\iota_{\vec{m}}\beta))$

1.4 Dot products

- The dot product of a p-form and a q-form is zero if $p \neq q$, unless $p = 0$ or $q = 0$, in which case it is scalar multiplication
- The dot product of α and β is denoted $\alpha \cdot \beta$
- If $\alpha = \sum_{i=1}^{p} P_i \,\mathrm{d}x_i$ and $\beta = \sum_{j=1}^{p} Q_j \,\mathrm{d}x_j$ then $\alpha \cdot \beta = \sum_{k=1}^{p} P_k Q_k$
- The dot product is commutative, linear, and associative.

1.5 Hodge dual

- The Hodge dual of a p-form in $\mathbb{R}^n$ is a $(n-p)$-form, and is denoted by $\star\alpha$
- The Hodge dual relates to the dot product by: $\star(\alpha \wedge (\star\beta)) = \alpha \cdot \beta$
- The Hodge dual of a scalar is a pseudoscalar: $\star f = f \,\mathrm{d}V$
- The Hodge dual of a pseudoscalar is a scalar: $\star(f \,\mathrm{d}V) = f$
- The Hodge dual is idempotent: $\star\star\,\alpha = \alpha$
- The Hodge dual is distributive: $\star(\alpha + \beta) = (\star\alpha) + (\star\beta)$

1.6 Exterior derivative

- So far we have dealt with d operator applied to functions of the form $f = x_k \to \mathrm{d}f = \mathrm{d}x_k$
- The exterior derivative is linear: $\mathrm{d}(af + bg) = a(\mathrm{d}f) + b(\mathrm{d}g)$
- The exterior derivative is nilpotent: $\mathrm{d}(\mathrm{d}\alpha) = 0$
- The exterior derivative over wedge products: $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|}(\alpha \wedge \mathrm{d}\beta)$
- A form α is closed if $\mathrm{d}\alpha = 0$.
- A closed form α is exact if there exists some β such that $\alpha = \mathrm{d}\beta$
- Useful properties: $\mathrm{d}f = \widetilde{\nabla f}$ (f is a 0-form)

Chapter 2

Problems

2.1 Electrodynamics

- The Minkowski metric has a signature of $(-, +, +, +)$
- Thus, we define our basis forms: $\mathrm{d}t = -\,\mathrm{d}x_1$, $\mathrm{d}x = \mathrm{d}x_2$, $\mathrm{d}y = \mathrm{d}x_3$, $\mathrm{d}z = \mathrm{d}x_4$
- Also, terms with $\mathrm{d}t$ in the dot product of two forms are now negative
- For example, $\widetilde{P} \cdot \widetilde{Q} = -P_0Q_0 + P_1Q_1 + P_2Q_2 + P_3Q_3$, where $\mathbf{P}$ and $\mathbf{Q}$ are vector fields
- Additionally, for some vector field $\mathbf{V}$, it is now the case that $\widetilde{V} = V_0\,\mathrm{d}t + V_1\,\mathrm{d}x + V_2\,\mathrm{d}y + V_3\,\mathrm{d}z$
- The Maxwell 2-form is defined by:
  $F = E_1\,\mathrm{d}t \wedge \mathrm{d}x + E_2\,\mathrm{d}t \wedge \mathrm{d}y + E_3\,\mathrm{d}t \wedge \mathrm{d}z + B_1\,\mathrm{d}y \wedge \mathrm{d}z + B_2\,\mathrm{d}z \wedge \mathrm{d}x + B_1\,\mathrm{d}x \wedge \mathrm{d}y$
- Problem 1: Calculate $\mathrm{d}F$
- Problem 2: Restate $\mathrm{d}F = 0$ in terms of vector calculus
- Problem 3: If $\mathrm{d}F = 0$, there is probably an A such that $F = \mathrm{d}A$. Find this A.
- Problem 4: Restate $F = \mathrm{d}A$ in terms of vector calculus
- Problem 5: Redefine $\star\alpha$ in terms of the new (Minkowski metric) dot product.
- Problem 6: Find the Hodge dual of all the basis forms.
- Problem 7: Calculate $\widetilde{J} = \star\,\mathrm{d}(\star F)$
- Problem 8: Restate $\widetilde{J} = \star\,\mathrm{d}(\star F)$ in terms of vector calculus
- For some particle with a velocity of $\vec{v} = (v_1, v_2, v_3)$, its 4-velocity is $\vec{u}$, where:
  $\vec{u} = \gamma(1, v_1, v_2, v_3)$
  $\gamma = \frac{1}{1-|\vec{v}|^2}$
- At a sufficiently low velocity, $\gamma = 1$
- Problem 9: If a particle has a charge of q and a mass of m, calculate $\widetilde{w} = -q \cdot \iota_{\vec{u}} F$
- Problem 10: What does the time component of $\vec{w}$ correspond to? The spatial components?

2.2 TODO

- Add generalized n-dimensional Stokes'/Green's/divergence theorem
- Tensor shit?