8. Complete Metric Spaces and sets

8.1. Definition and properties.

Definition 8.1. *Let (X, d) be a metric space. We say that the metric space X is complete, or that X is a complete space, iff every Cauchy sequence in X converges (to a point in X).*

Definition 8.2. *Let (X, d) be a metric space and $A \subseteq X$. We say that A is complete if each Cauchy sequence $(x_n)_{n=1}^{\infty} \subseteq A$ converges to a point in A.*[12]

Examples.

1. Cauchy's convergence criterion (see Theorem 7.2)) shows that $\mathbb{R}$ with the usual metric is complete.

2. Let $n \in \mathbb{N}$. The Euclidean space $\mathbb{R}^n$ is complete.
Proof. Recall that the Euclidean metric on $\mathbb{R}^n$ is defined by

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

for all $x = (x_1, \dots x_n)$ and $y = (y_1, \dots y_n)$ in $\mathbb{R}^n$.

Suppose that $(x_r)_{r=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^n$, where

$$x_r = (x_{r,1}, \dots x_{r,n}), \qquad r = 1, 2, \dots$$

For each $i \in \{1, \dots, n\}$, the sequence $(x_{r,i})_{r=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ with the usual metric, since

$$|x_{r,i} - x_{s,i}| \le d(x_r, x_s) = \sqrt{\sum_{i=1}^{n} (x_{r,i} - x_{s,i})^2},$$

for any $r, s \in \mathbb{N}$, by the definition of Euclidean metric on $\mathbb{R}^n$. Hence, by using the Cauchy's criterion of convergence, $(x_{r,i})_{r=1}^{\infty}$ converges to x_i say (on $\mathbb{R}$ with the usual metric). We shall prove that

$$x_r \longrightarrow x = (x_1, \dots, x_n), \qquad \text{as} \quad r \longrightarrow \infty$$

on $\mathbb{R}^n$ with the Euclidean metric. Indeed, given $\varepsilon > 0$, since $x_{r,i} \longrightarrow x_i$ as $r \longrightarrow \infty$ for each $i \in \{1, \dots, n\}$, then for each $i \in \{1, \dots, n\}$ there exists $N_i \in \mathbb{N}$ such that

$$|x_{r,i} - x_i| < \frac{\varepsilon}{\sqrt{n}}, \qquad \forall\, r \ge N_i.$$

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[12] So, A subset of X is complete iff the metric subspace A of X is a complete space.

Now, take $N = \max\{N_1, \ldots, N_n\}$. Then, if $r \geq N$ we have that

$$d(x_r, x) = \sqrt{\sum_{i=1}^{n} (x_{r,i} - x_i)^2} < \sqrt{(\varepsilon/\sqrt{n})^2 + \cdots + (\varepsilon/\sqrt{n})^2} = \sqrt{n\left(\frac{\varepsilon^2}{n}\right)} = \varepsilon.$$

Thus, $x_r \longrightarrow x$ as $r \longrightarrow \infty$ in $\mathbb{R}^n$ with the Euclidean metric.

3. $\mathbb{Q}$ with the usual metric is not complete: any sequence in $\mathbb{Q}$ which converges in $\mathbb{R}$ to $\sqrt{2}$ is a Cauchy sequence in $\mathbb{Q}$ which does not converge to any point in $\mathbb{Q}$.

4. Any discrete metric space is complete (see exercise in problem sheet).

5. Each closed <u>interval</u> of $\mathbb{R}$ is complete (with the usual metric). More generally, any closed <u>subset</u> of $\mathbb{R}$ is complete. This is a consequence of the following general result.

Proposition 8.3. *Suppose that (X, d) is a <u>complete</u> metric space, and $A \subseteq X$.*

If A is a closed set, then A is complete.

The above proposition states that: A closed subset A of a <u>complete</u> metric space X is complete.

Proof. Suppose that X is complete and that A is closed in X. Then, by Corollary 6.5

$$\overline{A} = A. \tag{8.1}$$

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $A \subseteq X$, by the completeness of X, the sequence $(x_n)_{n=1}^{\infty}$ converges to a point $x \in X$. Now, by Theorem 6.6, we obtain that $x \in \overline{A}$ and from (8.1) we get that $x \in A$.

The above argument shows that every Cauchy sequence in A converges to a point in A. Thus, A is complete. □

The following result shows that the converse of the Proposition 8.3 also hold in any general metric space.

Proposition 8.4. *Let (X, d) be a metric space, and $A \subseteq X$.*

If A is complete, then A is closed in X.

In words: A complete subset A in a metric space is closed in X.

Proof. Suppose that $A \subseteq X$ is complete and let $x \in \overline{A}$, we want to show that $x \in A$.

By Theorem 6.6, there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq A$ converging to x. Since $(x_n)_{n=1}^{\infty}$ is a convergent sequence, in particular it is a Cauchy sequence (see Proposition 7.4), so by the completeness of A the sequence $(x_n)_{n=1}^{\infty}$ converges to a point in A. By uniqueness of limits of convergent sequences (see Theorem 5.2) in metric spaces, this point mus be x. Hence $x \in A$.

The above arguments shows that $\overline{A} \subseteq A$, and since we always have that $A \subseteq \overline{A}$, we conclude that $\overline{A} = A$, or equivalently (see Corollary 6.5) A is closed in X. $\square$

Remark. Since $\mathbb{R}$ with the usual metric is complete, Proposition 8.3 and Proposition 8.4 imply that:

The complete subsets of $\mathbb{R}$ are precisely the closed subsets.

8.2. **Some complete metric spaces.**

Consider

$$\mathcal{B}([a,b],\mathbb{R}) = \{f : [a,b] \longrightarrow \mathbb{R} \ : \ f \text{ bounded}\}^{13}$$

with the supremum metric

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

for all f and g in $\mathcal{B}([a,b],\mathbb{R})$.

