METRIC SPACES

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Autumn, 2017[1]

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[1]Last LaTeXed on: 2/10/2017.

0. Some notation and terminology

1. Some set theory.

- If an element a belongs to a set A, we write $a \in A$; and if not we write $a \notin A$.
- If A is a subset of B (perhaps equal to B), we write

$$A \subseteq B \quad (\text{or} \quad B \supseteq A).$$

- Let A and B two subsets of a set X. Then,

$$A = B \qquad \text{iff}^1 \qquad A \subseteq B \qquad \text{and} \qquad B \subseteq A.$$

- Let A and B be two sets. Then,
  (1) The *union* of A and B, $A \cup B$, is the set defined by

$$A \cup B = \{x \in X \ : \ x \in A \quad \text{or} \quad x \in B\}.$$

  (2) The *intersection* of A and B, $A \cap B$, is the set defined by

$$A \cap B = \{x \in X \ : \ x \in A \quad \text{and} \quad x \in B\}.$$

  If $A \cap B = \emptyset$, then A and B have no points in common and A and B are said to be *disjoint*.

- Let A be a subset of a set X. Then, the *complement of A in X*, A^c (also $X - A$ or $X \backslash A$) is the set

$$A^c = \{x \in X \quad \text{such that} \quad x \notin A\}.$$

- Let I be any set (finite or infinite). If for each $i \in I$, we are given a subset A_i of a set X, then the collection of sets $\{A_i\}_{i \in I}$ is called an *indexed family of subsets of X*, and I is called an *index set* (or indexing set).

Given a indexed family of sets $\{A_i\}_{i \in I}$ (or $\{A_i \ : \ i \in I\}$), we denote the union and intersection over all indexes $i \in I$ respectively by

$$\bigcup_{i \in I} A_i, \qquad \text{and} \qquad \bigcap_{i \in I} A_i{}^2$$

the union and intersection over all $i \in I$.

The following identities are true:

$$(1) \qquad \left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c.$$

$$(2) \qquad \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

The identities (1) and (2) are known as De Morgan's laws.

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[1] Iff means "if and only if".

[2] Sometimes we will write just $\cap A_i$ and $\cup A_i$ for $\cap_{i \in I} A_i$ and $\cup_{i \in I} A_i$, respectively.

- Let A and B be sets. The *Cartesian product* of A and B, $A \times B$, is the set defined by

$$A \times B = \{(a,b) \ : \ a \in A \quad \text{and} \quad b \in B\}.$$

This definition generalizes to the cartesian product of a finite number of sets in the following way. If $\{A_1, \dots, A_N\}$ is a finite collection of $N \in \mathbb{N}$ sets, then the cartesian product of $A_1, \dots, A_N$ is the set

$$A_1 \times \cdots \times A_N = \{(a_1, \dots, a_N) \ : \ a_i \in A_i, \ \text{for each } i = 1, \dots, N\}.$$

2. Funtions/Map.

Let A and B be sets.

- A *function* or *map* f between the sets A and B, denoted by $f : A \longrightarrow B$, is a correspondence that associates with each element $a \in A$ an (unique) element $f(a)$ in B. We call A the *domain* of f (Dom(f)) and B the *codomain* of f (Codom(f)).

- Let $f : A \longrightarrow B$ be a function.

- For any $C \subseteq A$, the image of C under f, $f(C)$, is the set
$$f(C) = \{f(c) \ : c \in C\},$$
or equivalently
$$f(C) = \{b \in B \ : \ b = f(c) \quad \text{for some} \quad c \in C\}.$$
- For any $D \subseteq B$, the inverse image of D under f, $f^{-1}(D)$, is the set defined by
$$f^{-1}(D) = \{a \in A \ : \ f(a) \in D\}^3$$

- Let $f : A \longrightarrow B$ be a function, $\{C_i\}_{i \in I}$ be an indexed family of subsets of A, and $\{D_i\}_{i \in I}$ be an indexed family of subsets of B. Then, the following formulae are true

$$f\left(\textstyle\bigcap_{i \in I} C_i\right) \subseteq \textstyle\bigcap_{i \in I} f(C_i) \qquad f\left(\textstyle\bigcup_{i \in I} C_i\right) = \textstyle\bigcup_{i \in I} f(C_i)$$

$$f^{-1}\left(\textstyle\bigcap_{i \in I} D_i\right) = \textstyle\bigcap_{i \in I} f^{-1}(D_i) \qquad f^{-1}\left(\textstyle\bigcup_{i \in I} D_i\right) = \textstyle\bigcup_{i \in I} f^{-1}(D_i)$$

$$f^{-1}(B - D_i) = A - f^{-1}(D_i).$$

The equality does not necessarily hold in the first of these formulae.

- Let $f : A \longrightarrow B$ be a function. Then,

f is *injective* if and only if whenever $f(a) = f(a')$ for $a, a' \in A$, then $a = a'$.
f is *surjective* (or onto) if and only if $f(A) = B$, that is

$$\forall \, b \in B \quad \exists \, a \in A \quad \text{such that} \quad f(a) = b.$$

[3]Do not confuse with the notation of the inverse function of a function f!!

f is a *bijection* (or one-one) if f is injective and surjective.

- Finally, we use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ to denote the sets of natural numbers, integers, rational numbers and real numbers. We often refer to $\mathbb{R}$ as the *real line* and to $\mathbb{R}^2$ as the *plane*.

1. Introduction

Most of the ideas and concepts we will be covering in this part of the course arise from extending the notions of continuity and convergence on the real line to more general spaces called *metric spaces*, a class of spaces which includes many of the spaces used in analysis and geometry.

We will see that many of the classical notions and results for real numbers and functions of a real variable can be made sense of in a much wider context - For example, we might want to consider continuous functions on a more general set X (the elements of X could now be functions themselves!). What do we mean by continuity in this context?

The broadening of perspective provided by the study of topology, in particular of *metric spaces* is huge, and the underpinning concepts are fundamental to most of modern mathematics.

We will begin by reviewing and discussing some of the concepts, properties and results of real numbers and continuous real functions of one real variable. Then, we will abstract these ideas considerably by considering much more general sets X.

Let us reflect on the definition of continuity of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point. Recall the following definition:

Definition. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that f is continuous at a point $a \in \mathbb{R}$ if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \qquad |x - a| < \delta.$$

Roughly speaking, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$ if we can make the distance between $f(x)$ and $f(a)$ as small as we please by choosing x so that its distance from a is sufficiently small, in other words, to say a function f is continuous at $a \in \mathbb{R}$ is to say that we can make $f(x)$ and $f(a)$ arbitrarily "close" by making x and a "close".

What is it that makes the notion of continuity really work? The notion of "closeness", that is the notion of "distance".

Notice that the above definition makes no reference to the fact that f is defined on $\mathbb{R}$ with values in $\mathbb{R}$ except for the definition of "distance" in $\mathbb{R}$. Thus, if we can find an appropriate notion of "distance" on $\mathbb{R}^n$, we may give the definition of what it means for a function f to be a continuous function on $\mathbb{R}^n$, or any other spaces rather than $\mathbb{R}$.

How would one proceed? By analogy with the case of real functions of one real variable, we could say that:

- A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous at the point $(a_1, a_2) \in \mathbb{R}^2$ if we can make the distance between $f(x_1, x_2)$ and $f(a_1, a_2)$ (this is $|f(x_1, x_2) - f(a_1, a_2)|$) as small as we please by choosing (x_1, x_2) so that the distance from (a_1, a_2) (that is $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$) sufficiently small.

- A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous at a point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ if we can make the distance between $f(x)$, with $x = (x_1 \ldots x_n) \in \mathbb{R}^n$, and $f(a)$ (that is $|f(x) - f(a)|$) as small as we please by choosing the distance between $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$ sufficiently small.

We see that, in order to generalise the definition of continuity at a point for real-valued functions of n-real variables, we need a definition of "distance" between $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$.

Observe that

$$
\begin{array}{lll}
n = 1 & \text{line} & |x - y| \\
n = 2 & \text{plane} & \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} \\
n = 3 & \text{space} & \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} \\
\vdots & \vdots & \vdots
\end{array}
$$

It is natural to define the "Euclidean distance" in $\mathbb{R}^n$ to be

$$\sqrt{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2}$$

for any $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. In what follows, very often we refer to the Euclidean metric on $\mathbb{R}^n$ as the *usual or standard metric/distance* on $\mathbb{R}^n$.

With this notion of distance in $\mathbb{R}^n$, we have the following definition

Definition. A real-valued function of n real variables $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous at $a = (a_1, \ldots, a_n)$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \qquad \sqrt{(x_1 - a_1)^2 + \ldots (x_n - a_n)^2} < \delta.$$

This suggests that we can get a sensible definition of continuity of a function

$$f : X \longrightarrow Y$$

at a point $a \in X$ provided that we have a sensible idea about the distance between points in X and likewise in Y.

