A DYNAMIC MODEL OF DIRECTED NETWORK FORMATION

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Introduction

In this article, we focus on directed network formation in non-cooperative games. The model we draw on Bala and Goyal's (2000) static game model, and we thought it would be interesting to explore a dynamic model under similar circumstances. In addition, we chose to base on their model because, while complicated, they gave three probabilities – $p \in \{0.2, 0.5, 0.8\}$ – and we observed very different results between them. Therefore, we set out to find the probability of inertia that leads to the lowest number of games until the network converges.

Our main results highlight two aspects: the first is *network stability*, meaning, the equilibrium of network formation in which no player wants to change his connections. We show that there are two strict Nash equilibria in the dynamic model identical to those found in the static model. The second aspect is *convergence*; interestingly, we found that in the dynamic process the network usually converges faster than in the static process.

The field of network formation can be divided into several areas. We chose to focus on a game with multiple players where each has an individual choice. There are many economic situations that may be interesting to look at from the perspective of social network formation. These range from seeking job opportunities to creating peace agreements between countries, and even developing countries' efforts to cooperate with wealthier countries. Montgomery (1991) states that most job opportunities arise from personal relationships. Given that social networks are prevalent in many areas of our lives, it is important to understand how these networks form.

From an economic perspective, it is worth addressing several relevant questions concerning networks and their importance of network relationships to determining the outcome of economic situations. In particular, how can we predict the equilibrium of a network or the structure of the outcome network? A second question, from a self-centered perspective, is how can a single player maximise his utility? There are many ways to model networks, and we will now describe some of them as they appear in the literature.

The modeling of network formation as a game was first introduced by Aumann and Myerson (1988). They focused on value distribution amongst players in a network. To examine the process they were interested in, they first had to model periods that withheld the network formation itself. Watts (1997) was the first to model a dynamic game; she looked at the stable condition as a situation where there is a time t, after which no player will change his links. Later, Watts (2003) offered a dynamic framework for network formation, where she shows to which structure the network converges. Her model requires consent of both players to form a link. This is helpful to understand social connections or political alliances. But, to the best of our knowledge, no model offers a dynamic game where links are formed unilaterally.

A social network is formed by a set of players. Each player can be linked, or not, to any other player. There are numerous characteristics that differentiate the network's dynamics from one another. One is *direction*: the network can be either *directed* or *non-directed*. In a non-directed network two player are either both connected by a link, or both disconnected from one another. A directed network means a player is able to connect to another player allowing the latter to stay unconnected (Jackson 2003). Some models use this idea to indicate that if there is a benefit to being connected - when the network is non-directed the benefits flow in both directions. When a network is directed, benefits flow only in one direction – towards the connected player. Bala and Goyal refer to directed and non-directed models as *one-way flow* and *two-way flow* models, respectively.

Another characteristic is *cooperation*: formation of links can be *cooperative*, meaning that players being approached must accept the link being formed. Or it could be *non-cooperative*, and then there is no need for agreement. Our model applies the non-cooperative feature.

A third characteristic is the game *property*: some models assume that network formations are *static* processes, which implies that all players decide who they want to connect to without knowing the other players' decisions in each game. Other models assume that network formations are *dynamic* processes, whereby players play sequentially, meaning that the decisions of link formation are known to all other players once any player acts.

The last characteristic we acknowledge here is the player's utility function, as linear or complex (nonlinear). This function might depend on the permutation of links or simply on the number of connected players. Yet other functions include a decay parameter, whereby a player benefits from a direct link more than indirectly through other players (Jackson 2003). The utility function in our model signifies the information (also referred to as payoff) from which the player benefits via all his links.

We chose to focus on a non-cooperative directed model because it is under-explored in the literature despite its relevance to several real-life situations and potential economic implications. It explains the behaviour of internet influencers and blog writers, as well as security intelligence organisations in today's world, for example. These and other groups attempt to obtain as much information as possible at the lowest possible cost. Bloggers 'play' in a network where the sharing of information is an agreed-upon principle (involving a second or third party, or more), while in intelligence, one organisation infiltrates another without asking for consent – both modeled as the same game. We model such networks in this paper, and in doing so explain entities' utility by their cost and benefit. Benefit will therefore be also related to as information.

The Model

The main difference between our model and Bala and Goyal's (2000) model (henceforth, the original model) concerns the properties of the game. In the original model, the game was static; we converted the model to a dynamic game. We describe our model using the original model definitions and notations. For certain results we use the original model to expand on Bala and Goyal's research findings. It should be noted that we will only relate to the one-way flow version of the game.

