Technology Choice, Spillovers, and the Concentration of R&D

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Abstract

The direction of innovation shapes both current technologies and future innovation opportunities, as firms acquire expertise and create public knowledge through discovery. But how do firms choose which technologies to develop, and why might they fail to exploit new technological paradigms? I study these questions in a tractable new model of directed innovation and firm dynamics, highlighting a novel connection between market structure, the direction of innovation, and economic growth: Expertise in a current technology gives incumbents a comparative advantage at innovating it relative to entrants, who instead favor a new technology with higher growth potential. Each firm's innovation decisions influence others through knowledge spillovers, which can inefficiently delay or prevent the emergence of the new technology. Concentrating R&D resources in a small number of firms can exacerbate this problem by amplifying the influence of incumbents, even though it accelerates growth in the absence of a technology choice. I provide empirical evidence for the theory using data on firm patenting and R&D expenditures, and I apply it to explain the historical development of mRNA vaccines and to draw implications for the rising concentration of R&D in artificial intelligence.

JEL Classification: L16, L25, O31, O33, O41

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1 Introduction

Rapid progress in artificial intelligence (AI) has sparked intense debate about how firms are developing this technology and whether governments should intervene. Two issues feature prominently: First, AI is a general-purpose technology that can be applied to a wide variety of tasks, from predicting the structure of proteins to personalizing online ads. Firms innovating AI must choose which applications to target, but there is broad concern that the resulting *directon of innovation* may be not be in society's best interest.[1] Second, many decisions about the direction of innovation in AI are made by just a handful of large, incumbent technology firms. These firms in particular may face different innovation incentives than society or smaller entrants, and they also control vast research and development (R&D) resources that lend them influence over the direction of innovation. Figure 1 shows that the *concentration of R&D* is a recent and growing macro phenomenon: In the past decade, the share of US R&D expenditures accounted for by the top five US public firms has nearly doubled to 22.5%. This trend coincides exactly with the rise of the "Big Five" technology firms to the top of the R&D rankings.

The issues above raise several fundamental questions about the relationship between market structure and innovation, all of which apply more broadly to many innovative industries:

(i) What incentives drive the innovation direction of incumbents, and how do they differ from the incentives faced by entrants or a welfare-minded social planner?

(ii) How do the innovation decisions of individual firms collectively determine the overall direction of innovation?

(iii) How does the growing concentration of R&D resources within large firms affect the direction of innovation?

The central argument of this paper is that our current answers to these questions are incomplete, because they focus primarily on how competition between firms or technologies affects innovation incentives. In doing so, they overlook how a more basic feature of the innovation process — the accumulation of knowledge — independently drives incumbents and entrants to innovate different technologies, while linking these decisions through spillovers to determine the aggregate direction of innovation. I develop a new model of directed innovation and firm dynamics to clarify this mechanism. The model reveals how the interaction between market structure and knowledge accumulation can drive innovation in a socially inefficient

[1] For example, see Acemoglu (2021), Brynjolfsson (2023), and un.org/sg/en/content/sg/statement/2024-01-17/secretary-generals-special-address-the-world-economic-forum-delivered.