A metric space consists of a (non-empty) set X and a "distance" function on X

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying some properties.[4]

[4]Recall that $X \times X$ is the *Cartesian product* of X with itself defined as the set

$$X \times X = \{(x, x') \ : x \in X \text{ and } x' \in X\}.$$

2. Metric spaces: definition and examples

The precise definition of a metric space is the following:

Definition 2.1. *A metric space (X, d) consists of a (non-empty set) X together with a function*

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following properties

(M1) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$
(M2) $d(x, y) = d(y, x)$
(M3) $d(x, y) \leq d(x, z) + d(z, y)$ *(Triangle inequality)*

for all x, y and z in X.

A function $d : X \times X \longrightarrow \mathbb{R}$ satisfying (M1)-(M3) is called a metric or distance function on X. The elements of X are called *points*. Finally, the condition (M3) is the so-called *triangle inequality*.

We will sometimes ignore the pair (X, d) and write "X with distance d", or just "Let X be a metric space".

We continue giving some examples of different metrics and metric spaces.

Examples.

1. $\mathbb{R}$ with the distance $d(x, y) = |x - y|$, for any x and y in $\mathbb{R}$.

2. $\mathbb{R}^2$ with the distance

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

for all (x_1, x_2), (y_1, y_2) in $\mathbb{R}^2$.

3. **Euclidean n-space**. Let $X = \mathbb{R}^n$ and d be the function defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$.

It is easy to see that (M1) and $(M2)$ are satisfied (both properties are an immediate consequence of the definition of d). To check (M3): let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, and let all the summations be over $i = 1, 2, \ldots n$. Then we have to prove

$$\sqrt{\sum (x_i - y_i)^2} \leq \sqrt{\sum (x_i - z_i)^2} + \sqrt{\sum (z_i - y_i)^2}. \tag{2.1}$$

To this end, define $r_i = x_i - z_i$ and $s_i = z_i - y_i$ for $i = 1, 2, \ldots n$, so that $r_i + s_i = x_i - y_i$. With this definition (2.1) rewrites

$$\sqrt{\sum (r_i + s_i)^2} \leq \sqrt{\sum r_i^2} + \sqrt{\sum s_i^2},$$

which, after expending both sides, developing the square $(r_i+s_i)^2$ and squaring both sides of the obtained inequality becomes

$$\begin{aligned}
\sum (r_i+s_i)^2 &\le \sum r_i^2 + \sum s_i^2 + 2\sqrt{\left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right)} \Leftrightarrow \\
\sum r_i^2 + \sum s_i^2 + 2\sum r_i s_i &\le \sum r_i^2 + \sum s_i^2 + 2\sqrt{\left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right)} \Leftrightarrow \\
\sum r_i s_i &\le \sqrt{\left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right)} \Leftrightarrow
\end{aligned}$$

$$\boxed{\left(\sum r_i s_i\right)^2 \le \left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right)}$$

The framed inequality is the so-called *Cauchy-Schwarz's Inequality.*

Proof of the Cauchy-Schwarz's inequality. Define the function f by

$$f(x) = \sum (r_i + x s_i)^2 = \sum r_i^2 + 2x \sum r_i s_i + x^2 \sum s_i^2, \qquad \forall\, x \in \mathbb{R}.$$

Notice that $f(x)$ and is a polynomial function of degree two in the variable x and $f(x) \ge 0$. Therefore, $f(x)$ can only have one real root, thus the associated discriminant has to be ≤ 0, that is

$$b^2 - 4ac = \left(2\sum r_i s_i\right)^2 - 4\left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right) \le 0,$$

or equivalently

$$\left(\sum r_i s_i\right)^2 \le \left(\sum r_i^2\right)\cdot\left(\sum s_i^2\right)$$

as desired.

$\square$

We call $\mathbb{R}^n$ with this metric the Euclidean n-space and the distance d is called the Euclidean distance on $\mathbb{R}^n$. We will usually refer to the Euclidean distance on $\mathbb{R}^n$ as the *usual* or *standard metric* on $\mathbb{R}^n$.

4. **Discrete metric spaces.** Let X be any non-empty set, and define d by

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y, \end{cases}$$

for any x and $y \in X$.

For any set X, this metric d is called the discrete metric.

5. Different metrics on the same set. The following examples show that in general there may be many possible distinct metrics defined on the same set.

Consider $\mathbb{R}^n$, and the functions d_1, d_2 and d_∞ defined as follows

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d_\infty(x,y) = \max\{|x_1 - y_1|,\ |x_2 - y_2|\}$$

for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

The functions d_1, d_2 and d_∞ are metrics on $\mathbb{R}^2$.

We have already seen that d_2 is a metric on $\mathbb{R}^2$. We continue to check that $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfied the conditions (M1)-(M3) in the definition of a metric (see Definition 2.1).

Indeed, notice that conditions (M1) and (M2) easily follow from the definition of d_1. Only (M3) remains to be proved. To this end, let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ be arbitrary points in $\mathbb{R}^2$. Then, by using the triangle inequality for real numbers, we get that

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)| \\ &\le |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &= d_1(x,z) + d_1(z,y). \end{aligned}$$

Thus, d_1 is a metric on $\mathbb{R}^2$. We leave as an exercise to check that d_∞ defines also a metric on $\mathbb{R}^2$.

Notice, that from the definition of d_1, d_2 and d_∞, it is clear how to define analogues of these metrics on $\mathbb{R}^n$ for any $n \in \mathbb{N}$. The proofs that the axioms (M1)-(M3) hold are similar to the above.

6. Metric subspaces. Let (X, d) be a metric space and $Y \subseteq X$ be a (non-empty) subset of X. Let

$$d\,|_Y : Y \times Y \longrightarrow \mathbb{R}$$

be the restriction of d to $Y \times Y$, that is

$$d\,|_Y\,(x, x') = d(x, x') \qquad \forall\, x, x' \in Y.$$

Notice that the metric space axioms hold for $d\,|_Y$ since they hold for d.

The metric space $(Y, d\,|_Y)$ is called <u>a subspace of X</u>, and $d\,|_Y$ is called <u>the metric on Y induced by d</u>.

Sometimes, with some abuse of notation, we will only write d for $d\,|_Y$.

The above examples of metrics and metric spaces are not the only examples. More interesting examples, in particular from the point of view of applications, come from considering the space X to the a collection of functions[5].

7. A metric on a space of functions. Let X be the set of all bounded functions $f : [a, b] \longrightarrow \mathbb{R}$. Let d be defined as follows

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \tag{2.2}$$

for any f and g in X. Then (X, d) is a metric space.

First, note that the righthand side of (2.2) exists: Given any f and g bounded (real-valued) functions on the interval $[a, b]$, there exists constants K_1 and K_2 such that

$$|f(x)| \leq K_1 \qquad \text{and} \qquad |g(x)| \leq K_2 \qquad \forall\, x \in [a, b].$$

Therefore, by the triangle inequality,

$$|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq K_1 + K_2 \qquad \forall\, x \in [a, b],$$

so that the set

$$\{|f(x) - g(x)| \mid x \in [a, b]\}$$

is a (non-empty) set bounded from above by $K_1 + K_2$. Thus, by using the Completeness axiom, this set has a supremum.

To see that d is a metric on X, we have to check that d satisfies the conditions (M1)-(M3) in the definition of a metric. Once again, from the definition of d it is a simple exercise to check that d satisfies the conditions (M1) and $M2$. To check (M3):

Let f, g and h in X and $x \in [a, b]$. Then, by using the triangle inequality for real numbers, we have the following chain of inequalities

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - h(x)| + |h(x) - g(x)| \\ &\leq \sup_{x \in [a,b]} |f(x) - h(x)| + \sup_{x \in [a,b]} |h(x) - g(x)| \\ &= d(f, h) + d(h, g). \end{aligned}$$

Taking the supremum over all $x \in [a, b]$ gives

$$d(f, g) \leq d(f, h) + d(h, g).$$

The metric d is called the <u>supremum metric</u> (sup-metric for short) or <u>the uniform metric</u>. We write[6] $\mathcal{B}([a, b], \mathbb{R})$ for X, that is

$$\mathcal{B}([a, b], \mathbb{R}) = \{f : [a, b] \longrightarrow \mathbb{R} \mid f \text{ is a bounded function }\}.$$

[5] We can take some collections of functions and decide to treat it as a space, calling the individual functions "points" and putting a metric on the collection.

[6] This notation is not uniformly accepted.

We refer the reader to the problems sheets for another example of a metric defined on a space of functions.

3. Balls, open, closed and bounded sets

When studying the notions of continuity or differentiability for real-valued functions of one-real variable we often talked about open or closed intervals or closed bounded intervals etc (recall, for example, the statement of the Intermediate Value Theorem, or the result that asserts that any continuous function defined on a closed bounded interval is bounded).

What are the corresponding notions in general in a metric space (where we do not have intervals)? What about the notion of open/closed set?