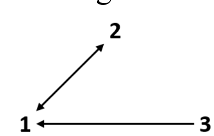
A game comprises agents (players) and a sequence of actions (or decisions). The first element of the game is an agent who seeks information throughout different actions. A game always has at least three agents for reasons that will be understood later; the set of agents will be denoted as $N = \{1, \dots, n\} \subseteq \mathbb{N}$.

A *strategy* of agent $i \in N$ is a vector $g_i = \left(g_{i,1}, \dots, g_{i,i-1}, g_{i,i+1}, \dots, g_{i,n}\right)$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. We say agent i has a *link* with j if $g_{i,j} = 1$. The link allows the information to flow only one-way, meaning that if $g_{i,j} = 1$, agent i can access the information of agent j, but not vice versa. The set of strategies of agent i is denoted by $\mathcal{G}_i$. We include only pure strategies. Since agent i can decide either to form or not form a link with each of the remaining $n-1$ agents, the number of strategies of agent i is $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times \dots \times \mathcal{G}_n$ is the space of pure strategies of all the agents.

One-Way Flow

The link $g_{i,j} = 1$ is represented by a line between agents i and j with arrowhead pointing at agent i. Figure 1 shows an example with three agents. Agent 1 is connected to agents 2 and 3, agent 2 is connected to agent 1 while agent 3 has no links.

Figure 1:



The set $N^d(i; g) = \left\{k \in N | g_{i,k} = 1\right\}$ shows all the agents with whom i formed a link. A *path* from j to i in g means that either $g_{i,j} = 1$ or other agents $j_1, \dots, j_m$ excluding i and j exist, such that $g_{i,j_1} = g_{j_1,j_2} = \dots = g_{j_m,j} = 1$. For example, in figure 1 there is a path from agent 3 to agent 2. We denote a path as "$j \xrightarrow{g} i$". The set $N(i; g) = \left\{k \in N | k \xrightarrow{g} i\right\} \cup \{i\}$, indicates all agents that i has access to their information. We use the convention where i always has a path to himself. Let $\mu_i^d : \mathcal{G} \to \{0, \dots, n-1\}$ and $\mu_i : \mathcal{G} \to \{1, \dots, n\}$ be defined as $\mu_i^d(g) \equiv \left|N^d(i; g)\right|$ and $\mu_i(g) \equiv |N(i; g)|$ for $g \in \mathcal{G}$. Here, $\mu_i^d(g)$ is the number of links formed by i while $\mu_i(g)$ is the number of agents observed by i. Denote the set of nonnegative integers as $\mathbb{Z}_+$, let $\Phi : \mathbb{Z}_+^2 \to \mathbb{R}$ be a payoff functions such that $\Phi(x, y)$ strictly increases in x and strictly decreases in y. The payoff function for every agent $\Pi_i : \mathcal{G} \to \mathbb{R}$ is:

(2.1) $\Pi_i(g) = \Phi\left(\mu_i(g), \mu_i^d(g)\right)$

Considering the assumptions of the payoff function, $\mu_i(g)$ is the *benefit* i gets from all agents he observes, and $\mu_i^d(g)$ is the *cost* of maintaining (establish or reestablish later) the links.

We assume no information decay in the transmission of information. Let $V > 0$ be the *value* of information and $c > 0$ be the cost of one link formation. A special case of (2.1) is a linear payoff function. Without loss of generality, we normalise $V = 1$, hence the function is given by:

(2.2) $\Pi_i(g) = \mu_i(g) - c \cdot \mu_i^d(g)$

This indicates that an agent's payoff is the number of agents he observes minus the total cost of links formed. Note that If $c \in (0, 1)$, then agents will form a link even for the sake of one other agent's information alone. When $c \in (1, n-1)$, to form a beneficial link agents will need to know they will access information of additional agents, If $c > n-1$, then there is no link that can produce a benefit that can overcome the cost. Hence, it is a dominant strategy to not form any links.

