Notation

- Differences in between common set-theoretic notation and what is used in this document:
  - Usually, $f(-)$ is used to denote $x \mapsto f(x)$, where x is a fresh variable.
    This document instead uses $f(\circ)$, since it can be less confusing in some cases.
  - $\mathsf{Fin}(n)$ for any $n \in \mathbb{N}$ denotes the set $\{k \in \mathbb{N} \mid 0 \leq k < n\}$.

Computer Science

- Maximum cut
  - A **cut** in a weighted graph $G = (V, E, w)$ is a pair of sets $(V_x, V_y) \in \mathbb{P}(V)^2$ such that $V_x \cup V_y = V$, and $V_x \cap V_y = \varnothing$. In other words, it is a partition of the vertices of G into two disjoint sets. The edges $(s, t) \in E$ that have $s \in V_x$ and $t \in V_y$ or vice versa are said to **span** the cut.

    In a graph whose edges are weighted by nonnegative real numbers, the **weight** of a cut is the sum of the weight of all edges that span the cut.

    Cuts can also be considered on weighted graphs, in which case we treat all edges as having weight 1. This means that the weight of a cut of an unweighted graph is the number of edges that span the cut.

    A maximum cut is a cut whose weight is at least as large as any other cut.

  - The problem of finding a maximum cut in a graph is an NP-complete optimization problem. This problem is sometimes denoted MAX-CUT. This problem is also APX-hard, which means that no polynomial-time approximation scheme can exist for it.

  - The simplest approximation algorithm for MAX-CUT is a randomized algorithm which is a 0.5-approximation. For each vertex, you flip a coin and put that vertex in V_x if it is heads, V_y if it is tails.

    Every edge consists of two vertices, so there are four cases: both vertices are heads, both are tails, one is heads and one is tails, or one is tails and one is heads. In half of those cases, the edge is part of the cut, so on average the weight of the cut will be half of the total weight of all edges. The total weight of all the edges is an upper bound on the weight of the maximum cut, therefore this is a 0.5-approximation algorithm.

    This algorithm can be derandomized using the method of conditional probabilities.

  - A symmetric real matrix $M \in \mathbb{R}^{n \times n}$ is said to be **positive semidefinite** iff $v^\top M v \geq 0$ for every vector $v \in \mathbb{R}^n$. Equivalently, a matrix is positive semidefinite iff all of its eigenvalues are nonnegative. This is denoted by $M \succeq 0$.

  - **Semidefinite programming** is a convex optimization problem in which we are trying to minimize $\mathrm{tr}(C^\top \mathbf{X})$ subject to $\mathbf{X} \succeq 0$ and $\mathrm{tr}(A_k^\top \mathbf{X}) \leq b_k$ for all $0 \leq k < m$, where $\mathbf{X}$ is a Hermitian $n \times n$ matrix variable and $A = [A_0, \ldots, A_{m-1}]$ is a sequence of Hermitian $n \times n$ matrices, $b = [b_0, \ldots, b_{m-1}]$ is a sequence of n-vectors, and C is a Hermitian $n \times n$ matrix.

  - Semidefinite programming can be solved via a variety of methods, including interior point methods, conic optimization algorithms, the spectral bundle method, and the augmented Lagrangian method. All of these algorithms return a solution to the SDP problem that has a user-provided additive error ϵ, and they have a runtime that is polynomial in $n^2 m + \log(\frac{1}{\epsilon})$.

  - MAX-CUT can be expressed as the following integer quadratic programming problem: maximize $\frac{1}{2} \sum_{(i,j) \in E} w(i, j)(1 - \mathbf{v}_i \mathbf{v}_j)$ subject to $\mathbf{v}_k^2 = 1$ for all $0 \leq k < |V|$.

    If there is a bijection $f : V \to [0, |V| - 1]$ then a vertex v is in V_x iff $\mathbf{v}_{f(v)} = -1$ and it is in V_y iff $\mathbf{v}_{f(v)} = 1$.

  - This integer quadratic programming problem can be relaxed to a semidefinite programming problem by replacing the indicator variables ($\mathbf{v}$) with vector variables (of dimension $|V|$) and then doing randomized rounding. This is the Goemans-Williamson algorithm for MAX-CUT, and it achieves an approximation ratio of 0.878, which is the best known approximation ratio of any approximation algorithm for MAX-CUT. Assuming the unique games conjecture is true, this is the best possible approximation ratio for any MAX-CUT approximation algorithm. No approximation algorithm for MAX-CUT can have an approximation ratio over 0.941.

  - The problem of partitioning a set of n-dimensional points into two clusters can be reduced to MAX-CUT (though there are good greedy algorithms for this).

    The problem of finding an assignment of spins to particles in a spin glass model that minimizes the Hamiltonian can be reduced to MAX-CUT. This is probably why spin glasses exhibit metastability: MAX-CUT is not a convex optimization problem.

- Shortest Common Superstring
  - The **shortest common superstring** of a set $S \subset \Sigma^\star$ of strings, denoted $\mathrm{scs}(S)$, is defined to be the shortest string that contains every element of S as a substring.

    For example, if S is {`alf`, `ate`, `half`, `lethal`, `alpha`, `alfalfa`}, then a shortest common superstring is `lethalphalfalfate`.

  - If $G = (V, E)$ is an undirected graph and $(s, t) \in E$ is an edge, the **edge contraction** of G with respect to (s, t) and a semigroup $\diamond : V^2 \to V$ is defined to be the graph whose vertices are

    $$V' = (V - \{s, t\}) \cup (s \diamond t)$$

    and whose edges are

    $$E' = \{(x, y) \in E \mid x \neq s \wedge y \neq t\} \cup \{(x, s \diamond t) \mid (x, y) \in (E \cap V'^2) \wedge y \in \{s, t\}\}$$

    In other words, the result of applying edge contraction to a graph G is that the two vertices of the edge are combined using a semigroup, and any edges going into either vertex of the edge now point at the combined vertex.

    This operation is denoted by $\mathrm{contract}_\diamond(G, (s, t)) = (V', E')$.

  - If $s, t \in \Sigma^\star$, then the **overlap** of s and t is the length of the longest $y \in \Sigma^\star$ such that there exist $x, z \in \Sigma^\star$ such that $(s, t) = (x \cdot y, y \cdot z)$ or $(s, t) = (y \cdot x, z \cdot y)$. The value $x \cdot y \cdot z$ is known as the **merger** of s and t. The overlap of two strings is denoted by $\mathrm{overlap} : \Sigma^\star \times \Sigma^\star \to \mathbb{N}$, while their merger is denoted by $\mathrm{merger} : \Sigma^\star \times \Sigma^\star \to \Sigma^\star$.

