3Top – Topology

lecture notes

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0 Preliminary remarks and notation

MS will denote the "Metric Spaces" lecture notes from the past term.

If $\mathcal{A}$ is a collection of sets, then $\bigcup \mathcal{A}$ denotes its *union*, i.e., $\bigcup \mathcal{A}$ is the set of the elements of elements of $\mathcal{A}$:

$$\bigcup \mathcal{A} = \{x : \exists A \in \mathcal{A} : x \in A\}.$$

Analogously, when $\mathcal{A}$ is nonempty, we denote by $\bigcap \mathcal{A}$ the *intersection* of $\mathcal{A}$, i.e., $\bigcap \mathcal{A}$ is the set of elements in common to all elements of $\mathcal{A}$:

$$\bigcap \mathcal{A} = \{x : \forall A \in \mathcal{A} : x \in A\}.$$

When $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ is an indexed family, an alternative notation is available[1]:

$$\bigcup \mathcal{A} = \bigcup_{\alpha \in I} A_\alpha, \qquad \bigcap \mathcal{A} = \bigcap_{\alpha \in I} A_\alpha.$$

Note that, when $\mathcal{A} = \{A\}$ has a single element, then

$$\bigcup \{A\} = A, \qquad \bigcap \{A\} = A;$$

more generally, when $\mathcal{A} = \{A_1, \dots, A_n\}$ is a finite family, then

$$\bigcup \mathcal{A} = A_1 \cup \dots \cup A_n, \qquad \bigcap \mathcal{A} = A_1 \cap \dots \cap A_n.$$

Many examples are provided throughout the text. Not all the statements in the *Examples* sections are proved (this is often remarked by the presence of "why?" or "how?" in parentheses) and the missing proofs are meant as exercises for the interested reader.

[1] See [**MS**, Section 0.1]

1 Topologies

Definition 1.1. Let X be a set. A *topology* on X is a collection τ of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \tau$;

(ii) for all subcollections $\mathcal{A} \subseteq \tau$, the union $\bigcup \mathcal{A} \in \tau$;

(iii) if $A, B \in \tau$, then $A \cap B \in \tau$.

Examples. The following are examples of topologies on a set X.

1. The collection $\{\emptyset, X\}$ is a topology on X, called the *trivial topology* on X.
2. The collection $\mathcal{P}(X)$ of all subsets of a set X is a topology on X, called the *discrete topology* on X.
3. The collection
$$\{A \subseteq X : A = \emptyset \text{ or } X \setminus A \text{ is finite}\}$$
is a topology on X (why?), called the *cofinite topology* on X.
4. Let (X, d) be a metric space. Then the collection τ_d of open subsets[2] of X is a topology[3] on X, which is said to be *induced* by the metric d.
5. Let $X = \mathbb{R}^n$. The topology induced by the Euclidean distance[4] is called *Euclidean topology* or *standard topology* on $\mathbb{R}^n$.

Definition 1.2. A *topological space* is a pair (X, τ), where X is a set and τ is a topology on X.

In other words, a topological space is a set X equipped with a topology τ. Sometimes, when the topology τ is understood from the context, we will just say "the topological space X" and omit the topology τ.

Every metric space will be thought of as a topological space, with the topology induced by the metric. The following terminology is consistent with the one introduced for metric spaces[5].

Definition 1.3. Let (X, τ) be a topological space.

(i) The elements of X are called *points* of X.

(ii) A subset $A \subseteq X$ is called *open* if $A \in \tau$.

(iii) A subset $C \subseteq X$ is called *closed*, if its complement $X \setminus C$ is open.

(iv) A subset $U \subseteq X$ is called *neighbourhood* of a point $x \in X$ if there exists an open set A such that $x \in A \subseteq U$.

Proposition 1.4 (properties of closed sets)**.** *Let (X, τ) be a topological space.*

(i) $\emptyset$ and X are closed subsets of X.

[2] In the sense of [**MS**, Definition 3.2].
[3] See [**MS**, Theorem 3.4(i)].
[4] See [**MS**, Section 2, Example 3].
[5] See [**MS**, Sections 2 and 3].

(ii) For all (nonempty) collections $\mathcal{C}$ of closed subsets of X, the intersection $\bigcap \mathcal{C}$ is closed.

(iii) If C and D are closed subsets of X, then $C \cup D$ is closed.

Proof. This follows immediately from the definitions and De Morgan's laws. See also [**MS**, Theorem 3.4(ii)]. ☐

Proposition 1.5 (properties of neighbourhoods)**.** *Let (X, τ) be a topological space.*

(i) If U and V are neighbourhoods of a point $x \in X$, then $U \cap V$ is a neighbourhood of x as well.

(ii) If U is a neighbourhood of $x \in X$ and $U \subseteq V \subseteq X$, then V is a neighbourhood of x as well.

(iii) A subset $A \subseteq X$ is open if and only if, for all $x \in A$, A is a neighbourhood of x.

Proof. (i). If U and V are neighbourhoods of x, then there exist open sets A and B in X such that $x \in A \subseteq U$ and $x \in B \subseteq V$. In particular $A \cap B$ is open and $x \in A \cap B \subseteq U \cap V$. This proves that $U \cap V$ is a neighbourhood of x.

(ii). If U is a neighbourhood of x, then there exists an open set A such that $x \in A \subseteq U$. Since $U \subseteq V$, we also have $x \in A \subseteq V$ and consequently V is a neighbourhood of x.

(iii). Suppose that A is open. Then, for all $x \in A$, we have that $x \in A \subseteq A$ and A is open; consequently, by definition, A is a neighbourhood of x.

Vice versa, suppose that A is a neighbourhood of all its point. Then, for all $x \in A$, there exists an open set B_x such that $x \in B_x \subseteq A$. Let $B = \bigcup_{x \in A} B_x$. Then B is open, being a union of open sets. Moreover B contains all points of A (since, for all $x \in A$, $x \in B_x \subseteq B$), i.e., $A \subseteq B$. On the other hand, $B \subseteq A$ (since every point of B is a point of some B_x and $B_x \subseteq A$ for all $x \in A$). In conclusion, $A = B$ and A is open. ☐

2 Topologies and metric spaces

Proposition 2.1 (neighbourhoods in metric spaces)**.** *Let (X,d) be a metric space. A subset $U \subseteq X$ is a neighbourhood of a point $x \in X$ if and only if there exists $\epsilon > 0$ such that the ball $B_\epsilon(x)$ is contained in U.*

Proof. Suppose that U is a neighbourhood of x. Then by Definition 1.3 there exists an open set A such that $x \in A \subseteq U$. Since A is open (in the topology induced by the metric), by [**MS**, Definition 3.2] there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$ and, since $A \subseteq U$, in particular $B_\epsilon(x) \subseteq U$.

Conversely, suppose that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Then $B_\epsilon(x)$ is open by [**MS**, Proposition 3.3], contains x and is contained in U, therefore U is a neighbourhood of x by Definition 1.3. □

Definition 2.2. Let X be a set. Let d and d' be metrics on X.

(i) d and d' are said to be *equivalent* if there exist constants $C_1, C_2 \in (0,\infty)$ such that, for all $x, y \in X$,

$$d(x,y) \leq C_1 d'(x,y) \qquad \text{and} \qquad d'(x,y) \leq C_2 d(x,y). \tag{2.1}$$

(ii) d and d' are said to be *topologically equivalent* if they induce the same topology on X.

Proposition 2.3. *Let X be a set and d, d' be metrics on X. If d and d' are equivalent, then they are topologically equivalent.*

Proof. Let τ and τ' be the topologies on X induced by d and d' respectively. Let $C_1, C_2 \in (0,\infty)$ be the constants such that the inequalities (2.1) hold for all $x, y \in X$.

We show now that $\tau \subseteq \tau'$. Let $A \in \tau$. Then, for all $x \in A$, there exists $\epsilon > 0$ such that $B^d_\epsilon(x) \subseteq A$. On the other hand, $B^{d'}_{\epsilon/C_1}(x) \subseteq B^d_\epsilon(x)$: indeed, for all $y \in X$, if $d'(x,y) < \epsilon/C_1$, then $d(x,y) \leq C_1 d'(x,y) < \epsilon$ by (2.1). In other words, for all $x \in A$, we have found a ball $B^{d'}_{\epsilon/C_1}(x)$ relative to the metric d' which is contained in A. Hence, by definition, $A \in \tau'$.

The opposite inclusion $\tau' \subseteq \tau$ is proved analogously. Hence $\tau = \tau'$. □

Examples. 1. The metrics[6] d_1, d_2, d_∞ on $\mathbb{R}^n$ are equivalent, since

$$d_\infty \leq d_2 \leq d_1 \leq n\, d_\infty.$$

Hence they all induce the standard topology on $\mathbb{R}^n$.

2. Let (X,d) be a metric space and define d' by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Then d' is a metric on X (why?) and is topologically equivalent to d (why?). However d' is always bounded, since $d' \leq 1$. In particular d' is not equivalent to d whenever d is unbounded (think, e.g., of d being the standard metric on $\mathbb{R}$).

[6] See [**MS**, Section 2, Example 5]

Definition 2.4. A topology is called *metrisable* if it is induced by a metric. A topological space is called *metrisable* if its topology is metrisable.

Examples. Let X be a set.

1. The discrete topology on X is induced by the discrete metric[7] on X (why?), hence it is metrisable.
2. If X has more than one point, then the trivial topology on X is not metrisable (why?).
3. If X is infinite, then the cofinite topology on X is not metrisable (why?).

The last two statements can be more conveniently proved by the use of separation axioms, that will be discussed later in the lectures.

[7] See [**MS**, Section 2, Example 4].

3 Continuous functions and homeomorphisms

Proposition 3.1 (metric space continuity and neighbourhoods)**.** *Let (X, d_X), (Y, d_Y) be metric spaces, $x \in X$ and $f : X \to Y$. The following are equivalent:*

(i) the function f is continuous[8] *at the point $x \in X$;*

(ii) for all neighbourhoods V of $f(x)$ in Y, the preimage $f^{-1}(V)$ is a neighbourhood of x in X.

Proof. (i) ⇒ (ii). Let V be a neighbourhood of $f(x)$ in Y. By Proposition 2.1, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq V$. Since f is continuous at x, by [**MS**, Definition 4.1] there exists $\delta > 0$ such that, for all $x' \in X$, $d_Y(f(x), f(x')) < \epsilon$ whenever $d_X(x, x') < \delta$. In other words, $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ and consequently $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))) \subseteq f^{-1}(V)$. This shows, by Proposition 2.1, that $f^{-1}(V)$ is a neighbourhood of x.

(ii) ⇒ (i). Let $\epsilon > 0$. Then $B_\epsilon(f(x))$ is a neighbourhood of $f(x)$ by Proposition 2.1. Hence the preimage $f^{-1}(B_\epsilon(f(x)))$ is a neighbourhood of x and consequently, again by Proposition 2.1, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$. This means that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$, i.e., for all $x' \in X$, if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have proved that f is continuous at x according to [**MS**, Definition 4.1]. □

Condition (ii) of Proposition 3.1 refers only to neighbourhoods and not directly to the metric. Hence it can be used as a definition of continuity in the more general context of topological spaces.

Definition 3.2. Let (X, τ_X), (Y, τ_Y) be topological spaces.

(i) Let $x \in X$. A function $f : X \to Y$ is said to be *continuous at x* if, for all neighbourhoods V of $f(x)$ in Y, the preimage $f^{-1}(V)$ is a neighbourhood of x in X.

(ii) A function $f : X \to Y$ is said to be *continuous* if f is continuous at all points of X.

Composition is a very common way of constructing functions from other functions, so it is an important fact that continuity is preserved by composition.

Proposition 3.3 (continuity and composition)**.** *Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces. Let $f : X \to Y$, $g : Y \to Z$.*

(i) Let $x \in X$. If f is continuous at x and g is continuous at $f(x)$, then their composition $g \circ f$ is continuous at x.

(ii) If f and g are continuous, then their composition $g \circ f$ is continuous.

Proof. (i). Let V be a neighbourhood of $g \circ f(x) = g(f(x))$ in Z. Since g is continuous at $f(x)$, $g^{-1}(V)$ is a neighbourhood of $f(x)$ in Y. Therefore, since f is continuous at x, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is a neighbourhood of x in X.

(ii). Apply part (i) to all points $x \in X$. □

As already seen in the case of metric spaces (see [**MS**, Theorem 4.2]), continuity can be also characterised in terms of open sets.

[8] In the sense of [**MS**, Definition 4.1].

Proposition 3.4 (characterisations of continuity). *Let (X, τ_X), (Y, τ_Y) be topological spaces. Let $f : X \to Y$. Then the following are equivalent:*

(i) f is continuous;

(ii) for all open sets $A \subseteq Y$, the preimage $f^{-1}(A)$ is open in X;

(iii) for all closed sets $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

Proof. (i) $\Rightarrow$ (ii). Let $A \subseteq Y$ be open. Then A is neighbourhood of all its points by Proposition 1.5. In particular, for all $x \in f^{-1}(A)$, A is a neighbourhood of $f(x)$; therefore, since f is continuous, $f^{-1}(A)$ is a neighbourhood of x. This shows that $f^{-1}(A)$ is a neighbourhood of all its points, which means, by Proposition 1.5, that $f^{-1}(A)$ is open.

(ii) $\Rightarrow$ (i). We must show that f is continuous at all points $x \in X$. Let $x \in X$ and V be a neighbourhood of $f(x)$ in Y. Then there exists an open subset A of Y such that $f(x) \in A \subseteq V$. Since A is open in Y, by our assumption $f^{-1}(A)$ is open in X, and moreover $x \in f^{-1}(A) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is a neighbourhood of x in X.

(ii) $\Leftrightarrow$ (iii). See [**MS**, Corollary 4.3] (the same proof works here). □

Examples. Let (X, τ_X), (Y, τ_Y) be topological spaces.

1. If τ_X is the discrete topology on X or τ_Y is the trivial topology on Y, then all functions $f : X \to Y$ are continuous.

2. If τ_X is the trivial topology on X and τ_Y is the discrete topology on Y, then all continuous functions $f : X \to Y$ are constant.

3. All constant functions $f : X \to Y$ are continuous (independently of the topologies on X and Y).

4. Suppose that $X = Y$, $\tau_X = \tau_Y$. Then the *identity function* $\mathrm{id}_X : X \to X$, defined by $\mathrm{id}_X(x) = x$ for all $x \in X$, is continuous.

The following definitions introduce some fundamental notions in the study of topology.

Definition 3.5. Let (X, τ_X), (Y, τ_Y) be topological spaces.

(i) A function $f : X \to Y$ is said to be a *homeomorphism* if f is continuous, bijective, and moreover its inverse $f^{-1} : Y \to X$ is continuous.

(ii) The two topological spaces (X, τ_X) and (Y, τ_Y) are said to be *homeomorphic* if there exists a homeomorphism $f : X \to Y$.

Definition 3.6. A *topological property* (also known as *topological invariant*) is a property of topological spaces that is invariant by homeomorphisms.

"Topology" can be thought of as the study of topological properties. One of the main objectives of these lectures is to learn how to distinguish between homeomorphic and non-homeomorphic spaces. Topological properties can be used to do this: if two topological spaces differ by a topological property, then they cannot be homeomorphic.

Examples. In the following examples, subsets of $\mathbb{R}^n$ are thought of as metric subspaces of $\mathbb{R}^n$ and given the topology induced by the Euclidean metric.

1. The function $f : [0,1] \to [0,2]$ defined by $f(x) = 2x$ is a homeomorphism. Indeed it is continuous and bijective, and its inverse $f^{-1} : [0,2] \to [0,1]$, which is given by $f^{-1}(y) = y/2$, is continuous as well.
2. The function $g : \mathbb{R} \to (-\pi/2, \pi/2)$ defined by $g(x) = \arctan x$ is a homeomorphism. Indeed it is continuous and bijective, and its inverse $g^{-1} : (-\pi/2, \pi/2) \to \mathbb{R}$, which is given by $g^{-1}(y) = \tan y$, is continuous as well.
3. The function $h : [0,1) \cup [2,3] \to [0,2]$, defined by
$$h(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ x-1, & \text{if } x \in [2,3], \end{cases}$$
is continuous and bijective, but it is not a homeomorphism (why?).
4. Let $P = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$ be a parabola in $\mathbb{R}^2$. Then the function $k : \mathbb{R} \to P$ given by $k(x) = (x, x^2)$ is a homeomorphism. Indeed it is continuous and bijective, and its inverse $k^{-1} : P \to \mathbb{R}$, which is given by $k^{-1}(x,y) = x$, is continuous as well.

The above examples show that properties such as “length” or “curvature” are not topological properties.

Proposition 3.7 (properties of homeomorphisms)**.** *Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces.*

(i) The identity function $\mathrm{id}_X : X \to X$, defined by $\mathrm{id}_X(x) = x$ for all $x \in X$, is a homeomorphism.

(ii) If $f : X \to Y$ is a homeomorphism, then its inverse $f^{-1} : Y \to X$ is a homeomorphism as well.

(iii) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then their composition $g \circ f : X \to Z$ is a homeomorphism as well.