Nash Networks

Given a network $g \in \mathcal{G}$, We denote the network without agent i's links as g_{-i}. Thus we can say that $g = g_{-i} \oplus g_i$ where $\oplus$ means that g is the union of g_{-i} and g_i. Strategy g_i is considered as the *Best Response (BR)* of agent i to g_{-i} if

(2.3) $\Pi_i(g_{-i} \oplus g_i) \geq \Pi_i(g_{-i} \oplus g_i') \ \forall g_i' \in \mathcal{G}_i$

All the BRs to g_{-i} is denoted by the set $BR_i(g_{-i})$. Network $g = (g_1, ..., g_n)$ is a *Nash network* only if $g_i \in BR_i(g_{-i}) \ \forall i$. A *strict* Nash network exists when every agent receives a strictly higher payoff playing his current strategy than for any other strategy.

A *dynamic game* can be defined as a sequence of n decisions. Each agent in turn chooses a strategy that maintains $g_i \in BR_i(g_{-i})$. We assume agents can only observe the process that precedes their own turn in the decision-making process. We also explore what occurs when agents predict future decisions, i.e., backward induction.

The Process

From here on, a *process* refers to a game being played repeatedly in each time period $t = 1, 2, \cdots, T$. When more than one game is played repeatedly, it is reasonable to assume that an agent will sometimes choose comfort over the search of a BR. Let $r \in (0, 1)$ be the probability of agent i exhibiting *inertia*, i.e., maintaining the previous strategy. With probability $p = 1 - r$, agent i acts, therefore chooses a BR randomly from the set $BR_i(g_{-i})$. This assumption rules out the possibility of a weak Nash equilibrium as a steady state.

Results

Game Properties

Let g be a network, we call $C \subset N$ a *component* of g if for every pair of agents $i, j \in C$ we have $j \overset{g}{\to} i$ ($j \in N(i; g)$) and $\nexists\, \overline{C} \supset C$ for which this is true. C is *minimal* if C is not a component after breaking a link between two agents. A network is connected if it has a unique component. A network is called *minimally connected* if the unique component is minimal. A network can be *empty* if $N(i; g) = \{i\}$ and it can be *complete* if $N^d\{i; g\} = N \backslash \{i\}$ for all $i \in N$. A *wheel* network is one where all agents are arranged as $\{i_1, ..., i_n\}$ with $g_{i_2,i_1} = ... = g_{i_n,i_{n-1}} = g_{i_1,i_n} = 1$ and there are no other links.
Bala and Goyal proved two propositions regarding properties of Nash equilibria of the networks. The propositions, presented below, deal with the state of the network within the process and not of the process itself, which means the propositions apply in our model as well.

Proposition 1. *Let the payoffs be given by (2.1). A Nash network is either empty or minimally connected.*

Proposition 2. *Let the payoffs be given by (2.1). A strict Nash network is either the wheel or the empty network.* (a) *If* $\Phi(\hat{x}+1, \hat{x}) > \Phi(1, 0)$ *for some* $\hat{x} \in \{1, ..., n-1\}$*, then the wheel is the unique strict Nash.* (b) *If* $\Phi(x+1, x) < \Phi(1, 0)$ *for all* $x \in \{1, ..., n-1\}$ *and* $\Phi(n, 1) > \Phi(1, 0)$*, then the empty network and the wheel are both strict Nash.* (c) *If* $\Phi(x+1, x) < \Phi(1, 0)$ *for all* $x \in \{1, ..., n-1\}$ *and* $\Phi(n, 1) < \Phi(1, 0)$*, then the empty network is the unique strict Nash.*

SPE - Wheel Network in One Game

We demonstrate that, assuming an agent is able to conduct backward induction, and it is not beneficial to connect to another agent for only their information, the network reaches the wheel network which is the unique subgame perfect equilibrium (SPE).

Lemma 1. *Let the payoffs be given by (2.1). If* $\Phi(n, 1) > \Phi(1, 0)$ *and* $\exists\, k \in \{3, \dots, n\}$ *such that if* $\mu_i(g) \geq k$ *then* $\Phi(\mu_i(g), 1) > \Phi(1, 0)\ \forall\, i \in N$ *then the BR of every agent* $i \in N \backslash \{n\}$ *is* (a) $g'_i = g_{i,j} = 0\ \forall j \in N \backslash \{i\}$ *or* (b) $g''_i = g_{i,i+1} = 1$ *and no other link.*