Proposition 8.5. *The space $\mathcal{B}([a,b],\mathbb{R})$, of bounded real-valued functions on $[a,b]$, with the sup-metric is complete.*

Proof. In order to simplify the notation, in what follows we denote $\mathcal{B}([a,b],\mathbb{R})$ by X. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in X and let $\underline{\varepsilon > 0}$. Then, there exists $N \in \mathbb{N}$ such that

$$d(f_n, f_m) = \sup_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon, \qquad \forall\, n, m \geq N,$$

so in particular

$$|f_n(x) - f_m(x)| < \varepsilon, \qquad \forall\, n, m \geq N \qquad \text{and all} \quad x \in [a,b]. \tag{8.2}$$

Then, for each $x \in [a,b]$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (with the usual metric), and by Cauchy's principle of convergence (see Theorem 7.2), $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$ say, that is

$$f_n(x) \longrightarrow f(x), \qquad \text{as} \quad n \longrightarrow \infty \quad \text{in } \mathbb{R}. \tag{8.3}$$

We can do this for each $x \in [a,b]$. This defines a function $f : [a,b] \longrightarrow \mathbb{R}$. We will continue to show that

i) $f_n \longrightarrow f$, as $n \longrightarrow \infty$ in (X,d), and
ii) $f \in X$.

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[13] Here, a and $b \in \mathbb{R}$ and $a < b$.

Proof of i) Indeed, from (8.2), we know that for each $x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \varepsilon, \qquad \forall\, n, m \geq N.$$

Then, by letting $m \longrightarrow \infty$ and using (8.3), for each $x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \geq N.$$

Here, we have used that the function $g(y) = |f_n(x) - y|$ is continuous and $f_m(x) \longrightarrow f(x)$ as $m \longrightarrow \infty$ for each $x \in [a, b]$.

By taking the supremum over all $x \in [a, b]$,

$$d(f_n, f) = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \geq N.$$

Thus, we conclude that $f_n \longrightarrow f$, as $n \longrightarrow \infty$ in (X, d), as required.

Proof of ii) Finally, we will show that $f \in X = \mathcal{B}([a, b], \mathbb{R})$. Indeed, since $f_n \longrightarrow f$, as $n \longrightarrow \infty$, given $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$d(f_N, f) = \sup_{x \in [a,b]} |f_N(x) - f(x)| < 1,$$

so

$$|f_N(x) - f(x)| < 1, \qquad \text{for all} \quad x \in [a, b]. \tag{8.4}$$

Now, since $f_N \in X = \mathcal{B}([a, b], \mathbb{R})$, there exists $M \in \mathbb{R}$ such that

$$|f_N(x)| \leq M, \qquad \text{for all} \quad x \in [a, b]. \tag{8.5}$$

From (8.4) and (8.5), we conclude that for each $x \in [a, b]$

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M.$$

Thus $f \in \mathcal{B}([a, b], \mathbb{R})$, as desired. This concludes the proof. □

Consider

$$\mathcal{C}([a, b], \mathbb{R}) = \{f : [a, b] \longrightarrow \mathbb{R} \ : \ f \text{ is continuous } \}^{14}.$$

Recall the following result from the Real Analysis-modules: If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$ and attains its bounds (that is there exists x_1 and $x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$, for all $x \in [a, b]$). Thus

$$\mathcal{C}([a, b], \mathbb{R}) \subseteq \mathcal{B}([a, b], \mathbb{R}).$$

Moreover,

Proposition 8.6. *$\mathcal{C}([a, b], \mathbb{R})$ is a closed subset of $\mathcal{B}([a, b], \mathbb{R})$ endowed with the supremum metric.*

We have the following:

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[14] Here, a and $b \in \mathbb{R}$ and $a < b$.

Corollary 8.7. *The space $\mathcal{C}([a,b],\mathbb{R})$, of real-valued continuous functions on $[a,b]$, with the sup-metric is complete.*

Corollary 8.7 is a immediate consequence of Proposition 8.5, Proposition 8.6 and Proposition 8.3.

Proof of Proposition 8.6. We prove that $\mathcal{C}([a,b],\mathbb{R})$ is closed in $\mathcal{B}([a,b],\mathbb{R})$ by proving that: for every convergent sequence of functions of $\mathcal{C}([a,b],\mathbb{R})$, the limit is also in $\mathcal{C}([a,b],\mathbb{R})$ (see Proposition 6.7).

Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{C}([a,b],\mathbb{R})$ convergent to f say with respect to the sup-metric, we want to prove that f is continuous, that is f is a continuous function at any arbitrary point $c \in [a,b]$.

Let $c \in [a,b]$ and $\varepsilon > 0$. Since $f_n \longrightarrow f$ in $\mathcal{B}([a,b],\mathbb{R})$, there exists $N \in \mathbb{N}$ such that

$$d(f_n, f) = \sup_{x\in[a,b]} |f_n(x) - f(x)| < \frac{\varepsilon}{3}, \qquad \forall\, n \geq N,$$

so, in particular

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}, \qquad \text{for all} \quad x \in [a,b]. \tag{8.6}$$

By hypothesis, f_N is continuous at c, therefore there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}, \qquad \text{whenever} \quad |x-c| < \delta \qquad \text{and} \quad x \in [a,b]. \tag{8.7}$$

Therefore from (8.6) and (8.7) it follows that

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

whenever $|x-c| < \delta$ and $x \in [a,b]$. This shows that f is continuous at c, and since $c \in [a,b]$ is arbitrary we conclude that f is continuous on $[a,b]$. □

8.3. **Uniform convergence. Pointwise convergence.**

The convergence in $\mathcal{B}([a,b],\mathbb{R})$ or $\mathcal{C}([a,b],\mathbb{R})$ with the sup-metric may be re-interpreted as uniform convergence for sequences of functions in these spaces.

We need the following definition

Definition 8.8. *A sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions defined on $D \subseteq \mathbb{R}$ converges to a function f uniformly on D if given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N \qquad \textit{and all} \qquad x \in D.$$

If this is full-filled, we write '$f_n \longrightarrow f$ uniformly on D', and call f the 'uniform limit of f_n'.

Now the convergence of a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{B}([a,b],\mathbb{R})$ or $\mathcal{C}([a,b]\mathbb{R})$ with the sup-metric to a point/function f means that: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(f_n, f) = \sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \geq N$$

so

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall\, n \geq N \qquad \text{and} \quad x \in [a,b].$$

By Definition 8.8, this means precisely that

$$f_n \longrightarrow f \qquad \text{uniformly on} \qquad [a,b].$$

How does the uniform convergence compare with "pointwise convergence"?

Definition 8.9. *A sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions defined on $D \subseteq \mathbb{R}$ converges to a function f pointwise on D if for each $x \in D$, the real number sequence $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$*

Remark. Obviously, if $f_n \longrightarrow f$ uniformly on D, then $f_n \longrightarrow f$ pointwise on D[15].