Example. What is the difference about the intervals $(0,1)$, $(0,1]$ and $[0,1]$?

Every point a in the interval $(0,1)$ has an open interval about it included in $(0,1)$, say $(a-\varepsilon, a+\varepsilon)$ for some small enough $\varepsilon > 0$. To this end, it suffices to take $\varepsilon < \max\{|a-1|, |a|\}$. Notice that the open interval $(a-\varepsilon, a+\varepsilon)$ is the set

$$(a-\varepsilon, a+\varepsilon) = \{x \in \mathbb{R} \mid |x-a| < \varepsilon\} = \{x \in \mathbb{R} \mid d(x,a) < \varepsilon\}$$

where d denotes the standard/usual metric on $\mathbb{R}$.

The following definition generalises the notion of open interval about a point on the real line to the setting of metric spaces.

Definition 3.1. *Let (X,d) be a metric space. Let $a \in X$ and $r > 0$. The open ball centred at a with radius r is the set*

$$B_r(a) = \{x \in X \mid d(x,a) < r\}.$$

Notice that the definition of $B_r(a)$ depends on d: different metrics will have different open ball. When we want to make clear the metric we are working with we will write $B_r^d(a)$ for $B_r(a)$.

Let us continue to see some examples to clarify the above definition.

Examples.

1. **Euclidean n-space.** Consider $\mathbb{R}^n$ with the usual metric

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$ and $r > 0$.

- If $n = 1$, then

$$B_r(a) = \{x \in \mathbb{R} \mid |x-a| < r\} = (a-r, a+r),$$

that is $B_r(a)$ is the open interval about the point a of radius r.

- If $n = 2$ and $a = (a_1, a_2)$, then

$$B_r(a) = \{(x_1, x_2) \in \mathbb{R} \mid \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < r\},$$

that is $B_r(a)$ is the open disc in $\mathbb{R}^2$ with centre at $a = (a_1, a_2)$ of radius r.

2. $\mathbb{R}^2$ with the metric defined by

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

for all (x_1, x_2), (y_1, y_2) in $\mathbb{R}^2$.

Given $a = (a_1, a_2)$ and $r > 0$. The open ball centred at a with radius r is the set

$$B_r((a_1, a_2)) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1 - a_1| + |x_2 - a_2| < r\}$$

Hint: Think about the case when $a = (0, 0)$. Can you represent in the plane $B_r((0, 0))$, for any given radius $r > 0$?

3. $\mathbb{R}^2$ with the metric defined by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

for all (x_1, x_2), (y_1, y_2) in $\mathbb{R}^2$.

Given $a = (a_1, a_2)$ and $r > 0$. The open ball centred at a with radius r is the set

$$B_r((a_1, a_2)) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \max\{|x_1 - a_1|, |x_2 - a_2|\} < r\}$$

Hint: Think about the case when $a = (0, 0)$. Can you represent in the plane $B_r((0, 0))$, for any given radius $r > 0$?

4. Let X be a (non-empty) set with the discrete metric, that is

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases} \qquad \forall\, x, y \in X.$$

Given any $a \in X$ and $r > 0$,

$$B_r(a) = \{x \in X \mid d(x, a) < r\}.$$

Notice that if $r > 1$, every point in X has distance less than r from a, so that in this case $B_r(a) = X$. On the other hand, if $0 < r \leq 1$, the only point in X whose distance from a is less than r is a, so that in this case $B_r(a) = \{a\}$. Therefore,

$$B_r(a) = \begin{cases} X, & \text{if } r > 1 \\ \{a\}, & \text{if } 0 < r \leq 1. \end{cases}$$

We continue to define the notions of open, closed and bounded sets.

Definition 3.2. *Let (X, d) be a metric space.*

i) A subset $U \subseteq X$ is open (in X) iff for every $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$.
ii) A subset $F \subseteq X$ is closed (in X) iff the complement of F in X, $X \backslash F$, is open.
iii) [7] A subset $Y \subseteq X$ is bounded iff there exists $a \in X$ and $K \in \mathbb{R}$ such that

$$d(x, a) \leq K, \qquad \text{for all } x \in Y.$$

Remark. In *i)*, notice that ε might depend on x, that is to say that ε need not be the same for all $x \in U$.

Remark. Notice that for any subset $A \subseteq X$, we have that $X \backslash (X \backslash A) = A$. From this observation and part *(i)* it follows the following characterization of closed sets:

$$\text{A subset } F \subseteq X \quad \text{is closed iff} \quad X \backslash F \quad \text{is open.}$$

Examples.

1. Consider $\mathbb{R}$ with the standard metric. Let $a < b$ be real numbers.

- An open interval (a, b) is open in $\mathbb{R}$: Given any $x \in (a, b)$, we can take $\varepsilon = \min\{|x - a|, |x - b|\} = \min\{x - a, b - x\}$, so that

$$B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subseteq (a, b).$$

- The interval $[a, b]$ is not open: To see this it suffices to find a point x in the interval $[a, b]$ such that any open ball centred at the point x contains points outside the set $[a, b]$. Consider the point b in $[a, b]$, then for any $\varepsilon > 0$

$$B_\varepsilon(b) = (b - \varepsilon, b + \varepsilon)$$

contains points which are not in the set $[a, b]$.
- The interval $[a, b]$ is a closed set: notice that $\mathbb{R} \backslash [a, b] = (-\infty, a) \cup (b, \infty)$ is an open set.
- The interval $(a, b]$ is neither open nor closed (Exercise).
- All of the above intervals are bounded sets.

2. In any metric space (X, d), $\emptyset$ and X are open and closed sets.

[7] iii) is equivalent to say that there exists $K \in \mathbb{N}$ and $a \in Y$ such that $Y \subseteq B_K(x)$.

3. Let (X, d) be a discrete metric space. Then every subset of X is open, closed and bounded.

Recall that, for any given $a \in X$ and $r > 0$, the open ball centred at a with radius r (with respect to the discrete metric) is given by

$$B_r(a) = \begin{cases} X, & \text{if } r > 1 \\ \{a\}, & \text{if } 0 < r \leq 1 \end{cases}$$

Given any $U \subseteq X$ and $x \in U$, we can take ε to be any number $0 < \varepsilon \leq 1$ (say $\varepsilon = 1/2$). Then

$$B_\varepsilon(x) = \{x\} \subseteq U \qquad \text{since } x \in U.$$

Therefore U is open.

Notice also that any $F \subseteq X$ is closed, since $X \backslash F$ is open (because, we have just seen that every subset of X is open).

Finally, given any $Y \subseteq X$. Take any point $a \in Y$, and consider $K > 1$. Then, since the discrete distance between two points of X is either 0 or 1, we have that

$$d(a, x) < K \qquad \forall\, x \in Y.$$

Hence Y is bounded.

The following proposition asserts that any open ball $B_r(a)$ in a metric space X is open (in X).

Proposition 3.3. *Let (X, d) be a metric space, $a \in X$ and $r > 0$. Then*

$$B_r(a) \qquad \textit{is open in } X.$$

Proof. Recall that, by definition,

$$B_r(a) = \{x \in X \mid d(x, a) < r\}.$$

Let $x \in B_r(a)$. Then, $d(x, a) < r$ (and, in particular $r - d(x, a) > 0$). We want to show that there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq B_r(a).$$

Take $\varepsilon = r - d(x, a)$ (> 0). We will prove that

$$B_\varepsilon(x) \subseteq B_r(a).$$

Indeed, let $y \in B_\varepsilon(x)$, so that $d(x, y) < \varepsilon$. Then, by the triangle inequality (M3) and (M2), we get that

$$\begin{aligned} d(y, a) &\leq d(y, x) + d(x, a) < \varepsilon + d(x, a) \\ &= r - d(x, a) + d(x, a) = r. \end{aligned}$$

This shows that $y \in B_r(a)$ as required. $\square$

The next results look at how open and closed sets behave under union and intersections.

Theorem 3.4. *Let (X, d) be a metric space.*

(i) *A union of arbitrarily many open subsets of X is open.*
An intersection of finitely many open subsets of X is open.
(ii) *An intersection of arbitrarily many closed subsets of X is closed.*
A union of finitely many closed subsets of X is closed.

Proof.

Proof of part (i)

- Let I be a set and for each $i \in I$ let U_i be an open subset of X. We want to show that

$$\bigcup_{i \in I} U_i \qquad \text{is open.}$$

To this end, given any $x \in \bigcup_{i \in I} U_i$, then there exists $i \in I$ such that $x \in U_i$, and since by hypothesis U_i is open, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq U_i.$$

Since clearly $U_i \subseteq \bigcup_{i \in I} U_i$, we get that

$$B_\varepsilon(x) \subseteq \bigcup_{i \in I} U_i.$$

Since $x \in \bigcup_{i \in I} U_i$ is an arbitrary point, we conclude that $\bigcup_{i \in I} U_i$ is open.