  - If $S \subset \Sigma^\star$ is a set of strings, it gives rise to a weighted graph known as the **overlap graph** of S, which is defined by $\mathsf{Overlap}(S) = (S, \{(s, t) \in S^2 \mid \mathrm{overlap}(s, t) > 0\}, \mathrm{overlap})$.

    There is also a closely related class of graphs, called the class of **overlap graphs**, which are graphs that can be realized as intersection graphs of a set of intervals such that no two intervals are related by an inclusion. The overlap graph of a set of strings is in this class; the SCS of the set of strings is essentially an interval model.

  - If $S \subset \Sigma^\star$ is a set of strings, then S^{R} is the **reduced form** of S, which is defined as the smallest set that has the same SCS as S such that no two elements of S^{R} are substrings of one another.

  - The GREEDY approximation algorithm for shortest common superstring of a set S involves finding the edge in $\mathsf{Overlap}(S^{\mathrm{R}})$ that has the largest weight, contracting that edge, and repeating until the graph only has one vertex.

    In other words, if f is defined by

    $$\begin{aligned} f((\{v\}, \varnothing, w)) &= v \\ f((V, E, w)) &= f(\mathrm{contract}_{\mathrm{merger}}((V, E, w), \operatorname*{argmax}_{e \in E} w(e))) \end{aligned}$$

    then $\mathsf{GREEDY}(S) = f(\mathsf{Overlap}(S^{\mathrm{R}}))$.

    This has been proven to be a 4-approximation algorithm. It is conjectured to be a 2-approximation algorithm. This approximation results from a reduction to the longest Hamiltonian path problem on overlap graphs.

    A conjectured worst case input for this algorithm is $\{\mathtt{c}(\mathtt{ab})^k, (\mathtt{ba})^k, (\mathtt{ab})^k\mathtt{c}\}$; the shortest common superstring of which is $\mathtt{c}(\mathtt{ab})^{k+1}\mathtt{c}$; GREEDY will output $\mathtt{c}(\mathtt{ab})^k\mathtt{c}(\mathtt{ba})^k$.

- Graph coloring
  - A **coloring** of a graph $G = (V, E)$ is a map $\phi : V \to \mathbb{N}$ such that for any $(s, t) \in E$, $\phi(s) \neq \phi(t)$. In other words, it is an assignment of numbers ("colors") to vertices in a graph such that no two neighboring vertices are assigned the same number.

    The set of all colorings of G is denoted $\mathsf{Coloring}(G)$.

    The **order** of a coloring ϕ is the maximum element of $\mathbb{N}$ assigned to any vertex of the graph; i.e.: $\mathrm{order}(\phi) = \max(\mathsf{Im}(\phi))$.

    The **chromatic number** $\chi(G)$ of a graph G is defined to be the order of the smallest coloring of G; i.e.: $\chi(G) = \min\{\mathrm{order}(\phi) \mid \phi \in \mathrm{Coloring}(G)\}$.

    A coloring ϕ of a graph G is said to be **optimal** if the order of ϕ is equal to the chromatic number of G.

  - For any graph $G = (V, E)$, an ordering $\sigma : \mathsf{Fin}(|V|) \leftrightarrow V$ induces a coloring $\gamma_\sigma : V \to \mathbb{N}$ of G called the **greedy coloring**, which is defined by

    $$\gamma_\sigma(v) = \min(\mathbb{N} - \{\gamma_\sigma(\sigma(k)) \mid k \in \mathsf{Fin}(\sigma^{-1}(v)) \wedge (\sigma(k), v) \in E\})$$

    This coloring iterates through the vertices via the ordering σ, assigning each vertex the smallest color that is not used by any of its neighbors.

  - A **perfect elimination ordering** of a graph is an ordering of the vertices of a graph such that for each vertex v in the ordering, v and all of its neighbors that occur after it in the ordering form a clique.

  - **Theorem**: greedily coloring a graph using the reverse of a perfect elimination ordering always gives an optimal coloring of that graph.

  - A **chordal graph** is a graph where every induced cycle has exactly three vertices; i.e.: every cycle of length 4 or more has a **chord**: an edge that is not part of the cycle but connects two vertices in the cycle.

  - **Theorem**: a perfect elimination ordering of a chordal graph can be computed in polynomial time using the **lexicographic BFS** algorithm.

  - Corollary: chordal graphs can be optimally colored in polynomial time.

  - An **interval graph** is a graph $G = (V, E)$ for which there exists an assignment $\iota : V \to \mathbb{R} \times \mathbb{R}$ called an **interval model** such that $(s, t) \in E$ iff $I(\iota(s)) \cap I(\iota(t)) \neq \varnothing$ where $I : \mathbb{R} \times \mathbb{R} \to \mathbb{P}(\mathbb{R})$ is the mapping from the data of an interval to the actual subset of $\mathbb{R}$ that that interval represents.

  - **Theorem**: every interval graph is a chordal graph.

  - **Theorem**: if you have an interval model of an interval graph, then sorting the intervals by their left endpoint gives a perfect elimination ordering of the interval graph.

  - A **unit disk graph** is a graph that is generated by a set of points in $\mathbb{R}^2$ such that two vertices have an edge iff their corresponding points are within some distance of each other.

  - **Theorem**: there is an algorithm that, given a unit disk graph G, computes a coloring ϕ of G such that $\mathrm{order}(\phi) \leq 3\chi(G)$, i.e.: it is a 3-approximation algorithm for coloring unit disk graphs.