"Uniform convergence is stronger in that, given any $\varepsilon > 0$, there must exist an N which does the necessary job for all x in D simultaneously, while for pointwise convergence, given $\varepsilon > 0$ we may use a different N_x for each $x \in D$. Thus uniform convergence is global in D" (direct quotation from Sutherland's book).

In general, it is not true that the pointwise limit of continuous functions is a continuous function (See problem sheet 4- Q1)!

The following result asserts that "the uniform limit of continuous functions is a continuous function".

Proposition 8.10. *If $f_n : [a,b] \longrightarrow \mathbb{R}$ is continuous for each $n \in \mathbb{N}$, and if $f_n \longrightarrow f$ uniformly on $[a,b]$, then f is continuous.*

Proof. See the proof of Proposition 8.6, and reinterpret the convergence in $\mathcal{C}([a,b],\mathbb{R})$ with the sup-metric in terms of uniform convergence of sequences of functions. $\square$

[15] This is a direct consequence of the definitions of uniform convergence and pointwise convergence. See Definitions 8.8 and 8.9

9. Contraction Mapping Theorem

Concern: Applications of the theory of metric spaces to algebra and analysis.

One of the most attractive theorems about metric spaces is the so-called Contraction Mapping Theorem, also referred to as Banach Fixed Point Theorem or Contraction Mapping Principle.

(1) On the theoretical side, it unifies the proofs of several theorems on the existence of solutions of algebraic, differential, integral and functional equations.
(2) When it works, the theorem tell us that the solution is unique as well.
(3) The proof of the result is constructive, provides a constructive method to find the fixed point (see definition below).

9.1. **Contraction Mapping Theorem.**

Before stating the result, we need to introduce some terminology.

Definition 9.1. *Let (X, d) be a metric space. A map/function $f : X \longrightarrow X$ is a contraction if there exists a non-negative real number $K < 1$ such that*

$$d(f(x), f(y)) \leq K\, d(x, y), \qquad \textit{for all} \qquad x, y \in X.$$

In words: the distance between $f(x)$ and $f(y)$ is definitely less than the distance between x and y, for any x and y in X; this is why the map/function is called a contraction. How does the property of a function being a contraction relates with the property of continuity of the function? we have the following:

Proposition 9.2. *Let (X, d) be a metric space, and $f : X \longrightarrow X$ be a function.*

If f is a contraction, then f is continuous

In words: Every contraction is continuous.

Proof. (Exercise) □

Examples.

1. Let $X = \mathbb{R}$ with the usual metric, $d(x, y) = |x-y|$, $\forall\, x, y \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \frac{x}{2} \qquad \forall\, x \in \mathbb{R}.$$

Then, for any $x, y \in \mathbb{R}$

$$|f(x) - f(y)| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}\,|x - y|,$$

so f is a contraction on $\mathbb{R}$.

2. Let $X = [0,1]$ with the usual metric and f be the function on X defined by

$$f(x) = \frac{1}{7}(x^3 + x^2 + 1), \qquad \forall\, x \in [0,1].$$

Notice that for any $x \in [0,1]$,

$$f(x) = \frac{1}{7}(x^3 + x^2 + 1) \qquad \text{and} \qquad f(x) \leq \frac{1}{7}(1+1+1) = \frac{3}{7} \leq 1,$$

so $f(x) \in [0,1]$, and therefore $f : [0,1] \longrightarrow [0,1]$.

Also, for any $x \in [0,1]$,

$$\begin{aligned} |f(x) - f(y)| &= \frac{1}{7}|(x^3 + x^2 + 1) - (y^3 + y^2 + 1)| \\ &= \frac{1}{7}|(x^3 - y^3) + (x^2 - y^2)| \\ &= \frac{1}{7}|(x-y)(x^2 + xy + y^2) + (x-y)(x+y)| \\ &\underbrace{\leq}_{x,y\in[0,1]} \frac{1}{7}(3|x-y| + 2|x-y|) \\ &= \frac{5}{7}|x-y|, \end{aligned}$$

so f is a contraction on $[0,1]$.

3. If $f : [a,b] \longrightarrow [a,b]$ is continuous on $[a,b]$, and differentiable on (a,b), and there exists $0 < K < 1$ such that $|f'(x)| \leq K$ for all $x \in (a,b)$. Then f is a contraction on $[a,b]$.

This follows from the Mean Value Theorem. Let $x, y \in [a,b]$, and assume without loss of generality that $x < y$. Consider the interval $[x,y]$. Then, by the Mean Value Theorem,

$$|f(x) - f(y)| \leq |f'(c)|\,|x-y|,$$

for some $c \in (x,y)$, so by using the hypothesis, we conclude that

$$|f(x) - f(y)| \leq K|x-y|,$$

for some $0 < K < 1$. Therefore, f is a contraction on $[a,b]$.[16]

In the proof of the Contraction Mapping Theorem, we will also make use of the following important lemma, which gives us a characterization of the continuity of a function at a point in term of the convergence of sequences (see Proposition 5.3)).

Lemma 9.3. *Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \longrightarrow Y$ be a function and $a \in X$. Then*

f is continuous at a if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in X converging to a, then $f(x_n) \longrightarrow f(a)$ as $n \longrightarrow \infty$.

[16] Example 2 is a particular case of such this function f.

Proof.

The proof of this lemma was given in the the proof of Proposition 5.3 in Chapter 5. $\square$

Theorem 9.4 (The Contraction Mapping Theorem/ Banach Fixed Point Theorem)**.** *Let (X, d) be a complete metric space, and let $f : X \longrightarrow X$ be a contraction. Then, there exists a unique point $x^\star \in X$ such that*

$$f(x^\star) = x^\star.$$

Remark. A point $x^\star \in X$ such that $f(x^\star) = x^\star$ is called a fixed point of f in X.

Theorem 9.4 states that if f is a contraction on a complete metric space, then f has precisely one fixed point in X.