- Let $U_1, \ldots, U_n$ be open subsets of X. We want to show that

$$\bigcap_{i=1}^{n} U_i \qquad \text{is open.}$$

Let $x \in \bigcap_{i=1}^n U_i$, then $x \in U_i$ for all $i = 1, \ldots, n$, and since U_i is open for each $i = 1, \ldots n$, then for each $i = 1, \ldots, n$ there exists $\varepsilon_i > 0$ such that

$$B_{\varepsilon_i}(x) \subseteq U_i.$$

Take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq U_i, \qquad \forall\, i = 1, \ldots, n.$$

Thus

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} U_i.$$

Since $x \in \bigcap_{i=1}^n U_i$ is an arbitrary point, we conclude that $\bigcap_{i=1}^n U_i$ is open, as required.

Proof of part (ii) follows by using de Morgan's Laws and part (i) (Exercise). $\square$

Examples.

- $(0,+\infty)$ is an open set in $\mathbb{R}$ with the usual metric. Indeed, notice that
$$(0,+\infty)=\bigcup_{n=1}^{\infty}(0,n)$$
and for each $n\in\mathbb{N}$, the subset $(0,n)$ is an open set in $\mathbb{R}$ (because is an open interval). The result is now a consequence of Theorem 3.4.
- For any given $a\in X$, and $r>0$, define the *sphere centred at the point a of radius r* by
$$S_r(a)=\{x\in X\ :\ d(x,a)=r\}.$$
Then, $S_r(a)$ is a closed set.

Indeed, the result is a consequence of Theorem 3.4 and the fact that
$$S_r(a)=A\cap B,$$
with
$$A=\{x\in X\ :\ d(x,a)\le r\}\quad \text{closed (see Problem sheet 2).}$$
and
$$B=\{x\in X\ :\ d(x,a)\ge r\}\quad \text{closed}$$
(since its complementary in X, $X\backslash B=B_r^d(a)$ is open (see Proposition 3.3)).

Some remarks are in order.

Remarks.

1. Arbitrary intersections of open sets need not be open.
Example: Let $\mathbb{R}$ with the usual metric. For each $n\in\mathbb{N}$, define the set
$$U_n=\left(0,1+\frac{1}{n}\right).$$
We have that for each $n\in\mathbb{N}$, U_n is open, BUT
$$\bigcap_{n=1}^{\infty}U_n=\bigcap_{n=1}^{\infty}\left(0,1+\frac{1}{n}\right)=(0,1]$$
is not open (since for all $\varepsilon>0$ $B_\varepsilon(1)\not\subseteq\ (0,1]$).

2. Arbitrary unions of closed sets need not be closed.
Example. Consider $\mathbb{R}$ with the usual metric. For each $n\in\mathbb{N}$, define the set
$$F_n=\left[0,1-\frac{1}{n}\right].$$
We have that for each $n\in\mathbb{N}$, F_n is closed BUT
$$\bigcup_{n=1}^{\infty}F_n=\bigcup_{n=1}^{\infty}\left[0,1-\frac{1}{n}\right]=[0,1)$$
which is not a closed set.

What about bounded sets? how does the property of boundedness behave with respect to the union and intersections of sets?

Recall the definition of bounded set: Let (X, d) be a metric space. A set $Y \subseteq X$ is bounded iff there exists $a \in X$ and $K \in \mathbb{R}$ such that

$$d(x, a) \leq K, \qquad \text{for all} \quad x \in Y.$$

Lemma 3.5. *Let (X, d) be a metric space and $A, B \subseteq X$. If both A and B are bounded sets, then $A \cup B$ is a bounded set.*

Proof. Since A is bounded, there exists $x_1 \in X$ and $K_1 \in \mathbb{R}$ such that

$$d(x, x_1) \leq K_1, \qquad \forall\, x \in A. \tag{3.1}$$

Since B is bounded, there exists $x_2 \in X$ and $K_2 \in \mathbb{R}$ such that

$$d(x, x_2) \leq K_2, \qquad \forall\, x \in B. \tag{3.2}$$

Let $K = K_1 + K_2 + d(x_1, x_2)$. Let $x \in A \cup B$, then

$$x \in A \qquad \text{or} \qquad x \in B.$$

If $x \in A$, then

$$d(x, x_1) \underbrace{\leq}_{(3.1)} K_1 \underbrace{\leq}_{\text{trivially}} K.$$

If $x \in B$, then

$$d(x, x_1) \underbrace{\leq}_{(M3)} d(x, x_2) + d(x_2, x_1) \underbrace{\leq}_{(3.2)} K_2 + d(x_1, x_2) \underbrace{\leq}_{\text{trivially}} K.$$

The above argument shows that there exist $x_1 \in X$ and $K = K_1 + K_2 + d(x_1, x_2)$ (real) such that

$$d(x, x_1) \leq K_1, \qquad \forall\, x \in A \cup B.$$

Thus, $A \cup B$ is bounded. $\square$

An immediate consequence of the above lemma is the following:

Corollary 3.6. *A union of finitely many bounded sets in a metric space X is bounded.*

Remark. Arbitrary unions of bounded sets need not be bounded.

Example. Consider $\mathbb{R}$ with the standard metric. For each $n \in \mathbb{N}$, define the set

$$Y_n = [1, n].$$

We have that, for each $n \in \mathbb{N}$, $Y_n = [1, n]$ is a bounded set BUT

$$\bigcup_{n=1}^{\infty} [1, n] = [1, \infty) \qquad \text{which is not a bounded set.}$$

4. Continuity in metric spaces

We begin this section by recalling the following definition of what it means to say that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - f(a)|}_{d(f(x),f(a))} < \varepsilon \qquad \text{whenever} \qquad \underbrace{|x - a|}_{d(x,a)} < \delta.$$

Intuitively, this means that we can make the distance between $f(x)$ and $f(a)$ as small as we please by choosing the distance between x and a sufficiently small.

The following definition generalizes the notion of continuity to the more general setting of metric spaces.

Definition 4.1. *Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \longrightarrow Y$ be a function.*

- *We say that f is continuous at $a \in X$ if $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(a)) < \varepsilon, \qquad \textit{whenever} \qquad d_X(x, a) < \delta.$$

- *Given $A \subseteq X$. We say that f is continuous on A if f is continuous at a, for all $a \in A$.*
- *We say that f is continuous if f is continuous at all $x \in X$ (i.e. if f is continuous on X).*

Examples.

1. Let $X = Y = \mathbb{R}$ with the usual metric. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } \quad x \neq 0 \\ 0, & \text{if } \quad x = 0. \end{cases}$$

The function f is not continuous at 0.

2. Let $X = (0, \infty)$, $Y = \mathbb{R}$ and d_X and d_Y be the usual metric. Consider the function $f : (0, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(x) = 1/x, \qquad \text{for all} \qquad x \in (0, \infty).$$

Then, f is continuous (as $0 \notin (0, \infty)$).

3. Let (X, d_X) be a discrete metric space, so

$$d_X(x, y) = \begin{cases} 1, & \text{if } \quad x \neq y \\ 0, & \text{if } \quad x = y, \end{cases} \qquad \forall\, x, y \in X.$$

Let (Y, d_Y) be any metric space, and $f : X \longrightarrow Y$ be a function. Then f is continuous; that is every function from a discrete metric space to another metric space is continuous.

Indeed, let $a \in X$ and $\varepsilon > 0$. Take $\delta = 1/2 > 0$. Notice that

$$d_X(x, a) < \delta \qquad \text{iff} \qquad x = a.$$

Therefore, whenever $d_X(x, a) < \delta = 1/2$ (that is, whenever $x = y$),

$$d_Y(f(x), f(a)) = d_Y(f(a), f(a)) \underbrace{=}_{(M1)} 0 < \varepsilon.$$

4. Consider the space

$$X = \mathcal{C}([a, b]; \mathbb{R}) = \{f \ : \ [a, b] \longrightarrow \mathbb{R} \ : \ f \text{is continuous}\}$$

with the supremum metric, that is

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Define the map $T : X \longrightarrow \mathbb{R}$ as follows

$$T(f) = \int_a^b f(x)\, dx, \qquad \text{for all} \qquad f \in X.$$

Show that T is continuous.

Solution. We need to show that T is a continuous map at any given $f_0 \in X$. Let $f_0 \in X$ and $\varepsilon > 0$. Notice that

$$\begin{aligned} |T(f) - T(f_0)| &= \left| \int_a^b f(x)\, dx - \int_a^b f_0(x)\, dx \right| \\ &= \left| \int_a^b (f(x) - f_0(x))\, dx \right| \leq \int_a^b |f(x) - f_0(x)|\, dx \\ &\leq \int_a^b \sup_{x \in [a,b]} |f(x) - f_0(x)|\, dx = d(f, f_0) \left(\int_a^b dx \right) \\ &= d(f, f_0)(b - a) < \delta(b - a), \end{aligned} \tag{4.1}$$

whenever $d(f, f_0) < \delta$. Therefore, by taking (say) $\delta = \varepsilon/(b - a) > 0$, from (4.1) and this choice of *delta*, it follows that

$$|T(f) - T(f_0)| < \delta(b - a) = \varepsilon,$$

whenever $d(f, f_0) < \delta$. Thus, T is continuous at $f_0 \in X$, and since f_0 is an arbitrary function in X, we conclude that T is continuous on X.