Proof. (i). Clearly $\mathrm{id}_X^{-1}(A) = A$ is open for all open subset A of X, so id_X is continuous. Since $\mathrm{id}_X^{-1} = \mathrm{id}_X$, id_X is invertible, with continuous inverse. Hence id_X is a homeomorphism.

(ii). If $f : X \to Y$ is a homeomorphism, then it is invertible. So its inverse f^{-1} is bijective and $(f^{-1})^{-1} = f$. Since f is a homeomorphism, we know that f and f^{-1} are both continuous, hence we conclude that f^{-1} is a homeomorphism as well.

(iii). Composition of bijective functions is bijective, so $g \circ f$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Since f and g are homeomorphisms, we know that f, g, f^{-1}, g^{-1} are all continuous, hence their compositions $g \circ f$ and $f^{-1} \circ g^{-1}$ are continuous as well, by Proposition 3.3. Therefore $g \circ f$ is a homeomorphism. $\square$

Corollary 3.8. *“Being homeomorphic” is an equivalence relation between topological spaces.*

Proof. Parts (i), (ii) and (iii) of Proposition 3.7 correspond to the fact that “being homeomorphic” is a reflexive, symmetric and transitive relation. $\square$

4 Convergence of sequences, separation axioms

As in the case of continuity, the notion of convergence of a sequence, already introduced for metric spaces, can be rephrased in terms of neighbourhoods.

Proposition 4.1 (convergence in metric spaces)**.** *Let (X, d) be a metric space. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of of elements of X. The following are equivalent:*

(i) the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x;

(ii) for all neighbourhoods U of x in X, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $x_n \in U$.

Proof. (i) ⇒ (ii). Let U be a neighbourhood of x. By Proposition 2.1 there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Since $(x_n)_n$ converges to x, by [**MS**, Definition 5.1] there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Therefore, for all $n \geq N$, $x_n \in B_\epsilon(x) \subseteq U$.

(ii) ⇒ (i). Let $\epsilon > 0$. Then $B_\epsilon(x)$ is a neighbourhood of x. Hence there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. That is, $d(x_n, x) < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we have proved that $(x_n)_n$ converges to x by [**MS**, Definition 5.1]. □

Again, this allows us to extend the notion of convergence of a sequence to the context of topological spaces.

Definition 4.2. Let (X, τ) be a topological space, let $z \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to the point z if, for all neighbourhoods U of z, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $x_n \in U$. In this case we also write $x_n \to z$ and say that z is the *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$.

By Proposition 4.1, the above definition is consistent with the one already given[9] for metric spaces, in the case the topology is induced by a metric.

As we know from the case of metric spaces, a sequence need not converge. However, in metric spaces, if a sequence converges, then the limit is unique.[10] This fact need not be true in a general topological space.

Examples. Let X be a set.

1. Let X be given the discrete topology. Then a sequence $(x_n)_{n\in\mathbb{N}}$ converges to the point x if and only if $x_n = x$ for all n sufficiently large (why?). In other words, the only convergent sequences are the ones that are eventually constant. So "most" sequences do not have a limit (as soon as X has more than one point). However, if the limit exists, then it is unique.

2. Let X be given the trivial topology. Then every sequence of elements of X converges to every point of X (why?). In particular, limits of sequences are never unique (as soon as X has more than one point).

3. Let X be given the cofinite topology. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X such that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ converges to every point of X (why?).

[9] See [**MS**, Definition 5.1].
[10] See [**MS**, Theorem 5.2].

To avoid the "pathological behaviour" in the last two examples, we introduce a particular class of topological spaces.

Definition 4.3. A topological space (X, τ) is said to be *Hausdorff* if, for all pairs of distinct points $x, y \in X$, there exist disjoint neighbourhoods U of x and V of y.

Examples. Let X be a set.

1. If (X, d) is a metric space, then it is a Hausdorff topological space. Indeed, if $x, y \in X$ are distinct points, then $\epsilon = d(x, y)/2 > 0$ and the balls $B_\epsilon(x), B_\epsilon(y)$ are disjoint neighbourhoods of x, y.

2. If X has more than one point, then X with the trivial topology is not Hausdorff (why?); in particular, the trivial topology it is not induced by a metric.

3. If X is infinite, then X with the cofinite topology is not Hausdorff (why?).

By assuming the Hausdorff property, we can prove that limits of sequences are unique.

Proposition 4.4 (uniqueness of limit)**.** *Let (X, τ) be a Hausdorff topological space and $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of X. Suppose that $x_n \to z$ and $x_n \to w$ for some $z, w \in X$. Then $z = w$.*

Proof. By contradiction, suppose that $z \neq w$. Then, since X is Hausdorff, there exist disjoint neighbourhoods U of z and V of w. Since $x_n \to z$ and U is a neighbourhood of z, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Similarly, there exist $M \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq M$. In particular, if we take $n = \max\{M, N\}$, then $x_n \in U \cap V$. This contradicts the disjointness of U and V. □

The definition of "being Hausdorff" is expressed in terms of the topology alone, so it is reasonable that it is a topological property. Here is a precise proof.

Proposition 4.5 ("being Hausdorff" is a topological property)**.** *Let (X, τ_X), (Y, τ_Y) be homeomorphic topological spaces. Suppose that (X, τ_X) is Hausdorff. Then (Y, τ_Y) is Hausdorff as well.*

Proof. Since X and Y are homeomorphic, there exists a homeomorphism $f : Y \to X$. Let y and y' be distinct points of Y. Since f is bijective, $f(y)$ and $f(y')$ are distinct points of X. Since X is Hausdorff, there exist disjoint neighbourhoods U of $f(y)$ and U' of $f(y')$ in X. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(U')$ are neighbourhoods of y and y' respectively, and they are disjoint because f is bijective. □

The Hausdorff property can be equivalently expressed in terms of open sets.

Proposition 4.6 (Hausdorff and open sets)**.** *Let X be a topological space. Then X is Hausdorff if and only if, for all pairs of distinct points $x, y \in X$, there exist disjoint open neighbourhoods A of x and B of y.*

Proof. Clearly, if all pairs of distinct points have disjoint open neighbourhoods, then X is Hausdorff by definition.

Conversely, assume that X is Hausdorff. Let $x, y \in X$ be any distinct points. Then, since X is Hausdorff, there exist a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \emptyset$. Since U is a neighbourhood of x, by definition there exists an open set A such that $x \in A \subseteq U$. Similarly there exists an open set B such that $y \in B \subseteq V$. Then A is an open neighbourhood of x and B is an open neighbourhood of y, and moreover $A \cap B \subseteq U \cap V = \emptyset$, so A and B are disjoint. □

Here is an important consequence of the Hausdorff property.

Proposition 4.7 (points are closed)**.** *Let (X, τ) be a Hausdorff topological space. Then, for all $x \in X$, the set $\{x\}$ is closed in X.*

Proof. Let $x \in X$. We must show that $X \setminus \{x\}$ is open. Let $y \in X \setminus \{x\}$. Then $x \neq y$, hence (since X is Hausdorff) there exist disjoint neighbourhoods U of x and V of y. Since $x \in U$ and $U \cap V = \emptyset$, it must be $x \notin V$, i.e., $V \subseteq X \setminus \{x\}$. Since V is a neighbourhood of y, by Proposition 1.5(ii) we deduce that $X \setminus \{x\}$ is a neighbourhood of y as well. This proves that $X \setminus \{x\}$ is a neighbourhood of all its points, hence $X \setminus \{x\}$ is open by Proposition 1.5(iii). □

Note that the point $x \in X$ is not the same thing as the set $\{x\}$: the former is an element of X, while the latter is a subset of X. However, with a slight abuse of language, we use the expression "points are closed", meaning that subsets formed by just one point (a.k.a. "singletons") are closed.

Examples.

1. Every metric space is Hausdorff, so "points are closed" there.
2. If X has more than one element, then "points are not closed" with respect to the trivial topology on X (why?).
3. If X is infinite, then the cofinite topology on X is not Hausdorff, nevertheless "points are closed" (why?).

Both "being Hausdorff" and the fact that "points are closed" are part of a larger family of so-called *separation axioms*.

Definition 4.8 (separation axioms)**.** Let (X, τ) be a topological space. We say that X is...

(i) ...T_0, if for all pairs of distinct points $x, y \in X$, at least one of the points has a neighbourhood that does not contain the other point;

(ii) ...T_1, if points are closed in X;

(iii) ...T_2, if X is Hausdorff;

(iv) ...T_3, if points are closed in X and moreover, for all closed subsets $C \subseteq X$ and all points $x \in X \setminus C$, there exist disjoint open subsets $A, B \subseteq X$ such that $x \in A$ and $C \subseteq B$;

(v) ...T_4, if points are closed in X and moreover, for all pairs of disjoint closed subsets $C, D \subseteq X$, there exist disjoint open subsets $A, B \subseteq X$ such that $C \subseteq A$ and $D \subseteq B$.

It is possible to prove that $\mathrm{T}_0, \mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_4$ are all topological properties, and that each is more restrictive than the previous ones. In other words, for a given topological space, $\mathrm{T}_4 \Rightarrow \mathrm{T}_3 \Rightarrow \mathrm{T}_2 \Rightarrow \mathrm{T}_1 \Rightarrow \mathrm{T}_0$. The implication $\mathrm{T}_2 \Rightarrow \mathrm{T}_1$ is shown in Proposition 4.7. We leave the proofs of the other implications as an exercise for the interested reader. It is also possible to prove that every metric space is T_4.

Note that the reverse implications do not hold in general. The above example about the cofinite topology shows that $\mathrm{T}_1 \not\Rightarrow \mathrm{T}_2$. A more difficult exercise is to find counterexamples to all the reverse implications.

5 Interior, closure, boundary, limit points

Definition 5.1. Let X be a topological space and $U \subseteq X$.

(i) A point $x \in X$ is called *interior point* of U if U is a neighbourhood of x.

(ii) The set $\mathring{U}$ of all interior points of U is called *interior* of U.

(iii) A point $x \in X$ is called *closure point* (or *adherent point*) of U if all neighbourhoods of x intersect U.

(iv) The set $\overline{U}$ of all closure points of U is called *closure* of U.

(v) The set $\partial U = \overline{U} \setminus \mathring{U}$ is called *boundary* of U.

(vi) The elements of ∂U are called *boundary points* of U.

We leave to the interested reader to check that the above definition of closure is consistent with the one[11] given for metric spaces.

Examples. Let $X = \mathbb{R}$ with the standard topology.

1. Let $U = [a, b)$ for some $a, b \in \mathbb{R}$ with $a < b$. Then
$$\overline{U} = [a, b], \qquad \mathring{U} = (a, b), \qquad \partial U = \{a, b\}.$$

2. Let $U = \{1/n : n \in \mathbb{N}\}$. Then
$$\overline{U} = \{0\} \cup \{1/n : n \in \mathbb{N}\}, \qquad \mathring{U} = \emptyset, \qquad \partial U = \{0\} \cup \{1/n : n \in \mathbb{N}\}.$$

Let now $X = \mathbb{R}^2$ with the standard topology.

3. Let $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then
$$\overline{U} = \{(x, y) : x^2 + y^2 \leq 1\}, \qquad \mathring{U} = U, \qquad \partial U = \{(x, y) : x^2 + y^2 = 1\}.$$

Despite the apparently very different definitions of interior and closure, these notions are strictly related.

Proposition 5.2 (interior, closure, boundary and complements)**.** *Let X be a topological space and $U \subseteq X$.*

(i) $\overline{X \setminus U} = X \setminus \mathring{U}$.

(ii) $(X \setminus U)^\circ = X \setminus \overline{U}$.

(iii) $\partial(X \setminus U) = \partial U$.

Proof. (i). If $x \in \overline{X \setminus U}$, i.e., x is a closure point of $X \setminus U$, then all neighbourhoods of x intersect $X \setminus U$. Therefore there is no neighbourhood of x that is contained in U. In particular U is not a neighbourhood of x and $x \notin \mathring{U}$, i.e., $x \in X \setminus \mathring{U}$.

Suppose instead that $x \notin \overline{X \setminus U}$, i.e., x is not a closure point of $X \setminus U$. Then there is a neighbourhood V of x that does not intersect $X \setminus U$. Therefore $V \subseteq U$ and, by Proposition 1.5(ii), U is a neighbourhood of x as well. Hence $x \in \mathring{U}$, i.e., $x \notin X \setminus \mathring{U}$.

(ii). By part (i), $X \setminus (X \setminus U)^\circ = \overline{X \setminus (X \setminus U)} = \overline{U}$, whence $(X \setminus U)^\circ = X \setminus \overline{U}$.

(iii). By parts (i) and (ii), $\partial(X \setminus U) = \overline{X \setminus U} \setminus (X \setminus U)^\circ = (X \setminus \mathring{U}) \setminus (X \setminus \overline{U}) = \overline{U} \setminus \mathring{U} = \partial U$. □

[11] See [**MS**, Definition 6.3].

Proposition 5.3 (characterisation of interior and closure)**.** *Let X be a topological space and $U \subseteq X$.*

(i) $\mathring{U}$ is the union of all open subsets of X that are contained in U. Hence $\mathring{U}$ is open and in fact is the largest open subset of X contained in U.

(ii) $\overline{U}$ is the intersection of all closed subsets of X that contain U. Hence $\overline{U}$ is closed and in fact is the smallest closed subset of X containing U.

Proof. (i). Let $\mathcal{A}$ be the collection of the open sets that are contained in U and let $U' = \bigcup \mathcal{A}$. Then clearly U' is open and $U' \subseteq U$. In fact, U' is the largest open set contained in U (because all elements of $\mathcal{A}$ are subsets of U'). Moreover, for all $x \in U'$, we have $x \in U' \subseteq U$ and U' is open, therefore U is a neighbourhood of x and consequently $x \in \mathring{U}$. Vice versa, if $x \in \mathring{U}$, then U is a neighbourhood of x, therefore there exists an open set A such that $x \in A \subseteq U$; so $A \in \mathcal{A}$ and consequently $x \in A \subseteq \bigcup \mathcal{A} = U'$. This shows that $\mathring{U} = U' = \bigcup \mathcal{A}$.

(ii). Let $\mathcal{C}$ be the collection of all closed sets that contain U. Let $\mathcal{A}$ be the collection of all open sets that are contained in $X \setminus U$. Then $C \in \mathcal{C}$ if and only if $X \setminus C \in \mathcal{A}$. In particular $\bigcap \mathcal{C} = X \setminus \bigcup \mathcal{A}$ by De Morgan's laws. On the other hand $\bigcup \mathcal{A} = (X \setminus U)^{\circ}$ by part (i) and $(X \setminus U)^{\circ} = X \setminus \overline{U}$ by Proposition 5.2(ii). Therefore $\bigcap \mathcal{C} = X \setminus \bigcup \mathcal{A} = \overline{U}$. □

Corollary 5.4 (characterisation of open and closed sets)**.** *Let X be a topological space and $U \subseteq X$.*

(i) U is open if and only if $\mathring{U} = U$.

(ii) U is closed if and only if $\overline{U} = U$.

Proof. (i). If $U = \mathring{U}$, then U is open by Proposition 5.3. Vice versa, if U is open, then U is the largest open set contained in U and therefore $U = \mathring{U}$ by Proposition 5.3 again.

(ii). The proof is analogous. □

Proposition 5.5 (another characterisation of continuity)**.** *Let X and Y be topological spaces and $f : X \to Y$. The following are equivalent:*

(i) f is continuous;

(ii) $f(\overline{V}) \subseteq \overline{f(V)}$ for all $V \subseteq X$.

Proof. (i) $\Rightarrow$ (ii). Let $V \subseteq X$. Then $\overline{f(V)}$ is closed in Y. Hence $f^{-1}(\overline{f(V)})$ is closed in X, because f is continuous (see Proposition 3.4(iii)). Moreover $V \subseteq f^{-1}(f(V)) \subseteq f^{-1}(\overline{f(V)})$. That is, $f^{-1}(\overline{f(V)})$ is a closed subset of X that contains V, and consequently $\overline{V} \subseteq f^{-1}(\overline{f(V)})$ by Proposition 5.3.

(ii) $\Rightarrow$ (i). Let C be a closed subset of Y. Then, by (ii), $f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))}$, and moreover, by Proposition 5.3, $\overline{f(f^{-1}(C))} \subseteq C$ (because C is closed and $f^{-1}(f(C)) \subseteq C$). This implies that $f(\overline{f^{-1}(C)}) \subseteq C$, that is, $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$. Since the opposite inclusion is trivial (see Proposition 5.3), we have that $\overline{f^{-1}(C)} = f^{-1}(C)$, that is, $f^{-1}(C)$ is closed by Corollary 5.4. Since C was an arbitrary closed subset of Y, by Proposition 3.4(iii) we conclude that f is continuous. □

Definition 5.6. Let X be a topological space and $U \subseteq X$. We say that U is *dense* in X if $\overline{U} = X$.

Examples. Let $X = \mathbb{R}$ with the standard topology. Then the sets $\mathbb{Q}$ of rational numbers and $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers are both dense in $\mathbb{R}$ (why?).

Definition 5.7. Let X be a topological space, $x \in X$ and $U \subseteq X$. We say that x is a *limit point* (or *accumulation point*, or *cluster point*) of U if all neighbourhoods of x intersect $U \setminus \{x\}$.