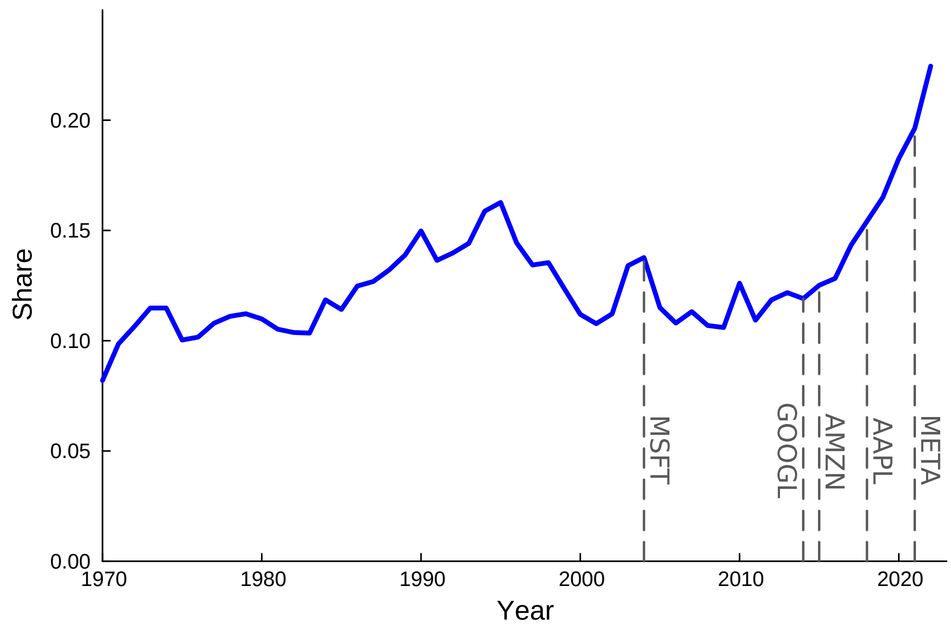


Figure 1: Share of US R&D Expenditures, Top 5 US Public Firms 1970-2022

Notes: Amazon (AMZN), Alphabet (GOOGL), Meta (META), Apple (AAPL), and Microsoft (MSFT) comprise the top five firms by R&D expenditures in 2022, in decreasing order. The vertical dashed lines mark the first year each firm appeared in the top five. All firms except for Microsoft remain in the top five after entry, while Microsoft has continuously appeared in the top five since 2011. Data on firm R&D expenditures come from Compustat North America, while data on aggregate US R&D expenditures come from the National Science Board.

direction, and it shows that the concentration of R&D across firms crucially mediates the collective technology choice. I provide evidence for the theory through an empirical analysis of firm patenting: Firms generally build on their own discoveries, and incumbents with greater patenting experience are slower to innovate emerging technologies. I also discuss a case study on the development of mRNA vaccines, where my theory offers a simple explanation of firm innovation decisions. This case demonstrates the advantages of decentralized R&D in promoting the exploration of new technologies, contrasting sharply with the current pattern of innovation in AI.[2]

I formulate the theory in an endogenous growth model with two key ingredients. First, there are *two technologies* used by a large number of firms to produce two different goods. Each firm can employ a team of scientists to innovate and raise its technology-specific qualities (or productivities). But each technology builds on a distinct body of knowledge, so the research useful for one technology is not useful for the other: Firms must choose *which* technology their

[2]See Klinger, Mateos-Garcia, and Stathoulopoulos (2020), Jurowetzki et al. (2021), and Ahmed, Wahed, and Thompson (2023) for suggestive evidence that growing concentration of AI research in corporations is driving falling diversity in research topics.

scientists will innovate. In this sense, the technologies represent not just production processes, but *technological paradigms* that structure how firms can produce and innovate (Dosi, 1982). The concentration of R&D across firms is controlled by the size of the team of scientists that each firm can manage. With the aggregate number of scientists fixed, larger team sizes allow for more rapid innovation but within fewer firms.

Second, innovations produce knowledge that accumulates within each technology, raising the productivity of future research efforts directed toward it. Some of this knowledge accumulates publicly and can be used by any firm (Romer, 1990), but crucially I assume that knowledge also accumulates *within the firm*, embodied in its qualities. For example, firms may specialize on particular research lines within technologies, or firms may work as a means to internalize knowledge spillovers between scientists. As a result, a firm's current research productivity for each technology depends disproportionately on its own past innovation decisions; this will imply that *the firm* per se plays an important role in the growth process. The assumption of knowledge accumulation within firms is consistent with recent evidence from innovation in the auto industry (Aghion et al., 2016), and I provide evidence across a broader range of industries in Section 5.

In the model, all firms initially innovate for an "old" technology, and incumbent firms exit at random and are replaced by new entrants. As in existing models of the firm-size distribution (e.g., Luttmer, 2007), the combination of innovation and stochastic entry and exit generates heterogeneity — incumbents with a long track record of innovation have higher qualities than recent entrants. Starting from the stationary distribution, I consider the unanticipated arrival of a "new" technology. The new technology allows for faster growth than the old technology in the long run, but with little accumulated knowledge it offers slower growth in the short run. For example, in the context of AI the old technology might represent integration with existing products like office productivity software or internet search, while the new technology might represent more radical applications like automated drug discovery or adaptive learning. At each time, firms must choose which technology to innovate so as to maximize their discounted value of profits. The bulk of the paper analyzes the resulting equilibrium innovation decisions.