The following important result gives an open sets characterization of continuity. Theorem 4.2 and Corollary 4.3 allows to define the continuity of a function using open or closed sets.

Theorem 4.2 (Open sets characterization of continuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \longrightarrow Y$ be a function. Then,*

$$f \textit{ is continuous if and only if for any open subset } U \subseteq Y,\ f^{-1}(U) \textit{ is open in } X.$$

Proof. Recall that $f^{-1}(U) = \{x \in X \ : \ f(x) \in U\}$.

$\Rightarrow$) Suppose hat f is continuous and $U \subseteq Y$ open. We want to show that $f^{-1}(U)$ is open.

Let $x \in f^{-1}(U)$, then $f(x) \in U$ and since U is open in Y, there exists $\varepsilon > 0$ such that

$$B_\varepsilon^{d_Y}(f(x)) \subseteq U. \tag{4.2}$$

Since f is continuous at x, there exists $\delta > 0$ such that

$$d_Y(f(x), f(x')) < \varepsilon \qquad \text{whenever} \qquad d_X(x, x') < \delta,$$

or equivalently,

$$f(x') \in B_\varepsilon^{d_Y}(f(x)) \qquad \text{whenever} \qquad x' \in B_\delta^{d_X}(x).$$

Therefore,

$$B_\delta^{d_X}(x) \subseteq f^{-1}\left(B_\varepsilon^{d_Y}(f(x))\right) \underbrace{\subseteq}_{(4.2)} f^{-1}(U).$$

$\Leftarrow$) Suppose that for any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X. Let $a \in X$, we want to show that f is continuous at a.

Take any $\varepsilon > 0$, and consider

$$B_\varepsilon^{d_Y}(f(a)).$$

By Proposition 3.3, $B_\varepsilon^{d_Y}(f(a))$ is an open set in Y. Then, by assumption

$$f^{-1}\left(B_\varepsilon^{d_Y}(f(a))\right) \text{ is open in } X,$$

and $a \in f^{-1}\left(B_\varepsilon^{d_Y}(f(a))\right)$, since $f(a) \in B_\varepsilon^{d_Y}(f(a))$.

Now, since $f^{-1}\left(B_\varepsilon^{d_Y}(f(a))\right)$ is open in X, from the definition of open set, we get that there exists $\delta > 0$ such that

$$B_\delta^{d_X}(a) \subseteq f^{-1}\left(B_\varepsilon^{d_Y}(f(a))\right),$$

so for any $x \in B_\delta^{d_X}(a)$, then $f(x) \in B_\varepsilon^{d_Y}(f(a))$ or, equivalently,

$$d_Y(f(x), f(a)) < \varepsilon \qquad \text{whenever} \qquad d_X(x, a) < \delta.$$

□

Remarks.

1. If $f : X \longrightarrow Y$ is continuous and $U \subseteq X$ is open, then $f(U)$ need not be open. Example. Consider $\mathbb{R}$ with the usual metric, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ the function defined by

$$f(x) = 0, \qquad \forall x \in \mathbb{R}.$$

In this example, f is a continuous function. Moreover, given any non-empty set $U \subseteq \mathbb{R}$, then $f(U) = \{0\}$ which is not an open set in $\mathbb{R}$ with the usual metric.

2. If $f : X \longrightarrow Y$ is not continuous, given $U \subseteq Y$ open , then $f^{-1}(U)$ need not be open.

Example. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if} \quad x = 0 \\ x, & \text{if} \quad x \neq 0. \end{cases}$$

In this example f is not continuous (since it is not continuous at 0). If we consider the open set $(0, \infty)$, then

$$f^{-1}((0,\infty)) = \{x \in \mathbb{R} \;:\; f(x) > 0\} = [0, \infty)$$

which is not an open set in $\mathbb{R}$ with the usual metric.

The following result gives us a closed sets characterization of the continuity of a function.

Corollary 4.3. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \longrightarrow Y$ be a function. Then*

f is continuous if and only if for any closed subset $F \subseteq Y$, $f^{-1}(F)$ is closed in X.

Proof.

$\Rightarrow$) Suppose that f is continuous and let $F \subseteq Y$ be a closed subset in Y. Then

$$X \setminus f^{-1}(F) =^{8} f^{-1}(Y \setminus F)$$

is open, since $Y \setminus F$ is open (because F is closed), f is continuous and Theorem 4.2. Thus,

$$f^{-1}(F)$$

is closed in X, as required.

$\Leftarrow$) Conversely, suppose that f is such that

$$\text{for any closed } F \subseteq Y, \quad f^{-1}(F) \text{ is closed in } X. \tag{4.3}$$

We want to show that f is continuous, by Theorem 4.2, it suffices to prove that for any open subset $U \subseteq Y$, $f^{-1}(U)$ is open in X.

[8] Notice that $x \in X \setminus f^{-1}(F) \Leftrightarrow x \notin f^{-1}(F) \Leftrightarrow f(x) \notin F \Leftrightarrow f(x) \in Y \setminus F \Leftrightarrow x \in f^{-1}(Y \setminus F)$.

Let $U \subseteq Y$ be an open set in Y. Then $Y \setminus U$ is closed, and from (4.3) it follows that

$$X \setminus f^{-1}(U) = f^{-1}(Y \setminus U) \qquad \text{is closed in } X,$$

therefore

$$f^{-1}(U) \qquad \text{is open in } X,$$

as desired. $\square$

Important remark. Theorem 4.2 and Corollary 4.3 are very useful tools to show that sets are open or closed.

Example.

1. Show that

$$U = \{x \in \mathbb{R} \ : \ x^2 - 1 > 0\}$$

is an open set in $\mathbb{R}$ with the usual metric.

Solution. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2 - 1$ for all $x \in \mathbb{R}$ is continuous *Why?*, and

$$U = \{x \in \mathbb{R} \ : \ x^2 - 1 > 0\} = \{x \in \mathbb{R} \ : \ f(x) > 0\} = f^{-1}((0, \infty))$$

Since f is continuous and $(0, \infty)$ is open in $\mathbb{R}$ then, by Theorem 4.2, $U = f^{-1}((0, \infty))$ is open.

Alternatively, notice that $U = (-1, 1)$ which is an open set in $\mathbb{R}$ (with the usual metric).

2. Is the set

$$A = \{x \in \mathbb{R} \ : \ 1 < \cos^2 + \sin x + e^x < 7\}$$

an open set in $\mathbb{R}$ with the usual metric?

Solution. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \cos^2 x + \sin x + e^x$ for all $x \in \mathbb{R}$ is continuous (since is the sum of the continuous functions $\cos^2 x$, $\sin x$ and e^x), and

$$\begin{aligned} A &= \{x \in \mathbb{R} \ : \ 1 < \cos^2 + \sin x + e^x < 7\} \\ &= \{x \in \mathbb{R} \ : \ 1 < f(x) < 7\} = f^{-1}((1, 7)). \end{aligned}$$

Since f is continuous and $(1, 7)$ is open in $\mathbb{R}$ (because it is an open interval, and we already know that open intervals are open sets in $\mathbb{R}$ with the usual metric) then, by Theorem 4.2, $A = f^{-1}((1, 7))$ is open.

5. Convergence of sequences in metric spaces

Recall the following definition of convergence of a sequence of real numbers.

Definition. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=1}^{\infty}$ converges to a point x in $\mathbb{R}$ if and only if

$$\forall\, \varepsilon > 0, \quad \text{there exists} \quad N \in \mathbb{N} \quad \text{such that} \quad |x_n - x| < \varepsilon, \quad \forall\, n \geq N.$$

We write $x_n \longrightarrow x$, as $n \longrightarrow \infty$ or $\lim_{n\to\infty} x_n = x$.

Informally: $(x_n)_{n=1}^{\infty}$ converges to x if the distance between x_n and x is as small as we want provided that we look at sufficiently large indeces n.

The above definition easily generalises to the following definition of convergence of a sequence to a point in a metric space.

Definition 5.1. *Let (X, d) be a metric space. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in X, and $x \in X$. We say that $\underline{(x_n)_{n=1}^{\infty} \text{ converges to } x}$ iff*

$$\forall\, \varepsilon > 0, \qquad \textit{there exists} \quad N \in \mathbb{N} \quad \textit{such that} \quad d(x_n, x) < \varepsilon, \qquad \textit{whenever} \quad n \geq N.$$

If $(x_n)_{n=1}^{\infty}$ converges to x in (X, d), we will write $x_n \longrightarrow x$, as $n \longrightarrow \infty$.

Examples.

