CPSC 2101: Programming Methodology Lab

A Brief Introduction to Matrix Operations

1 What is a Matrix?

A matrix is an array representation of data grouped in rows and columns. We can write a matrix as follows:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To refer to an element a in matrix M we denote it by row i, and column j, where $0 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 2$. For example, $a_{0,0} = 1$ and $a_{2,2} = 9$. Next, we will discuss various operations that can be performed on a matrix.

2 Matrix Operations

There are several matrix operations that we will look at. Specifically we will discuss matrix addition, subtraction, multiply and transpose.

2.1 Matrix Addition

Matrix addition is a matrix x matrix operation meaning that it requires two matrices.

$$M = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} N = \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix}$$

$$M + N = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5+3 & 2+1 & 1+6 \\ 4+5 & 2+2 & 4+2 \\ 1+4 & 3+3 & 1+3 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 7 \\ 9 & 4 & 6 \\ 5 & 6 & 4 \end{bmatrix}$$

Note the two matrices above. $M + N$ is the sum from an element a in M and b in N. So, for each i, j in the result matrix is $a_{i,j} + b_{i,j}$ or:

$$M + N = \begin{bmatrix} a_{0,0} + b_{0,0} & a_{0,1} + b_{0,1} & a_{0,2} + b_{0,2} \\ a_{1,0} + b_{1,0} & a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,0} + b_{2,0} & a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{bmatrix}$$

2.2 Matrix Subtraction

Matrix subtraction is very similar to matrix addition. We simply perform a subtraction operation on each element instead of addition.

$$M = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} N = \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix}$$

$$M - N = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5-3 & 2-1 & 1-6 \\ 4-5 & 2-2 & 4-2 \\ 1-4 & 3-3 & 1-3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -5 \\ -1 & 0 & 2 \\ -3 & 0 & -2 \end{bmatrix}$$

The corresponding matrix operations that are performed can be seen below:

$$M + N = \begin{bmatrix} a_{0,0} - b_{0,0} & a_{0,1} - b_{0,1} & a_{0,2} - b_{0,2} \\ a_{1,0} - b_{1,0} & a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} \\ a_{2,0} - b_{2,0} & a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} \end{bmatrix}$$

2.3 Matrix Multiplication

Matrix multiplication is different from matrix addition and subtraction in that we do not simply multiply each element in a matrix M by each element in matrix N. We can write matrix multiplication for a single element as:

$$(MN)_{i,j} = \sum_{k=1}^{n} M_{i,k} N_{k,j}$$

If we look at the examples we have been using we get the resulting matrix:

$$M = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} N = \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix}$$

$$M * N = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} * \begin{bmatrix} 3 & 1 & 6 \\ 5 & 2 & 2 \\ 4 & 3 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} (5*3)+(2*5)+(1*4) & (5*1)+(2*2)+(1*3) & (5*6)+(2*2)+(1*3) \\ (4*3)+(2*5)+(4*4) & (4*1)+(2*2)+(4*3) & (4*6)+(2*2)+(4*3) \\ (1*3)+(3*5)+(1*4) & (1*1)+(3*2)+(1*3) & (1*6)+(3*2)+(1*3) \end{bmatrix} =$$

$$\begin{bmatrix} 9 & 12 & 37 \\ 38 & 20 & 40 \\ 22 & 10 & 15 \end{bmatrix}$$

We can write the matrix as follows:

$$\begin{bmatrix} (a_{0,0}*b_{0,0})+(a_{0,1}*b_{1,0})+(a_{0,2}*b_{2,0}) & (a_{0,0}*b_{0,1})+(a_{0,1}*b_{1,1})+(a_{0,2}*b_{2,1}) & (a_{0,0}*b_{0,2})+(a_{0,1}*b_{1,2})+(a_{0,2}*b_{2,2}) \\ (a_{1,0}*b_{0,0})+(a_{1,1}*b_{1,0})+(a_{1,2}*b_{2,0}) & (a_{1,0}*b_{0,1})+(a_{1,1}*b_{1,1})+(a_{1,2}*b_{2,1}) & (a_{1,0}*b_{0,2})+(a_{1,1}*b_{1,2})+(a_{1,2}*b_{2,2}) \\ (a_{2,0}*b_{0,0})+(a_{2,1}*b_{1,0})+(a_{2,2}*b_{2,0}) & (a_{2,0}*b_{0,1})+(a_{2,1}*b_{1,1})+(a_{2,2}*b_{2,1}) & (a_{2,0}*b_{0,2})+(a_{2,1}*b_{1,2})+(a_{2,2}*b_{2,2}) \end{bmatrix}$$

2.4 Matrix Scale

The final operation we will discuss is matrix scaling. In matrix scaling, we multiply a matrix by a single scalar value.

$$M = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} s = 2$$

$$M * s = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} * 2 = \begin{bmatrix} 5*2 & 2*2 & 1*2 \\ 4*2 & 2*2 & 4*2 \\ 1*2 & 3*2 & 1*2 \end{bmatrix} = \begin{bmatrix} 10 & 4 & 2 \\ 8 & 4 & 8 \\ 2 & 6 & 2 \end{bmatrix}$$

We write this matrix as follows:

$$M * s = \begin{bmatrix} a_{0,0} * s & a_{0,1} * s & a_{0,2} * s \\ a_{1,0} * s & a_{1,1} * s & a_{1,2} * s \\ a_{2,0} * s & a_{2,1} * s & a_{2,2} * s \end{bmatrix}$$

2.5 Matrix Transposition

Matrix transposition is a unary operation that requires only a single matrix. For a matrix M, a matrix transposition denoted as M^T, replaces an arbitrary element $a_{i,j}$ with element $a_{j,i}$. For example let's use the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

If we apply the transpose operation on M, we get:

$$M^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

So, a general matrix transpose operation would look like this:

$$M = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{bmatrix}, M^T = \begin{bmatrix} a_{0,0} & a_{1,0} & a_{2,0} \\ a_{0,1} & a_{1,1} & a_{2,1} \\ a_{0,2} & a_{1,2} & a_{2,2} \end{bmatrix}$$

2.6 Matrix Symmetry

Matrix symmetry is a property of a square matrix. If a matrix is symmetric, this means that a matrix $M = M^T$. For example:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is not symmetric. If we take M^T, it does not equal M.

$$M' = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 5 & 6 \\ 7 & 6 & 9 \end{bmatrix}$$

However, matrix M' is symmetric. Note that the elements on either side of the diagonal for matrix M' are the same on both sides.