Probability Theory

- These notes are primarily based on the following sources:
  - *Radically Elementary Probability Theory* by Edward Nelson
  - The notes I wrote when I took UIUC IE 300: Analysis of Data.
- A **finite probability space** is a tuple $(\Omega \in \mathsf{Set}, \mathrm{pr} : \Omega \to \mathbb{R})$ such that $\sum\{\mathrm{pr}(\omega) \mid \omega \in \Omega\} = 1$ and $\forall \omega \in \Omega \,.\, \mathrm{pr}(\omega) > 0$.
- A **random variable** on Ω is a function $X : \Omega \to \mathbb{R}$.
  - The **expectation** of a random variable X, denoted $\mathbb{E}(X)$, is defined by
$$\mathbb{E}(X) = \sum\{X(\omega) \cdot \mathrm{pr}(\omega) \mid \omega \in \Omega\}$$
- An **event** is a subset $A \subseteq \Omega$ of the set underlying a finite probability space.
  - The **probability** of an event is defined by
$$\mathbb{P}(A) = \sum\{\mathrm{pr}(\omega) \mid \omega \in A\}$$
  - For any event A, the **indicator function** of A, denoted χ_A, is a random variable defined by
$$\chi_A(\omega) = \left\{ \begin{array}{ll} 1 & \text{when } \omega \in A \\ 0 & \text{when } \omega \notin A \end{array} \right\}$$
    We can think of the probability of an event as being the expectation of the indicator function for that event: $\mathbb{P}(A) = \mathbb{E}(\chi_A)$.
  - The **complementary event** for any event A, denoted A^c, is defined by $A^c = \Omega \setminus A$.
- The set $\Omega \to \mathbb{R}$ of all random variables on Ω is an n-dimensional vector space, where $n = |\Omega|$.
- FIXME: If $Z_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Z_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, then $Z_1 + Z_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Graph Theory

Introduction

- In the following definitions, $(R, [0_R, 1_R], [\mathrm{neg}_R], [+_R, \times_R])$ will be an arbitrary rig (semiring). The rig of $m \times n$ R-valued matrices will be denoted by $\mathrm{Mat}_{m\times n}(R)$ or alternatively $n \multimap_R m$. Since $+_R$ is commutative, we can denote the arbitrary sum of a set $S \subseteq R$ by $\sum_R S$. In some cases, we may assume that R is a commutative rig, in which case we are allowed to take the arbitrary product of a set $S \subseteq R$, denoted $\prod_R S$.

- A **digraph** (also called a **directed graph** or for the purposes of these notes simply a **graph**) is a pair $G = (V, E)$ of a set of **vertices** $V \subset \mathsf{Set}$ (also called **nodes**) and a set of **edges** $E \subseteq V^2$. V is the **vertex set** of G and E is the **edge set** of G. Since E is a set of pairs, it is also sometimes called the **edge relation** of G.

- An **undirected graph** is a graph $G = (V, E)$ such that if $(x, y) \in E$ then $(y, x) \in E$. Equivalently, a graph is undirected iff its edge relation is symmetric.

- An R-**weighted graph** is a pair $G = ((V, E), \delta)$ of a graph (V, E) and a function $\delta : E \to R$. We will sometimes call a graph "unweighted" to emphasize that it is not a weighted graph. We will sometimes refer to (V, E) as the **underlying (unweighted) graph** of G. Similarly, δ is called the **(edge) weight function** of G.

  If $(M, +_M, 0_M)$ is a monoid, the **pair weight function** of an M-weighted graph $G = ((V, E), \delta)$, denoted $\pi_G : V^2 \to M$, is defined by:

  $$\pi_G(x, y) = \left\{ \begin{array}{ll} \delta(x, y) & \text{when } (x, y) \in E \\ 0_M & \text{when } (x, y) \notin E \end{array} \right\}$$

  This should be obvious from the notation, but if R is a semiring, the pair weight function of an R-weighted graph will use the additive monoid of R.

- The **complement** of an unweighted graph $G = (V, E)$ is defined by $\overline{G} = (V, \{e \in V^2 \mid e \notin E\})$.

Algebraic Graph Theory

- The **adjacency matrix** of an R-weighted graph $G = ((V = [1, n] \subset \mathbb{N}, E \subseteq V^2), \delta \in E \to R)$ is the unique $n \times n$ R-valued matrix $\mathsf{Adj}(G)$ such that for all $(i, j) \in V^2$, $\mathsf{Adj}(G)_{ij} = \pi_G(i, j)$.

  - An adjacency matrix of an unweighted graph $G = (V, E)$ is just defined to be an adjacency matrix of $(V, E, (x, y) \mapsto 1_R)$ for any commutative ring R.

  - **Theorem**: for any graph G, if A and B are both adjacency matrices of P, then there exists a permutation matrix P such that $B = P^\top AP$.

  - If, for some graph $G = (V \subset \mathsf{Set}, E \subseteq V^2)$, there is an obvious total order relation available for V, we will define the adjacency matrix of G to be the adjacency matrix of the graph given by the unique bijection between V and $[1, \mathrm{card}(V)]$ generated by the total order (i.e.: the sequence generated by sorting the elements of V under the total order).

    Henceforth, when defining terminology relating to graphs, we will assume that the graph has $V = [1, n] \subset \mathbb{N}_+$ for some $n \in \mathbb{N}_+$, and then implicitly extend the defined notion to any graph with a totally ordered vertex set in the way we just did for $\mathsf{Adj}(-)$.

  - **Theorem**: if a graph G is undirected, then $\mathsf{Adj}(G)$ is a symmetric matrix.

    Theorem: if a graph G is unweighted, then $\mathsf{Adj}(G)$ is a binary matrix.

    Theorem: if a graph G has no edges ($G = ((V, \varnothing), \varnothing)$), then $\mathsf{Adj}(G)$ is a zero matrix.

    Theorem: if an unweighted graph G is a "complete graph", that is, if there exists a set V such that $G = (V, V^2)$, then $\mathsf{Adj}(G)$ is a matrix full of 1_R.

- The **weighted outdegree** of a node x in an R-weighted graph $G = ((V, E), \delta)$ is defined to be the weighted sum of the set of edges that *start* at x: $\mathrm{odeg}^R_G(x) = \sum_R \{\delta(x, b) \mid (x, b) \in E\}$.

  The **weighted indegree** of a node x in an R-weighted graph $G = ((V, E), \delta)$ is defined to be the weighted sum of the set of edges that *end* at x: $\mathrm{ideg}^R_G(x) = \sum_R \{\delta(a, x) \mid (a, x) \in E\}$.

  The **weighted degree** of a node x in a weighted *undirected* graph G is the same as its weighted indegree or its weighted outdegree: $\deg^R_G(x) = \mathrm{ideg}^R_G(x) = \mathrm{odeg}^R_G(x)$.

  The **outdegree** and **indegree** in an unweighted graph $G = (V, E)$ are given by the weighted outdegree and weighted indegree of $((V, E), (x, y) \mapsto 1 \in \mathbb{N})$. They are denoted by $\mathrm{odeg}_G(\circ)$ and $\mathrm{ideg}_G(\circ)$ respectively.