We leave to the interested reader to check that the above definition is consistent with the one given for metric spaces[12].

Proposition 5.8. *Let X be a topological space and $U \subseteq X$. Let U' be the set of limit points of U. Then*

$$\overline{U} \setminus U \subseteq U' \subseteq \overline{U}.$$

Proof. From the respective defintions, it is clear that any limit point of U is a closure point of U, and therefore $U' \subseteq \overline{U}$.

Suppose now that $x \in \overline{U} \setminus U$. Let V be any neighbourhood of x. Since $x \in \overline{U}$, x is a closure point of U and therefore $V \cap U \neq \emptyset$. Take $y \in V \cap U$. Since $x \notin U$, it must be $x \neq y$ and therefore $y \in V \cap (U \setminus \{x\})$, so $V \cap (U \setminus \{x\}) \neq \emptyset$. Since V was an arbitrary neighbourhood of x, this shows that x is a limit point of U, i.e., $x \in U'$. Since x was an arbitrary element of $\overline{U} \setminus U$, this shows that $\overline{U} \setminus U \subseteq U'$. □

The following statement extends [**MS**, Theorem 6.2] to all topological spaces.

Corollary 5.9. *Let X be a topological space and $U \subseteq X$. Then U is closed if and only if U contains all its limit points.*

Proof. Let U' be the set of limit points of U.

Suppose that U is closed. Then $\overline{U} = U$ by Corollary 5.4 and $U' \subseteq \overline{U}$ by Proposition 5.8; hence $U' \subseteq U$, that is, U contains all its limit points.

Conversely, suppose that $U' \subseteq U$. Since $\overline{U} \setminus U \subseteq U'$ by Proposition 5.8, we conclude that $\overline{U} \setminus U \subseteq U$. But $\overline{U} \setminus U$ is clearly disjoint from U, hence $\overline{U} \setminus U = \emptyset$ and $\overline{U} = U$. This means that U is closed by Corollary 5.4. □

[12] See [**MS**, Definition 6.1].

6 Subspace topology

Proposition 6.1. *Let (X, τ) be a topological space and $U \subseteq X$. Then the collection $\tau|_U$ defined by*

$$\tau|_U = \{A \cap U : A \in \tau\}. \tag{6.1}$$

is a topology on U.

Proof. We must check that the collection $\tau|_U$ of subsets of U satisfies parts (i)-(iii) of Definition 1.1.

(i). Note that $\emptyset = \emptyset \cap U$ and $U = X \cap U$. Since $\emptyset, X \in \tau$, then $\emptyset, U \in \tau|_U$.

(ii). Let $\mathcal{A}$ be any subcollection of $\tau|_U$. Then, for all $A \in \mathcal{A}$, there exists $\tilde{A} \in \tau$ such that $A = \tilde{A} \cap U$. In particular $\tilde{\mathcal{A}} = \{\tilde{A}\}_{A \in \mathcal{A}}$ is a subcollection of τ, hence its union $\bigcup \tilde{\mathcal{A}}$ belongs to τ. On the other hand

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (\tilde{A} \cap U) = \left(\bigcup_{\tilde{A} \in \mathcal{A}} A \right) \cap U = \left(\bigcup \tilde{\mathcal{A}} \right) \cap U$$

and $\bigcup \tilde{\mathcal{A}} \in \tau$, therefore $\bigcup \mathcal{A} \in \tau|_U$.

(iii). Let $A, B \in \tau|_U$. Then there exist $\tilde{A}, \tilde{B} \in \tau$ such that $A = \tilde{A} \cap U$ and $B = \tilde{B} \cap U$. In particular $A \cap B = (\tilde{A} \cap U) \cap (\tilde{B} \cap U) = (\tilde{A} \cap \tilde{B}) \cap U$. But $\tilde{A} \cap \tilde{B} \in \tau$. Hence $A \cap U \in \tau|_U$. □

Definition 6.2. Let (X, τ) be a topological space and $U \subseteq X$. The topology $\tau|_U$ defined by (6.1) is called the *subspace topology* or *relative topology* on U. It is also called the topology on U *induced* by (X, τ).

The subspace topology is a particularly convenient way to construct new topologies from a given one and we will use it systematically. Note however that this notion may be a bit confusing, since the notion of "open set" and "closed set" are *relative* to the ambient space that we take as a reference.

Examples. Let $X = \mathbb{R}$ be given the standard topology.

1. Let $U = [0, \infty)$ be given the subspace topology. Then $[0, 1)$ is not open in X. However $[0, 1)$ is open in U, because $[0, 1) = (-1, 1) \cap U$ and $(-1, 1)$ is open in X.

2. Let $[0, 1)$ be given the subspace topology. Then $[0, 1)$ is neither open nor closed in $\mathbb{R}$. However $[0, 1)$ is both open and closed in itself.

3. The subspace topology on $\mathbb{N}$ coincides with the discrete topology. Indeed, for all $A \subseteq \mathbb{N}$, the set $\tilde{A} = A + (-1/4, 1/4) = \bigcup_{n \in A}(n - 1/4, n + 1/4)$ is open in $\mathbb{R}$ and $A = \tilde{A} \cap \mathbb{R}$. Therefore all subsets of $\mathbb{N}$ are open in $\mathbb{N}$, but (apart from the empty set) none of them is open in $\mathbb{R}$.

There is one case, however, where "openness" is less ambiguous.

Proposition 6.3 (open in open is open)**.** *Let X be a topological space. Let $U \subseteq X$ be given the subspace topology. Let $V \subseteq U$.*

(i) If V is open in X, then V is open in U.

(ii) If V is open in U and U is open in X, then V is open in X.

Proof. (i). If V is open in X, then $V = V \cap U$ is open in U.

(ii). If V is open in U, then $V = V' \cap U$ for some open subset V' of X. But U is open in X as well. So V is the intersection of two open subsets of X and therefore V is open in X as well. □

Closed subsets of a subspace can be characterised analogously.

Proposition 6.4 (closed subsets of a subspace)**.** *Let X be a topological space. Let $U \subseteq X$ be given the subspace topology. Then $C \subseteq U$ is closed in U if and only if there exists a closed subset C' of X such that $C = C' \cap U$.*

Proof. If C is closed in U, then $U \setminus C$ is open in U. Therefore $U \setminus C = A \cap U$ for some open subset A of X. Let $C' = X \setminus A$. Then C' is closed in X and moreover $C' \cap U = (X \setminus A) \cap U = U \setminus (U \cap A) = C$.

Vice versa, suppose that $C = C' \cap U$ for some closed subset C' of X. Then $A = X \setminus C'$ is open in X. Moreover $A \cap U = (X \setminus C') \cap U = U \setminus (C' \cap U) = U \setminus C$. This shows that $U \setminus C$ is open in U and therefore C is closed in U. □

Proposition 6.5 (closed in closed is closed)**.** *Let X be a topological space. Let $U \subseteq X$ be given the subspace topology. Let $V \subseteq U$.*

(i) If V is closed in X, then V is closed in U.

(ii) If V is closed in U and U is closed in X, then V is closed in X.

Proof. Analogous to the proof of Proposition 6.3. □

Proposition 6.6 (neighbourhoods in a subspace)**.** *Let X be a topological space. Let $U \subseteq X$ be given the subspace topology. Let $x \in U$. Then $V \subseteq U$ is a neighbourhood of x in U if and only if there exists a neighbourhood V' of x in X such that $V = V' \cap U$.*

Proof. Let V be a neighbourhood of x in U. Then there exists an open subset A of U such that $x \in A \subseteq V$. Since A is open in U, there exists an open subset A' of X such that $A = A' \cap U$. Let $V' = A' \cup V$. Then $x \in A' \subseteq V'$ and therefore V' is a neighbourhood of x in X. Moreover $V' \cap U = (A' \cup V) \cap U = (A' \cap U) \cup (V \cap U) = A \cup V = V$.

Vice versa, suppose that $V = V' \cap U$ for some neighbourhood V' of x in X. Then there exists an open subset A' of X such that $x \in A' \subseteq V'$. So $A = A' \cap U$ is open in U and moreover $x \in A \subseteq V' \cap U = V$. Therefore V is a neighbourhood of x in U. □

The above characterisation of neighbourhoods in the subspace topology immediately yields that the Hausdorff property is preserved when passing to a subspace.

Corollary 6.7. *A subspace of a Hausdorff topological space is Hausdorff.*

Proof. Let X be a Hausdorff topological space. Let $U \subseteq X$ be given the subspace topology. We must show that U is Hausdorff as well.

Let x, x' be distinct points of U. Then x, x' are distinct points of X. Since X is Hausdorff, there exist disjoint neighbourhoods V of x and V' of x' in X. By Proposition 6.6, the sets $W = V \cap U$ and $W' = V' \cap U$ are neighbourhoods of x and x' in U respectively and moreover they are clearly disjoint. □

A subset of a subset is a subset. Therefore for a subset of a subset of a topological space we have two ways of defining a topology and the following proposition shows that they coincide.

Proposition 6.8 (subspace of subspace)**.** *Let* (X, τ) *be a topological space and let* $V \subseteq U \subseteq X$*. Then* $\tau|_V = (\tau|_U)|_V$*.*

Proof. Let $A \in \tau|_V$. Then there exists $A' \in \tau$ such that $A = A' \cap V$. Hence $A' \cap U \in \tau|_U$ by definition and $(A' \cap U) \cap V \in (\tau|_U)|_V$. On the other hand $(A' \cap U) \cap V = A' \cap (U \cap V) = A' \cap V = A$, so $A \in (\tau|_U)|_V$.

Vice versa, suppose that $A \in (\tau|_U)|_V$. Then there exists $A' \in \tau_U$ such that $A = A' \cap V$. Therefore there exists $A'' \in \tau$ such that $A' = A'' \cap U$. On the other hand, $A = A' \cap V = (A'' \cap U) \cap V = A'' \cap (U \cap V) = A'' \cap V$. This shows that $A \in \tau|_V$. □

For a subset of a metric space, we have two different ways of defining a topology. The following proposition shows that they concide.

Proposition 6.9 (subspace of a metric space)**.** *Let* (X, d) *be a metric space and* $U \subseteq X$*. Let* τ_d *be the topology on* X *induced by the metric and* $\tau_d|_U$ *the corresponding subspace topology on* U*. Let* d_U *be the restriction of the metric* d *to* U*, and let* τ_{d_U} *be the metric topology on* U *induced by* d_U*. Then* $\tau_{d_U} = \tau_d|_U$*.*

Proof. Let $A \in \tau_{d_U}$. Then, for all $x \in A$, there exists $\epsilon_x > 0$ such that $B^{d_U}_{\epsilon_x}(x) \subseteq A$. Therefore $A = \bigcup_{x \in A} B^{d_U}_{\epsilon_x}(x)$. Moreover

$$B^{d_U}_{\epsilon_x}(x) = \{x' \in U : d_U(x, x') < \epsilon_x\} = \{x' \in X : d(x, x') < \epsilon_x\} \cap U = B^d_{\epsilon_x}(x) \cap U.$$

Hence $A = \bigcup_{x \in A} (B^d_{\epsilon_x}(x) \cap U) = A' \cap U$, where $A' = \bigcup_{x \in A} B^d_{\epsilon_x}(x)$. Since A' is a union of open sets, it is open in (X, d), that is, $A' \in \tau_d$, and consequently $A = A' \cap U \in \tau_d|_U$.

Vice versa, let $A \in \tau_d|_U$. Then there exists $A' \in \tau_d$ such that $A = A' \cap U$. If $x \in A$, then $x \in A'$ and consequently there exists $\epsilon > 0$ such that $B^d_\epsilon(x) \subseteq A'$. Hence $B^{d_U}_\epsilon(x) = B^d_\epsilon(x) \cap U \subseteq A' \cap U = A$. Since $x \in A$ was arbitrary, we have proved that A is open in the metric space (X, d_U), i.e., $A \in \tau_{d_U}$. □

The following propositions are particularly important, because they show that the subspace topology "behaves well" with respect to continuous functions.

Proposition 6.10 (inclusion map)**.** *Let* (X, τ) *be a topological space. Let* $U \subseteq X$ *be given the subspace topology and let* $i : U \to X$ *be the* inclusion map *(i.e.,* $i(x) = x$ *for all* $x \in U$*). Then* i *is continuous.*

Proof. Let $A \subseteq X$. Then it is easy to see that $i^{-1}(A) = A \cap U$. In particular, if A is open in X, then $i^{-1}(A)$ is open in U by definition of subspace topology. Since A was arbitrary, we conclude that i is continuous. □

Proposition 6.11 (range extension/restriction)**.** *Let* (X, τ_X)*,* (Y, τ_Y) *be topological spaces. Let* $Z \subseteq Y$ *be given the subspace topology and let* $i : Z \to Y$ *be the inclusion map. Then a function* $f : X \to Z$ *is continuous if and only if* $i \circ f : X \to Y$ *is continuous.*

Proof. By Proposition 6.10, $i : Z \to Y$ is continuous. So, if $f : X \to Z$ is continuous, then $i \circ f : X \to Y$ is continuous by Proposition 3.3.

Vice versa, suppose that $i \circ f : X \to Y$ is continuous. Let $A \subseteq Z$ be open. Then $A = A' \cap Z$ for some open subset A' of Y. In particular

$$f^{-1}(A) = f^{-1}(A' \cap Z) = f^{-1}(i^{-1}(A')) = (i \circ f)^{-1}(A').$$

Since A' is open in Y and $i \circ f : X \to Y$ is continuous, $f^{-1}(A)$ is open in X. Therefore, since A was an arbitrary open subset of Z, we conclude that $f : X \to Z$ is continuous. $\square$

Proposition 6.12 (domain restriction)**.** *Let (X, τ_X), (Y, τ_Y) be topological spaces. Let $U \subseteq X$ be given the subspace topology. For all continuous functions $f : X \to Y$, the restriction $f|_U : U \to Y$ is continuous.*

Proof. Note that $f|_U = f \circ i$, where $i : U \to X$ is the inclusion map. But i is continuous by Proposition 6.10. So $f|_U = f \circ i$ is continuous by Proposition 3.3. $\square$

A common way of defining functions is the "definition by cases". In other words, functions are often defined by "gluing together" other functions defined on smaller subsets. We now show that, under suitable assumptions, by gluing together continuous functions, one obtains another continuous function.

Definition 6.13. Let (X, τ) be a topological space.

(i) A *cover* of X is a collection $\mathcal{A}$ of subsets of X such that $\bigcup \mathcal{A} = X$.[13]

(ii) A cover $\mathcal{A}$ of X is said to be *open* if the elements of $\mathcal{A}$ are open subsets of X.

(iii) A cover $\mathcal{A}$ of X is said to be *closed* if the elements of $\mathcal{A}$ are closed subsets of X.

Proposition 6.14 (gluing continuous function together)**.** *Let (X, τ_X), (Y, τ_Y) be topological spaces. Let $\mathcal{A}$ satisfy one of the following conditions:*

(i) $\mathcal{A}$ is an open cover of X;

(ii) $\mathcal{A}$ is a finite closed cover of X.

Then a function $f : X \to Y$ is continuous if and only if, for all $A \in \mathcal{A}$, the restriction $f|_A : A \to X$ is continuous.

Proof. If f is continuous, then its restrictions $f|_A$ are continuous by Proposition 6.12. What we need to prove is the reverse implication.

Suppose then that $f|_A$ is continuous for all $A \in \mathcal{A}$. Suppose moreover that (i) holds. Let $U \subseteq Y$ be open. Then

$$f^{-1}(U) = f^{-1}(U) \cap X = f^{-1}(U) \cap \bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (f^{-1}(U) \cap A) = \bigcup_{A \in \mathcal{A}} f|_A^{-1}(U).$$

[13] Warning: The word "cover" is often given a broader definition in the literature. Namely, one defines a "cover" of a subset $C \subseteq X$ as a collection $\mathcal{A}$ of subsets of X such that $C \subseteq \bigcup \mathcal{A}$ (note that only containment and not equality is required). We will not use this more extensive notion in these lectures.

For all $A \in \mathcal{A}$, since $f|_A$ is continuous, the set $f|_A^{-1}(U)$ is open in A, therefore it is open in X as well (by Proposition 6.3). Hence $f^{-1}(U)$ is union of a collection of open sets, so $f^{-1}(U)$ is open. Since U was an arbitrary open subset of Y, we have proved that $f : X \to Y$ is continuous.

In the case (ii) holds, one can prove in a similar way that $f^{-1}(U)$ is closed in X for all closed subsets U of Y: Proposition 6.5 is used in place of Proposition 6.3 and the finiteness of $\mathcal{A}$ is used to show that the union of the family of closed sets $f|_A^{-1}(U)$ is closed as well. □

Examples. Let $\mathbb{R}$ be given the standard topology. The following examples demonstrate the importance of the conditions (i)-(ii) of Proposition 6.14.

1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x^5 - 3x^2 & \text{otherwise,} \end{cases}$$

is continuous. Indeed note that

$$\lim_{x \to 0-} \sin x = \lim_{x \to 0+} (x^5 - 3x^2) = 0 = f(0).$$

We can also use Proposition 6.14 to prove that f is continuous: the collection $\{A, B\}$, where $A = (-\infty, 0]$ and $B = [0, \infty)$, is a finite closed cover of $\mathbb{R}$ and $f|_A$, $f|_B$ are both continuous functions, since

$$f(x) = \begin{cases} \sin x & \text{if } x \in A, \\ x^5 - 3x^2 & \text{if } x \in B, \end{cases}$$

so Proposition 6.14 applies.