Proof. Consider the strategy of the last agent n:
$g_{n,i} = 1\ \forall i \in \{1, ..., n\}\ if\ |N(i; g) \backslash N(n; g)| \geq k-1$. Now consider agent $n-1$'s strategy. Suppuse $\exists\, \hat{g}_{n-1} \neq g'_{n-1} \neq g''_{n-1}$ such that $\hat{g}_{n-1}$ is also a BR. This implies $\exists\, j \in \{1, ..., n-2\}$ such that $\hat{g}_{n-1,j} = 1$. This implies $\mu_j(\hat{g}) \geq k-1$. But then either (a) $j \in N(n; g)$ or that

exists agent $m \neq j, n \in N$ such that
(b) $j \notin N(n; g)$ & $m \in N(n; g)$ & $|N(m; g)| > |N(j; g)|$.
if (a) then we obtain $\Phi_{n-1}(\mu_n(\hat{g}) + 1, 1) \geq \Phi_{n-1}(\mu_j(\hat{g}) + 2, 1) > \Phi_{n-1}(\mu_j(\hat{g}) + 1, 1)$
if (b) then we obtain $\Phi_{n-1}(\mu_n(\hat{g}) + 1, 1) \geq \Phi_{n-1}(\mu_m(\hat{g}) + 2, 1) > \Phi_{n-1}(\mu_m(\hat{g}) + 1, 1) > \Phi_{n-1}(\mu_j(\hat{g}) + 1, 1)$. Either way, we show that the payoff from maintaining a link with agent n is strictly higher than the payoff from any other link. Agent $n-1$ will not maintain both links since $|N(j; g) \backslash N(n; g)| < k - 1$. Now consider agent $n-2$. Due to the same argument it is deduced that the payoff from maintaining a link with agent $n-1$ is strictly higher than the payoff from any other link. With the same argument the proof follows up to the first agent. □

Theorem 1. *Let the payoffs be given by (2.1). If* $\Phi(n, 1) > \Phi(1, 0)$ *and* $\exists\, k \in \{3, \ldots, n\}$ *such that if* $\mu_i(g) \geq k$ *then* $\Phi(\mu_i(g), 1) > \Phi(1, 0)\ \forall\, i \in N$ *then the wheel network is unique SPE. If* $\Phi(n, 1) < \Phi(1, 0)$ *then the empty network is unique SPE.*

Proof. Consider the last agent's (n) strategy:
$g_{n,i} = 1\ \forall i\ \in N\ if\ |N(i; g) \backslash N(n; g)| \geq k - 1$.
Now consider every agent $i \in N \backslash \{n\}$, it follows from lemma 1 that his strategy is one of the follwing:
1. $g'_i = (g_{i,1} = 0, \ldots, g_{i,i-1} = 0, g_{i,i+1} = 0, \ldots)$ if he knows that agent $i+1$ will not connect to anyone.
2. $g''_i = (g_{i,1} = 0, \ldots, g_{i,i-1} = 0, g_{i,i+1} = 1, \ldots)$ if he knows that agent $i+1$ will connect to someone.
We will now give an example before we present the rest of the proof with $k = 3$; in that case, agent 1 knows that if he chooses to connect with any other agent, then in agent n's turn the latter will have at least one agent (agent 1) worth connecting to. This implies that agent n will definitely connect to someone and it follows from proposition 1 that every other agent will connect to the next agent.
Consider agent 1: if he chooses to connect to any other agent (e.g. agent 2) he knows that the approached agent (2) will know that if he chooses to connect to a different disconnected agent (e.g. agent 3), his approached agent (3) will know that if he chooses… and so on for $k-2$ times. The $k-2$ agent will know that if he chooses to connect to any disconnected agent, in agent n's turn he will have at least one agent (agent 1) worth connecting to. This implies that agent n will definitely connect to someone, and it follows from lemma 1 that every other agent will connect to the following agent. This implies $\forall j \in \{2, \ldots, n\}\ N(1; g) \supset N(j; g)$, then agent n's unique BR is $g_n = (g_{n,1} = 1,\ g_{n,2} = 0, \ldots, g_{n,n-2} = 0,\ g_{n,n-1} = 0)$. In this way, we obtained the wheel network.

When $\Phi(n,1) < \Phi(1,0)$ it is follows from proposition 2 that the only strict Nash is the empty network. □

These last results are highly significant. They suggest the network converges into a steady state within one game. This is valuable in many situations. At the same time, the backward induction assumption is strong, which is why we chose to present the process relaxing it.