  Similarly, the **degree** in an unweighted undirected graph $G = (V, E)$ is given by the weighted degree in $((V, E), (x, y) \mapsto 1 \in \mathbb{N})$. The degree is denoted by $\deg_G(\circ)$.

  Let's define the outdegree/indegree/degree of a node in a W-weighted graph to be the outdegree/indegree/degree of the node in the underlying unweighted graph.

  A graph G is n-**regular** iff every node in G has indegree n and outdegree n.

  A graph G is **regular** iff there exists an $n \in \mathbb{N}$ such that G is n-regular.

- The **indegree matrix** of an order-n graph G is given by $\mathsf{D}^{\mathrm{in}}(G) = \mathrm{diag}(\mathrm{ideg}_G(1), \ldots, \mathrm{ideg}_G(n))$.

  The **outdegree matrix** of an order-n graph G is given by $\mathsf{D}^{\mathrm{out}}(G) = \mathrm{diag}(\mathrm{odeg}_G(1), \ldots, \mathrm{odeg}_G(n))$.

  The **degree matrix** of an undirected graph $G = (([1, n], E), \delta)$ is the same as its indegree matrix and its outdegree matrix: $\mathsf{D}(G) = \mathsf{D}^{\mathrm{in}}(G) = \mathsf{D}^{\mathrm{out}}(G) = \mathrm{diag}(\deg_G(1), \ldots, \deg_G(n))$.

- The **Laplacian matrix** of an undirected graph $G = (V, E)$ is $\Delta(G) = \mathsf{D}(G) - \mathsf{Adj}(G)$.

  On a directed graph $G = (V, E)$, the **indegree Laplacian matrix** is $\Delta^{\mathrm{in}}(G) = \mathsf{D}^{\mathrm{in}}(G) - \mathsf{Adj}(G)$. The **outdegree Laplacian matrix** $\Delta^{\mathrm{out}}(G)$ is defined the same way.

  - **Theorem**: $\Delta(G)$ is a symmetric positive-semidefinite matrix.

    Theorem: The smallest eigenvalue of $\Delta(G)$ is always 0.

    Theorem: The product of the nonzero eigenvalues of $\Delta(G)$ is equal to $|G|$ times the number of spanning trees in G.

    Theorem: The number of connected components of G, n, is equal to the dimension of the nullspace of $\Delta(G)$. Furthermore, the multiplicity of the 0 eigenvalue in the spectrum of $\Delta(G)$ is also equal to n.

    Theorem: $\Delta(G)$ is always singular.

  - The Laplacian matrix is like a discrete version of the Laplace operator.

- If M is $n \times n$ $\mathbb{C}$-valued matrix, then the **spectrum** of M is its set of eigenvalues. The spectrum of M is denoted by $\mathsf{Spec}(M) \in \{A \in \mathbb{P}(\mathbb{C}) \mid \mathrm{card}(A) = n\}$. If R is a subring of $\mathbb{C}$, then the spectrum of an R-valued matrix is defined by the inclusion map into $\mathbb{C}$.

  The function mapping the spectrum of an $n \times n$ R-matrix M to the corresponding eigenvectors is denoted by $\eta_M : \mathsf{Spec}(M) \to \mathbb{C}^n$. The function mapping the spectrum of M to the corresponding eigenvalue multiplicities is denoted by $\kappa_M : \mathsf{Spec}(G) \to \mathbb{N}$.

  Define $\mathrm{sort}(S \subseteq \mathbb{C}) : [1, \mathrm{card}(S)] \to \mathbb{C}$ to be the function that sorts a set of complex numbers in increasing order of absolute value (i.e.: under the order induced by the total ordering of the reals when mapping with $(z \mapsto z\bar{z}) \in \mathbb{C} \to \mathbb{R}$).

  As shorthand, we will define the spectrum of a R-weighted graph $G = ((V, E), \delta)$, where $R \subseteq \mathbb{C}$, to be the spectrum of its adjacency matrix; that is to say $\mathsf{Spec}(G) \equiv \mathsf{Spec}(\mathsf{Adj}(G))$. The $\eta_\circ$ and $\kappa_\circ$ functions are lifted to graphs in this way as well.

  If A and B have the same spectrum, we say that A is **isospectral** to B; this is true whether A and B are graphs or matrices.

  - **Note**: Most of the following statements come from [1].

  - **Theorem**: FIXME: spectral theorem, maybe in terms of a C^*-algebra

  - **Theorem**: If $G = ((V, E), \delta)$, then $\mathsf{Spec}(\overline{G}) = \{\mathrm{card}(V) - v \mid v \in \mathsf{Spec}(\Delta(G))\}$.

References

[1] Mario Kummer. *Spectra of graphs*. URL: https://www.win.tue.nl/~aeb/2WF02/spectra.pdf.

Differential Geometry

Point-Set Topology

- A **topological space** is a pair $(S, \Omega_S \subseteq \mathbb{P}(S))$ such that:
  - $\{\varnothing, S\} \subseteq \Omega$
  - Ω is closed under arbitrary unions: $\forall \omega \subseteq \Omega \,.\, \cup(\omega) \in \Omega$
  - Ω is closed under finite intersections: $\forall (A, B) \in \Omega^2 \,.\, A \cap B \in \Omega$
- Ω is called the **topology** of S. Elements of Ω are called **open sets**.
- If (S, Ω) is a topological space and $X \in \Omega$ and $X \neq S$, then X is an **open subset** in (S, Ω).
- If (S, Ω) is a topological space, then a subset $U \subset S$ **neighborhood** of $p \in S$ if there exists an open subset of X of (S, Ω) such that $p \in X$ and $X \subset U$.

  If a open subset is also a neighborhood of p, then it is said to be an **open neighborhood** of p.
- An **open cover** of a topological space is a collection of open subsets such that the union of all the open subsets is the entire space.
- An open cover $C = \{U_i \subset S\}_{i \in I}$ is **locally finite** if for every $p \in S$ there exists a neighborhood N of p such that the cardinality of $\{U_i \in C \mid U_i \cap N \neq \varnothing\}$ is finite.

  In other words, an open cover is locally finite if every neighborhood intersects only finitely many elements of the cover.
- A **refinement** of an open cover C of a space (S, Ω) is an open cover R of (S, Ω) such that for every $U_R \in R$ there exists a $U_C \in C$ such that $U_R \subset U_C$.
- A **continuous map** f from a topological space (X, Ω_X) to a topological space (Y, Ω_Y) is a function $f : X \to Y$ such that, for every open subset $U \subset Y$, the preimage of U under f is an open subset of X.
- A **homeomorphism** is a continuous map that is also an isomorphism.