1. Consider $\mathbb{R}$ with the usual/standard metric, that is

$$d(x, y) = |x - y|, \qquad \forall\, x, y \in \mathbb{R}.$$

Then, the above definition is nothing else that the "usual" definition of convergence of sequences of real numbers.

2. Let (X, d) be a discrete metric space, so

$$d(x, y) = \begin{cases} 1, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Assume that $(x_n)_{n=1}^{\infty} \longrightarrow x \in X$, as $n \longrightarrow \infty$. Then, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \qquad \text{whenever} \quad n \geq N.$$

In particular, for $\varepsilon = 1/2$ (say), we have that there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{1}{2}, \qquad \text{whenever} \quad n \geq N,$$

or, equivalenly (using the above definition of the discrete metric),

$$x_n = x \qquad \text{whenever} \qquad n \geq N.$$

Therefore, there exists $N \in \mathbb{N}$ such that $x_n = x$ whenever $n \geq N$.

The above argument shows that convergent sequences in a discrete metric space are those which are eventually stationary.

Theorem 5.2. *Let (X,d) be a metric space. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X, and x,y be two points in X.*

$$\text{If } x_n \longrightarrow x \text{ and } x_n \longrightarrow y, \text{ as } n \to \infty, \text{ then } x = y.$$

Theorem 5.2 states that a convergent sequence in a metric space (X,d) has a unique limit.

Proof. Let (X,d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X. Suppose that

$$x_n \longrightarrow x \quad \text{and} \quad x_n \longrightarrow y \quad , \text{as} \quad n \to \infty \quad (x,y \in X). \tag{5.1}$$

We want to show that $x = y$.

Given $\varepsilon > 0$, from (5.1), there exist N_1 and $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} d(x_n,x) &< \frac{\varepsilon}{2}, \qquad \forall\, n \geq N_1 \qquad \text{and} \\ d(x_n,y) &< \frac{\varepsilon}{2}, \qquad \forall\, n \geq N_2. \end{aligned}$$

Then, by using propeties (M2) and (M3) for the metric d and the above inequalities, we have that

$$\begin{aligned} d(x,y) &\underbrace{\leq}_{(M3)} d(x,x_n) + d(x_n,y) \\ &\underbrace{=}_{(M2)} d(x_n,x) + d(x_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall\, n \geq \max\{N_1,N_2\}. \end{aligned}$$

Since $\varepsilon > 0$ and arbitrary, we conclude that $d(x,y) \leq 0$, and using the property (M1) of the metric d we get that $d(x,y) = 0$, so $x = y$ as required. $\square$

Proposition 5.3. *Let (X,d_X) and (Y,d_Y) be metric spaces, $f : X \longrightarrow Y$ be a function and $a \in X$. Then*

$$\begin{gathered} f \text{ is continuous at } a \text{ if and only} \\ \text{if for every sequence } (x_n)_{n=1}^{\infty} \text{ in } X \text{ converging to } a, \text{ then } f(x_n) \longrightarrow f(a) \text{ as } n \longrightarrow \infty. \end{gathered}$$

In words: A function is continuous at a point a if and only if for every sequence converging to the point a, the sequence of the images under the function f converges to the image of a under the function f.

Proof.

$\Rightarrow$) Suppose that f is continuous at $a \in X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that

$$x_n \longrightarrow a, \qquad \text{as} \qquad n \longrightarrow \infty.$$

Let $\varepsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \varepsilon \qquad \text{whenever} \qquad d_X(x, a) < \delta. \tag{5.2}$$

Since $x_n \longrightarrow a$, as $n \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that

$$d_X(x_n, a) < \delta, \qquad \text{for all} \qquad n \geq N. \tag{5.3}$$

From (5.2) and (5.3), we conclude that there exists $N \in \mathbb{N}$ such that

$$d_Y(f(x_n), f(a)) < \varepsilon \qquad \text{whenever} \qquad n \geq N.$$

Hence $f(x_n) \longrightarrow f(a)$, as $n \longrightarrow \infty$ in (Y, d_Y).

$\Leftarrow$) Suppose (as a contradiction) that f is not continuous at a, that is there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in X$ such that

$$d_Y(f(x), f(a)) \geq \varepsilon \qquad \text{but} \qquad d_X(x, a) < \delta.$$

So, for all $n \in \mathbb{N}$, we can find $x_n \in X$ such that

$$d_Y(f(x_n), f(a)) \geq \varepsilon \qquad \text{but} \qquad d_X(x_n, a) < \frac{1}{n}. \tag{5.4}$$

Then, since $1/n \longrightarrow 0$, as $n \to \infty$, then from (5.4) $x_n \longrightarrow a$, as $n \longrightarrow \infty$ but $f(x_n)$ does not converge to $f(a)$ as $n \longrightarrow \infty$. This is a contradiction. $\square$

More examples. Let $X = \mathcal{B}([0,1], \mathbb{R})$, the space of real-valued functions defined on $[0,1]$ which are bounded, with the sup-metric

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

for all $f, g \in X$.

For each $n \in N$, define the function f_n in X by

$$f_n(x) = \frac{x}{n}, \quad \forall\, x \in [0,1].$$

We will continue to show that $f_n \longrightarrow f$ with $f \equiv 0$, as $n \longrightarrow \infty$, in X with the sup-metric. Indeed, first notice that

$$d(f_n, f) = \sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} |\frac{x}{n} - 0| = \sup_{x \in [0,1]} |\frac{x}{n}|$$

and, for fixed $x \in [0,1]$

$$\left|\frac{x}{n}\right| = \frac{|x|}{n} \leq \frac{1}{n}$$

so, by taking the supremum over all $x \in [0,1]$, we get that

$$d(f_n, f) \leq \frac{1}{n}$$

Let $\varepsilon > 0$. Since $1/n \longrightarrow 0$, as $n \longrightarrow \infty$ in $\mathbb{R}$ (with the usual metric), there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon, \qquad \forall\, n \geq N.$$

Thus, from the above two inequalities we get that for fixed $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(f_n, f) \leq \frac{1}{n} < \varepsilon, \qquad \forall\, n \geq N,$$

so that $f_n \longrightarrow f$, as $n \longrightarrow \infty$ in X with the sup-metric.

Proposition 5.4. *Let $X = \mathcal{B}([a,b], \mathbb{R})$ with the supremum metric. Let $(f_n)_{n=1}^{\infty} \subseteq X$ and $f \in X$.*

If $f_n \longrightarrow f$ in X, then for any $x \in [a,b]$, $f_n(x) \longrightarrow f(x)$, as $n \to \infty$ in $\mathbb{R}$[9].

Proof. Let $x \in [a,b]$ and $\varepsilon > 0$. Since by hypothesis $f_n \longrightarrow f$ in X with the sup-metric, there exists $N \in \mathbb{N}$ such that

$$d(f_n, f) = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N,$$

and since

$$|f_n(x) - f(x)| \leq \sup_{x \in [a,b]} |f_n(x) - f(x)|, \quad \text{for all} \qquad x \in [a,b],$$

we get that

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \geq N.$$

Thus $f_n(x) \longrightarrow f(x)$ in $\mathbb{R}$ (with the usual metric). □

Remark. We continue to show that the converse of Proposition 5.4 is not true (in general).

Example. For each $n \in \mathbb{N}$ define the function f_n, and the function f by

$$f_n(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ 1, & \text{if} \quad x \in \left(0, \frac{1}{n}\right) \\ 0, & \text{if} \quad x \in \left[\frac{1}{n}, 1\right] \end{cases} \qquad \text{and} \qquad f(x) = 0, \qquad \forall\, x \in [0,1].$$

Then, we have that for all $x \in [0,1]$, $f_n(x) \longrightarrow f(x) = 0$, as $n \to \infty$ (in $\mathbb{R}$ with the usual metric)(*why?*) BUT $(f_n)_{n=1}^{\infty}$ does not converge to f in X with the sup-metric, since

$$d(f_n, f) = \sup_{x \in [0,1]} |f_n(x) - f(x)| = 1, \quad \text{for all} \quad n \in \mathbb{N}.$$

[9] With the usual metric.

6. Limit points of sets. Closure of sets.

Motivation. Think of the sequences $\{1/n \,:\, n \in N\}$ and $\{1-\frac{1}{n} \,:\, n \in \mathbb{N}\}$. Clearly, 0 is important in the first one, in the second one 1 is playing some role. What is it?

Somehow the sets are arbitrarily close to 0 and 1, respectively. How do we measure closeness? By ε-open balls.

The above example motivates the following:

Definition 6.1. *Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$, We say that* *<u>x is a limit point of A in X</u>* *if*

$$\textit{given any} \quad \varepsilon > 0 \qquad (B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset.$$[10]

In words: A point $x \in X$ is a limit point of a set if <u>any</u> open ball centred at x contains some point of A other than x.

Examples. Consider $\mathbb{R}$ with the usual (euclidean) metric.