The model yields four key theoretical insights. First, greater concentration of R&D always increases the economy's growth rate along its "steady states," or *balanced growth paths* (Proposition 1). This holds because all firms must innovate the same technology along a balanced growth path, and when knowledge accumulates within firms, greater concentration of R&D allows them to build more quickly on their past innovations. This result highlights an underappreciated but potentially important role played by *the firm* in economic growth: Larger firms can accelerate growth by focusing more R&D resources on their own research lines, catalyzing internal knowledge accumulation that makes their research even more productive in the future.[3] It also suggests an optimistic interpretation of Figure 1, that the rising concentration of R&D might simply allow the largest firms to innovate more quickly. But this finding only applies when all firms innovate the same technology, or in the context of AI target similar applications, which as noted above is often not an appropriate assumption.

To assess the robustness of this finding, I study the model's dynamic equilibria after the introduction of the new technology. The model's second main result demonstrates that innovation decisions are both *path-dependent* and *forward-looking* at the firm level, and that these forces arise because knowledge is technology-specific and accumulates within the firm (Proposition 2). Under these assumptions, firms have incentives to continue innovating a technology in which they have developed expertise, but they also consider whether to pursue faster long-run growth by innovating specifically for the new technology. The resulting trade-off is different for each firm: Incumbents with substantial expertise in the old technology choose to continue innovating for it, while younger firms and entrants instead embrace the new technology. In this way, the knowledge accumulation mechanism generates endogenous comparative advantage *in innovation* and predicts different innovation incentives for incumbents and entrants — large technology firms have natural incentives to incorporate AI into their existing products, even if other applications hold greater long-run promise.

This explanation for heterogeneous innovation incentives differs fundamentally from most work on the relationship between market structure and innovation. A large literature following Arrow (1962) studies how competition for monopoly rents can incentivize incumbents to innovate more or less than entrants, depending on whether innovation primarily cannibalizes incumbents' existing products or allows them to escape competition from entrants.[4] A distinct literature following Henderson and Clark (1990) emphasizes that incumbent firms may have an *absolute* disadvantage at innovating new technologies because of organizational rigidities.[5] As a result, the arrival of new technologies that compete with old ones can trigger the failure of established incumbents. The knowledge accumulation mechanism I identify is independent of these effects. I show this by deliberately constructing the model to rule out competition

[3]This result also offers a novel foundation for Schumpeter's (1942) claim that larger firms boost growth, but independent of his original justification that size enables greater surplus extraction in the goods market.

[4]The "replacement effect" is formalized by Arrow (1962) and Reinganum (1983), while the "escape-competition effect" is formalized by Gilbert and Newbery (1982), Aghion et al. (2001), and Aghion et al. (2005). Acemoglu and Cao (2015) explore the implications of the replacement effect for incremental and radical innovations, while Igami (2017) provides an empirical analysis of these issues in the hard drive disk industry.

[5]See also Henderson (1993) and Christensen (1997).

between firms or technologies in the goods market. I also assume that incumbents do not have an absolute disadvantage at innovating the new technology when it arrives.

The third main result of the model concerns the aggregation of firm innovation decisions, and it highlights my distinct focus on the *direction* of innovation chosen by firms instead of the *rate*. In equilibrium, a firm's innovation also generates technology-specific public knowledge that can be used by other firms to innovate in the future. For example, entrants start by drawing some knowledge for each technology from the frontier, while incumbents can similarly build on growing public knowledge when they innovate. These knowledge spillovers generate complementarities between firms, which imply that initial innovation decisions — and hence the initial *market structure* (firm-quality distribution) — can have a persistent effect on the aggregate direction of innovation. However, strategic complementarities across incumbents introduce substantial technical challenges, with complex equilibrium dynamics and the potential for multiple equilibria. In a technical contribution of my analysis, I resolve these difficulties by defining and characterizing the class of *monotone equilibria* (Proposition 3). These equilibria allow for rich interactions between firms, reflecting the patterns of innovation unveiled in the case study and the empirical analysis, but they can be tractably described as the solutions to a second-order dynamical system. I provide conditions for the existence of monotone equilibria and find that a transition to the new technology can be sustained if and only if the initial gap to the old technology is not too large (Proposition 4).

