Graph Theory and Map Coloring

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MIT PRIMES Circle

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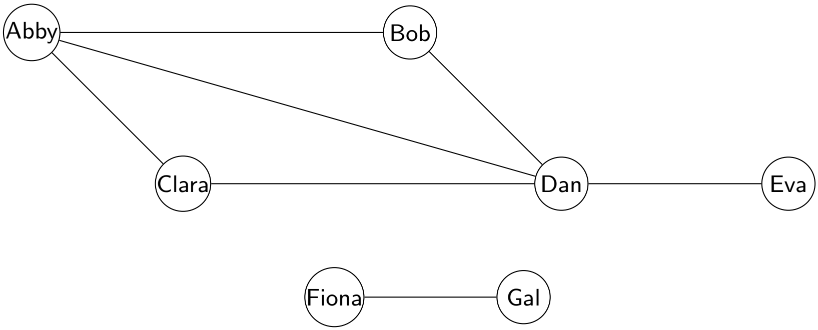
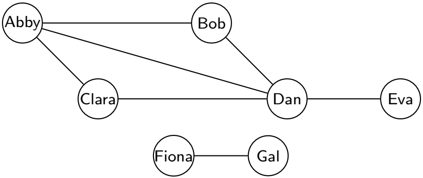
Modeling Facebook



Modeling Facebook



Definition

A *graph* G is a set of *vertices* $V(G)$ and *edges* $E(G)$.

Definition

The *order* and *size* of a graph G are the cardinality of its vertex set $|V(G)|$ and edge set $|E(G)|$, respectively.

Definition

For a graph G and vertex $v \in V(G)$, the *degree* $\deg_G(v)$ of v is the number of edges incident with v.

Fundamental Theorem of Graph Theory

Theorem

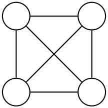
For any graph G of size m,

$$\sum_{v \in V(G)} \deg_G(v) = 2m.$$

Types of Graphs

Definition

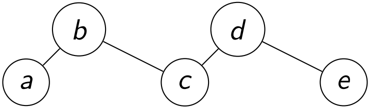
A *complete graph* is a graph G where every vertex $v \in V(G)$ is adjacent to every other vertex $u \in V(G)$. A complete graph is denoted by K_n where n is the order of G.



Types of Graphs

Definition

A *path* P_n is a sequence of n vertices where every two consecutive vertices have an edge between them.

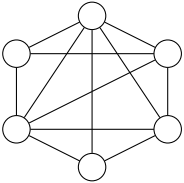


Definition

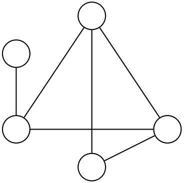
A graph $H = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$, and every edge $e \in E'$ has its endpoints in V'.

Types of Graphs

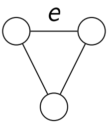
H:



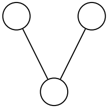
H':



G:



$G - e$:

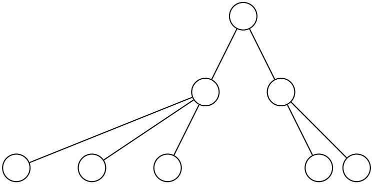


Definition

A graph G is *connected* if for any pair of vertices in $V(G)$ there is a path between them. A graph G is *disconnected* if there exists a pair of vertices in G which is not connected by a path.

Definition

A *bridge* is an edge e of a graph G such the graph $G - e$ is disconnected.



Definition

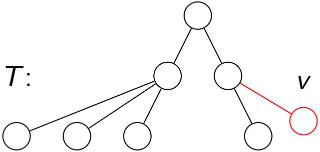
A *tree* is a connected graph where every edge is a bridge.

Theorem

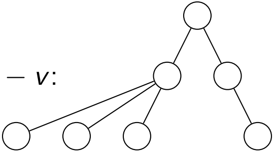
For every tree with order n and size m, we have $m = n - 1$.