2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x^5 - 3x^2 + 1 & \text{otherwise,} \end{cases}$$

is not continuous. Indeed note that

$$\lim_{x \to 0-} \sin x = 0 \neq 1 = \lim_{x \to 0+} (x^5 - 3x^2 + 1).$$

Observe that, if $C = (-\infty, 0)$ and $D = [0, \infty)$, then $\{C, D\}$ is a finite cover of $\mathbb{R}$ and $f|_C$, $f|_D$ are both continuous. However $\{C, D\}$ is neither an open cover nor a closed cover, so Proposition 6.14 does not apply here.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be any non-continuous function. Let $\mathcal{C} = \{\{x\} : x \in \mathbb{R}\}$ be the set of singletons of $\mathbb{R}$. Then $\mathcal{C}$ is an (infinite) closed cover of $\mathbb{R}$ and obviously $f|_{\{x\}} : \{x\} \to \mathbb{R}$ is continuous (it is constant) for all $x \in \mathbb{R}$. However $\mathcal{C}$ is neither an open cover nor a finite closed cover.

7 Base of a topology

Definition 7.1. Let τ be a topology on a set X and let $\mathcal{B}$ be a collection of subsets of X. We say that $\mathcal{B}$ is a *base* of the topology τ if

$$\tau = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}. \tag{7.1}$$

In other words, if (X, τ) is a topological space, then a base of the topology τ is a collection $\mathcal{B}$ of open sets such that every open subset of X is the union of some subcollection of $\mathcal{B}$.

Examples. Here are some examples of bases of topologies.

1. Let (X, τ) be any topological space. The topology τ itself is a base of the topology τ.

2. Let X be a metric space. The collection $\mathcal{B} = \{B_r(x) : x \in X, r \in (0, \infty)\}$ of the balls in X is a base of the topology of X. Indeed, for all open subsets A of X and all $x \in A$, there exists $r_x \in (0, \infty)$ such that $B_{r_x}(x) \subseteq A$, and therefore $A = \bigcup_{x \in A} B_{r_x}(x)$.

3. The collection $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ of open intervals with rational endpoints is a base of the standard topology of $\mathbb{R}$ (why?).

4. Let X be given the discrete topology. The collection $\mathcal{B} = \{\{x\} : x \in X\}$ of singletons of X is a base of the topology of X. Indeed, every subset of X is open and we can write $A = \bigcup_{x \in A} \{x\}$ for all $A \subseteq X$.

Proposition 7.2. *Let $\mathcal{B}$ be a collection of subsets of a set X such that:*

(i) $X = \bigcup \mathcal{B}$;

(ii) for all $A, B \in \mathcal{B}$, there exists a subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that $A \cap B = \bigcup \mathcal{C}$.

Then the collection τ of subsets of X defined by (7.1) *is a topology on X and $\mathcal{B}$ is a base of τ.*

Vice versa, if $\mathcal{B}$ is a base of a topology τ on X, then $\mathcal{B}$ satisfies the properties (i) and (ii).

Proof. Suppose first that $\mathcal{B}$ satisfies the properties (i) and (ii) and let τ be defined by (7.1). We must show that τ is a topology on X.

Indeed by (i) we have $X = \bigcup \mathcal{B}$ and moreover $\emptyset = \bigcup \emptyset$; since $\mathcal{B}$ and $\emptyset$ are both subcollections of $\mathcal{B}$, we conclude that $X, \emptyset \in \tau$.

Let $A, A' \in \tau$. Let us show that $A \cap A' \in \tau$. Indeed, by definition of τ, $A = \bigcup \mathcal{U}$ and $A' = \bigcup \mathcal{U}'$ for some subcollections $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{B}$. So

$$A \cap A' = \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{U' \in \mathcal{U}'} U' \right) = \bigcup_{U \in \mathcal{U}} \bigcup_{U' \in \mathcal{U}'} (U \cap U')$$

On the other hand, for all $U, U' \in \mathcal{B}$, by (ii) there exists a subcollection $\mathcal{A}_{U,U'}$ of $\mathcal{B}$ such that $U \cap U' = \bigcup \mathcal{A}_{U,U'}$. Therefore

$$A \cap A' = \bigcup_{U \in \mathcal{U}} \bigcup_{U' \in \mathcal{U}'} \bigcup \mathcal{A}_{U,U'} = \bigcup \mathcal{A},$$

where $\mathcal{A} = \bigcup_{U \in \mathcal{U}} \bigcup_{U' \in \mathcal{U}'} \mathcal{A}_{U,U'}$ is a subcollection of $\mathcal{B}$. Hence $A \cap A' \in \tau$.

Finally, let $\mathcal{A}$ be any subcollection of τ. Let us show that $\bigcup \mathcal{A} \in \tau$. Indeed, by definition of τ, for all $A \in \mathcal{A}$ there exists a subcollection $\mathcal{U}_A \subseteq \mathcal{B}$ such that $A = \bigcup \mathcal{U}_A$. So

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} \bigcup \mathcal{U}_A = \bigcup \mathcal{U},$$

where $\mathcal{U} = \bigcup_{A \in \mathcal{A}} \mathcal{U}_A$ is a subcollection of $\mathcal{B}$. Hence $\bigcup \mathcal{A} \in \tau$.

We have then proved that τ is a topology on X. Moreover, since (7.1) holds by construction, $\mathcal{B}$ is a base of τ.

Vice versa, suppose that $\mathcal{B}$ is a base of a topology τ on X. Then (7.1) holds. In particular, since $X \in \tau$, there exists a subcollection $\mathcal{A}$ of $\mathcal{B}$ such that $X = \bigcup \mathcal{A}$, hence a fortiori $X = \bigcup \mathcal{B}$. This proves that (i) holds.

Let now $A, B \in \mathcal{B}$. Then $\{A\}$ and $\{B\}$ are subcollections of $\mathcal{B}$, hence $A = \bigcup \{A\}$ and $B = \bigcup \{B\}$ are elements of τ by (7.1). Therefore $A \cap B \in \tau$ as well, because τ is a topology. So $A \cap B$ is the union of a subcollection of $\mathcal{B}$ by (7.1). This proves that (ii) holds. $\square$

Proposition 7.3 (continuity and bases)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a base of τ_Y. Then the following are equivalent:*

(i) f is continuous;

(ii) $f^{-1}(A)$ is open in X for all $A \in \mathcal{B}$.

Proof. (i) $\Rightarrow$ (ii). Note that all elements A of $\mathcal{B}$ are open in Y. Therefore, if $f : X \to Y$ is continuous, then $f^{-1}(A)$ is open in X by Proposition 3.4.

(ii) $\Rightarrow$ (i). Let U be any open set in Y. Then, since $\mathcal{B}$ is a base of the topology of Y, there exists a subcollection $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$. In particular

$$f^{-1}(U) = f^{-1}\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f^{-1}(A).$$

By (ii), $f^{-1}(A)$ is open in X for all $A \in \mathcal{A}$. So $f^{-1}(U)$ is a union of open sets and therefore it is open in X. Since U is an arbitrary open set in Y, we conclude that f is continuous by Proposition 3.4. $\square$

Corollary 7.4. *Let X be a set. Let τ and τ' be topologies on X. Let $\mathcal{B}$ be a base of τ. Then the following are equivalent:*

(i) $\tau \subseteq \tau'$;

(ii) $\mathcal{B} \subseteq \tau'$.

Proof. Note that $\tau \subseteq \tau'$ if and only if the identity map $\mathrm{id}_X : X \to X$ is continuous from (X, τ') to (X, τ). Similarly, $\mathcal{B} \subseteq \tau'$ if and only if $\mathrm{id}_X^{-1}(A) \in \tau'$ for all $A \in \mathcal{B}$. Therefore the equivalence of (i) and (ii) follows from Proposition 7.3 applied to the topological spaces (X, τ') and (X, τ) and the map id_X. $\square$

8 Product topology

Given two topological spaces X and Y, we would like to give the set $X \times Y$ a topology. How can we construct a topology on $X \times Y$, starting from the topologies on X and Y?

It seems natural to declare open in $X \times Y$ the products $A \times B$ of open subsets A of X and B of Y, the so-called *open rectangles*. However in general the collection of open rectangles is not a topology on $X \times Y$ (a union of rectangles need not be a rectangle). Therefore we need a more refined construction.

Proposition 8.1. *Let (X, τ_X), (Y, τ_Y) be topological spaces. The collection*

$$\mathcal{B} = \{A \times B : A \in \tau_X,\ B \in \tau_Y\} \tag{8.1}$$

of open rectangles is a base of a topology on $X \times Y$.

Proof. Let us check that $\mathcal{B}$ satisfies the conditions of Proposition 7.2.

(i). $X \times Y \in \mathcal{B}$, so $X \times Y$ is the union of the subcollection $\{X \times Y\}$ of $\mathcal{B}$.

(ii). Let $U, U' \in \mathcal{B}$. Then $U = A \times B$ and $U' = A' \times B'$ for some $A, A' \in \tau_X$ and $B, B' \in \tau_Y$. Hence

$$U \cap U' = (A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Since $A \cap A' \in \tau_X$ and $B \cap B' \in \tau_Y$, we conclude that $U \cap U' \in \mathcal{B}$. In particular $U \cap U'$ is the union of the subcollection $\{U \cap U'\}$ of $\mathcal{B}$. $\square$

Definition 8.2. Let (X, τ_X), (Y, τ_Y) be topological spaces. The topology whose base is given by (8.1) is called *product topology* on $X \times Y$.

In case X and Y are metric spaces, the product topology on $X \times Y$ turns out to be a metric topology.

Proposition 8.3 (product of metric topologies)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let d be the metric*[14] *on the product $X \times Y$ defined by*

$$d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\} \tag{8.2}$$

for all $(x, y), (x', y') \in X \times Y$. Then the metric topology on $(X \times Y, d)$ coincides with the product topology of the metric topologies on X and Y.

Proof. Let τ be the product topology of the metric topologies on (X, d_X) and (Y, d_Y). Let τ_d be the metric topology on $(X \times Y, d)$. We must show that $\tau = \tau_d$.

Let $\mathcal{B}_d$ be the collection of balls in $(X \times Y, d)$. Since $\mathcal{B}_d$ is a base of the topology τ_d, in order to show that $\tau_d \subseteq \tau$, by Corollary 7.4 it is enough to prove that $\mathcal{B}_d \subseteq \tau$. On the other hand, for all $(x, y) \in X \times Y$ and $r > 0$,

$$\begin{aligned} B_r^d((x, y)) &= \{(x', y') \in X \times Y : \max\{d_X(x, x'), d_Y(y, y')\} < r\} \\ &= B_r^{d_X}(x) \times B_r^{d_Y}(y), \end{aligned}$$

so in particular all balls in $(X \times Y, d)$ are open rectangles and therefore $\mathcal{B}_d \subseteq \tau$.

Let $\mathcal{B}$ be the collection of open rectangles in $X \times Y$. As before, since $\mathcal{B}$ is a base of τ, in order to show that $\tau \subseteq \tau_d$ it is sufficient to prove that $\mathcal{B} \subseteq \tau_d$. In

[14]The proof that d is indeed a metric on $X \times Y$ is left as an exercise to the reader.

other words, we must show that $A \times B$ is open in the metric space $(X \times Y, d)$ for all open sets A in X and B in Y.

Let $(x, y) \in A \times B$. Since A is open in X, there exists $r_X > 0$ such that $B^{d_X}_{r_X}(x) \subseteq A$. Similarly there exists $r_Y > 0$ such that $B^{d_Y}_{r_Y}(y) \subseteq B$. In particular, if $r = \min\{r, r'\}$, then $r > 0$ and

$$B^d_r((x, y)) = B^{d_X}_r(x) \times B^{d_Y}_r(y) \subseteq B^{d_X}_{r_X}(x) \times B^{d_Y}_{r_Y}(y) \subseteq A \times B.$$

Since $(x, y) \in A \times B$ was arbitrary, this proves that $A \times B$ is open in the metric space $(X \times Y, d)$. □

Examples. Here are a few examples of product topologies.

1. Let $\mathbb{R}$ be given the standard topology. Then the product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the standard topology on $\mathbb{R}^2$. Indeed the Euclidean metric d_2 on $\mathbb{R}^2$ is equivalent to the metric d_∞ (see example on page 4), which is the same as the "product metric" (8.2) constructed from the Euclidean metrics on the factors $\mathbb{R}$ and $\mathbb{R}$.

2. More generally, if $\mathbb{R}^n$ and $\mathbb{R}^m$ are given the standard topology, then the product topology on $\mathbb{R}^n \times \mathbb{R}^m$ coincides with the standard topology on $\mathbb{R}^{n+m}$, up to the identification

$$((x_1, \dots, x_n), (y_1, \dots, y_m)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_m)$$

of $\mathbb{R}^n \times \mathbb{R}^m$ with $\mathbb{R}^{n+m}$. Indeed the Euclidean metric on $\mathbb{R}^{n+m}$ is equivalent to the "product metric" (8.2) constructed from the Euclidean metrics on $\mathbb{R}^n$ and $\mathbb{R}^m$.

3. Let X and Y be given the discrete topology. Then the product topology on $X \times Y$ is the discrete topology on $X \times Y$ (why?).

Proposition 8.4 (projections)**.** *Let (X, τ_X), (Y, τ_Y) be topological spaces. Let $X \times Y$ be given the product topology. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the* canonical projections *(i.e., $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$). Then π_1 and π_2 are continuous.*

Proof. Let us prove that $\pi_1 : X \times Y \to X$ is continuous. Let A be an open subset of X. Then $\pi_1^{-1}(A) = A \times Y$ is an open rectangle, hence it is open in $X \times Y$. Since A is an arbitrary open set in A, we conclude that π_1 is continuous by Proposition 3.4.

The proof that $\pi_2 : X \times Y \to Y$ is continuous is analogous. □

Proposition 8.5 (continuity and components)**.** *Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces. Let $X \times Y$ be given the product topology. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the canonical projections. Let $f : Z \to X \times Y$. The following are equivalent:*

(i) $f : Z \to X \times Y$ is continuous;

(ii) the components $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continuous.

Proof. (i) $\Rightarrow$ (ii). Note that π_1 and π_2 are continuous by Proposition 8.4. Hence $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous by Proposition 3.3.

(ii) $\Rightarrow$ (i). Recall that the open rectangles form a base of the topology of $X \times Y$. So, by Proposition 7.3, it is sufficient to prove that $f^{-1}(A \times B)$ is open in Z for all open subsets A of X and B of Y. On the other hand

$$A \times B = (A \times Y) \cap (X \times B) = \pi_1^{-1}(A) \cap \pi_2^{-1}(B),$$

so

$$f^{-1}(A \times B) = f^{-1}(\pi_1^{-1}(A)) \cap f^{-1}(\pi_2^{-1}(B)) = (\pi_1 \circ f)^{-1}(A) \cap (\pi_2 \circ f)^{-1}(B).$$

Since $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, $(\pi_1 \circ f)^{-1}(A)$ and $(\pi_2 \circ f)^{-1}(B)$ are open in Z. Hence $f^{-1}(A \times B)$ is the intersection of two open sets and therefore it is open in Z as well. $\square$

Examples. Proposition 8.5 states in great generality a principle that has been implicitly used many times, e.g., to check whether a function taking values in $\mathbb{R}^2$ is continuous. Let $\mathbb{R}$ and $\mathbb{R}^2$ be given the standard topology and let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections onto the first and second factor respectively (i.e., $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ for all $(x,y) \in \mathbb{R}^2$). Recall that the standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the product topology.

1. The function $f : \mathbb{R} \to \mathbb{R}^2$, defined by $f(t) = (t^2, \sin t)$ for all $t \in \mathbb{R}$, is continuous. Indeed its components $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ are given by

$$f_1(t) = t^2, \qquad f_2(t) = \sin t,$$

for all $t \in \mathbb{R}$, and $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are continuous.

2. Let $g : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$g(t) = \begin{cases} (1, \sin t) & \text{if } t \geq 0, \\ (-3, t) & \text{if } t < 0, \end{cases}$$

for all $t \in \mathbb{R}$. Note that the components $g_1 = \pi_1 \circ g$ and $g_2 = \pi_2 \circ g$ of g are given by

$$g_1(t) = \begin{cases} -1 & \text{if } t \geq 0, \\ 3 & \text{if } t < 0, \end{cases} \qquad g_2(t) = \begin{cases} \sin t & \text{if } t \geq 0, \\ t & \text{if } t < 0, \end{cases}$$

for all $t \in \mathbb{R}$. Here $g_2 : \mathbb{R} \to \mathbb{R}$ is continuous, but $g_1 : \mathbb{R} \to \mathbb{R}$ is not. Hence $g : \mathbb{R} \to \mathbb{R}^2$ is not continuous.

Proposition 8.6. *Let X and Y be Hausdorff topological spaces. Then $X \times Y$ (with the product topology) is Hausdorff.*

Proof. Let $(x,y), (x',y')$ be distinct points of $X \times Y$. Then $x \neq x'$ or $y \neq y'$.