To develop additional intuition about the relationship between market structure and the direction of innovation, I consider the special case of the model without spillovers across incumbents. This case is particularly tractable, featuring a unique equilibrium with dynamics that can be solved in closed form. The final main result of the model shows that an increase in R&D concentration has two opposing effects on the equilibrium direction of innovation (Proposition 5): Greater concentration allows firms to innovate more rapidly, producing greater expertise for the old technology before the new technology arrives. As a result, more initial incumbents continue innovating for the old technology, and the resulting spillovers can sway future entrants away from the new technology. However, greater concentration also allows firms to innovate more rapidly for either technology; this particularly benefits the new technology because it allows for faster long-run growth. The former composition effect dominates the latter growth effect when the discount rate is sufficiently high, so that an increase in R&D concentration can induce lock-in to the old technology. Lock-in may be inefficient because of knowledge spillovers, and in general a social planner would always transition to the new technology more quickly than in equilibrium (Proposition 6).

This result demonstrates that R&D concentration can have critically different implications for growth when firms choose the direction of innovation. While it always raises growth within a technology (Proposition 1), it can also delay or prevent the emergence of a new one by amplifying the influence of incumbents that prefer the old technology. This second finding suggests an alternative interpretation to Figure 1: The rising concentration of R&D may drive lower diversity in research as firms follow in the footsteps of large incumbents, potentially *reducing* long-run growth as alternative directions remain unexplored. I provide a simple calibration of the model in Section 6 to show that this can hold for reasonable parameter values, drawn from existing work on firm dynamics and my own empirical analysis. The implications for innovation and antitrust policy are clear: Complementarities in firm innovation decisions provide an entirely new reason why industry consolidation might be harmful, completely independent of standard concerns about competition.

To provide evidence for the theory, I conduct a case study and an empirical analysis of US patent data. The case study traces the development of mRNA vaccine technology, which was critical to the social and economic recovery from the COVID-19 pandemic of 2020-2021. Historical accounts emphasize that small biotechnology firms were largely responsible for innovating this technology, while large pharmaceutical incumbents with expertise in conventional vaccines decided against it (Dolgin, 2021); these decisions are also reflected in the patent record. Competition- and organization-based theories of innovation incentives have difficulty explaining this pattern because conventional and mRNA vaccines are often not competing for applications. My theory instead offers a simple explanation based on firms' comparative advantage in innovation, and it rationalizes why several incumbent firms have recently begun to explore mRNA technology. This case also suggests the potential stakes of the concentration of R&D: Had incumbent pharmaceutical firms exercised greater control over R&D resources, we might have been left without a crucial tool to fight the COVID-19 pandemic.

To show that the lessons of the case study generalize, I study the innovation decisions made by firms using a panel dataset of US patents matched to US public firms over 1980-2021 (Arora et al. (2024)). I establish three facts consistent with the model. First, a firm's current patenting is highly correlated with its previous patenting, controlling for a variety of other firm-level determinants of innovation including R&D expenditures. This provides evidence that technological knowledge accumulates within firms, raising their research productivities. Second, after clustering patents according to the new technologies identified by Kalyani et al. (2023), I find that a firm's patenting for a technology is better predicted by previous patenting *within that technology* than patenting in general. This suggests that knowledge is not only cumulative within firms, but technology-specific – two critical assumptions of the theory. Finally, I show that incumbents with greater patenting experience innovate *less* for new technologies than less-experienced firms, though this gap shrinks as the technologies mature. This finding supports the theory's main prediction that experienced incumbents are reluctant to embrace new technologies, and it reflects the same pattern of incumbent innovation exhibited in the case study. The empirical analysis broadly supports both the assumptions and implications of the theory, and it also provides parameter estimates used to calibrate the quantitative example in Section 6.

Related Literature

This paper contributes most directly to an expansive literature on the relationship between market structure and innovation. In addition to the seminal contributions discussed above, recent work assesses how incumbents can attempt to block innovation by competitors through acquistions (Cunningham, Ederer, and Ma, 2021), defensive patenting (Argente et al., 2020), and pre-emptive hiring of inventors (Akcigit and Goldschlag, 2023). The knowledge accumulation mechanism I study works independently of competitive pressures, and hinges instead on how complementarities between firms lend incumbents influence over the direction of innovation. In a potentially surprising contrast to these studies, I find that incumbents can reduce long-run growth *precisely by innovating according to their expertise*, generating knowledge spillovers on other firms that raise the opportunity cost of exploring new technologies.