Proof. (Constructive method)

Existence. Choose $x_0 \in X$ (arbitrarily), and consider the sequence $(x_n)_{n=1}^\infty$ defined recursively by

$$x_n = f(x_{n-1}), \qquad \text{for all} \qquad n \in \mathbb{N}.$$

We shall prove that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X. First, since f is a contraction, there exists a non-negative real constant $K < 1$ such that

$$d(f(x), f(y)) \leq K d(x, y), \qquad \forall\, x, y \in X,$$

so

$$\begin{array}{rcl} d(x_2, x_1) & = & d(f(x_1), f(x_2)) \leq K d(x_1, x_0) \\ d(x_3, x_2) & = & d(f(x_2), f(x_1)) \leq K d(x_2, x_1) \leq K^2 d(x_1, x_0) \\ \vdots & \vdots & \end{array}$$

by induction

(9.1) $$d(x_r, x_{r-1}) \leq K^{r-1} d(x_1, x_0), \qquad \text{for all} \qquad r \geq 2$$

Now let $m > n \geq 1$, by repeated use of the triangle inequality and (9.1), we have that

$$\begin{array}{rcl} d(x_m, x_n) & \leq & d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ & \leq & (K^{m-1} + K^{m-2} + \ldots K^n) d(x_1, x_0) \\ & = & K^n (K^{m-n-1} + K^{m-n-2} + \ldots + 1) d(x_1, x_0) \\ & \leq & \dfrac{K^n}{1-K}\, d(x_1, x_0). \end{array}$$

Since $0 \leq K < 1$, $K^n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, $(x_n)_{n=1}^\infty$ is a Cauchy sequence. Since X is complete $(x_n)_{n=1}^\infty$ converges to some point $x^\star$ in X, that is there exists $x^\star \in X$ such that

(9.2) $$x_n \longrightarrow x^\star, \qquad \text{as} \qquad n \longrightarrow \infty.$$

Since f is a contraction, then f is continuous (see Proposition 9.2), and since $x_n \longrightarrow x^\star$ as $n \longrightarrow \infty$, by Lemma 9.3,

$$x_{n+1} = f(x_n) \longrightarrow f(x^\star) \qquad \text{as} \qquad n \longrightarrow \infty. \tag{9.3}$$

By the uniqueness of the limit of sequences in metric spaces (see Theorem 5.2), from (9.2) and (9.3), we conclude that

$$f(x^\star) = x^\star.$$

Uniqueness.

If $f(x^\star) = x^\star$ and $f(y^\star) = y^\star$, with $x^\star, y^\star \in X$ then

$$d(x^\star, y^\star) = d(f(x^\star), f(y^\star)) \leq K d(x^\star, y^\star).$$

Since $K < 1$, this is a contradiction unless $d(x^\star, y^\star) = 0$. Hence, by property (M1) in the definition of a metric, $x^\star = y^\star$, and the fixed point of f is unique. □

Remark. The condition that X is complete cannot be removed. Exercise in problem sheet:

Let $f : (0, 1/4) \longrightarrow (0, 1/4)$ defined by $f(x) = x^2$. Show that f is a contraction without a fixed point.

9.2. **Some applications. Examples.**

This subsection will be devoted to give some applications of the Contraction Mapping Theorem to the existence and uniqueness of solutions of algebraic, differential and integral equations.

Example 1. (Algebraic equations)

1. Use the Contraction Mapping Theorem to show that the equation

$$x = \frac{1}{7}(x^3 + x^2 + 1)$$

has precisely one solution in the interval $[0, 1]$.

Consider $[0, 1]$ with the standard metric.

First, notice that $[0, 1]$ is complete (since is a closed set of $\mathbb{R}$ with the standard metric, and $\mathbb{R}$ with the standard metric is complete).

Second, define the function f by

$$f(x) = \frac{1}{7}(x^3 + x^2 + 1), \qquad x \in [0, 1].$$

We have already seen that f is a contraction on $[0,1]$. By using the Contraction Mapping Theorem, we conclude that there exists a unique $x \in [0,1]$ such that

$$f(x) = x,$$

or, equivalently, there exists a unique $x \in [0,1]$ such that

$$x = \frac{1}{7}(x^3 + x^2 + 1).$$

2. let $K \in [1/2, 1)$. Use the Contraction Mapping Theorem to show that the equation

$$x = K\left(x + \frac{1}{x}\right)$$

has a unique solution in $[1,\infty)$. Consider $[1,\infty)$ with the standard metric.

First, observe that $[1,\infty)$ is a closed subset of $\mathbb{R}$, and $\mathbb{R}$ with the standard metric is a complete space, therefore $[1,\infty)$ is complete.

On the other hand, define the function f on $[1,\infty)$ by

$$f(x) = K\left(x + \frac{1}{x}\right), \qquad x \in [1,\infty).$$

It is easy to see that f is an increasing function on the interval $[1,\infty)$, so

$$f(x) \geq f(1) = 2K \geq 1, \qquad \forall x \in [1,\infty).$$

since we are assuming that $K \geq 1/2$. Thus, $f : [1,\infty) \longrightarrow [1,\infty)$.

On the other hand,

$$|f'(x)| = K\left(1 - \frac{1}{x^2}\right) < K, \qquad \forall\, x \in [1,\infty),$$

from which it follows that

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq K|x - y|$$

for some c between x and y, by using the Mean Value Theorem. Since by hypothesis $K < 1$, from the above inequality, it follows that f is a contraction on $[1,\infty)$.

Using The Contraction Mapping Theorem, from the above observations we conclude that there exists a unique $x^\star \in [1,\infty)$ such that

$$f(x^\star) = x^\star,$$

or, equivalently, there exists a unique solution $x^\star$ of the equation

$$x^\star = K\left(x^\star + \frac{1}{x^\star}\right)$$

in the interval $[1,\infty)$.

Example 2. (Existence and uniqueness of solutions to first order differential equations: Picard's Theorem)

Consider the initial value problem given by

$$\begin{cases} \dfrac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{9.4}$$

where F is continuous (real-valued) and

$$|F(t, y_1) - F(t, y_2)| \leq K\,|y_1 - y_2|, \qquad \text{for some} \qquad K < 1. \tag{9.5}$$

(that is, F is a contraction on the second variable)

We look for solutions y of (9.4) for $|t| \leq 1$, say.

By the Fundamental Theorem of Calculus (by integrating both sides of the differential equation in (9.4) between 0 and t, with $t \in [-1, 1]$, and using the initial condition $y(0) = y_0$), any solution of (9.4) is a solution of

$$y(t) = y_0 + \int_0^t F(s, y(s))\, ds.^{17} \tag{9.6}$$

Conversely, any continuous solution of (9.6) is a solution of (9.4)[18] Therefore, roughly speaking, solving the initial value problem (9.4) is equivalent to solving the integral equation (9.6).

Now, let us look for solutions to (9.6) within the metric space $\mathcal{C}([-1, 1], \mathbb{R})$ with the sup-metric.