If $x \neq x'$, then by Proposition 4.6 we can find disjoint open neighbourhoods A of x and A' of x' in X. So $A \times Y$ and $A' \times Y$ are open in $X \times Y$, $(x,y) \in A \times Y$, $(x',y') \in A' \times Y$ and $(A \times Y) \cap (A' \times Y) = (A \cap A') \times Y = \emptyset$. Hence $A \times Y$ and $A' \times Y$ are disjoint neighbourhoods of (x,y) and (x',y') respectively.

If $y \neq y'$, the proof is analogous (swap the roles of X and Y). $\square$

Proposition 8.7 (products and subspaces)**.** *Let X and Y be topological spaces. Let $U \subseteq X$ and $V \subseteq Y$ be given the subspace topologies induced by X and Y respectively. Then the product topology on $U \times V$ is the same as the subspace topology induced by $X \times Y$.*

Proof. Let τ be the product topology on $X \times Y$ and σ be the product topology on $U \times V$. We must show that $\sigma = \tau|_{U \times V}$.

To show that $\sigma \subseteq \tau|_{U\times V}$, by Corollary 7.4 it is sufficient to prove that $\mathcal{B} \subseteq \tau|_{U\times V}$, where $\mathcal{B}$ is the collection of open rectangles in $U \times V$. Let A be open in U and B be open in V. Then $A = \tilde{A} \cap U$ and $B = \tilde{B} \cap V$ for some open sets $\tilde{A}$ in X and $\tilde{B}$ in Y. In particular $A \times B = (\tilde{A} \cap U) \times (\tilde{B} \cap U) = (\tilde{A} \times \tilde{B}) \cap (U \times V)$. Since $\tilde{A} \times \tilde{B}$ is an open rectangle in $X \times Y$, we conclude that $\tilde{A} \times \tilde{B} \in \tau$ and $A \times B \in \tau|_{U\times V}$.

It remains to be shown that $\tau|_{U\times V} \subseteq \sigma$, i.e., that $C \cap (U \times V) \in \sigma$ for all $C \in \tau$. Since $C \cap (U \times V) = i^{-1}(C)$, where $i : U \times V \to X \times Y$ is the inclusion map, what we must show is that $i^{-1}(C) \in \sigma$ for all $C \in \tau$, i.e., that i is continuous from $(U \times V, \sigma)$ to $(X \times Y, \tau)$. By Proposition 7.3, it is then sufficient to prove that $i^{-1}(A \times B) \in \sigma$ for all open rectangles $A \times B$ in $X \times Y$. On the other hand, $i^{-1}(A \times B) = (A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$ is an open rectangle in $U \times V$, whenever A is open in X and B is open in Y. □

So far we have treated the product of two topological spaces. In fact, is not difficult to extend the above definitions and results to the case of the product of three, four, ..., or more generally a finite number of topological spaces. However we can avoid introducing new definitions by means of an "iterative approach".

Namely, suppose that we want to give a topology to the product $X \times Y \times Z$ of three topological spaces X, Y and Z. As a set, we can identify $X \times Y \times Z$ with $(X \times Y) \times Z$ via the bijection $(x, y, z) \mapsto ((x, y), z)$. We can now give $X \times Y$ the product topology of the topologies of X and Y, and then give $(X \times Y) \times Z$ the product topology of the topologies of $X \times Y$ and Z.

Note that we could have chosen a different identification, i.e., we could have identified $X \times Y \times Z$ with $X \times (Y \times Z)$. The following proposition shows that the resulting topology on $X \times Y \times Z$ does not depend on this choice.

Proposition 8.8 ("associativity" of product)**.** *Let X, Y, Z be topological spaces. Let $X \times Y$ and $Y \times Z$ be given the product topologies. Let $(X \times Y) \times Z$ and $X \times (Y \times Z)$ be given the product topologies. Then the map*

$$F : (X \times Y) \times Z \to X \times (Y \times Z),$$
$$((x, y), z) \mapsto (x, (y, z))$$

is a homeomorphism.

Proof. F is bijective: its inverse is given by $F^{-1}(x, (y, z)) = ((x, y), z)$.

Let us show that F is continuous. Let $\pi_1 : X \times (Y \times Z) \to X$ and $\pi_2 : X \times (Y \times Z) \to Y \times Z$ the canonical projections. Then, by Proposition 8.5, it is sufficient to show that $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous.

On the other hand, $\pi_1 \circ F((x, y), z) = x$ for all $x \in X$, $y \in Y$, $z \in Z$. In other words, $\pi_1 \circ F : (X \times Y) \times Z \to X$ is the composition of the projections $(X \times Y) \times Z \to X \times Y$ and $X \times Y \to X$ and consequently is continuous by Propositions 8.4 and 3.3.

Moreover, $\pi_2 \circ F((x, y), z) = (y, z)$ for all $x \in X$, $y \in Y$, $z \in Z$. If $\pi_3 : Y \times Z \to Y$ and $\pi_4 : Y \times Z \to Z$ are the canonical projections, then, by Proposition 8.5, in order to show that $\pi_2 \circ F$ is continuous it is sufficient to show that $\pi_3 \circ \pi_2 \circ F$ and $\pi_4 \circ \pi_2 \circ F$ are continuous.

Note that $\pi_3 \circ \pi_2 \circ F((x, y), z) = y$ and $\pi_4 \circ \pi_2 \circ F((x, y), z) = z$ for all $x \in X$, $y \in Y$, $z \in Z$. Therefore $\pi_3 \circ \pi_2 \circ F$ is the composition of the projections $(X \times Y) \times Z \to X \times Y$ and $X \times Y \to Y$, while $\pi_4 \circ \pi_2 \circ F$ is just the projection $(X \times Y) \times Z \to Z$. Hence $\pi_3 \circ \pi_2 \circ F$ and $\pi_4 \circ \pi_2 \circ F$ are both continuous by Propositions 8.4 and 3.3.

The proof that F^{-1} is continuous is analogous. $\square$

9 Connectedness

Definition 9.1. Let (X, τ) be a topological space.

(i) X is called *disconnected* if there exist two nonempty disjoint open subsets $A, B \subseteq X$ such that $X = A \cup B$.

(ii) X is called *connected* if X is not disconnected.

(iii) A subset $U \subseteq X$ is called *connected* if U, with the subspace topology, is a connected topological space.

Examples. Let X be a topological space.

1. If X has at most one point, then X is connected.
2. If X has the trivial topology, then X is connected.
3. If X has the discrete topology and more than one point, then X is disconnected.

If $\mathbb{R}$ is given the standard topology, then the connected subsets of $\mathbb{R}$ have a particularly explicit characterisation. Recall that an *interval* in $\mathbb{R}$ is a subset $I \subseteq \mathbb{R}$ such that, for all $x, y \in I$ and $z \in \mathbb{R}$, if $x < z < y$ then $z \in I$ (i.e., if I contains two points of $\mathbb{R}$, then it contains all points in between). It is easily seen (how?) that all intervals in $\mathbb{R}$ have one of the following forms:

$$\mathbb{R},\ (-\infty, b),\ (-\infty, b],\ (a, \infty),\ [a, \infty),\ (a, b),\ [a, b],\ (a, b],\ [a, b),\ \{a\},\ \emptyset,$$

where $a, b \in \mathbb{R}$ and $a < b$.

Proposition 9.2 (connected subsets of $\mathbb{R}$)**.** *Let $\mathbb{R}$ be given the standard topology and $C \subseteq \mathbb{R}$. The following are equivalent:*

(i) C is connected;

(ii) C is an interval.

In particular $\mathbb{R}$ is connected.

Proof. If C is not an interval, then there exist two points $x, y \in I$ and a point $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin I$. In particular we can write $C = A \cup B$, where $A = C \cap (-\infty, z)$ and $B = C \cap (z, \infty)$. Note that A and B are open in C (by definition of subspace topology) and are clearly disjoint. Moreover $x \in A$ and $y \in B$, so A and B are both nonempty. This proves that C is not connected.

Vice versa, if C is not connected, then we can write $C = A \cup B$ for some disjoint, nonempty open subsets A, B of C. In particular $\{A, B\}$ is an open cover of C, and therefore the function $f : C \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & \text{if } t \in A, \\ 1 & \text{if } t \in B, \end{cases}$$

is continuous by Proposition 6.14. Take $x \in A$ and $y \in B$ (recall that A and B are nonempty). Up to swapping A and B, we may assume that $x < y$. If C were an interval, then $[x, y] \subseteq C$ and $f|_{[x,y]} : [x, y] \to \mathbb{R}$ would be a continuous function that only takes the values 0 and 1, but this would contradict the Intermediate Value Theorem. So C cannot be an interval. $\square$

The following proposition is one of the main tools to prove that a given topological space is connected.

Proposition 9.3 (continuous image of connected is connected)**.** *Let X and Y be topological spaces and $f : X \to Y$ be continuous. For all connected subsets $C \subseteq X$, the image $f(C)$ is a connected subset of Y.*

Proof. Note that the function $\tilde{f} : C \to f(C)$ defined by $\tilde{f}(x) = f(x)$ for all $x \in C$ is continuous by Propositions 6.11 and 6.12. Hence, up to replacing f by $\tilde{f}$, we may assume that $C = X$ (so X is connected) and $f(X) = Y$.

By contradiction, suppose that Y is disconnected. Then there exist disjoint, nonempty open sets A, B in Y such that $Y = A \cup B$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty open sets in X, because f is continuous, and moreover

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset,$$
$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X.$$

Therefore X is disconnected, contradiction. □

Examples. Let $\mathbb{R}$ and $\mathbb{R}^2$ be given the standard topology.

1. Let the parabola $P = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ be given the topology induced by $\mathbb{R}^2$. Then P is connected, because $P = f(\mathbb{R})$, where $f : \mathbb{R} \to \mathbb{R}^2$ is the continuous map defined by $f(t) = (t, t^2)$ for all $t \in \mathbb{R}$.

2. Let the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be given the topology induced by $\mathbb{R}^2$. Then S^1 is connected, because $S^1 = g(\mathbb{R})$, where $g : \mathbb{R} \to \mathbb{R}^2$ is the continuous map defined by $f(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$.

Proposition 9.3, together with the characterisation of connected subsets of $\mathbb{R}$, yields a very general version of the Intermediate Value Theorem.

Corollary 9.4 (Intermediate Value Theorem (enhanced))**.** *Let X be a connected topological space. Let $\mathbb{R}$ be given the standard topology. Let $f : X \to \mathbb{R}$ be continuous. For all $x_1, x_2 \in X$ and $t \in \mathbb{R}$, if $f(x_1) < t < f(x_2)$, then there exists $x \in X$ such that $f(x) = t$.*

Proof. Since X is connected and f is continuous, $f(X)$ is connected as well (by Proposition 9.3), hence $f(X)$ is an interval in $\mathbb{R}$ (by Proposition 9.2). Suppose that $x_1, x_2 \in X$ and $t \in \mathbb{R}$ are such that $f(x_1) < t < f(x_2)$. Since $f(X)$ is an interval and $f(x_1), f(x_2) \in f(X)$, this implies that $t \in f(X)$ as well. Hence there exists $x \in X$ such that $f(x) = t$. □

Here is another important consequence of Proposition 9.3.

Corollary 9.5 (connectedness is a topological property)**.** *Let X and Y be homeomorphic topological spaces. If X is connected, then Y is connected as well.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then f is continuous and surjective, i.e., $f(X) = Y$, so Y is connected by Proposition 9.3. □

Since connectedness is a topological property, it may be used to prove that two given topological spaces are not homeomorphic.

Examples. Let $\mathbb{R}$ and $\mathbb{R}^2$ be given the standard topology.

1. Let $X = \mathbb{R} \setminus \{0\}$ be given the subspace topology. Then $\mathbb{R}$ and X are not homeomorphic. Indeed $\mathbb{R}$ is connected, while $X = (-\infty, 0) \cup (0, \infty)$ is disconnected.

2. Let the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be given the subspace topology. Then S^1 and $\mathbb{R}$ are not homeomorphic (despite the fact that they are both connected). Indeed, suppose by contradiction that there exists a homeomorphism $f : \mathbb{R} \to S^1$, and set $u = f^{-1}((1, 0))$. Then $\mathbb{R} \setminus \{u\}$ and $S^1 \setminus \{(1, 0)\}$ are homeomorphic as well (via the homeomorphism obtained by restricting domain and range of f). But $\mathbb{R} \setminus \{u\} = (-\infty, u) \cup (u, \infty)$ is disconnected, while $S^1 \setminus \{(1, 0)\}$ is connected, because it is the image of the interval $(0, 2\pi)$ via the continuous function $g : \mathbb{R} \to \mathbb{R}^2$ defined by $g(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. So $\mathbb{R} \setminus \{u\} = (-\infty, u) \cup (u, \infty)$ and $S^1 \setminus \{(1, 0)\}$ cannot be homeomorphic, contradiction.

Proposition 9.6 (connectedness and unions)**.** *Let X be a topological space. Suppose that there exists a collection $\mathcal{C}$ of connected subsets of X such that:*

(i) $\bigcup \mathcal{C} = X$ *(i.e., $\mathcal{C}$ is a cover of X), and*

(ii) $C \cap D \neq \emptyset$ *for all $C, D \in \mathcal{C}$.*

Then X is connected.

Proof. By contradiction, suppose that X is disconnected. Then there exist nonempty, disjoint open sets A, B in X such that $X = A \cup B$.

We now show that, for all $C \in \mathcal{C}$, either $C \subseteq A$ or $C \subseteq B$. Indeed, if this were not the case, then $A \cap C$ and $B \cap C$ would be nonempty, disjoint open subsets of C such that $C = (A \cap C) \cup (B \cap C)$, but this would contradict the fact that C is connected.

So we can split $\mathcal{C}$ into two subcollections $\mathcal{A} = \{C \in \mathcal{C} : C \subseteq A\}$ and $\mathcal{B} = \{C \in \mathcal{C} : C \subseteq B\}$, so that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Clearly $\bigcup \mathcal{A} \subseteq A$ and $\bigcup \mathcal{B} \subseteq B$, and moreover

$$\bigcup \mathcal{A} \cup \bigcup \mathcal{B} = \bigcup \mathcal{C} = X.$$

Therefore it must be $\bigcup \mathcal{A} = A$ and $\bigcup \mathcal{B} = B$, because A and B are disjoint. Since both A and B are nonempty, this implies that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty.

In particular, we can take $C \in \mathcal{A}$ and $D \in \mathcal{B}$. Since $C \subseteq A$ and $D \subseteq B$ and $A \cap B = \emptyset$, we conclude that C and D are disjoint, which contradicts (ii). □

Example. Let $\mathbb{R}^2$ be given the standard topology and $X = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ be given the subspace topology. Then X is connected. To prove this, we observe that $X = C_1 \cup C_2$, where $C_1 = \{(x, y) \in R^2 : y = 0\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$. Note that C_1 and C_2 are connected, because $C_1 = F_1(\mathbb{R})$ and $C_2 = F_2(\mathbb{R})$, where $F_1, F_2 : \mathbb{R} \to \mathbb{R}^2$ are the continuous functions defined by $F_1(t) = (t, 0)$ and $F_2(t) = (0, t)$ for all $t \in \mathbb{R}$, and $\mathbb{R}$ is connected. Moreover $C_1 \cap C_2 = \{(0, 0)\}$ is nonempty. Therefore Proposition 9.6 applied to the cover $\mathcal{C} = \{C_1, C_2\}$ of X gives that X is connected.

Proposition 9.7 (connectedness and products)**.** *Let X and Y be connected topological spaces. Then $X \times Y$ (with the product topology) is connected.*

Proof. Note that, if either X or Y is empty, then the product $X \times Y$ is empty and therefore it is trivially connected. So we may assume that both X and Y are nonempty.

Let us first show that, for all $y \in Y$, $X \times \{y\}$ is a connected subset of $X \times Y$. Indeed $X \times \{y\} = f(X)$, where $f : X \to X \times Y$ is defined by $f(x) = (x, y)$. Note that f is continuous by Proposition 8.5, because both its components are continuous (one component is the identity map id_X, the other component is constant). Hence $X \times \{y\}$ is the continuous image of a connected set and therefore it is connected by Proposition 9.3.

In the same way we can prove that, for all $x \in X$, $\{x\} \times Y$ is a connected subset of $X \times Y$.

Choose a point $x_0 \in X$ and define $H_y = (\{x_0\} \times Y) \cup (X \times \{y\})$ for all $y \in Y$. Then H_y is the union of two connected sets whose intersection $\{(x_0, y)\}$ is nonempty, and therefore H_y is connected by Proposition 9.6, for all $y \in Y$.

Note now that $X \times Y = \bigcup_{y \in Y} H_y$, and moreover $\bigcap_{y \in Y} H_y = \{x_0\} \times Y$ is nonempty. Therefore $X \times Y$ is connected, again by Proposition 9.6 (applied to the cover $\{H_y : y \in Y\}$ of $X \times Y$). ☐

Examples. Let $\mathbb{R}^k$ be given the standard topology for all $k \in \mathbb{N}$.