A related literature in macroeconomics explores how firms grow and compete through innovation. Most of the work in this literature is based on the path-breaking contribution of Klette and Kortum (2004), who develop a model of firm dynamics wherein firms grow by innovating improved versions of other firms' existing products. The baseline version of this model is explicitly designed to aggregate: Knowledge accumulates only at the product level, so that firm boundaries and market structure play no role in the innovation process. Two recent papers introduce assumptions that break this aggregation. Akcigit and Kerr (2018) assume that incumbent firms can conduct external innovation to improve other firms' products or internal innovation to improve their own, and they estimate the corresponding model to assess how the firm-size distribution affects economic growth. They do not consider an innovation choice between technological paradigms, so the issues of path dependence and complementarities that occupy my analysis do not arise. The most closely related paper to mine is a contemporaneous contribution by Aghion et al. (2024), who explore issues related to firm-level path dependence in a model with innovation for clean and dirty technologies. They build on the framework of Klette and Kortum (2004) and break aggregation by assuming that firms of different ages face different credit constraints. As a result, firm boundaries and market structure affect innovation only insofar as credit constraints bind, and they use their model to quantitatively explore how credit conditions can disproportionately affect green innovation. My framework instead builds on existing models of the firm-productivity distribution that imply an explicit role for *the firm* in accumulating knowledge (Luttmer, 2007; Lucas and Moll, 2014; Benhabib, Perla, and Tonetti, 2021). My analysis clarifies how the innovation process then generates a link between market structure and growth, mediated by the concentration of R&D across firms.

I also build on an older literature concerned with the role that technological paradigms and increasing returns to scale play in the development and adoption of new technologies (Nelson and Winter, 1982; Dosi, 1982; Katz and Shapiro, 1986; Farrell and Saloner, 1986; Arthur, 1989). To maintain tractability in the face of non-convexities, much of this literature focuses on settings with learning-by-doing or non-optimizing adoption decisions. I contribute by developing an analytically tractable model with forward-looking firms, better suited to assess how market structure and the concentration of R&D shape the direction of innovation. Finally, this work also relates to the literature on directed technological change, and particularly recent work related to the clean transition (Acemoglu, 2002; Acemoglu et al., 2012). This literature primarily focuses on how the demand for different goods mediates the direction of innovation. My focus is instead on features of the "supply side" of innovations, and in particular how the accumulation of knowledge through innovation links market structure to economic growth.

The rest of this paper is organized as follows: Section 2 presents the case study on mRNA vaccines. Section 3 sets up the baseline model of directed innovation and firm dynamics and characterizes its balanced growth paths. Section 4 analyzes the equilibria of the model when firms choose their direction of innovation. Section 5 describes the empirical analysis, while Section 6 uses these results to calibrate a simple quantitative example. Section 7 concludes.