We define a map T on $\mathcal{C}([-1, 1], \mathbb{R})$ in the following way: For $y \in \mathcal{C}([-1, 1], \mathbb{R})$, consider the function denoted by Ty defined by

$$(Ty)(t) = y_0 + \int_0^t F(s, y(s))\, ds. \tag{9.7}$$

We will continue to see that

i) $T : \mathcal{C}([-1, 1], \mathbb{R}) \longrightarrow \mathcal{C}([-1, 1], \mathbb{R})$, that is to say that if y is a continuous real-valued function defined on $[-1, 1]$, then the function Ty defined in (9.7) is also a continuous real-valued function defined on $[-1, 1]$.

ii) T is a contraction on $\mathcal{C}([-1, 1], \mathbb{R})$ (with the sup-metric).

---

[17] Duhamel's formula.

[18] Indeed, by the Fundamental Theorem of Calculus, if $y(t)$ is continuous solution of (9.6), then $\frac{dy}{dt}$ exists and it is equal to $F(t, y(t))$. Recall that F is continuous.

Indeed, given $y \in \mathcal{C}([-1,1],\mathbb{R})$, for fixed $t_0 \in [-1,1]$ we have that

$$\begin{aligned}
|(Ty)(t)-(Ty)(t_0)| &= \left|\left(y_0+\int_0^t F(s,y(s))\,ds\right)-\left(y_0+\int_0^{t_0} F(s,y(s))\,ds\right)\right| \\
&= \left|\int_t^{t_0} F(s,y(s))\,ds\right| \le |t-t_0| \sup_{s\in[-1,1]} |F(s,y(s))| \\
&= C|t-t_0|.
\end{aligned}$$

Since $C|t-t_0|\longrightarrow 0$, as $t\longrightarrow t_0$, from the above inequality we get that Ty is continuous at t_0, and since $t_0 \in [-1,1]$ is arbitrary, we conclude that $Ty \in \mathcal{C}([-1,1],\mathbb{R})$. Thus, $T : \mathcal{C}([-1,1],\mathbb{R})\longrightarrow\mathcal{C}([-1,1],\mathbb{R})$.

Next, we continue to see that T is a contraction on $\mathcal{C}([-1,1],\mathbb{R})$. Indeed, let y_1 and $y_2 \in \mathcal{C}([-1,1],\mathbb{R})$, we need to study

$$d(Ty_1,Ty_2) = \sup_{t\in[-1,1]} |(Ty_1)(t) - Ty_2(t)|.$$

Let $t \in [-1,1]$ (fixed but arbitrary). From the definition in (9.7), the fact that y_1 $y_2 \in \mathcal{C}([-1,1],\mathbb{R})$ and (9.5) , it follows that:

$$\begin{aligned}
|(Ty_1)(t)-(Ty_2)(t)| &= \left|\int_0^t [F(s,y_1(s))-F(s,y_2(s))]\,ds\right| \\
&\le K\int_0^t |y_1(s)-y_2(s)|\,ds \\
&\le K \sup_{s\in[-1,1]} |y_1(s)-y_2(s)|\,|t| = K\,|t|\,d(y_1,y_2) \\
&\le K\,d(y_1,y_2),
\end{aligned}$$

for some $K<1$, and by taking the supremum over all possible $t\in[-1,1]$ we get that

$$d(Ty_1,Ty_2) \le Kd(y_1,y_2).$$

Thus, T is a contraction on $\mathcal{C}([-1,1],\mathbb{R})$.

Since $\mathcal{C}([-1,1],\mathbb{R})$ is complete (with the sup-metric), from the above observations and using the Contraction Mapping Theorem, we conclude that there exists a unique $y \in \mathcal{C}([-1,1],\mathbb{R})$ such that

$$Ty = y$$

or, equivalently, a unique solution $y \in \mathcal{C}([-1,1],\mathbb{R})$ such that

$$y(t) = y_0 + \int_0^t F(s,y(s))\,ds, \qquad \forall\, t\in[-1,1].$$

Example 3. (Integral equations)

Use the Contraction Mapping Theorem to show that there exists a unique solution $f \in \mathcal{C}([0,1], \mathbb{R})$ of the integral equation

$$f(x) = x^2 + \frac{1}{2} \int_0^1 \cos(xt)\, f(t)\, dt, \qquad x \in [0,1]. \tag{9.8}$$

To simplify the notation, in what follows we will denote the space $\mathcal{C}([0,1], \mathbb{R})$ by X.

Define the map T on $X = \mathcal{C}([0,1], \mathbb{R})$ as follows: given $f \in X$, define Tf as the function on $[0,1]$ given by

$$(Tf)(x) = x^2 + \frac{1}{2} \int_0^1 \cos(xt)\, f(t)\, dt, \tag{9.9}$$

for any $x \in [0,1]$.

We will continue to show that

i) $T : X \longrightarrow X$, and
ii) $T : X \longrightarrow X$ is a contraction on X, where we consider in X the sup-metric.

i) Let $f \in X$. We want to show that the function Tf defined in (9.9) is a continuous function on $[0,1]$.

For any fixed $a \in [0,1]$, using the definition of Tf, we have that

$$\begin{aligned}
|(Tf)(x) - (Tf)(a)| &= \left| \left(x^2 + \frac{1}{2} \int_0^1 \cos(xt) f(t)\, dt \right) - \left(a^2 + \frac{1}{2} \int_0^1 \cos(at) f(t)\, dt \right) \right| \\
&= \left| (x^2 - a^2) + \frac{1}{2} \int_0^1 [\cos(xt) - \cos(at)] f(t)\, dt \right| \\
&\leq |x^2 - a^2| + \frac{1}{2} \int_0^1 |\cos(xt) - \cos(at)|\, |f(t)|\, dt \\
&\leq |x - a|(|x| + |a|) + \frac{1}{2} \int_0^1 |x - a|\, |t|\, |f(t)|\, dt \\
&\leq |x - a| \left(2 + \frac{1}{2} \int_0^1 |t|\, |f(t)|\, dt \right).
\end{aligned}$$

Here, we have used the Mean Value Theorem in obtaining the second of the above inequalities, and that x and a are point in $[0,1]$ in obtaining the third one.