1. Let I, J be intervals in $\mathbb{R}$. Then $I \times J$ is connected in $\mathbb{R}^2$ (by Propositions 8.7, 9.2 and 9.7). In particular $\mathbb{R}^2$ itself is connected.

2. More generally, let $I_1, \dots, I_n$ be intervals in $\mathbb{R}$. Then $I_1 \times \cdots \times I_n$ is connected in $\mathbb{R}^n$. In particular $\mathbb{R}^n$ itself is connected.

3. The ball $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ are connected in $\mathbb{R}^2$. Indeed the map $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$G(r, t) = (r \cos t, r \sin t)$$
is continuous and $B = G([0, 1) \times \mathbb{R})$, $D = G([0, 1] \times \mathbb{R})$; since $[0, 1) \times \mathbb{R}$ and $[0, 1] \times \mathbb{R}$ are connected, B and D are connected by Proposition 9.3.

4. The sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is connected in $\mathbb{R}^3$. Indeed the map $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by
$$F(t, s) = (\cos t, \sin t \, \cos s, \sin t \, \sin s)$$
is continuous and $F(\mathbb{R}^2) = S^2$, so S^2 is connected by Proposition 9.3.

The previous examples about balls and spheres have higher dimensional generalisations.

Proposition 9.8. *Let $\mathbb{R}^n$ be given the standard topology. Let the punctured space $X_n = \mathbb{R}^n \setminus \{0\}$ and the sphere*
$$S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$$
be given the subspace topology. If $n > 1$, then X_n and S^{n-1} are connected.

Proof. Consider first the case $n = 2$. Let $F : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ be defined by
$$F(r, \theta) = (r \cos \theta, r \sin \theta).$$

Then F is continuous and moreover $F(\{1\} \times \mathbb{R}) = S^1$ and $F((0,\infty) \times \mathbb{R}) = X_2$. Since $\{1\} \times \mathbb{R}$ and $(0,\infty) \times \mathbb{R}$ are connected (they are products of intervals), S^1 and X_2 are connected as well by Proposition 9.3.

The proof in the case $n > 2$ is analogous and uses the map $F : (0,\infty) \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by

$$F(r, (\theta_1, \dots, \theta_{n-1})) = \begin{pmatrix} r \cos\theta_1 \\ r \sin\theta_1 \cos\theta_2 \\ \vdots \\ r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\ r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1} \end{pmatrix}$$

instead. □

Corollary 9.9. *If $n > 1$, then $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. By contradiction, suppose that there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$ and let $u = f^{-1}(0)$. Then $\mathbb{R} \setminus \{u\}$ and $\mathbb{R}^n \setminus \{0\}$ are homeomorphic (via the homeomorphism obtained by restricting domain and range of f). However $\mathbb{R} \setminus \{u\} = (-\infty, u) \cup (u, \infty)$ is disconnected, while $\mathbb{R}^n \setminus \{0\}$ is connected by Proposition 9.8, so they cannot be homeomorphic, contradiction. □

The previous result is a particular instance of the "invariance of dimension theorem" (according to which $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$), whose proof in its generality is beyond the reach of these lectures.

Proposition 9.10 (connectedness and closure)**.** *Let X be a topological space and $C, D \subseteq X$. If C is connected and $C \subseteq D \subseteq \overline{C}$, then D is connected as well.*

Proof. By contradiction, suppose that D is disconnected, i.e., $D = A \cup B$ for some nonempty, disjoint open subsets of D. Note that A and B are complement of each other in D, so A and B are also closed in D. Therefore there exist closed subsets $\tilde{A}$ and $\tilde{B}$ of X such that $A = \tilde{A} \cap D$ and $B = \tilde{B} \cap D$.

We now show that $C \not\subseteq \tilde{A}$. Indeed, if $C \subseteq \tilde{A}$, then $\overline{C} \subseteq \tilde{A}$ by Proposition 5.3, because $\tilde{A}$ is closed in X; hence $D = \overline{C} \cap D \subseteq \tilde{A} \cap D = A$, which is impossible because $D = A \cup B$ and A, B are disjoint and nonempty. Similarly we can show that $C \not\subseteq \tilde{B}$.

On the other hand, $C \subseteq D \subseteq \tilde{A} \cup \tilde{B}$. Therefore neither $C \cap \tilde{A}$ nor $C \cap \tilde{B}$ is empty (otherwise C would be contained in the other one, which we have excluded). Moreover $C = (C \cap \tilde{A}) \cup (C \cap \tilde{B})$ and $(C \cap \tilde{A}) \cap (C \cap \tilde{B}) \subseteq A \cap B = \emptyset$. Since $\tilde{A}$ and $\tilde{B}$ are closed in X, the sets $C \cap \tilde{A}$ and $C \cap \tilde{B}$ are closed in C and therefore they are also open in C (they are complement of each other in C). So C is disconnected, contradiction. □

Not all topological spaces are connected. However, as we shall see, every topological space can be split into "maximal connected pieces".

Definition 9.11. Let (X, τ) be a topological space. A subset C of X is called a *connected component* of X if

(i) C is a nonempty connected subset of X, and

(ii) for all connected subsets D of X, if $C \subseteq D$ then $C = D$.

In other words, a connected component of X is a maximal nonempty connected subset of X.

Definition 9.12. Let X be a set. A *partition* of X is a collection $\mathcal{C}$ of nonempty subsets of X such that:

(i) $X = \bigcup \mathcal{C}$;

(ii) $A \cap B = \emptyset$ for all distinct $A, B \in \mathcal{C}$.

Proposition 9.13. *Let X be a topological space.*

(i) The collection of the connected components of X is a partition of X.

(ii) For all points $x \in X$, the connected component of X containing x is the union of all the connected subsets of X containing x.

(iii) X is connected if and only if X is empty or has exactly one connected component.

Proof. Let us show that, if C and C' are distinct connected components, then they are disjoint. Indeed, if they were not disjoint, then $C \cup C'$ would be connected by Proposition 9.6, and moreover $C \cup C'$ contains both C and C'; but then we would have $C = C \cup C'$ and $C' = C \cup C'$ (because C and C' are maximal connected), and in particular $C = C'$, contradiction.

So connected components of X are pairwise disjoint, and moreover they are nonempty by definition. In order to show that connected components form a partition of X, it remains only to prove that their union is the whole X, i.e., that every element of x is contained in a connected component of X.

For all $x \in X$, let $\mathcal{C}_x$ be the collection of all connected subsets of X containing x, and define $C_x = \bigcup \mathcal{C}_x$. Note that $\mathcal{C}_x$ is nonempty, since $\{x\} \in \mathcal{C}_x$, and moreover $x \in \bigcap \mathcal{C}_x$, so $\bigcap \mathcal{C}_x \neq \emptyset$. Therefore $C_x = \bigcup \mathcal{C}_x$ is connected by Proposition 9.6. Moreover C_x is nonempty, because $x \in C_x$. Finally, if $D \subseteq X$ is connected and $C_x \subseteq D$, then $x \in D$ and therefore $D \in \mathcal{C}_x$; so $D \subseteq \bigcup \mathcal{C}_x = C_x$ and consequently $D = C_x$. This proves that C_x is a maximal nonempty connected subset of X, i.e., C_x is a connected component of X, and clearly C_x contains x.

Since $x \in X$ was arbitrary, part (i) is proved. Moreover part (ii) is proved as well, because C_x is the connected component containing x and is also the union of all connected subsets of X containing x. So it remains to prove (iii).

If X is connected, then X is clearly the only maximal connected subset of X. So X is the only connected component of X (unless X is empty).

If X is empty, then X is trivially connected. If X has exactly one connected component, then this connected component must be X itself (because the union of the connected components is X, by part (i)), hence X is connected. □

The following proposition shows that connected components are preserved by homeomorphisms.

Proposition 9.14. *Let X and Y be topological spaces and $f : X \to Y$ be a homeomorphism. A subset C of X is a connected component of X if and only if its image $f(C)$ is a connected component of Y.*

Proof. Suppose that C is a connected component of X. Since C is nonempty, its image $f(C)$ is nonempty as well. Moreover, since C is connected in X and $f : X \to Y$ is continuous, $f(C)$ is connected in Y by Proposition 9.3. In order to conclude that $f(C)$ is a connected component of Y, it remains to show that $f(C)$ is a maximal connected subset of Y.

Let $D \subseteq Y$ be connected and such that $f(C) \subseteq D$. Then $C \subseteq f^{-1}(D)$. Moreover $f^{-1} : Y \to X$ is continuous (because f is a homeomorphism) and D is connected in Y, hence $f^{-1}(D)$ is connected in X by Proposition 9.3. Since C is a connected component of X, it is a maximal connected subset of X and therefore, since $f^{-1}(D)$ is connected and $f^{-1}(D) \supseteq C$, we conclude that $f^{-1}(D) = C$. Since f is bijective, this implies that $D = f(C)$. Since D was an arbitrary connected subset of Y containing $f(C)$, this proves that $f(C)$ is a maximal connected subset of Y, and therefore $f(C)$ is a connected component of Y.

Conversely, suppose that $f(C)$ is a connected component of Y. Then $C = f^{-1}(f(C))$, because f is bijective. Moreover $f^{-1} : Y \to X$ is a homeomorphism. In other words, C is the image of the connected component $f(C)$ of Y via the homeomorphism $f^{-1} : Y \to X$. Hence, from what we have just proved, it follows that C is a connected component of X. □

Corollary 9.15 (the number of connected components is a topological property)**.** *Let X and Y be homeomorphic topological spaces. Then X has as many connected components as Y.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be the collections of the connected components of X and Y respectively. Let $f : X \to Y$ be a homeomorphism. Then, by Proposition 9.14, $f(C) \in \mathcal{D}$ for all $C \in \mathcal{C}$ and, conversely, $f^{-1}(D) \in \mathcal{C}$ for all $D \in \mathcal{D}$. Since $f : X \to Y$ is bijective, this shows that the correspondence $C \mapsto f(C)$ is a bijection from $\mathcal{C}$ to $\mathcal{D}$. Therefore the collections $\mathcal{C}$ and $\mathcal{D}$ have the same cardinality, that is, $\mathcal{C}$ has as many elements as $\mathcal{D}$. □

Here are a few other properties of connected components.

Proposition 9.16. *Let X be a topological space.*

(i) Each connected component of X is closed in X.

(ii) If X has finitely many connected components, then each connected component is open in X.

Proof. (i). Let C be a connected component of X. Then $\overline{C}$ is connected by Proposition 9.10 and $\overline{C} \supseteq C$, therefore $\overline{C} = C$ because C is maximal connected.

(ii). By Proposition 9.13, the connected components of X form a partition of X. Therefore the complement of a connected component C of X is the union of all the other connected components, which are closed by part (i). If X has finitely many connected components, then $X \setminus C$ is the union of finitely many closed sets in X and therefore $X \setminus C$ is closed as well. □

The following statement is useful for identifying connected components.

Proposition 9.17. *Let X be a topological space. Let $\mathcal{U}$ be a partition of X, made of connected nonempty open subsets of X. Then $\mathcal{U}$ is the collection of the connected components of X.*

Proof. Let us first prove that $X \setminus C$ is open in X for all $C \in \mathcal{U}$. Indeed $X \setminus C = \bigcup\{C' \in \mathcal{U} : C' \neq C\}$, because $\mathcal{C}$ is a partition of X. This proves that $X \setminus C$ is a union of open sets, since every element of $\mathcal{C}$ is open, and therefore $X \setminus C$ is open as well.

We now prove that every element $C \in \mathcal{U}$ is a connected component of X. Since C is nonempty and connected by the assumption on $\mathcal{U}$, it is enough to prove that C is a maximal connected subset of X. Let $D \subseteq X$ be connected and such that $C \subseteq D$; since C and $X \setminus C$ are open in X and disjoint, the sets $C = D \cap C$ and $D \setminus C = D \cap (X \setminus C)$ are open in D (by definition of subspace topology) and disjoint, and moreover $C \cup (D \setminus C) = D$; since D is connected and $C \neq \emptyset$, it must be $D \setminus C = \emptyset$, that is, $D = C$. Since D was an arbitrary connected subset of X containing C, this shows that C is a maximal connected subset of X, hence C is a connected component of X.

In order to conclude, it remains to show that every connected component C of X is an element of $\mathcal{U}$. Indeed, since C is nonempty (by definition of connected component) and $X = \bigcup \mathcal{U}$ (because $\mathcal{U}$ is a partition of X), there exists $C' \in \mathcal{U}$ such that $C \cap C' \neq \emptyset$. On the other hand, we have just proved that all elements of $\mathcal{U}$ are connected components of X; so both C and C' are connected components of X. Since the collection of the connected components of X is a partition of X (by Proposition 9.13(i)) and $C' \cap C \neq \emptyset$, we conclude that $C = C' \in \mathcal{U}$. $\square$

Examples. Let $\mathbb{R}$ be given the standard topology. In the following examples, subsets of $\mathbb{R}$ are given the subspace topology.

1. $\mathbb{R} \setminus \{1, 2\}$ has three connected components, namely $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. They are all open and closed in $\mathbb{R} \setminus \{1, 2\}$.

2. The connected components of $\mathbb{R} \setminus \mathbb{Z}$ are the intervals $(n, n+1)$ where $n \in \mathbb{Z}$ (so they are infinitely many). They are all open and closed in $\mathbb{R} \setminus \mathbb{Z}$.

3. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. The connected components of X are the singletons $\{x\}$ with $x \in X$, since X does not contain any interval of $\mathbb{R}$ with more than one point. Clearly all the singletons $\{x\}$ are closed. Moreover the singletons of the form $\{1/n\}$, where $n \in \mathbb{N}$, are also open in X, because $\{1/n\} = X \cap (1/(n+1), 1/(n-1))$ when $n > 1$ and $\{1\} = X \cap (1/2, 2)$. However $\{0\}$ is not open in X, because every ball centred at 0 contains points of X other than 0.

4. The connected components of $\mathbb{Z}$ are the singletons $\{n\}$ where $n \in \mathbb{Z}$ (so they are infinitely many). They are all open and closed in $\mathbb{Z}$.

5. The connected components of $\mathbb{Q}$ are the singletons $\{q\}$ where $q \in \mathbb{Q}$ (so they are infinitely many). They are all closed, but none is open in $\mathbb{Q}$.

10 Compactness

Definition 10.1. Let X be a topological space and $\mathcal{A}$ be a cover of X. A *subcover* of $\mathcal{A}$ is a subcollection $\mathcal{B} \subseteq \mathcal{A}$ which is a cover of X as well.

Definition 10.2. Let X be a topological space.

(i) X is called *compact* if every open cover of X has a finite subcover.

(ii) A subset $C \subseteq X$ is called *compact* if C, with the subspace topology, is a compact topological space.

Examples. Let X be a topological space.

1. If X has finitely many points, then X is compact. Indeed any cover of X is a subset of the power set of X and therefore it is finite.
2. If X has the trivial topology, then X is compact. Indeed the trivial topology has only two element ($\emptyset$ and X), so any open cover of X is finite.
3. If X has the discrete topology and X is compact, then X is finite. Indeed the collection $\mathcal{A} = \{\{x\} : x \in X\}$ of singletons of X is an open cover of X; if X is compact, then $\mathcal{A}$ has a finite subcover, so X is the union of finitely many singletons, i.e., X is finite.

A different definition of compactness was given in **MS** for metric spaces (see [**MS**, Definition 10.1]), in terms of sequences and subsequences (instead of open covers and subcovers), which here we refer to as "sequential compactness".

Definition 10.3. A topological space X is said to be *sequentially compact* if every sequence $(x_n)_{n=1}^{\infty}$ taking its values in X has a subsequence that converges to a point of X.

Compactness and sequential compactness are different properties for general topological spaces. However they happen to coincide in the case of metric spaces (see also [**MS**, Section 10.3]).

Proposition 10.4. *Let X be a metric space. Then the following are equivalent:*

(i) X is compact;

(ii) X is sequentially compact.

The proof of Proposition 10.4 is fairly long and complicated and is omitted. In view of Proposition 10.4, the above definition of compactness is consistent, in the case of metric spaces, with the one given in **MS**. In particular, since $\mathbb{R}^n$ and its subspaces are metric spaces (with the Euclidean metric), from [**MS**, Theorem 10.4] we obtain a characterisation of the compact subsets of $\mathbb{R}^n$.

Corollary 10.5 (Heine–Borel Theorem)**.** *Let $\mathbb{R}^n$ be given the standard topology. Let $C \subseteq X$. Then the following are equivalent:*

(i) C is compact in $\mathbb{R}^n$;

(ii) C is closed and bounded in $\mathbb{R}^n$.

This characterisation is particularly important and is extensively used. At the end of this section we will give an alternative proof of Corollary 10.5 that refers directly to Definition 10.1 and does not go through Proposition 10.4.

In addition to the above characterisation, another important tool to prove that a given space is compact is given by the following result.

Proposition 10.6 (continuous image of compact is compact)**.** *Let X and Y be topological spaces and $f : X \to Y$ be continuous. For all compact subsets $C \subseteq X$, $f(C)$ is a compact subset of Y.*

Proof. As in the proof of Proposition 9.3, it is not restrictive to assume that $C = X$ and $f(X) = Y$.