Process Convergence

Bala and Goyal's main theorem concerning the convergence of the process of the game. The theorem is:

Theorem. *Let the payoff functions be gien by equation* (2.1) *and let* g *be the initial network.* (a) *If there is some* $\hat{x} \in \{1,...,n-1\}$ *such that* $\Phi(\hat{x}+1,\hat{x}) \geq \Phi(1,0)$*, then the dynamic process conerges to the wheel network, with probability 1.* (b) *If instead,* $\Phi(x+1,x) < \Phi(1,0)$ *for all* $x \in \{1,...,n-1\}$ *and* $\Phi(n,1) > \Phi(1,0)$*, then the process conerges to either the wheel or the empty network, with probability 1.* (c) *Finally, if* $\Phi(x+1,x) < \Phi(1,0)$ *for all* $x \in \{1,...,n-1\}$ *and* $\Phi(n,1) < \Phi(1,0)$*, then the process conerges to the empty network, with probability 1.*

The proof uses the inertia parameter to show that in a given period t, there is a positive probability that all agents but one - i, will exhibit inertia. That allows agent i to play his BR - $(\hat{g})$. At period $t+1$, the network will be $g^t_{-i} \oplus \hat{g}$ i.e., the network g^{t+1} will be identical to the network at period t (g^t) except for i's links. This idea is possible in our settings, too. If positive inertia exists in a process, there is a possibility that all agents but one will exhibit inertia in a certain game which is the reason the proof of the theorem applies in our model as well. In a game with no inertia there is a posibility that the network will not converge to any steady state, because each agent plays every turn and constantly affects all other agents' BRs.

Speed of Convergence

In the dynamic process, there is less miscoordination – when each agent plays, he is aware of all preceded actions and not the end of the last game. Thus, we hypothesise networks will converge faster. To test just this hypothesis, we created a code that attained Bala and Goyal's results in the one-way flow model (see table 1) using (2.1) payoff function. We ran the code simulating probabilities of action (and inertia) $p \in \{0.2, 0.5, 0.8\}$ for $c \in \{0.5, 1.5\}$ one cost in each category: $(0,1)$ and $(1,2)$ based on the number of agents, as they did. We then adjusted the simulation to fit the dynamic process, too, simulating 500 games for each condition in both models. In the static process, we followed Bala and Goyal's settings. In the dynamic process, we counted the number of games played until no agent changed his connections, as suggested by Watts (1997). We presumed that a network, in which all agents did not change their preferences after two games, had converged. In games with high inertia two games were not enough to declare convergence. To insure at least 99% of all processes converged, we required the number of games in which the network remained the same to equal[1] $max\,(2, log_r 0.01)$. In any case, the process was stopped after 1,000 games due to the interest in rapid convergence and the limit of computational power.

Table 1: Replication of Bala and Goyal's Simulations[2]

N	$c \in (0,1)$						$c \in (1,2)$					
	$p = 0.2$		$p = 0.5$		$p = 0.8$		$p = 0.2$		$p = 0.5$		$p = 0.8$	
	BG	R	BG	R	BG	R	BG	R	BG	R	BG	R
3	15.29	15.52	7.05	7.58***	6.19	6.44	8.58	9.67***	4.50	6.39***	5.51	7.41***
	(0.53)	(0.52)	(0.19)	(0.20)	(0.19)	(0.19)	(0.35)	(0.33)	(0.17)	(0.18)	(0.24)	(0.28)
4	23.23	22.66	12.71	12.36	13.14	13.49	11.52	14.21***	5.98	7.75***	6.77	8.30***
	(0.68)	(0.74)	(0.37)	(0.39)	(0.42)	(0.50)	(0.38)	(0.38)	(0.18)	(0.19)	(0.22)	(0.23)
5	28.92	29.77	17.82	19.05***	28.99	30.26	15.19	18.24***	9.16	10.35***	14.04	14.76
	(0.89)	(0.83)	(0.54)	(0.57)	(1.07)	(1.18)	(0.40)	(0.42)	(0.27)	(0.26)	(0.59)	(0.48)
6	38.08	36.97	26.73	27.27	55.98	56.22	19.93	22.83***	12.68	14.09***	28.81	27.51
	(1.02)	(0.99)	(0.91)	(0.96)	(2.30)	(2.17)	(0.57)	(0.58)	(0.41)	(0.37)	(1.16)	(1.05)
7	45.90	46.66	35.45	38.49***	119.57	110.27*	25.46	25.62	18.51	19.42	57.23	57.16
	(1.3)	(1.34)	(1.19)	(1.37)	(5.13)	(4.65)	(0.71)	(0.58)	(0.57)	(0.59)	(2.29)	(2.46)
8	57.37	55.11	54.02	55.60	245.70	229.59	27.74	31.4***	26.24	27.54	121.99	133.36*
	(1.77)	(1.65)	(2.01)	(1.97)	(10.01)	(10.09)	(0.70)	(0.80)	(0.89)	(0.90)	(5.62)	(5.87)