Since $|x - a| \longrightarrow 0$ as $x \longrightarrow a$, from the above inequality it follows that Tf is continuous at a. Alternatively, defining the constant M as

$$M = 2 + \frac{1}{2} \int_0^1 |t|\, |f(t)|\, dt,$$

from the above inequality we have that: for any given $\varepsilon > 0$ there exists $\delta = \varepsilon / M$ (say) such that

$$|(Tf)(x) - (Tf)(a)| \leq M|x - a| < M\delta = \varepsilon.$$

whenever $|x - a| < \delta$ and $x \in [0,1]$. Thus Tf is continuous at a.

Since a is an arbitrary point in $[0,1]$ we get that Tf is continuous (on $[0,1]$) as required.

ii) Next, we will show that $T : X \longrightarrow X$ is a contraction on X with the sup-metric.

Given any f_1 and $f_2 \in X$, we need to study

$$d(Tf_1, Tf_2) = \sup_{x\in[0,1]} |(Tf_1)(x) - (Tf_2)(x)|,$$

and compare it with $d(f_1, f_2)$. To this end, fix $x \in [0,1]$. Then, from the definition on Tf in (9.9) it follows that

$$\begin{aligned} |(Tf_1)(x) - (Tf_2)(x)| &= \left| \frac{1}{2} \int_0^1 \cos(xt) \left(f_1(t) - f_2(t)\right) dt \right| \\ &\leq \frac{1}{2} \int_0^1 |\cos(xt)| \, |f_1(t) - f_2(t)| \, dt \\ &\leq \frac{1}{2} \int_0^1 |f_1(t) - f_2(t)| \, dt \\ &\leq \frac{1}{2} \sup_{x\in[0,1]} |f_1(t) - f_2(t)| \left(\int_0^1 dt \right) \\ &= \frac{1}{2} d(f_1, f_2). \end{aligned}$$

Taking the supremum over all points x in $[0,1]$, from the above chain of inequalities, it follows that

$$d(Tf_1, Tf_2) \leq \frac{1}{2} d(f_1, f_2), \quad \text{and} \quad \frac{1}{2} < 1.$$

Therefore T is a contraction on X[19] .

Finally, since $X = \mathcal{C}([0,1], \mathbb{R})$ with the sup-metric is a complete metric space, and we have just shown that $T : X \longrightarrow X$ is a contraction, by applying the Contraction Mapping Theorem we conclude that there exists a unique $f \in \mathcal{C}([0,1], \mathbb{R})$ such that

$$Tf = f, \qquad (\text{i.e } (Tf)(x) = f(x), \qquad \forall x \in [0,1])$$

that is, there exists a unique solution $f \in \mathcal{C}([0,1], \mathbb{R})$ of the integral equation (9.8).

[19]Notice that, since f_1, $f_2 \in X$, then $f_1 - f_2 \in X$, and therefore this function attains its bounds in $[0,1]$, since it is a continuous function. As a consequence we can take the supremum over all $x \in [0,1]$ of $|f_1(t) - f_2(t)|$ outside the integral to get the last of the above inequalities.

10. Compact metric spaces. Compact sets

10.1. Definition and properties.

Definition 10.1. *Let X be a metric space, and $A \subseteq X$.*

> *A is a compact set if every sequence in A has a subsequence converging to a point in A.*
> *In particular, X is a compact space if every sequence in X has a convergent subsequence.*

Examples.

1. Consider $X = \mathbb{R}$ with the usual metric. Then,
   - $[0, 1]$ is compact (see Heine-Borel theorem below).
   - $[0, \infty)$ is not compact (the sequence $\{1, 2, 3, 4, \ldots\}$ has no convergent subsequences. *Why?*)
   - $\mathbb{R}$ is not a compact space (same example as the one given in previous line).

2. Let (X, d) be any metric space, and $A \subseteq X$. If A is a finite set, then A is a compact set. (*Why?*).

A motivation for the definition of compactness.

Recall the following theorem from $2RCA$-Real and Complex Analysis: If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then f is bounded on $[a, b]$ and attains its bounds, that is there exist points x_1 and $x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2), \qquad \forall\, x \in [a, b]$$

Aim: We would like to generalize the above result to the more general setting of metric spaces.

Let (X, d) be a metric space and let $f : X \longrightarrow \mathbb{R}$ be a continuous function.

Question: Which properties of X ensure that f is bounded?

- Is it true for all X? Answer: No
  Example: Consider $X = \mathbb{R}$ and $f(x) = x^2, \quad \forall\, x \in \mathbb{R}$.
- What if X is bounded? Answer: No
  Example: Consider $X = (0, 1)$ and $f(x) = \frac{1}{x}, \quad \forall\, x \in (0, 1)$.
- Try to find $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in X$.
  If $K = 1$ does not work, then there exists $x_1 \in X$ such that $|f(x_1)| > 1$.
  If $K = 2$ does not work, then there exits $x_2 \in X$ such that $|f(x_2)| > 2$
  $\ldots$

In general, for each $n \in N$, if $K = n$ does not work, then there exists $x_n \in X$ such that $|f(x_n)| > n$. Thus, we get a sequence (x_n) in X such that

$$|f(x_n)| \longrightarrow \infty, \qquad \text{as} \qquad n \longrightarrow \infty. \tag{10.1}$$

Now suppose that the sequence (x_n) has a convergent subsequence, say (x_{n_i}) such that

$$x_{n_i} \longrightarrow x \in X, \qquad \text{as} \qquad i \longrightarrow \infty$$

(i.e. suppose that X is compact).

Since f is continuous, then (see Proposition 5.3 or Lemma 9.3)

$$f(x_{n_i}) \longrightarrow f(x), \qquad \text{as} \qquad i \longrightarrow \infty,$$

so, by the continuity of the absolute value function $|.|$,

$$|f(x_{n_i})| \longrightarrow |f(x)|, \qquad \text{as} \qquad i \longrightarrow \infty,$$

which gives a contradiction (see 10.1).

The above argument shows that if X is compact and $f : X \longrightarrow \mathbb{R}$ is continuous, then f is bounded[20]. We will see later on that if X is compact, and $f : X \longrightarrow \mathbb{R}$ is continuous, then f also attains its bounds in X (see Theorem 10.6).

In what follows, we will see some properties related to compact sets.

Proposition 10.2. *Let (X, d) be a metric space and let $A \subseteq X$.*

If A is compact, then A is complete.