1. Consider the interval $(a, b]$, with $a < b$. Then a is a limit point of $(a, b]$ in $\mathbb{R}$: because for any $\varepsilon > 0$

$$(B_\varepsilon(a) \setminus \{a\}) \cap (a, b] = [(a-\varepsilon, a) \cup (a, a+\varepsilon)] \cap (a, b] \neq \emptyset.$$

2. Consider the set $A = \{a\} \cup [b, c]$, with $a < b < c$. Then a is not a limit point of A in $\mathbb{R}$: Take $\varepsilon = (b-a)/2$ (or any $0 < \varepsilon < (b-a)$). Then

$$(B_\varepsilon(a) \setminus \{a\}) \cap A = ((a-\varepsilon, a+\varepsilon) \setminus \{a\}) \cap A = \emptyset.$$

The following theorem relates limit points of sets and the property of a set being closed in a general metric space.

Theorem 6.2. *Let (X, d) be a metric space and $A \subseteq X$. Then,*

A is closed if and only if A contains all its limit points in X.

Proof. Let (X, d) be a metric space and $A \subseteq X$.

$\Leftarrow$) First, suppose that A contains all its limit points in X. To show that A is closed in X, we want to show that $X \setminus A$ is open in X.

To this end, let $x \in X \setminus A$. Then x is not a limit point of A (just by using the assumption that A contains all its limit points in X), so there exists $\varepsilon > 0$ such that

$$(B_\varepsilon(x) \setminus \{x\}) \cap A = \emptyset. \tag{6.1}$$

Also, since $x \in X \setminus A$, then

$$x \notin A \tag{6.2}$$

[10] Warning: There are other non-equivalent definitions of a limit point of a set in the literature.

From (6.1) and (6.2), we get that

$$B_\varepsilon(x) \cap A = \emptyset,$$

so that

$$B_\varepsilon(x) \subseteq X \setminus A.$$

Therefore, $X \setminus A$ is open in X, as required.

$\Rightarrow$) Conversely, suppose that A is closed in X. Take any $x \in X \setminus A$. We want to show that x is not a limit point of A in X.

Notice that since A is closed, then $X \setminus A$ is open and for any given $x \in X \setminus A$, we have that there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq X \setminus A,$$

and, in particular,

$$B_\varepsilon(x) \setminus \{x\} \subseteq X \setminus A,$$

so

$$(B_\varepsilon(x) \setminus \{x\}) \cap A = \emptyset.$$

Thus, x is not a limit point of A in X. □

Definition 6.3. *Let (X, d) be a metric space and $A \subseteq X$. The closure, $\overline{A}$, of A in X is the set defined by*

$$x \in X \text{ is in } \overline{A} \text{ if and only if given any } \varepsilon > 0, \quad B_\varepsilon(x) \cap A \neq \emptyset.$$

In words: a point $x \in X$ is in $\overline{A}$ iff any open ball centred at x contains some point in the set A.

Remark. Note that clearly, we always have that

$$A \subseteq \overline{A}.$$

We continue to revisit the previous examples.

Examples. Consider $\mathbb{R}$ with the usual metric.

1. Let $A = (a, b]$, with $a < b$. Then $a \in \overline{A}$. Indeed, for any $\varepsilon > 0$, $B_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$ and this set contains clearly points of $A = (a, b]$
2. Let $A = \{a\} \cup [b, c]$, with $a < b < c$. Then $a \in \overline{A}$, because $A \subseteq \overline{A}$ and $a \in A$ (but remember that a is not a limit point of A in $\mathbb{R}$).

More examples.

1. Consider $\mathbb{R}$ with the usual metric. Then,
   The closure of (a, b) in (a, ∞) is: $(a, b]$.
   The closure of (a, b) in $\mathbb{R}$ is: $[a, b]$.

This example shows that we have to be careful about what space closure is taken in.

2. The closure of (a,b) in $\mathbb{R}$ with the discrete metric is: (a,b).

Indeed, first recall that

$$B_\varepsilon(x) = \begin{cases} \mathbb{R}, & \varepsilon > 1 \\ \{x\}, & 0 < \varepsilon \leq 1. \end{cases}$$

Given any $x \notin (a,b)$ there exists $\varepsilon = 1/2$ say such that

$$(B_{\frac{1}{2}}(x) \setminus \{x\}) \cap (a,b) = \emptyset \cap (a,b) = \emptyset,$$

therefore x is not a limit point of (a,b) in $\mathbb{R}$ with the discrete metric, and since $x \notin (a,b)$, we conclude that $x \notin \overline{(a,b)}$.

Proposition 6.4. *Let (X,d) be a metric space and $A \subseteq X$. Then*

$$\overline{A} = A \cup \{ \textit{ limit points of } A \textit{ in } X\}.$$

Proof. We need to show the double inclusion.

$\subseteq$) Let $x \in A \cup \{$ limit points of A in X $\}$. Then

If $x \in A \subseteq \overline{A}$, then $x \in \overline{A}$.
If x is a limit point of A in X, then

$$\text{given any } \varepsilon > 0, \qquad (B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset,$$

so, in particular,

$$\text{given any } \varepsilon > 0, \qquad B_\varepsilon(x) \cap A \neq \emptyset.$$

Therefore $x \in \overline{A}$.

$\supseteq$) Let $x \in \overline{A}$. Then given any $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$.

We have two possible cases:

If $x \in B_\varepsilon(x) \cap A$, then

$$x \in A \subseteq A \cup \{\text{limit points of } A \text{ in } X\}.$$

If $x \notin B_\varepsilon(x) \cap A$, since $B_\varepsilon(x) \cap A \neq \emptyset$, we have that there exists $y \neq x \in B_\varepsilon(x) \cap A$, that is there exists $y \in (B_\varepsilon(x) \setminus \{x\}) \cap A$, and as a consequence

$$(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset.$$

The above argument gives that x is a limit point of A in X, so in this case

$$x \in A \cup \{ \text{ limit points of } A \text{ in } X \}.$$

$\square$

As a consequence of the above results we have the following

Corollary 6.5. *Let (X, d) be a metric space and $A \subseteq X$. Then,*

$$A \text{ is closed in } X \text{ if and only if } \overline{A} = A.$$

Proof. By Theorem 6.2 and Proposition 6.4, we have that

$$\begin{aligned} A \quad \text{is closed in } X \quad & \underbrace{\Leftrightarrow}_{Th.\ 6.2} \quad \{ \text{ limit points of } A \text{ in } X \ \} \subseteq A \\ & \underbrace{\Leftrightarrow}_{Pr.\ 6.4} \quad \overline{A} = A \cup \{\text{limit points of } A \text{ in } X \ \} = A \\ & \Leftrightarrow \quad \overline{A} = A, \end{aligned}$$

as required. □

The following result gives a characterization of points in the clousre of a set in terms of sequences.

Theorem 6.6. *Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. Then*

$x \in \overline{A}$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq A$ such that $x_n \longrightarrow x$, as $n \to \infty$.

Proof of Theorem 6.6.

$\Leftarrow$) Suppose that there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq A$ such that $x_n \longrightarrow x$, as $n \longrightarrow \infty$. Then, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \qquad \text{whenever} \quad n \geq N,$$

so

$$x_n \in B_\varepsilon(x) \qquad \text{whenever} \quad n \geq N,$$

and since $(x_n)_{n=1}^{\infty} \subseteq A$, we get that

$$x_n \in B_\varepsilon(x) \cap A, \qquad \text{whenever} \quad n \geq N,$$

and in particular $B_\varepsilon(x) \cap A \neq \emptyset$. Thus, x is a point of $\overline{A}$.

$\Rightarrow$) Suppose $x \in \overline{A}$, then

$$\text{given any } \varepsilon > 0, \qquad B_\varepsilon(x) \cap A \neq \emptyset \tag{6.3}$$

Take, for each $n \in \mathbb{N}$, $\varepsilon = 1/n$, then from (6.3)

$$B_{\frac{1}{n}}(x) \cap A \neq \emptyset,$$

so, for each $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x) \cap A$ or, equivalently,

$$\forall\, n \in \mathbb{N}, \quad \text{there exists } x_n \in A \quad \text{such that } d(x_n, x) < \frac{1}{n}. \tag{6.4}$$

Now, consider the sequence $(x_n)_{n=1}^{\infty} \subseteq A$ (since $x_n \in A$ for all $n \in \mathbb{N}$). Since $1/n \longrightarrow 0$ as $n \longrightarrow \infty$, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon \qquad \text{whenever} \quad n \geq N. \tag{6.5}$$

From (6.4) and (6.5), we get that given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) \underbrace{<}_{(6.4)} \frac{1}{n} \underbrace{<}_{(6.5)} \varepsilon, \qquad \text{whenever} \quad n \geq N.$$

Thus $x_n \longrightarrow x$ as $n \longrightarrow \infty$. □

Proposition 6.7. *Let (X, d) be a metric space and $A \subseteq X$. Then,*

A is closed in X if and only if
for each sequence $(x_n)_{n=1}^{\infty} \subseteq A$ such that $x_n \longrightarrow x$ as $n \to \infty$, then $x \in A$.