Appendix A[3] summarises the results of the average number of games until convergence is reached in both static and dynamic models. We found that (a) many of our results and Bala and Goyal's were in fact quite similar (in other words, the difference was not statistically significant), and amongst the statistically significant results the differnces were not economically significant (similar scale) – thus validating our results (see Appendix C). The results differ slightly, which is to be expected because the simulation randomises the initial network. (b) We found first-order stochastic dominance relationships between the convergence in the dynamic and static processes. The former is indeed faster than the latter.

We observe that a network with 6 players when $p = 0.5$ converges after 6.20 (0.15) games

[1] The higher value between two and the expression that represents the power in regard to r that leads to 1% error.

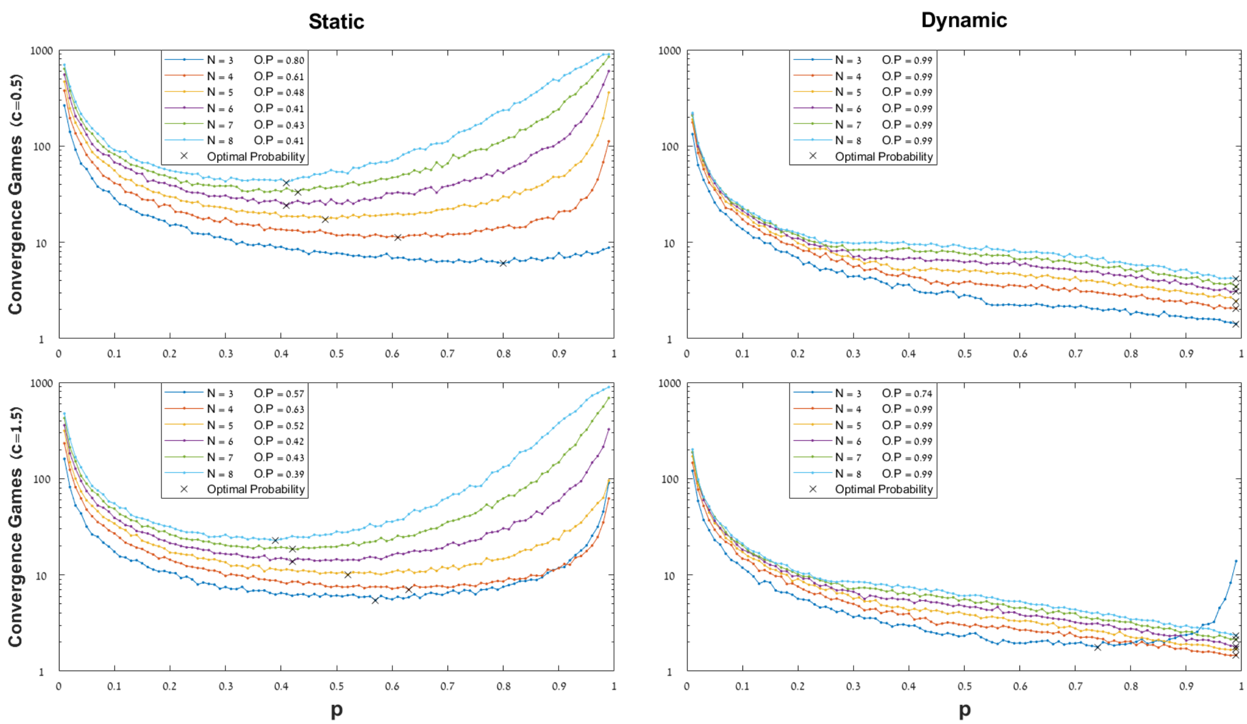
[2] BG and R refer to Bala and Goyal's simulations and the replicated simulations, respectively. Note that it is a positive result that the table has quite a low number of asterisks since we tried to replicate Bala and Goyal's simulations.

[3] See online appendices here.

in the dynamic process on average, but only after 27.27 (0.96) games in the static process.