Proof. Let (x_n) be a Cauchy sequence in A. Since A is compact, then there exists $a \in A$ and a subsequence (x_{n_i}) in A such that

$$x_{n_i} \longrightarrow a, \qquad \text{as} \qquad i \longrightarrow \infty. \tag{10.2}$$

Let $\varepsilon > 0$. From (10.2), there exists $I \in \mathbb{N}$ such that

$$d(x_{n_i}, a) < \frac{\varepsilon}{2} \qquad \forall\, i \geq I. \tag{10.3}$$

Since (x_{n_i}) is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{\varepsilon}{2}, \qquad \forall\, n, m \geq N. \tag{10.4}$$

Let $N' = \max(I, N)$. Then, if $n \geq N'$, from (10.3) and (10.4), we get that

$$\begin{aligned} d(x_n, a) \;\underbrace{\leq}_{(M3)}\; & \underbrace{d(x_n, x_{n_{N'}})}_{\text{since } n_{N'} \geq N' \geq N} + \underbrace{d(x_{n_{N'}}, a)}_{\text{since } N' \geq I} \\ < \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad \forall n \geq N', \end{aligned}$$

[20] A more detailed precise proof of this will be given later on.

so $x_n \longrightarrow a$, as $n \longrightarrow \infty$. Thus, every Cauchy sequence in A converges to a point in A. $\square$

The proposition below relates the properties of compactness and boundedness of a set. Recall that a set A of a metric space X is said to be *bounded* if there exists $x \in X$ and a K real such that

$$d(a, x) \leq K, \qquad \text{for all} \qquad a \in A.$$

Proposition 10.3. *Let (X, d) be a metric space, and let $A \subseteq X$.*

If A is compact, then A is bounded.

Proof. Choose $x \in X$ and suppose (by contradiction) that A is not bounded (i.e. $\forall K$ there exists $a \in A$ such that $d(x, a) > K$). Then, for each $n \in \mathbb{N}$ there exists $x_n \in A$ such that

$$d(x_n, x) > n \qquad \forall n \in \mathbb{N}. \tag{10.5}$$

Consider the sequence $(x_n)_{n=1}^{\infty} \subseteq A$. Since A is compact, there exists $a \in A$ and a subsequence (x_{n_i}) such that

$$x_{n_i} \longrightarrow a, \qquad \text{as} \qquad i \longrightarrow \infty,$$

so, in particular, (take $\varepsilon = 1$ in the definition of convergence of a sequence) there exists $I \in \mathbb{N}$ such that

$$d(x_{n_i}, a) < 1, \qquad \forall i \geq I.$$

Therefore,

$$d(x_{n_i}, x) \underbrace{\leq}_{(M3)} d(x_{n_i}, a) + d(a, x) < 1 + d(a, x), \qquad \forall i \geq I. \tag{10.6}$$

Choose $i \geq I$ such that $n_i > 1 + d(x, a)$, then from (10.5) and (10.6) it follows that

$$n_i \underbrace{<}_{(10.5)} d(x_{n_i}, x) \underbrace{<}_{(10.6)} 1 + d(a, x) < n_i,$$

which gives a contradiction. $\square$

Remark. Let (X, d) be a metric space and $A \subseteq X$. From Propositions 10.2, 10.3 and 8.4, we get that the following chain of implications hold true:

$$A \quad \text{compact} \quad \Rightarrow A \quad \text{complete} \quad \Rightarrow A \quad \text{closed}$$

and

$$A \quad \text{compact} \quad \Rightarrow A \quad \text{bounded},$$

so

$$\text{If } A \text{ is compact, then } A \text{ is closed and bounded.}$$

The converse of the above implication is not true in general. BUT if one considers $\mathbb{R}^n$ with the Euclidean metric, the converse is also true. The result is the so-called Heine-Borel Theorem, which gives a characterization of compact sets in $\mathbb{R}^n$ with the Euclidean/usual metric. More precisely, we have the following

Theorem 10.4. ***[Heine-Borel Theorem]*** *Let $\mathbb{R}^n$ with the Euclidean metric, and let $A \subseteq \mathbb{R}^n$. Then*

A is compact if and only if A is closed and bounded.

Proof.

$\Rightarrow$) See previous remark.

$\Leftarrow$) Assume[21] that $n = 1$.

Let A be a closed and bounded subset in $\mathbb{R}$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in A (we want to show that there exists a subsequence converging to a point in A). Notice that, since A is bounded, $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, thus[22] $(x_n)_{n=1}^{\infty}$ has a convergent subsequence. Moreover, since A is closed, the limit will be in A showing compactness. Here, we have used Proposition 6.7.

This concludes the proof. $\square$

10.2. **Compact sets and continuous functions.**

The aim of this section is to see how compact sets "related" to continuous functions. We will give the proof of two results.

Theorem 10.5. *Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \longrightarrow Y$ be a function and $A \subseteq X$.*

If A is compact and f is continuous, then $f(A)$ is a compact subset of Y.

In words: The above theorem states that the continuous image of a compact set is a compact set.

Recall that the image of $A \subseteq X$ under $f : X \longrightarrow Y$ is the set

$$f(A) = \{f(x) \ : \ x \in A\}.$$

Proof. Let (y_n) be a sequence in $f(A)$. Then, there exists $x_n \in A$ such that

$$y_n = f(x_n), \qquad \forall\, n \in \mathbb{N}. \tag{10.7}$$

Since $(x_n)_{n=1}^{\infty}$ is a sequence in A, and A is compact, then there exists $a \in A$ and a subsequence (x_{n_i}) such that

$$x_{n_i} \longrightarrow a, \qquad \text{as} \qquad i \longrightarrow \infty,$$

[21]The proof when $n \geq 2$ follows using similar arguments to the ones given in the proof when $n = 1$. Here, we will omit the details of the proof.

[22]Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

and since f is a continuous function (see Proposition 5.3 or Lemma 9.3)

$$y_{n_i} \underbrace{=}_{(10.7)} f(x_{n_i}) \longrightarrow f(a) \in f(A) \qquad \text{(since } a \in A\text{)}, \qquad \text{as} \qquad i \longrightarrow \infty.$$

We have shown that any sequence in $f(A)$ as a subsequence which converges to a point in $f(A)$, showing the compactness of $f(A)$. □

We will continue to see an example of application of Theorem 10.5.

Example. There exist <u>no</u> continuous function $f : [0,1] \longrightarrow (0,1)$ which is a surjection (that is $\forall\, y \in (0,1)$ there exists $x \in [0,1]$ such that $f(x) = y$, or equivalently $f([0,1]) = (0,1)$).