  In other words, a homeomorphism is a continuous map $f : X \to Y$ such that there exists a continuous map $g : Y \to X$ and $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

  If $f : X \simeq Y$, then f is a homeomorphism between X and Y.
- A **basis** of a topology is a collection of open subsets such that every open subset in the topology is an (arbitrary) union of elements in the basis.

  A **subbasis** of a topology is a collection of open subsets such that every open subset in the topology is an (arbitrary) union of finite intersections of elements in the subbasis.

  If B is a basis, then an element of B is called a **basic open subset**.

  If B is a subbasis, then an element of B is called a **subbasic open subset**.

  A topology Ω is **generated** by B if B is a basis of Ω.
- A **metric space** is a topological space (S, Ω) equipped with a **metric** $d : S \times S \to \mathbb{R}$, which is a function for which the following conditions all hold:
  - The **non-negativity** axiom: for every $(x, y) \in S^2$, $d(x, y) \geq 0$.
  - The **identity of indiscernables**: for every $(x, y) \in S^2$, $d(x, y) = 0$ iff $x = y$.
  - The **symmetry** axiom: for every $(x, y) \in S^2$, $d(x, y) = d(y, x)$.
  - The **triangle inequality** axiom: for every $(x, y, z) \in S^3$, $d(x, z) \leq d(x, y) + d(y, z)$.
- The **open ball** in a metric space $M = ((S, \Omega), d)$ with **center** $c \in S$ and **radius** $r \in \mathbb{R}_+$ is the open subset $B(c, r)$ of (S, Ω) defined by $B(c, r) = \{p \in S \mid d(c, p) < r\}$.
- A topological space (S, Ω) is said to be **metrizable** if it can be equipped with a metric d such that Ω is generated by the collection of open balls in S: $\{B(c, r) \subset S \mid c \in S \wedge 0 < r < \infty\}$.

Manifolds and Bundles

- A topological space (S, Ω) is a **topological manifold of dimension** n iff the following three conditions all hold:
  - (S, Ω) is **Hausdorff**: for any two points x and y in S, if, for every open neighborhood U of x in S and every open neighborhood V of y in S, $U \cap V \neq \varnothing$, then $x = y$.
  - (S, Ω) is **paracompact**: for every open cover C of (S, Ω), there exists a refinement of C that is locally finite.
  - (S, Ω) is **locally Euclidean**: for every $p \in S$, there exists an open neighborhood U such that there exists a homeomorphism $\phi : U \to \mathbb{R}^n$.
- **Theorem**: every topological manifold is metrizable.
- A **local coordinate chart** of an open subset U in a topological manifold of dimension n is a homeomorphism $\phi : U \to \mathbb{R}^n$.

  If $\phi_A : A \to \mathbb{R}^n$ and $\phi_B : B \to \mathbb{R}^n$ are two coordinate charts of a topological manifold M, then ϕ_A and ϕ_B are **compatible** iff there exists a **gluing homeomorphism** $g : \phi_A(A \cap B) \simeq \phi_B(A \cap B)$.

  An **atlas** is an open cover C of a topological manifold such that every $U \in C$ has a coordinate chart ϕ_U and for every $(A, B) \in C^2$ if $A \cap B \neq \varnothing$ then ϕ_A is compatible with ϕ_B.

  Every topological manifold of dimension n has at least one atlas.
- A k**-fold differentiable manifold of dimension** n is a topological manifold of dimension equipped with an atlas such that the gluing functions of the atlas are all k times differentiable.

  If $k = \infty$, then we call it a **smooth manifold of dimension** n.
- A **topological group** is a group G whose underlying set is equipped with a topology Ω_G such that the addition operator $+_G : G^2 \to G$ is a continuous map from $(G, \Omega_G)^2$ to (G, Ω_G) and the negation operator $-_G : G \to G$ is a continuous map from (G, Ω_G) to itself.

  Alternatively, a topological group is a group object in the category of topological spaces Top. From now on, we will define other types of groups in this way, to avoid redundancy.
- If $B = (S_B, \Omega_B)$ and $F = (S_F, \Omega_F)$ are a topological spaces, then a **fiber bundle** b is a pair of a topological space E_b, called the **total space** of b, and a continuous surjection $\pi_b : E \to B$, called the **projection map** of b, such that for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi_b(x)$ (called the **trivializing neighborhood** of x) and there exists a homeomorphism $\phi : \pi_b^{-1}(U) \simeq U \times F$, called a **local trivialization**, such that $((a, b) \mapsto a) \circ \phi = \pi_b$.

  In other words, a fiber bundle is a continuous surjection between two topological spaces E and B such that, for some topological space F, E locally looks like $B \times F$.
- If S is a set and G is a group, then a G**-action** is any function $a : G \times S \to S$ such that $a(0_G, x) = x$ and $a(p +_G q, x) = a(p, a(q, x))$ for all $(p, q) \in G^2$ and all $x \in S$.

  If $G \in \mathsf{Grp}$, then $\mathbf{B}(G)$ is the groupoid with one object $\star$ such that the automorphism group of $\star$ is isomorphic to G.

  Another way of thinking about group actions is that the action of a group G in a category C is a functor $\rho : \mathbf{B}(G) \to C$. The set S in the original definition of a group action is just $\rho(\star)$.

  A group action is **effective** if the functor ρ is faithful.

  Classically, an object of a (finitely cocomplete) category C is said to be **inhabited** iff it is not the initial object of C.

  A group action is **transitive** if C is cartesian closed and $\rho(\star)$ is inhabited and for every $x, y : 1 \to \rho(\star)$ in C, there exists a $g : \star \to \star$ such that $y = \rho(g) \circ x$.
- FIXME: Talk about vector bundles and tangent bundles here

Lie Theory

- A **Lie group** is a group object internal to Diff.
- A **Lie algebra** is a vector space $\mathfrak{g}$ over a field F equipped with an operator $[\circ, \circ] : \mathfrak{g}^2 \to \mathfrak{g}$, called the **Lie bracket**, that satisfies the following conditions:
  - The Lie bracket is **bilinear**: for all $a, b \in F$ and all $x, y, z \in \mathfrak{g}$, $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$.
  - The Lie bracket is **alternating**: for all $x \in \mathfrak{g}$, $[x, x] = 0_{\mathfrak{g}}$.
  - The **Jacobi identity** holds: for all $x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$
- FIXME: More Lie theory goes here