Let $\mathcal{A}$ be an open cover of Y. We must show that $\mathcal{A}$ has a finite subcover. Define $\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$. Then $\mathcal{B}$ is an open cover of X: indeed its elements are preimages of open subsets of Y via the continuous map $f : X \to Y$, hence they are open subsets of X, and moreover

$$\bigcup \mathcal{B} = \bigcup_{A\in\mathcal{A}} f^{-1}(A) = f^{-1}\left(\bigcup_{A\in\mathcal{A}} A\right) = f^{-1}\left(\bigcup \mathcal{A}\right) = f^{-1}(Y) = X.$$

Since X is compact, the open cover $\mathcal{B}$ has a finite subcover $\mathcal{B}'$.

For all elements $B \in \mathcal{B}'$, we can find $A \in \mathcal{A}$ such that $B = f^{-1}(A)$, and therefore, since f is surjective, $f(B) = f(f^{-1}(A)) = A \in \mathcal{A}$. In particular, the collection $\mathcal{A}' = \{f(B) : B \in \mathcal{B}'\}$ is a subcollection of $\mathcal{A}$, which is finite (because $\mathcal{B}'$ is finite) and moreover

$$\bigcup \mathcal{A}' = \bigcup_{B\in\mathcal{B}'} f(B) = f\left(\bigcup_{B\in\mathcal{B}'} B\right) = f(X) = Y,$$

because $\bigcup \mathcal{B}' = X$ and f is surjective. So $\mathcal{A}'$ is a finite subcover of $\mathcal{A}$. □

Examples. In the following examples, subsets of $\mathbb{R}^n$ are given the topology induced by the standard topology of $\mathbb{R}^n$.

1. Let $S^1 = \{(\cos t, \sin t) : t \in \mathbb{R}\}$ be the unit circle in $\mathbb{R}^2$. Then S^1 is compact. Indeed we can write $S^1 = f([0, 2\pi])$, where $f : \mathbb{R} \to \mathbb{R}^2$ is the continuous map defined by $f(t) = (\cos t, \sin t)$, and $[0, 2\pi]$ is compact.

2. Let $A = \{(x, y) \in \mathbb{R}^2 : y = x^2, x \in [-1, 2]\}$ be an arc of parabola. Then A is compact. Indeed $A = g([-1, 2])$, where $g : \mathbb{R} \to \mathbb{R}^2$ is the continuous function given by $g(t) = (t, t^2)$, and $[-1, 2]$ is compact.

Proposition 10.6, together with the characterisation of compact subsets of $\mathbb{R}$, yields a very general version of the Extreme Value Theorem.

Corollary 10.7 (Extreme Value Theorem, enhanced)**.** *Let X be a nonempty compact topological space. Let $\mathbb{R}$ be given the standard topology. Let $f : X \to \mathbb{R}$ be continuous. Then there exist $x_m, x_M \in X$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in X$.*

Proof. Since X is compact and $f : X \to \mathbb{R}$ is continuous, the image $f(X)$ is compact in $\mathbb{R}$ by Proposition 10.6. Hence, by Corollary 10.5, $f(X)$ is closed and bounded in $\mathbb{R}$. Let $t_m = \inf f(X)$ and $t_M = \sup f(X)$. Since $f(X)$ is bounded and nonempty, $-\infty < t_m \leq t_M < \infty$, i.e., t_m and t_M are elements of $\mathbb{R}$. From the definition of infimum and supremum it is easy to see that t_m and t_M are in the closure of $f(X)$. Since $f(X)$ is closed, we conclude that $t_m, t_M \in f(X)$, i.e., there exist $x_m, x_M \in X$ such that $f(x_m) = t_m$ and $f(x_M) = t_M$. In particular, for all $x \in X$, $f(x_m) = \inf f(X) \leq f(x) \leq \sup f(X) = f(x_M)$. □

Corollary 10.8 (compactness is a topological property)**.** *Let X, Y be homeomorphic topological spaces. If X is compact, then Y is compact as well.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then X is compact, f is continuous and $f(X) = Y$, so Y is compact by Proposition 10.6. □

Examples. Let $\mathbb{R}$ and $\mathbb{R}^2$ be given the standard topology. In the following examples, all subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are given the subspace topology.

1. $[0, 4]$ and $[-1, 3)$ are not homeomorphic. Indeed the former is compact (it is closed and bounded in $\mathbb{R}$), whereas the latter is not.

2. $\mathbb{R}$ and the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are not homeomorphic. Indeed the former is not compact, whereas the latter is compact.

Proposition 10.9 (closed in compact is compact)**.** *Let X be a compact topological space and let $C \subseteq X$ be closed. Then C is compact.*

Proof. Let C be given the subspace topology. Let $\mathcal{A}$ be an open cover of C. We must prove that $\mathcal{A}$ has a finite subcover.

For all $A \in \mathcal{A}$, A is open in C, so there exists $\tilde{A}$ open in X such that $A = \tilde{A} \cap C$. Let $\mathcal{B} = \{\tilde{A} : A \in \mathcal{A}\} \cup \{X \setminus C\}$. Note that $X \setminus C$ is open in X (since C is closed in X) and therefore $\mathcal{B}$ is a collection of open sets in X. Moreover

$$\bigcup \mathcal{B} = (X \setminus C) \cup \bigcup_{A \in \mathcal{A}} \tilde{A} \supseteq (X \setminus C) \bigcup_{A \in \mathcal{A}} A = (X \setminus C) \cup C = X,$$

hence $\mathcal{B}$ is an open cover of X.

Since X is compact, $\mathcal{B}$ has a finite subcover; in other words, there exists a finite subset $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{B}' = \{\tilde{A} : A \in \mathcal{A}'\} \cup \{X \setminus C\}$ is a cover of X. We now claim that $\mathcal{A}'$ is a cover of C. Indeed

$$\begin{aligned} C = C \cap X = C \cap \bigcup \mathcal{B}' &= C \cap \left((X \setminus C) \cup \bigcup_{A \in \mathcal{A}'} \tilde{A} \right) \\ &= (C \cap (X \setminus C)) \cup \bigcup_{A \in \mathcal{A}'} (C \cap \tilde{A}) = \emptyset \cup \bigcup_{A \in \mathcal{A}'} A = \bigcup \mathcal{A}'. \end{aligned}$$

So $\mathcal{A}'$ is a finite subcover of $\mathcal{A}$. □

Proposition 10.10 (compact in Hausdorff is closed)**.** *Let X be a Hausdorff topological space and let $C \subseteq X$ be compact. Then C is a closed subset of X.*

Proof. We must prove that $X \setminus C$ is open in X, i.e., that $X \setminus C$ is a neighbourhood of all its points (by Proposition 1.5(iii)).

Let $x \in X \setminus C$. For all $y \in C$, we have $x \neq y$ and therefore, since X is Hausdorff, by Proposition 4.6 there exist disjoint open neighbourhoods U_y of x and V_y of y in X.

The collection $\mathcal{A} = \{V_y \cap C : y \in C\}$ is then an open cover of C (here C is given the subspace topology): indeed all elements of $\mathcal{A}$ are open in C and every element $y \in C$ is contained in an element of $\mathcal{A}$, namely $V_y \cap C$. Since C is compact, $\mathcal{A}$ has a finite subcover. In other words, there exists a finite subset C' of C such that $\mathcal{A}' = \{V_y \cap C : y \in C'\}$ is a cover of C.

Take now $U = \bigcap_{y \in C'} U_y$. Note that U is the intersection of finitely many neighbourhoods of x in X and therefore U is a neighbourhood of x as well (by Proposition 1.5(i)). Moreover

$$U \cap C = U \cap \bigcup \mathcal{A}' \subseteq U \cap \bigcup_{y \in C'} V_y \subseteq \bigcup_{y \in C'} (U \cap V_y) \subseteq \bigcup_{y \in C'} (U_y \cap V_y) = \emptyset.$$

So U is disjoint from C, i.e., $U \subseteq X \setminus C$ and therefore $X \setminus C$ is a neighbourhood of x (by Proposition 1.5(ii)). □

Definition 10.11. Let X and Y be topological spaces.

(i) A function $f : X \to Y$ is said to be *closed* if, for all closed subsets $C \subseteq X$, the image $f(C)$ is a closed subset of Y.

(ii) A function $f : X \to Y$ is said to be *open* if, for all open subsets $A \subseteq X$, the image $f(A)$ is an open subset of Y.

Proposition 10.12. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If X is compact and Y is Hausdorff, then f is closed.*

Proof. Let $C \subseteq X$ be closed in X. We must show that $f(C)$ is closed in Y.

Since X is compact, C is compact as well by Proposition 10.9. Hence $f(C)$ is compact in Y, by Proposition 10.6. Since Y is Hausdorff, $f(C)$ is closed in Y by Proposition 10.10. □

Corollary 10.13. *Let X and Y be topological spaces and $f : X \to Y$ be continuous and bijective. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. Since f is continuous and bijective, it is invertible and it remains to show that the inverse $f^{-1} : Y \to X$ is continuous. By Proposition 3.4, for this it is sufficient to prove that the preimage $(f^{-1})^{-1}(C)$ of any closed subset C of X is closed in Y. On the other hand, $(f^{-1})^{-1}(C) = f(C)$ and $f : X \to Y$ is a closed map by Proposition 10.12, therefore $f(C)$ is closed in Y if C is closed in X. □

Lemma 10.14 (compactness and bases)**.** *Let (X, τ) be a topological space and $\mathcal{B}$ be a base of τ. The following are equivalent:*

(i) X is compact.

(ii) Every cover of X that is also a subcollection of $\mathcal{B}$ has a finite subcover.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, so we just need to prove the reverse implication.

Let us assume (ii) and let $\mathcal{A}$ be any open cover of X. We must show that $\mathcal{A}$ has a finite subcover. Note that $X = \bigcup \mathcal{A}$; therefore, for all $x \in X$, there exists $A_x \in \mathcal{A}$ such that $x \in A_x$. Since A_x is open and $\mathcal{B}$ is a base of the topology of X, A_x is union of a subcollection of $\mathcal{B}$ and in particular there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq A_x$.

Let now $\mathcal{U} = \{B_x : x \in X\}$. Then $\mathcal{U}$ is a subcollection of $\mathcal{B}$. Moreover every element $x \in X$ is contained in some element of $\mathcal{U}$, namely B_x; hence $\bigcup \mathcal{U} = X$, i.e., $\mathcal{U}$ is a cover of X. By (ii), $\mathcal{U}$ has a finite subcover; in other words, there exists a finite subset X' of X such that $\mathcal{U}' = \{B_x : x \in X'\}$ is a cover of X.

Take now $\mathcal{A}' = \{A_x : x \in X'\}$. Clearly $\mathcal{A}'$ is a finite subcollection of $\mathcal{A}$. Moreover $A_x \supseteq B_x$ for all $x \in X'$, hence $\bigcup \mathcal{A}' \supseteq \bigcup \mathcal{B}' = X$ and therefore $\mathcal{A}'$ is a cover of X as well. So $\mathcal{A}'$ is a finite subcover of $\mathcal{A}$. $\square$

Proposition 10.15 (compactness and products)**.** *Let X, Y be compact topological spaces. Then $X \times Y$ (with the product topology) is compact.*

Proof. Note that, if X or Y is empty, then $X \times Y$ is empty as well and in particular it is compact. Therefore in the rest of the proof we may assume that neither X nor Y is empty.

Recall that open rectangles form a base of the topology of $X \times Y$. Hence, by Lemma 10.14, in order to prove that $X \times Y$ is compact, it is sufficient to show that every cover of $X \times Y$ made of open rectangles has a finite subcover.

Let $\mathcal{U}$ be such a cover of $X \times Y$. Let $y \in Y$. For all $x \in X$, since $(x, y) \in X \times Y = \bigcup \mathcal{U}$, there exist open sets $A_{x,y}$ in X and $B_{x,y}$ in Y such that $(x, y) \in A_{x,y} \times B_{x,y} \in \mathcal{U}$. In particular $x \in A_{x,y}$ for all $x \in X$, so the collection $\mathcal{A}_y = \{A_{x,y} : x \in X\}$ is an open cover of X. Since X is compact, $\mathcal{A}_y$ has a finite subcover; in other words, there exists a finite subset X_y of X such that $\mathcal{A}'_y = \{A_{x,y} : x \in X_y\}$ is a cover of X.

Define now $B_y = \bigcap_{x \in X_y} B_{x,y}$. Note that B_y is the intersection of finitely many open sets in Y, so it is open in Y as well, and moreover $y \in B_y$ (because $y \in B_{x,y}$ for all x). In particular, if we define $\mathcal{B} = \{B_y : y \in Y\}$, then $\mathcal{B}$ is a collection of open sets in Y, such that every $y \in Y$ is contained in some element of $\mathcal{B}$ (namely, B_y). Hence $\mathcal{B}$ is an open cover of Y. Since Y is compact, $\mathcal{B}$ has a finite subcover, i.e., there exists a finite subset Y' of Y such that $\mathcal{B}' = \{B_y : y \in Y'\}$ is a cover of Y.

Finally define $Z = \{(x, y) : y \in Y', x \in X_y\}$. Note that Z is a finite set, since Y' is finite and each X_y is finite as well. Therefore the collection $\mathcal{U}' = \{A_{x,y} \times B_{x,y} : (x, y) \in Z\}$ is a finite subcollection of $\mathcal{U}$. In order to conclude, it is sufficient to show that $\mathcal{U}'$ is a cover of $X \times Y$.

Take any $(x^*, y^*) \in X \times Y$. Since $\mathcal{B}'$ is a cover of Y, we can find $y \in Y'$ such that $y^* \in B_y = \bigcap_{x \in X_y} B_{x,y}$. On the other hand, $\mathcal{A}'_y$ is a cover of X, therefore we can find $x \in X_y$ such that $x^* \in A_{x,y}$. So $(x^*, y^*) \in A_{x,y} \times B_{x,y} \in \mathcal{U}'$. Since $(x^*, y^*) \in X \times Y$ was arbitrary, this proves that every element of $X \times Y$ belongs to some element of $\mathcal{U}'$, i.e., $\bigcup \mathcal{U}' = X \times Y$. $\square$

We are now ready to give a proof of Corollary 10.5 about the characterisation of compact subsets of $\mathbb{R}^n$, that does not make use of the equivalence between compactness and sequential compactness.

Alternative proof of Corollary 10.5. (i) $\Rightarrow$ (ii). Let $C \subseteq \mathbb{R}^n$ be compact. Note that $\mathbb{R}^n$ is a Hausdorff space, therefore C is closed in $\mathbb{R}^n$ by Proposition 10.10. It remains to show that C is bounded.

Let $\mathcal{A} = \{C \cap B_r(0) : r \in (0, \infty)\}$, where $B_r(0)$ is the ball in $\mathbb{R}^n$ with radius r centred at 0. Then $\mathcal{A}$ is a collection of open sets in C (here C is given the subspace topology) and clearly $\bigcup \mathcal{A} = C$ (indeed, for all $x \in C$, if we take $r = 1 + d(0, x)$ then $x \in B_r(0)$). So $\mathcal{A}$ is an open cover of C. Since C is compact, $\mathcal{A}$ has an open subcover, i.e., there is a finite subset R of $(0, \infty)$ such that $\mathcal{A}' = \{C \cap B_r(0) : r \in R\}$ is a cover of C. Since R is finite, we can take $r^* = \max R$ and then

$$C = \bigcup \mathcal{A}' \subseteq \bigcup_{r \in R} B_r(0) = B_{r^*}(0).$$

This shows that C is bounded.

(ii) $\Rightarrow$ (i). Let C be a closed and bounded subset of $\mathbb{R}^n$. Since C is bounded, there exists $R \in (0, \infty)$ such that $C \subseteq [-R, R]^n$. Since C is closed in $\mathbb{R}^n$, it is also closed in $[-R, R]^n$ (see Proposition 6.5). So, if we knew that $[-R, R]^n$ is compact, by Proposition 10.9 we would deduce that C is compact. On the other hand, compactness of $[-R, R]^n$ would follow by Proposition 10.15 from compactness of $[-R, R]$. So we only need to show that $[-R, R]$ is compact in $\mathbb{R}$.

Let $\mathcal{A}$ be any open cover of $I = [-R, R]$. Let us call *$\mathcal{A}$-sequence* any finite increasing sequence $(x_0, \dots, x_N)$ of points of I such that $x_0 = -R$ and, for all $i = 1, \dots, N$, there exists $A \in \mathcal{A}$ such that $[x_{i-1}, x_i] \subseteq A$. A point of I will be called *$\mathcal{A}$-reachable* if it is the endpoint x_N of some $\mathcal{A}$-sequence $(x_0, \dots, x_N)$.

Let I' be the set of $\mathcal{A}$-reachable points of I. Note that the point $-R$ is $\mathcal{A}$-reachable (it is the endpoint of the 1-element $\mathcal{A}$-sequence $(-R)$), so $I' \neq \emptyset$.