In words: A subset A is closed is X if an only if for each sequence in A which converges, the limit is a point in the set A.

Proof. The result easily follows from Corollary 6.5 and Theorem 6.6. Indeed, A is closed iff $A = \overline{A}$ (by Corollary 6.5)), iff $\overline{A} \subseteq A$ (since it is always true that $A \subseteq \overline{A}$), that is if and only if for each $x \in \overline{A}$ then $x \in A$ which is equivalent to (from Theorem 6.6) for each sequence $(x_n)_{n=1}^{\infty} \subseteq A$ such that $x_n \longrightarrow x$ as $n \to \infty$ (in (X, d)) then $x \in A$. This concludes the proof. □

We conclude this section by stating and proving how taking the closure behaves with respect to the operations of union and intersections of sets.

Proposition 6.8. *Let (X, d) be a metric space.*

(i) *Let $\{A_i\}_{i \in I}$ be a (arbitrary) family of subsets of X indexed by the set I. Then,*

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}.$$

(ii) *Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of subsets of X. Then,*

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

Proof. The proof is left as an exercise.

Recall that for a subset A of X, the closure of A in X is the set

$$\overline{A} = \{x \in X \ : \ \forall \varepsilon > 0, \quad B_\varepsilon(x) \cap A \neq \emptyset\}.$$

□

The following example shows that the result stated in part (ii) of Proposition 6.8 can not be extended to consider an arbitrary collection of sets.

Example. Consider $\mathbb{R}$ with the usual metric. For each $n \geq 2$, define the set

$$A_n = [1/n, 1].$$

Then,

$$\overline{\bigcup_{n=2}^{\infty} A_n} = \overline{\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1\right]} = \overline{(0,1]} = [0,1],$$

while

$$\bigcup_{n=2}^{\infty} \overline{A_n} = \bigcup_{n=2}^{\infty} \overline{\left[\frac{1}{n}, 1\right]} = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1\right] = (0,1].$$

Thus,

$$\overline{\bigcup_{n=2}^{\infty} A_n} \neq \bigcup_{n=2}^{\infty} \overline{A_n}.$$

7. Cauchy Sequences

7.1. Cauchy sequences of real numbers.

Given a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$, can we decide whether $(x_n)_{n=1}^{\infty}$ converges, without knowing the limit?

- What if $(x_n)_{n=1}^{\infty}$ is such that $|x_n - x_{n+1}| \longrightarrow 0$, as $n \longrightarrow \infty$

Answer: No, the above condition does not imply convergence.

Example. Consider the sequence of real numbers

$$\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots\},$$

that is the sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ defined by $x_n = \sum_{i=1}^{n} \frac{1}{i}$, for $n = 1, 2 \ldots$. Then

$$|x_n - x_{n+1}| = \left| \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n+1} \frac{1}{i} \right| = \frac{1}{n+1} \longrightarrow 0, \qquad \text{as} \qquad n \longrightarrow \infty,$$

but $(x_n)_{n=1}^{\infty}$ does not converge as $n \longrightarrow \infty$, since we know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- What if $(x_n)_{n=1}^{\infty}$ is such that $|x_n - x_m| \longrightarrow 0$ as $n, m \longrightarrow \infty$?

Answer: The answer to this question is given by Cauchy's convergence criterion, which will be stated and proved below.

First, we need to introduce some terminology.

Definition 7.1. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence iff given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|x_n - x_m| < \varepsilon, \qquad \text{for all} \qquad n, m \geq N.$$ [11]

Examples.

1. Consider the sequence $(x_n)_{n=1}^{\infty}$ of real numbers defined by

$$x_n = \frac{1}{n}, \qquad \forall\, n \in \mathbb{N}.$$

Then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence: Let $\varepsilon > 0$. Notice that, whenever $n, m \geq N$ for some $N \in \mathbb{N}$,

$$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

Take N to be a natural number such that $N > \varepsilon/2$, then from the above inequality it follows that

$$|x_n - x_m| < \frac{2}{N} < \varepsilon, \qquad \forall\, n, m \geq N,$$

so $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

[11] We can make the distance between x_n and x_m small by choosing n and m sufficiently large.

The following result, known as Cauchy's principle/criterion of convergence, is one of the fundamental theorems of analysis. This result enables one to prove the convergence of a sequence of real numbers without having an apriori knowledge of what the limit should be.

First, we recall Bolzano-Weierstrass theorem from 1RCA-Real Analysis and the Calculus: Every bounded sequence of real numbers has a convergent subsequence.

Theorem 7.2 (Cauchy's convergence criterion or Cauchy principle of convergence)**.** *A sequence $(x_n)_{n=1}^{\infty}$ of real numbers converges if and only if it is a Cauchy sequence.*

Proof.

$\Rightarrow)$ Suppose that $(x_n)_{n=1}^{\infty}\longrightarrow x$, as $n\longrightarrow\infty$, that is $\forall\,\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$|x_n-x|<\frac{\varepsilon}{2},\qquad \forall\, n\geq N.$$

Then,

$$|x_n-x_m|\leq|x_n-x|+|x_m-x|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\qquad \forall\, n,m\geq N,$$

so $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

$\Leftarrow)$ Conversely, suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Claim 1: $(x_n)_{n=1}^{\infty}$ is bounded.

Indeed, let $\varepsilon=1$, since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists $N\in\mathbb{N}$ such that

$$|x_n-x_m|<1,\qquad \forall\, n,m\geq N, \tag{7.1}$$

so, in particular, by taking $m=N$, from (7.1) we get that

$$||x_n|-|x_N||\leq|x_n-x_N|<1,\qquad \forall\, n\geq N.$$

Thus,

$$|x_n|<1+|x_N|,\qquad \forall\, n\geq N. \tag{7.2}$$

Let $K=\max\{|x_1|,|x_2|,\dots,|x_{N-1}|,1+|x_N|\}$, then from (7.2) and the definition of K, we obtain that

$$|x_n|\leq K,\qquad \forall\, n\in\mathbb{N},$$

Hence, $(x_n)_{n=1}^{\infty}$ is bounded sequence.

Since $(x_n)_{n=1}^{\infty}$ is bounded, by Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ which converges, say

$$x_{n_i}\longrightarrow x,\qquad \text{as}\quad i\longrightarrow\infty.$$

Claim 2: $x_n\longrightarrow x$, as $n\longrightarrow\infty$.

Let $\varepsilon > 0$. Since $x_{n_i} \longrightarrow x$ as $n \to \infty$, there exists $I_1 \in \mathbb{N}$ such that

$$|x_{n_i} - x| < \frac{\varepsilon}{2}, \qquad \forall\, i \geq I_1. \tag{7.3}$$

Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2}, \qquad \forall\, n, m \geq N_1. \tag{7.4}$$

Choose $I_2 \in \mathbb{N}$ such that

$$I_2 \geq I_1 \qquad \text{and} \qquad n_{I_2} \geq N_1.$$

Define $N_2 = n_{I_2}$. Then, if $n \geq N_2$, from (7.3) and (7.4) we obtain that

$$|x_n - x| \leq |x_n - x_{N_2}| + |x_{N_2} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $x_n \longrightarrow x$, as $n \to \infty$. This finishes the proof. □

The definition of Cauchy sequence of real numbers generalizes to the following definition of Cauchy sequence in a general metric space.

7.2. **Cauchy sequences in metric spaces.**

Definition 7.3. *Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty} \subseteq X$ is a Cauchy sequence iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \varepsilon, \qquad \textit{for all} \qquad n, m \geq N.$$

The following result relates Cauchy sequences with convergent sequences.

Proposition 7.4. *Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X.*

If $(x_n)_{n=1}^{\infty}$ is a convergent sequence, then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in (X, d), such that $x_n \longrightarrow x$, as $n \longrightarrow \infty$, for some $x \in X$, that is: given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2}, \qquad \forall\, n \geq N.$$

Then, if $n, m \geq N$, from the properties $(M3)$ and $(M2)$ of the metric d and the above inequality, we get that

$$\begin{aligned} d(x_n, x_m) &\underbrace{\leq}_{(M3)} d(x_n, x) + d(x, x_m) \\ &\underbrace{=}_{(M2)} d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). □

What about the converse of Proposition 7.4? If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, does $(x_n)_{n=1}^{\infty}$ converge?

Answer: No in general.

Example. Consider $X = \mathbb{R}\backslash\{0\}$ with the usual metric, that is $d(x,y) = |x-y|$, for all $x, y \in X$. Define the sequence $(x_n)_{n=1}^{\infty}$ in X by

$$x_n = \frac{1}{n}, \qquad \forall\, n \in \mathbb{N}.$$

Then, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with $N > \varepsilon/2$ such that if $n, m \geq N$, then

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon.$$

However, $(x_n)_{n=1}^{\infty}$ does not converge to any point in X (since $0 \notin X$).

This leads to the definition of complete metric spaces.