Tensor/Exterior/Clifford Algebra

- FIXME

Riemannian Geometry

- FIXME: Define the dual of a vector space
- FIXME: Define the tangent space of a point on a manifold
- A **pseudo-Riemannian manifold** M is a smooth manifold equipped with an M-indexed collection of functions $g_p : T_p(M)^2 \to \mathbb{R}$, called the **metric tensor**, on the tangent vector space at every point p such that the following conditions hold:
  - The metric tensor is **smooth**: for all $p \in M$, g_p is a morphism in Diff.
  - The metric tensor is **bilinear**: for all $p \in M$ and all $a, b \in \mathbb{R}$ and all $x, y, z \in T_p(M)$, $g_p(ax + by, z) = ag_p(x, z) + bg_p(y, z)$ and $g_p(x, ay + bz) = ag_p(x, y) + bg_p(x, z)$.
  - The metric tensor is **symmetric**: for all $p \in M$ and all $x, y \in T_p(M)$, $g_p(x, y) = g_p(y, x)$.
  - The metric tensor is **non-degenerate**: for all $p \in M$ and all $x \in T_p(M)$, if $g_p(x, y) = 0$ for every $y \in T_p(M)$, then $x = 0$.
- A metric tensor is **positive definite** for every $p \in M$ and every $v \in T_p(M)$, $g_p(v, v) > 0$ or $v = 0$.

  A **Riemannian manifold** is a pseudo-Riemannian manifold with a positive definite metric tensor.
- A **(tangent) vector field** on a smooth manifold M is a smooth section of the tangent bundle of M. The set of all vector fields on M is denoted by $C^\infty(M, TM)$.

  A **scalar field** on a smooth manifold M is a smooth map from M to $\mathbb{R}$.
- Recall that the **directional derivative** of a scalar field f along a vector v represents the degree to which f changes along an infinitesimal vector pointing in the same direction as v. We will now generalize this notion to arbitrary smooth manifolds.
- A **Levi-Civita connection** on a pseudo-Riemannian manifold M is FIXME.
- The **fundamental theorem of Riemannian geometry** states that for any pseudo-Riemannian manifold M, there exists a unique Levi-Civita connection.

Linear Algebra

Matrices

- If s is a commutative ring and $(m, n) \in \mathbb{N}$, then $m \multimap_{\mathsf{s}} n$ denotes the s-bimodule of $n \times m$ matrices. Sometimes, we will use the alternative notation $\mathsf{M}_{m\times n}(\mathsf{s}) = n \multimap_{\mathsf{s}} m$.

- **Theorem**: If s is a field, then $m \multimap_{\mathsf{s}} n$ is a vector space with s as its field of scalars. Equivalently, $\mathsf{Vect}_{\mathsf{s}}$ is a closed category (which means that it has an internal hom).

- **Theorem**: Every matrix in $m \multimap_{\mathsf{s}} n$ corresponds to a linear map in $\mathsf{s}^m \to \mathsf{s}^n$. For a matrix M, this linear map is given by $v \mapsto M \cdot v$. Composition of these linear maps corresponds to the matrix product.

- There are several categories whose morphisms can be thought of as matrices:

  - $\mathsf{Mat}_{\mathsf{s}}$ is the category for which the objects are elements of the set $\{\mathsf{s}^n \mid n \in \mathbb{N}\}$, the hom-set between s^x and s^y is $\mathsf{M}_{y\times x}(\mathsf{s})$, and composition of morphisms is given by matrix multiplication.

  - $\mathsf{Vect}_{\mathsf{s}}$ is the category of vector spaces with linear maps as morphisms and composition of morphisms given by ordinary function composition.

    This might seem like a category relevant to matrices, but it isn't, as not only are there vector spaces in $\mathsf{Vect}_{\mathsf{s}}$ that are infinite-dimensional, but there are vector spaces in $\mathsf{Vect}_{\mathsf{s}}$ whose dimension is uncountable!

    For example, define Unc to be the vector space whose set of vectors is defined by $\{f \in \mathbb{R} \to \mathbb{R} \mid \mathrm{card}(\{x \in \mathbb{R} \mid f(x) \neq 0\}) \in \mathbb{N}\}$, whose addition is pointwise, and whose multiplication is pointwise. Since $\{(y \mapsto \delta(x, y)) \mid x \in \mathbb{R}\}$ is a basis for Unc, where $\delta(x, y) = [x = y]$ is the Kronecker delta function on $\mathbb{R}$, and $x \mapsto (y \mapsto \delta(x, y))$ is a bijection, and all the bases of a vector space must have the same cardinality as the dimension of the vector space, it must be the case that dimension of Unc is $|\mathbb{R}|$, which is uncountable.

    While the concept of infinite matrices might seem reasonable, the idea of a matrix with uncountably many columns is not reasonable. Thus, an exposition of $\mathsf{Mat}_{\mathsf{s}}$ needs to talk about $\mathsf{FinVect}_{\mathsf{s}}$, the full subcategory of $\mathsf{Vect}_{\mathsf{s}}$ consisting of only the finite-dimensional vector spaces (a full subcategory has the same hom-sets as its supercategory).

    Now we can describe $\mathsf{Mat}_{\mathsf{s}}$ as the quotient of $\mathsf{FinVect}_{\mathsf{s}}$ by the equivalence relation given by isomorphisms in $\mathsf{FinVect}_{\mathsf{s}}$. Thus, $\mathsf{Mat}_{\mathsf{s}}$ is the skeleton of $\mathsf{FinVect}_{\mathsf{s}}$, which is unique up to isomorphism of categories.

Matrix Product

- FIXME

Kronecker Product

- If $A \in m \multimap_{\mathsf{s}} n$ and $B \in p \multimap_{\mathsf{s}} q$, then the **Kronecker product** of A and B is defined by

$$A \otimes B \in mp \multimap nq$$
$$A \otimes B = \mathrm{flatten}\left(\begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}\right)$$

  where $\mathrm{flatten} \in \mathsf{M}_{a\times b}(\mathsf{M}_{x\times y}(\mathsf{s})) \to \mathsf{M}_{ax\times by}(\mathsf{s})$ is the function that "flattens" a block matrix into a matrix of scalars, and A_{xy} is the scalar coordinate of A at position (x, y).

- The Kronecker product can be used to make a tensor product for $\mathsf{Mat}_{\mathsf{s}}$, thus making it a (non-strict) monoidal closed category, in the following way:

  - The tensor unit is the FIXME: object in $\mathsf{Mat}_{\mathsf{s}}$.