The table concurs with the insights noted above, but also raises the question: what is the inertia parameter that will result in the fastest convergence? To answer this, we simulated the process for 99 different values of inertia $r = 1 - p \in \{0.01, 0.02, ..., 0.98, 0.99\}$ hoping to find the fastest convergence (see Figure 1).

Figure 1: Games until Convergence[4]



Since we assume complete information, the fastest convergence in the dynamic process occurs with minimum inertia ($p = 0.99$, or very close) because each agent can play his BR every turn, as expected. In the static process the inertia leading to the fastest convergence tends to rise with the number of players in the game. In other words, the higher the number of players, the harder it is for each one to play his BR since they all make decisions simultaneously – a decision in period t may not result in the outcome intended. For example, in the extreme case of $p = 1$ (no inertia), if in period t the network is empty when $c < 1$, all agents will choose to connect with all other agents in the network. But they will all realise this before their next move, and thus will choose to stay connected to only one other agent since all the agents are now connected to all the others in the network. The more agents participating in

[4]The graphs show the average number of games until convergence in both static and dynamic processes, on the left and the right respectively, and processes where cost equals 0.5 and 1.5, on the top and the bottom respectively.

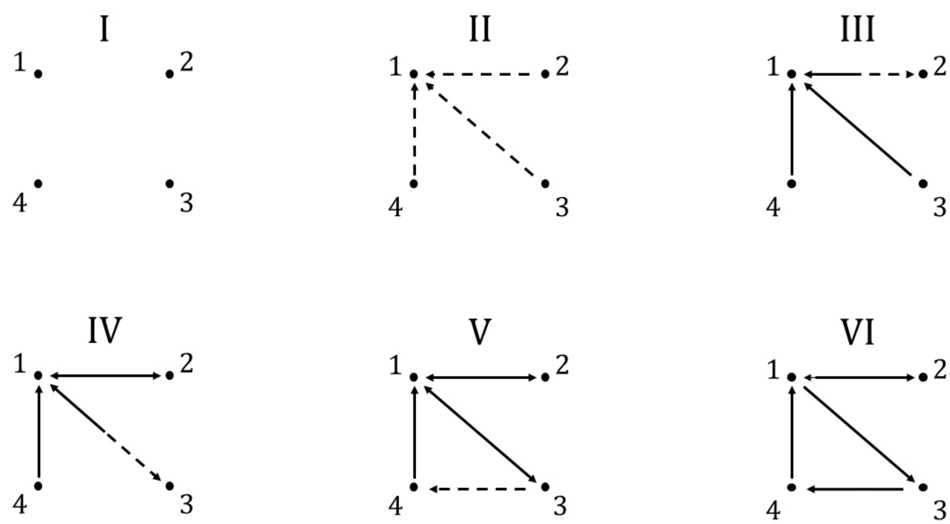
the game, the higher the chance that they will establish extra links, a process that could go on ad infinitum. As inertia increases, not all agents play every game and it becomes easier to coordinate. The logarithmic scale was chosen since the number of possible networks rises exponentially with the number of agents $\left(2^{n\cdot(n-1)}\right)$.

An anomaly is seen in the case of three agents and when the cost equals 1.5. As noted before when there is no inertia there is a chance that the network will not converge. When inertia is relatively low the process could take a rather long time until convergence is reached. In the three agents case there are only 64 possible networks to begin with. Thus, when 500 simulations were run each of the networks was raffled multiple times leading to longer processes which raised the average number of games until convergence. In the case of four agents, there are already 4,096 possible initial networks leading to a much higher chance of avoiding a network that results in long processes as seen above. This follows for five agents and more.

Payoff

In Figure 2 we discuss another result of the dynamic model, regarding the payoffs earned by the players as a function of their location in the game. For simplicity, let's think of a game with $|N| = 4$ that starts from the empty network, when $c = 0.5$, $p = 1$, and a payoff function given by equation (2.2) (linear payoff).

Figure 2: Game Example



In agent 1's first turn, his BR is to connect with all other agents in the network; his payoff will be 4–$0.5 * 3 = 2.5$. In the next agent's turn, the only BR is to connect exclusively with agent 1 since he is connected to all players in the network. Agent 3 now has two BRs: he can connect to either agent 1 or agent 2 to obtain all available information in the network. Agent

4 can then connect with any other agent for the same result. The payoffs of agents 2-4 will be $4–0.5 * 1 = 3.5$. In Figure 2, agent 3 chooses to connect with agent 1, and agent 4 chooses to connect with agent 3. This allows agent 1, in his next turn, to not reestablish (release) his link with agent 3 since he is now getting agent 3's information through agent 4. An agent's payoff is the sum of his payoff in each game. The difference between the agents' payoff is due to agent 1's BR in the first game, when he 'bears' the cost of connecting the entire network. We made three important assumptions here: first, that the process starts from the empty network, second, that there is no inertia ($p = 1$), and third, $c < 1$. We now ask: on average, is it worse to be the first agent in a network?