Proof

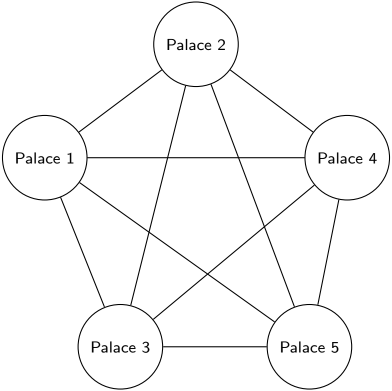
T:



$T - v$:



Problem of the Five Palaces



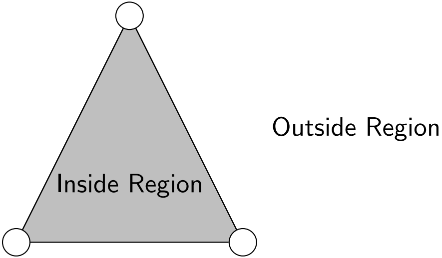
Planarity

Definition

A *planar graph* is a graph which can be drawn in a plane so that no edges intersect.

Definition

A *region* in a plane graph G is a section of the plane enclosed by edges of G.



Planarity

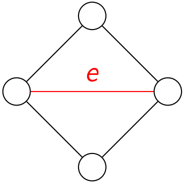
The Euler Identity

If G is a connected plane graph of order n, size m, and r regions, then $n - m + r = 2$.

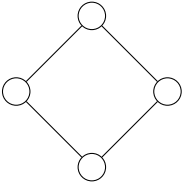
Proof

Trees: $k - (k-1) + 1 = 2$

G:



$G - e$:



$$n - (m-1) + (r-1) = 2$$

$$n - m + r = 2$$

Planarity

Theorem
For any planar graph G of size m and order $n \geq 3$, we have

$$m \leq 3n - 6.$$

Corollary
If G is a graph of order $n \geq 3$ and size m where $m > 3n - 6$, then G is nonplanar.

Planarity

Corollary

Every planar graph contains a vertex of degree 5 or less.

Theorem

The complete graph K_5 is nonplanar.

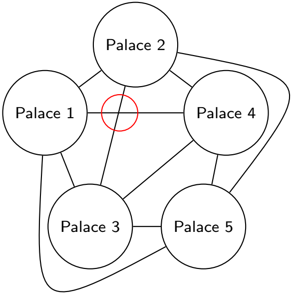
Proof

The size of K_5 is 10

The order of K_5 is 5

$$m > 3n - 6$$

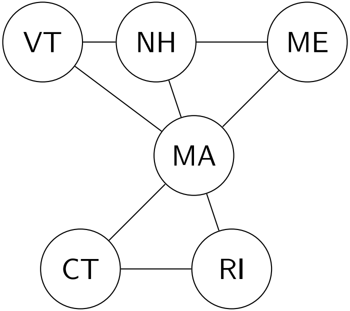
$$10 > 3(5) - 6 = 9$$



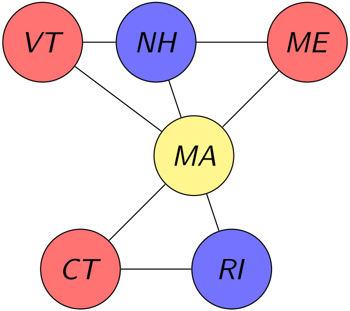
Graph Colorings

Figure: Blank Map of the New England States

Graph Colorings



Graph Colorings

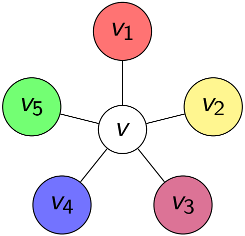


A 3-coloring of is the 3-chromatic graph G of the states ($\chi(G) = 3$).

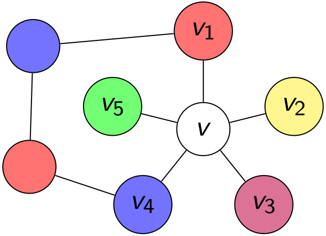
Four and Five Color Theorems

Theorem

Every planar graph is 5 - colorable



Four and Five Color Theorems



Four and Five Color Theorems

Theorem

Every planar graph is 4 - colorable

- Proven in 1976 by Kenneth Appel and Wolfgang Haken
- 1,936 cases
- 1,200 hours of computer time