  - The tensor product bifunctor $\otimes$ acts on objects of $\mathsf{Mat}_{\mathsf{s}}$ by

$$\mathsf{s}^m \otimes \mathsf{s}^n = (\mathsf{s}^m \times \mathsf{s}^n)/\phi$$
$$\phi((kx, y), (x, ky))$$

  - The tensor product bifunctor $\otimes$ acts on morphisms of $\mathsf{Mat}_{\mathsf{s}}$, which are matrices, via the Kronecker product.

  - The associator and unitors are the obvious ones.

  -

- There are a large number of identities involving the Kronecker product:

  - Since the Kronecker product is a tensor product, it is bilinear:

$$A \otimes (B + C) = A \otimes B + A \otimes C$$
$$(A + B) \otimes C = A \otimes C + B \otimes C$$
$$(rA) \otimes (sB) = rs(A \otimes B)$$

    and it is associative:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

  - Although the Kronecker product is *not* commutative, $A \otimes B$ is always permutation-equivalent to $B \otimes A$. In other words, for every pair of matrices $A \in m \multimap_{\mathsf{s}} n$ and $B \in p \multimap_{\mathsf{s}} q$, there always exist permutation matrices $P \in nq \multimap_{\mathsf{s}} nq$ and $Q \in mp \multimap_{\mathsf{s}} mp$ such that $A \otimes B = P \cdot (B \otimes A) \cdot Q$. If A and B are square, then $P = Q^{-1} = Q^{\top}$ and $A \otimes B$ is thus permutation similar to $B \otimes A$.

  - The matrix product $(\cdot)$ and Kronecker product $(\otimes)$ obey the **mixed product property**, which says that for any appropriately shaped matrices A, B, C, and D:

$$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

    Equivalently, if $\mathsf{Mat}_{\mathsf{s}}$ is thought of as a closed monoidal category with the Kronecker product as its tensor product, then $\mathsf{Mat}_{\mathsf{s}}$ is a symmetric monoidal closed category.

  - Similarly, the Hadamard product and the Kronecker product also have a mixed product property:

$$(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D)$$

  - If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

    Similarly, this holds for the pseudoinverse: $(A \otimes B)^{+} = A^{+} \otimes B^{+}$.

  - Transposition and conjugate transposition both distribute over the Kronecker product: $(A \otimes B)^{\top} = A^{\top} \otimes B^{\top}$ and $(A \otimes B)^{\star} = A^{\star} \otimes B^{\star}$ for all A and B.

  - If $A \in m \multimap m$ and $B \in n \multimap n$ and $\sigma(-)$ gives the set of eigenvalues (spectrum) of a matrix, then $\sigma(A \otimes B) = \{xy \mid x \in \sigma(A) \wedge y \in \sigma(B)\}$.

- The **Kronecker sum** is defined for $A \in m \multimap m$ and $B \in n \multimap n$ by $A \oplus B = A \otimes I_n + I_m \otimes B$, where I_k is the $k \times k$ identity matrix.

  - The matrix exponential transforms a Kronecker sum into a Kronecker product:

$$\exp(A \oplus B) = \exp(A) \otimes \exp(B)$$

Convolution

-

Abstract Algebra

Ideals

- An **ideal** I of a ring R should be thought of as a particular kind of subset of a ring.
- Specifically, for some ring $(R, +, \cdot)$, a set I is a **two-sided ideal** of R if it satisfies two conditions:
  - **Subgroup**: $(I, +)$ is a subgroup of $(R, +)$
  - **Absorption**: For any $r \in R$ and any $i \in I$, $r \cdot i$ and $i \cdot r$ are both elements of I.
- We can generalize this to the notion of a **left ideal** and a **right ideal**. A left ideal only requires $r \cdot i \in I$, while a right ideal only requires $i \cdot r \in I$.
- The canonical example of an ideal is that of the even integers $2\mathbb{Z}$, which are an ideal in $\mathbb{Z}$.
  - To see why this is, note that addition is closed over $2\mathbb{Z}$, and that the additive inverse of an even integer is also even. Thus, $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$. The other factor is that multiplying any number by an even number gives you another even number, so the absorption condition is satisfied.
- More generally, for any integer n in $\mathbb{Z}$, $n\mathbb{Z}$ (i.e.: the set $\{n \cdot x \mid x \in \mathbb{Z}\}$) is an ideal in $\mathbb{Z}$.
- In some sense, ideals generalize the idea of the set of all values divisible by a given value.
- Other examples:
  - For any ring R, R is trivially an ideal of R. This is called the **unit ideal** of R.
  - For any ring R, $\{0_R\}$ is an ideal of R. This is called the **zero ideal** of R.
  - Denote the ring of all univariate polynomials with real coefficients by $\mathbb{R}[x]$. Then the set of all such polynomials divisible by $x^2 + 1$ is an ideal in $\mathbb{R}[x]$.
- Some types of ideals:
  - A **proper ideal** is one that is not the unit ideal. A **nonzero ideal** is one that is not the zero ideal.
  - The **maximal ideal** of a ring is the largest possible proper ideal for that ring.
  - The **minimal ideal** of a ring is the smallest possible nonzero ideal for that ring.
  - A **prime ideal** is an ideal I such that for any $(a, b) \in R^2$, $a \cdot b \in I$ implies $a \in I$ or $b \in I$.
  - A **radical** or **semiprime ideal** is an ideal I such that for any $a \in R$, $a^n \in I$ implies $a \in I$.
  - A **principal ideal** is an ideal with one generator.
- The **quotient of a ring** R **by an ideal** I is $(R/\{(a, b) \in R^2 \mid (a - b) \in I\}, +, \cdot)$.
- Products and sums
  - Any two ideals I and J have a sum, defined as $I + J = \{a + b \mid a \in I \wedge b \in J\}$.
  - Any two ideals I and J have a product, defined as

    $$I \times J = \{\psi(0) + \cdots + \psi(n) \mid n \in \mathbb{N} \text{ and } \phi \in \mathbb{N} \to I \times J \text{ and } \psi = \pi \circ \phi\}$$

    where $\pi : R^2 \to R$ is defined as $\pi(a, b) = a \cdot b$
  - Note that $(I \cup J) \subseteq (I + J)$ and $(I \times J) \subseteq (I \cap J)$.
- The set of ideals of a ring R is denoted $\mathbb{I}_R$.
- For convenience, we will define a function $\Phi : (A^2 \to A) \times A \to \mathbb{P}(A) \to A$ by:

  $$\Phi(f, e)(\varnothing) = e \quad \text{and} \quad \Phi(f, e)(\{x\} \cup X) = f(x, \Phi(f, e)(X))$$

  where $x \in A$ and $X \subseteq \mathbb{P}(A)$. This is known as the **fold** of a binary operator f with a unit e.
- $(\mathbb{I}_R, \subseteq, \Phi(+, \{0_R\}), \Phi(\cap, R), \{0_R\}, R)$ is a complete modular lattice.