When the process starts from a randomised network, the costs and benefits vary rapidly according to the starting network. For example, if the initial network is one in which all agents are connected directly to all others, the agent who plays first is in the best position since he can release the links with all others except for one. In this case, the last agent to play gets the worst outcome. We, therefore, hypothesise that simulations will show no difference in the payoff between agents' positions.

If we start from the empty network, when inertia rises, there is a greater chance that agent 1 will not be the first to play. Here, the first agent to play will 'bear' the cost and therefore will get a smaller payoff than all the others who played, but he will still get a larger payoff than those who did not play in the first game. The chance of agent $i \in N$to play first is given by $(1 - p)^{(i-1)} \cdot p$. Since $p < 1$, agent i has a bigger chance than agent j to play first for all $i < j$. When many games are played, the number of games in which each agent exhibits inertia should be relatively similar. Hence, we hypothesise that when many games are played, on average agent 1 will be worse off than all other agents.

The last assumption we made regards $c \in (0, 1)$. When $c > 1$ and the initial network g is the empty network, it is already a strict Nash equilibrium (Proposition 2). This means all agents' payoffs will be equal to $\Pi_i(g_1) = \mu_i(g_1) - \mu_i^d(g_1) \cdot c = 1 - 0 \cdot c = 1$. When the process starts from a randomised network, as discussed above, the position of an agent makes no difference.

To test these hypotheses, we ran 1,000 simulations of initial randomised and empty networks in the dynamic process for each combination of $p \in \{0, 0.2, 0.5, 0.8\} \times |N| \in \{3, 4, 5, 6, 7, 8\}$ – a total of 480,000 simulations. We compared the averaged total payoff from the processes of each agent (see Appendix B). The simulation indeed shows a significant difference between agent 1 and all others when the process starts from the empty network (see Appendix D.1) and shows no significant differences when the process starts from a randomised network (see Appendix D.2). As shown, when $|N| = 6$ and $p = 0.5$ the average payoff of agent 1 is 50.89 (SE 0.64), and for all other agents' payoff are between 52.11-53.12

(SE 0.65), and when $|N| = 7$ and $p = 0.8$ the average payoff of agent 1 is 54.73 (SE 0.61), and for all other agents payoff are between 55.88-56.08 (SE 0.60-0.61) – these results are statistically significant (p-value << 0.001).

Limitations of the Research

The computational power rose exponentially with the number of agents, limiting both our research and the possible simultaneous paths we could take using the code at a given time. Another limitation in our model was the structure of the payoff equation, which did not consider any decay of the information passing through more links, amongst other issues.

Numerous assumptions were noted above that together comprised a frame for our model. On the one hand, this offered us a set of boundaries within which we could explore a range of possibilities; on the other, we were limited by those same boundaries from expanding our ideas in other directions. Should the research continue in the future, we believe next steps would include more complex (non-linear) payoff functions. Further research might also explore adding a condition of consent for the creation of the link by the agent approached. Another option could be to modify the original model to incorporate the dynamic process of the two-way flow model.

Conclusions

In this paper, we modified the work of Bala and Goyal on a non-cooperative model of network formation. Specifically, we explored the dynamic version of the game. We replicated Bala and Goyal's simulations and reported our results for the optimal probability of inertia that leads to the fastest convergence to a strict Nash equilibrium.

A major founding of ours was to prove that with the addition of one assumption in the dynamic process – agents being able to predict future decisions by others, there is always a unique SPE, if the cost is such that a link is not beneficial if it transfers information from only one other agent. We show that the SPE equilibrium is reached within a single game. We also found numerically that the dynamic process leads to faster convergence for any given p. As expected, the optimal p for the dynamic process is highest possible (0.99). We also showed that starting from an empty network the first agent's payoff will be lower. Starting from a randomised network we reach the conclusion that location in the network has no difference on the total payoff, on average. We believe our modified model has value in the realm of theoretical economic research, and it would be interesting to see it applied to a broader variety of real-life situations.

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Shared files can be found here.