Let $M = \sup I'$. Clearly $-R \leq M \leq R$, so $M \in I$. We now show that $M \in I'$, i.e., M is $\mathcal{A}$-reachable. Indeed, since $\mathcal{A}$ is an open cover of I, there exists an open set $A \in \mathcal{A}$ such that $M \in A$. So there is $\epsilon > 0$ such that $I \cap (M - \epsilon, M + \epsilon) \subseteq A$. Since $M = \sup I'$, there exists $x \in I'$ such that $M - \epsilon < x \leq M$. Now x is $\mathcal{A}$-reachable, so there exists an $\mathcal{A}$-sequence $(x_0, \dots, x_N)$ such that $x_N = x$. If we define $x_{N+1} = M$, then $[x_N, x_{N+1}] = [x_N, M] \subseteq I \cap (M - \epsilon, M + \epsilon) \subseteq A \in \mathcal{A}$. Hence the sequence $(x_0, \dots, x_{N+1})$ is an $\mathcal{A}$-sequence as well and therefore its endpoint M is $\mathcal{A}$-reachable.

Note that, in above argument, instead of M we could have chosen any point of $I \cap [M, M + \epsilon)$ as endpoint x_{N+1} of the new $\mathcal{A}$-sequence. Hence all points of $I \cap [M, M + \epsilon) = [-R, R] \cap [M, M + \epsilon)$ are $\mathcal{A}$-reachable. So, if M were less than R, then we would obtain $\mathcal{A}$-reachable points that are strictly larger than M, which contradicts the fact that $M = \sup I'$. Therefore it must be $M = R$.

In conclusion, we have proved that R is $\mathcal{A}$-reachable. Therefore there exist finitely many points $x_0, \dots, x_N \in I$ such that $-R = x_0 < \dots < x_N = R$ and, for all $i \in \{1, \dots, N\}$, an element $A_i \in \mathcal{A}$ such that $[x_{i-1}, x_i] \subseteq A_i$. So $\mathcal{A}' = \{A_1, \dots, A_N\}$ is a finite subcollection of $\mathcal{A}$ and $\bigcup \mathcal{A}' = \bigcup_{i=1}^N A_i \supseteq \bigcup_{i=1}^N [x_{i-1}, x_i] = I$, i.e., $\mathcal{A}'$ is a finite subcover of $\mathcal{A}$. $\square$

11 Quotient topology

Let us first recall, without proof, a few basic facts about equivalence relations and quotients, that do not involve topology.

Let X be a set and $\sim$ be an *equivalence relation* on X (i.e., $\sim$ is a binary relation on X that is reflexive, symmetric and transitive). Every element $x \in X$ belongs to an *equivalence class* $[x] = \{y \in X : y \sim x\}$ relative to $\sim$. Equivalence classes relative to $\sim$ form a partition of X. The collection $X/{\sim} = \{[x] : x \in X\}$ of equivalence classes is called *quotient* of X by $\sim$. The map $q : X \to X/{\sim}$ defined by $q(x) = [x]$ for all $x \in X$ is called *canonical projection.*

Proposition 11.1. *Let X be a set and $\sim$ be an equivalence relation on X. Let $q : X \to X/{\sim}$ be the canonical projection. Let Y be another set and $F : X \to Y$ be a function. Suppose that, for all $x, x' \in X$,*

$$x \sim x' \quad \Rightarrow \quad F(x) = F(x').$$

Then the following hold.

(i) The function $F : X \to Y$ descends to the quotient*, i.e., there exists a unique function $\tilde{F} : X/{\sim} \to Y$ such that*

$$\tilde{F} \circ q = F.$$

(ii) $\tilde{F} : X/{\sim} \to Y$ is surjective if and only if $F : X \to Y$ is surjective.

(iii) $\tilde{F} : X/{\sim} \to Y$ is injective if and only if, for all $x, x' \in X$,

$$F(x) = F(x') \quad \Rightarrow \quad x \sim x'.$$

Note that the discussion so far has been purely set-theoretic and has not involved any topology or continuity.

Suppose now that X is a topological space and $\sim$ is an equivalence relation on X. We want to define a topology on the quotient space $X/{\sim}$.

Proposition 11.2. *Let (X, τ) be a topological space, $\sim$ be an equivalence relation on X and $q : X \to X/{\sim}$ be the canonical projection. Then the collection*

$$\tau_\sim = \{A \subseteq X/{\sim} : q^{-1}(A) \in \tau\} \tag{11.1}$$

is a topology on $X/{\sim}$.

Proof. Let us check that $\tau_\sim$ satisfies the definition of a topology on the set $\tilde{X} = X/{\sim}$.

$\tilde{X} \in \tau_\sim$, because $q^{-1}(\tilde{X}) = X \in \tau$. Similarly, $\emptyset \in \tau_\sim$, since $q^{-1}(\emptyset) = \emptyset \in \tau$.

Let now A, B be any elements of $\tau_\sim$. We must prove that $A \cap B \in \tau_\sim$. On the other hand, $q^{-1}(A), q^{-1}(B) \in \tau$ (since $A, B \in \tau_\sim$), so

$$q^{-1}(A \cap B) = q^{-1}(A) \cap q^{-1}(B) \in \tau$$

(because τ is closed under finite intersections) and therefore $A \cap B \in \tau_\sim$.

Finally, let $\mathcal{A}$ be any subcollection of $\tau_\sim$. We must prove that $\bigcup \mathcal{A} \in \tau_\sim$. On the other hand, for all $A \in \mathcal{A}$, $q^{-1}(A) \in \tau$ (because $A \in \tau_\sim$), so

$$q^{-1}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} q^{-1}(A) \in \tau$$

(because τ is closed under arbitrary unions) and therefore $\bigcup \mathcal{A} \in \tau_\sim$. □

Definition 11.3. Let (X, τ) be a topological space and $\sim$ be an equivalence relation on X. The topology $\tau_\sim$ on the quotient $X/\sim$ defined by (11.1) is called *quotient topology.*

In other words, a subset A of the quotient $X/\sim$ is open in the quotient topology if and only if its preimage $q^{-1}(A)$ via the canonical projection q is open in X.

Proposition 11.4. *Let X be a topological space, $\sim$ be an equivalence relation on X. Let $X/\sim$ be given the quotient topology. Then the canonical projection $q : X \to X/\sim$ is continuous.*

Proof. Let A be open in $X/\sim$. By definition of quotient topology, $q^{-1}(A)$ is open in X. Since A was an arbitrary open set in $X/\sim$, this proves that q is continuous. □

Proposition 11.5. *Let X be a topological space, $\sim$ be an equivalence relation on X and $q : X \to X/\sim$ be the canonical projection. Let $X/\sim$ be given the quotient topology. Let Z be another topological space and $F : X \to Z$ be a function. Suppose that F descends to the quotient and defines a function $\tilde{F} : X/\sim \to Z$ such that $F = \tilde{F} \circ q$. Then the following are equivalent:*

(i) $\tilde{F} : X/\sim \to Z$ *is continuous;*

(ii) $F : X \to Z$ *is continuous.*

Proof. (i) $\Rightarrow$ (ii). Note that $\tilde{F}$ is continuous by (i) and q is continuous by Proposition 11.4. Hence $F = \tilde{F} \circ q$ is the composition of continuous functions and therefore it is continuous.

(ii) $\Rightarrow$ (i). Let A be an open set in Z. Since F is continuous, $F^{-1}(A)$ is open in X. However $F^{-1}(A) = (\tilde{F} \circ q)^{-1}(A) = q^{-1}(\tilde{F}^{-1}(A))$. So $q^{-1}(\tilde{F}^{-1}(A))$ is open in X and, by definition of quotient topology, $\tilde{F}^{-1}(A)$ is open in $X/\sim$. Since A was an arbitrary open set in Z, this proves that $\tilde{F} : X/\sim \to Z$ is continuous. □

Remark 11.6. Note that the above results about the quotient topology do not really use the fact that $X/\sim$ is a quotient of X and q is the canonical projection. In other words, it would be possible to replace $X/\sim$ with any other set Y and the canonical projection q with any other (surjective) function $f : X \to Y$, and define a topology on Y starting from the topology τ on X. Some references use the name "quotient topology" also for this more general construction.

Let X be a topological space and $\sim$ be an equivalence relation on X. Taking the quotient, i.e., passing from X to $X/\sim$, corresponds to the intuitive idea of "glueing together" all points that are in the same equivalence class relative to $\sim$. The above definition of quotient topology then gives us a way of defining a topology on the quotient space $X/\sim$, starting from the topology on X.

In these respects, an important example of quotients is obtained by identifying together all the points of a given subset of the original space. For this we introduce a particular piece of notation.

Definition 11.7. If X is a set and S is a subset of X, then we denote by X/S the quotient $X/\sim_S$, where $\sim_S$ is the equivalence relation on X defined by

$$x \sim_S x' \quad \Leftrightarrow \quad (x = x' \text{ or } x, x' \in S).$$

Warning: Despite the similarity of notation, please be careful not to confuse the set quotient X/S (the set of equivalence classes determined by the equivalence relation $\sim_S$) with the set difference $X \setminus S$ (the set of all elements of X which are not elements of S).

Examples. In the following examples, subsets of $\mathbb{R}^n$ are given the topology induced by the standard topology on $\mathbb{R}^n$. Moreover $\mathbb{C}$ is identified to $\mathbb{R}^2$ via the map $\mathbb{C} \ni x + iy \mapsto (x, y) \in \mathbb{R}^2$.

1. Let $I = [0, 1]$. Let $X = I/\{0, 1\}$ be given the quotient topology. In other words, X is obtained by glueing together the endpoints of I. Then X is homeomorphic to $S^1 = \{z \in \mathbb{C} : |z|^2 = 1\}$, i.e., the unit circle in $\mathbb{C}$.

   Indeed, let $H : I \to S^1$ be defined by $H(t) = e^{2\pi it}$. Then it is easily seen that $H : I \to S^1$ is surjective and moreover, for all $t, t' \in I$,

   $$t \sim_{\{0,1\}} t' \Leftrightarrow t - t' \in \mathbb{Z} \Leftrightarrow e^{2\pi it} = e^{2\pi it'} \Leftrightarrow H(t) = H(t').$$

   Therefore, by Proposition 11.1, H descends to the quotient and defines a bijective function $\tilde{H} : X \to S^1$ such that $\tilde{H} \circ q = H$, where $q : I \to X$ is the canonical projection. Since H is continuous, $\tilde{H}$ is continuous as well by Proposition 11.5. Moreover X is compact by Proposition 10.6 (because $X = q(I)$, q is continuous and I is compact) and S^1 is Hausdorff by Corollary 6.7 (because S^1 is a subspace of the Hausdorff space $\mathbb{C}$), so $H : X \to S^1$ is a homeomorphism by Corollary 10.13.

2. Let S^1 be the unit circle as above. Let $Y = S^1/\sim$, where $\sim$ is the equivalence relation defined by

   $$z \sim z' \quad \Leftrightarrow \quad z = \pm z'$$

   for all $z, z' \in S^1$. In other words, Y is obtained from S^1 by identifying antipodal points. Let $q : S^1 \to Y$ be the canonical projection. Let Y be given the quotient topology. Then Y is homeomorphic to S^1.

   Indeed, let $F : S^1 \to S^1$ be defined by $F(z) = z^2$. Then it is easily seen that $F : S^1 \to S^1$ is surjective (every complex number of modulus 1 has a square root of modulus 1) and moreover, for all $z, z' \in S^1$,

   $$z \sim z' \Leftrightarrow z = \pm z' \Leftrightarrow z^2 = (z')^2 \Leftrightarrow F(z) = F(z').$$

   Therefore, by Proposition 11.1, F descends to the quotient and defines a bijective map $\tilde{F} : Y \to S^1$ such that $\tilde{F} \circ q = F$. Since $F : S^1 \to S^1$ is continuous, $\tilde{F} : Y \to S^1$ is continuous as well by Proposition 11.5. Moreover Y is compact by Proposition 10.6 (because $Y = \pi(S^1)$, π is continuous and S^1 is compact) and S^1 is Hausdorff (by Corollary 6.7, because it is subspace of the Hausdorff space $\mathbb{R}^2$). So $\tilde{F}$ is a homeomorphism by Corollary 10.13.

3. Let $Q = [0, 1]^2 = [0, 1] \times [0, 1]$ be the unit square in $\mathbb{R}^2$. Let $\sim$ be the equivalence relation on Q defined by

   $$(x, y) \sim (x', y') \quad \Leftrightarrow \quad (x, y) = (x', y') \text{ or } (x, x' \in \{0, 1\} \text{ and } y = y').$$

Let $Z = Q/\sim$ be given the quotient topology. In other words, Z is obtained by glueing together one pair of opposite sides of the square Q. Then Z is homeomorphic to the cylinder $C = S^1 \times [0,1]$.

Indeed, define the map $G : Q \to C$ by $G(x,y) = (e^{2\pi i x}, y)$. Then it is easily seen that $G : Q \to C$ is surjective and moreover, for all $(x,y),(x',y') \in Q$,

$$\begin{aligned}(x,y) \sim (x',y') &\Leftrightarrow (y = y' \text{ and } x - x' \in \mathbb{Z}) \\ &\Leftrightarrow (y = y' \text{ and } e^{2\pi i x} = e^{2\pi i x'}) \Leftrightarrow G(x,y) = G(x',y').\end{aligned}$$

Therefore G descends to the quotient and defines a bijective map $\tilde{G} : Z \to C$ such that $\tilde{G} \circ q = G$, where $q : Q \to Z$ is the canonical projection. Since G is continuous, $\tilde{G}$ is continuous as well by Proposition 11.5. Moreover Z is compact by Proposition 10.6 (because $Z = p(Q)$, $p : Q \to Z$ is continuous and Q is compact) and C is Hausdorff by Corollary 6.7 (because C is a subspace of $\mathbb{R}^3$). So $\tilde{G}$ is a homeomorphism by Corollary 10.13.

In the previous examples, the quotient space could always be identified with some subspace of $\mathbb{R}^n$. This is not always possible. Indeed, the quotient of a Hausdorff space need not be Hausdorff. Hence the quotient construction is much less "well-behaved" than the previously discussed topological constructions (subspace topology and product topology) and may easily lead to "degenerate" examples.

Example. Let $X = \mathbb{R} \times \{0,1\}$ be given the topology induced by the standard topology on $\mathbb{R}^2$. Clearly X is Hausdorff. We can think of X as the "disjoint union" of two copies of $\mathbb{R}$ (indeed it is easily seen that $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ with the subspace topology are both homeomorphic to $\mathbb{R}$). Let $\sim$ be the equivalence relation on X defined by

$$(x,t) \sim (x',t') \quad \Leftrightarrow \quad x = x' \text{ and } (t = t' \text{ or } x \neq 0).$$

Let $Y = X/\sim$. In other words, Y is obtained from X by identifying all the pairs $(x,0)$ and $(x,1)$ where $x \in \mathbb{R} \setminus \{0\}$; however the pairs $(0,0)$ and $(0,1)$ are not identified and therefore correspond to two distinct points $[(0,0)]$ and $[(0,1)]$ of Y. Then $[(0,0)]$ and $[(0,1)]$ have no disjoint neighbourhoods (why?) and consequently Y is not Hausdorff.

We conclude with some other notable examples of quotients.

Examples. All subsets of $\mathbb{R}^n$ are given the topology induced by the standard topology of $\mathbb{R}^n$.

1. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $\sim$ be the equivalence relation on S^2 defined by

   $$(x,y,z) \sim (x',y',z') \quad \Leftrightarrow \quad (x,y,z) = \pm(x',y',z'),$$

   for all $(x,y,z),(x',y',z') \in S^2$. Let $P = S^2/\sim$. In other words, P is obtained from S^2 by identifying pairs of antipodal points. The topological space P (with the quotient topology) is known as the *(real) projective plane* and is the geometrical setting for the so-called *projective geometry.*

2. Let $Q = [0, 1]^2$ be the unit square. Let $\sim$ be the equivalence relation on Q defined by

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad (x, y) = (x', y') \text{ or } (|x - x'| = 1 \text{ and } y = 1 - y'),$$

for all $(x, y), (x', y') \in Q$. Let $M = Q/\sim$ be given the quotient topology. Then M is known as the *Möbius strip*. It is possible to find a subspace of $\mathbb{R}^3$ which is homeomorphic to M (this is left as an exercise to the reader).

3. Let $Q = [0, 1]^2$ be the unit square. Let $\sim$ be the equivalence relation on Q defined by

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad x - x' \in \mathbb{Z} \text{ and } y - y' \in \mathbb{Z},$$

for all $(x, y), (x', y') \in Q$. Let $T = Q/\sim$ be given the quotient topology. Then T is homeomorphic to $S^1 \times S^1$ (why?) and is known as the *(two-dimensional) torus*. It is also possible to find a subspace of $\mathbb{R}^3$ which is homeomorphic to T.

4. Let $Q = [0, 1]^2$ be the unit square. Let $\sim$ be the equivalence relation on Q defined by

$$\begin{aligned}(x, y) \sim (x', y') \quad \Leftrightarrow \quad & (x, y) = (x', y') \\ & \text{or } (|x - x'| = 1 \text{ and } y = 1 - y') \\ & \text{or } (|y - y'| = 1 \text{ and } x = x'),\end{aligned}$$

for all $(x, y), (x', y') \in Q$. Let $K = Q/\sim$ be given the quotient topology. Then K is known as the *Klein bottle*. It is possible to find a subspace of $\mathbb{R}^4$ that is homeomorphic to K. It is also possible (but beyond the reach of these lectures) to prove that there is no subspace of $\mathbb{R}^3$ which is homeomorphic to K.