Indeed, if such a function f would exists, since $[0,1]$ is compact and f is continuous, by using Theorem 10.5, we get that $f([0,1])$ is compact. Since f is a surjection, we conclude that $f([0,1]) = (0,1)$ is compact which is a contradiction (notice that $(0,1)$ is not compact since is not closed + Heine-Borel Theorem).

We conclude this subsection stating and proving the result that motivated our discussion on compact sets.

Theorem 10.6. *Let (X,d) be a (non-empty) metric space, and $f : X \longrightarrow \underline{\mathbb{R}}$ be a <u>continuous</u> function.*

If X is compact, then f is bounded and attains its bounds.

Recall that a function $f : X \longrightarrow \mathbb{R}$ is bounded if $f(X) \subseteq \mathbb{R}$ has an upper and lower bound, that is there exists K_1 and K_2 real such that

$$K_1 \leq f(x) \leq K_2, \qquad \forall\, x \in X,$$

in this situation K_1 is said to be a lower bound for $f(X)$, and K_2 is said to be an upper bound for $f(X)$.

Proof. The proof of Theorem 10.6 can be divided into two parts, the first part consists in proving that f is bounded, and the second part consists in proving that f attains its bounds on X.

i) f is bounded.

We will argue by contradiction: Assume that $f(X)$ has no upper bound, then for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$f(x_n) > n. \tag{10.8}$$

Since X is compact, there exists $x \in X$ and a subsequence (x_{n_i}) such that

$$x_{n_i} \longrightarrow x, \qquad \text{as} \qquad i \longrightarrow \infty.$$

Since f is continuous (see Proposition 5.3 or Lemma 9.3)

$$f(x_{n_i})\longrightarrow f(x), \qquad \text{as} \qquad i\longrightarrow\infty,$$

and in particular (take $\varepsilon = 1$ in the definition of convergence of a sequence) there exists $I \in N$ such that

$$|f(x_{n_i}) - f(x)| < 1, \qquad \forall\, i \geq I.$$

Then, for all $i \geq I$, we have that

$$|f(x_{n_i})| \underbrace{\leq}_{(M3)} |f(x_{n_i}) - f(x)| + |f(x)| < 1 + |f(x)|. \tag{10.9}$$

Now, choose $i \geq I$ such that $n_i > 1 + |f(x)|$, then from (10.8) and (10.9) it follows that

$$n_i \underbrace{<}_{(10.8)} |f(x_{n_i})| \underbrace{<}_{(10.9)} 1 + |f(x)| < n_1,$$

which is a contradiction. Therefore, $f(X)$ has an upper bound.

Similarly, one can prove that there exists a lower bound for $f(X)$. Here, we omit the details.

ii) f attains its bounds on X, that is to say that there exists x_1 and x_2 in X such that

$$f(x_1) \leq f(x) \leq f(x_2), \qquad \text{for all} \qquad x \in X.$$

Now, consider the element[23]

$$y_2 = \sup_{x\in X} f(x).$$

We claim that: There exists $x_2 \in X$ such that

$$f(x_2) = y_2,$$

and, as a consequence, there exists $x_2 \in X$ such that $f(x) \leq f(x_2)$ for all $x \in X$.

To this end, observe that, since $y_2 = \sup_{x\in X} f(x)$ (least upper bound), for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$|f(x_n) - y_2| < \frac{1}{n},$$

and since $\frac{1}{n}\longrightarrow 0$, as $n\longrightarrow\infty$, from the above inequality it follows that

$$f(x_n)\longrightarrow y_2, \qquad \text{as} \qquad n\longrightarrow\infty. \tag{10.10}$$

Since X is compact, there exists $\underline{x_2 \in X}$ and (x_{n_i}) subsequence such that

$$x_{n_i}\longrightarrow x_2, \qquad \text{as} \qquad i\longrightarrow\infty.$$

[23] This element exists by the Completeness Axiom: Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.

Since f is continuous (see Lemma 9.3),

$$f(x_{n_i}) \longrightarrow f(x_2), \qquad \text{as} \qquad i \longrightarrow \infty. \tag{10.11}$$

By the uniqueness of the limit of sequences, from (10.10) and (10.11), it follows that

$$y_2 = f(x_2) \qquad \text{with} \quad x_2 \in X,$$

as required.

Similarly, one can prove the "corresponding" statement about the lower bound. More precisely, consider

$$y_1 = \inf_{x \in X} f(x) \qquad \text{(greatest lower bound)}.$$

Then, there exists $x_1 \in X$ such that

$$f(x_1) = y_1,$$

and, as a consequence, there exists $x_1 \in X$ such that $f(x_1) \leq f(x)$, for all $x \in X$.

This concludes the proof. □

10.3 **Loose remark** (Not examinable)

There exists another (equivalent in the setting of metric spaces) definition of compactness which works for general topological spaces.

In general topology, our definition (that is, Definition 10.1) will be referred to as *sequential compactness*.

In the framework of topological spaces, both definitions are not in general equivalent. However it can be proved that, in the setting of metric spaces, both definitions (sequential compactness and the definition of compactness in topological spaces) are equivalent.

To be more precise, we introduce the following terminology:

Definition 10.7. *Let (X, d) be a metric space, and $A \subseteq X$.*

(a) *A collection of subsets $\{U_\alpha\}_{\alpha \in I}$ (indexed by the set I) is said to be a cover of A iff*

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha.$$

(b) *If $\{U_\alpha\}_{\alpha \in I}$ is a cover of A, a subcollection of $\{U_\alpha\}_{\alpha \in I}$ which also covers A is said to be a subcover of $\{U_\alpha\}_{\alpha \in I}$.*

(c) *A collection of sets $\{U_\alpha\}_{\alpha \in I}$ is said to be an open cover of A iff it is a cover of A consisting of open sets in X, that is if*
   - *U_α is an open set, for all $\alpha \in I$, and*
   - *$A \subseteq \bigcup_{\alpha \in I} U_\alpha$.*

With the above terminology, we give the following definition of compactness.

Definition 10.8. *A subset A of a metric space is said to be compact iff every cover of A has a finite subcover.*[24]

Remark. It can be seen that Definition 10.1 and Definition 10.8 are equivalent in the setting of metric spaces.

[24] We point out that Definition 10.7 and Definition 10.8 make sense also in a general topological spaces.