Locale Theory

- These notes are primarily based on the following sources:
  - *Topology via Logic* by Steven Vickers
- For some set A and relation $(\#) \subseteq A \times A$:
  - $(\#)$ is **reflexive** iff $\forall x \in A \;.\; x \# x$.
  - $(\#)$ is **transitive** iff $\forall (x, y, z) \in A^3 \;.\; ((x \# y) \wedge (y \# z)) \implies (x \# z)$.
  - $(\#)$ is **symmetric** iff for all $(x, y) \in A^2$, $(x \# y) \iff (y \# x)$.
  - $(\#)$ is **anti-symmetric** iff for all $(x, y) \in A^2$ such that $x \# y$ and $y \# x$, we have $x = y$.
- A **preorder** is a set P equipped with a relation $(\sqsubseteq)$ that is both reflexive and transitive.
  - Alternatively, a preorder can be thought of as a category C in which $\forall (X, Y) \in \mathrm{Ob}(C)^2 \;.\; |\mathrm{Hom}(X, Y)| \leq 1$.

    For this reason, preorders are sometimes called **thin categories**.
  - The **opposite preorder** for a preorder $P = (X, \sqsubseteq)$ is defined as $P^{\mathsf{op}} = (X, \sqsupseteq)$.
  - A function $f : (P, \sqsubseteq_P) \to (Q, \sqsubseteq_Q)$ is **monotone** iff for every $(x, y) \in P^2$, it is the case that $x \sqsubseteq_P y \implies f(x) \sqsubseteq_Q f(y)$.

    The category of thin categories has monotone functions as morphisms.

    In other words, a functor between thin categories is a monotone function.
- A **poset** is a preorder in which $(\sqsubseteq)$ is anti-symmetric.
  - Alternatively, a poset can be thought of as a category in which for any distinct pair of objects (X, Y), we have $|\mathrm{Hom}(X, Y) \uplus \mathrm{Hom}(Y, X)| \leq 1$.
  - Every preorder $(P, \sqsubseteq)$ gives rise to a poset $(P/{\equiv}, \sqsubseteq)$, where $a \equiv b \iff (a \sqsubseteq b) \wedge (b \sqsubseteq a)$.
  - In a given poset P, with $X \subseteq P$ and $a \in P$, a is a **lower bound** (resp. **upper bound**) for X iff for any $x \in X$, $a \sqsubseteq x$ (resp. $x \sqsubseteq a$).
  - A lower bound a of X is a **meet** if it is greater than or equal to any other lower bound.
  - A **join** is dual to a meet; if a is a join for X in P, then a is a meet for X in P^{op}.
- A **pseudolattice** is a poset in which every nonempty finite subset has a meet and a join.
- A **meet-/join-semilattice** is a poset in which every finite subset has a meet/join.
- A **lattice** is a poset that is both a meet-semilattice and a join-semilattice. Alternatively, a lattice is a pseudolattice with empty meets and joins (which correspond to unique maximal and minimal elements).
- **Theorem**: A poset P is a pseudolattice if it has binary meets and joins.
- **Corollary**: If P also has empty meets and joins, it is a lattice.
- Thus, we will heretofore denote all finite meets and joins by binary operators $(\sqcap)$ and $(\sqcup)$.
- Infinite meets and joins will look like $\sqcap(X)$ and $\sqcup(X)$ for some $X \subseteq P$.
- For lattices L_1 and L_2, a function $f : L_1 \to L_2$ is a **lattice homomorphism** if and only if for every (a, b) in L_1^2, $f(a \sqcap_1 b) = f(a) \sqcap_2 f(b)$ and $f(a \sqcup_1 b) = f(a) \sqcup_2 f(b)$.

  A lattice homomorphism is a monotone function that respects binary meets and binary joins.

  The category of lattices, Lat, has lattice homomorphisms as its morphisms.
- Note that the set of all pseudolattices is closed under $(\circ^{\mathsf{op}})$.
- A poset P is a **frame** iff every subset has join, every finite subset has a meet, and binary meets distribute over joins.
  - In general, if F is a frame, F^{op} is not a frame.
  - One notable exception is the **powerset frame** on a set X, $\mathbb{P}(X)$.
  - $\mathbf{1} = (\{\star\}, \varnothing)$, the **inconsistent frame**, so named because it has $\bot = \top$.
  - $\mathbf{2} = (\{\bot, \top\}, \{(\bot, \top)\})$, the **Sierpinski frame**.
- A **topology** Ω on a set X is a frame such that Ω is a subframe of $\mathbb{P}(X)$.
- A **topological space** is a set X equipped with a topology Ω.
- The elements of Ω are known as the **open subsets** of X.
- For a poset P where $x \in P$ and $S \subseteq P$:
  - $\uparrow(x) = \{y \in P \mid x \sqsubseteq y\}$ is the **upper closure** of x.
  - $\uparrow(S) = \{y \in P \mid (\exists x \in S \;.\; x \sqsubseteq y)\}$ is the **upper closure** of S.
  - The **lower closure** of x and S, denoted by $\downarrow(\circ)$, is just the upper closure in P^{op}.
- Examples of topologies:
  - The **discrete topology** is $\Omega = \mathbb{P}(X)$.
  - The **indiscrete topology** is $\Omega = \varnothing$.
  - The **Alexandrov topology** on a poset P is defined as $\Omega = \{\uparrow(x) \mid x \in P\}$.
- We will heretofore denote the topology associated with any given topological space X by Ω_X.
- The **interior** of a subset S of a topological space X is $\mathrm{Int}(S) = \cup\{A \in \Omega_X \mid A \subseteq S\}$
  - Note that $\mathrm{Int}(S)$ is always the largest open set contained in S.
- A subset $F \subseteq X$ is **closed** iff its complement is open.
- A subset is **clopen** iff it is both open and closed.
- The **topological closure** of a subset $S \subseteq X$ is $\mathrm{Cl}(S) = (\mathrm{Int}(S^c))^c$