2 Case Study: mRNA Vaccines

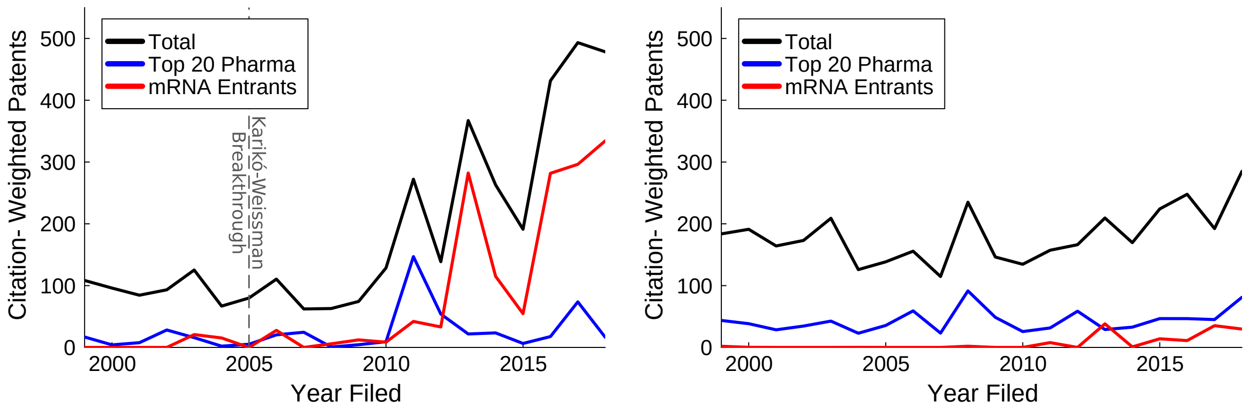
In this section, I provide a brief case study of the development of mRNA vaccines. This setting captures the basic features of the theory, with distinct old and new technologies and a clear difference in the direction of innovation chosen by incumbent and entrant firms. I argue that the competition- and organization-based theories of innovation incentives cannot explain this pattern, which is instead consistent with the knowledge accumulation mechanism outlined in my theory. This case also offers a sharp contrast to existing concerns about the concentration of R&D in AI: Pharmaceutical research is broadly decentralized across many firms, universities, and public and private research institutes. I argue that life-saving mRNA vaccines emerged precisely *because* of this decentralization, underscoring the substantial risks to concentrating R&D among a small number of firms in any industry.

2.1 Background

The spread of COVID-19 in early 2020 triggered one of the worst pandemics in a century and the sharpest economic contraction in the post-war era. Within a year, COVID-19 caused over half a million deaths in the United States, vaulting to third among the country's leading causes of mortality.[6] A combination of pandemic-driven uncertainty and strict public health interventions drove the unemployment rate in the United States to a high of nearly 15%, while real output fell by 7.5% in the second quarter of 2020 alone.[7] Given the severity of these initial impacts, the subsequent recovery is all the more remarkable: In just over three years, rapid growth in productivity, employment, and business creation returned the unemployment rate to its pre-pandemic level and real output to its pre-pandemic trend (de Soyres et al., 2024). Moreover, both the infection rate and the risk of hospitalization or death from COVID-19 have fallen dramatically from their peaks in 2021. A variety of policies and treatment advances supported these outcomes, but one critical innovation stands out: vaccines.

The race to develop vaccines for COVID-19 began at the onset of the pandemic, spearheaded by pharmaceutical companies with public backing through the US government's Operation Warp Speed program. Biotechnology firm Moderna and a joint venture between pharmaceutical giant Pfizer and biotechnology firm BioNTech produced the first two vaccines to receive US regulatory approval. These vaccines proved exceptionally effective, and they remain the most widely adopted COVID-19 vaccines in the United States.[8] Their underlying technology is particularly notable: Both vaccines are based on a novel technique for producing disease resistance that differs fundamentally from the mechanism used by conventional vaccines. Conventional vaccines stimulate immune resistance by confronting the immune system with a weakened version of a pathogen or one of its constituent proteins, both grown in a lab. The Moderna and Pfizer-BioNTech vaccines instead encode instructions for the production of a protein using a genetic material called "messenger RNA" (mRNA). *mRNA vaccines* ferry these instructions into human cells, where they are used to produce the protein en masse and stimulate immune resistance (Hedestam and Sandberg, 2023).

[6] See covid.cdc.gov/covid-data-tracker for COVID-19 statistics and Murphy et al. (2021) for causes of mortality.
[7] See fred.stlouisfed.org for all US macroeconomic statistics.
[8] See ourworldindata.org/grapher/covid-vaccine-doses-by-manufacturer?country= USA



(a) mRNA Therapies

(b) Conventional Vaccines

Figure 2: Citation-weighted Patents, 1999-2018

Notes: Figures 2(a) and 2(b) respectively include all mRNA therapy and conventional vaccine patents filed at the US Patent and Trademark Office (USPTO) over 1999-2018 and granted before 2023. I halt both figures in 2018 to mitigate truncation issues from the lag between patent filing and publication. The vertical dashed line in Figure 2(a) marks the year of the Karikó-Weissman breakthrough (Karikó et al., 2005). See Appendix D for full details about the data.

2.2 mRNA Innovation

The contrast between conventional and mRNA vaccines and the historical development of the latter offer an ideal illustration of my theory. This case features clearly identifiable old (conventional) and new (mRNA) technologies, both of which reflect the basic features of distinct *technological paradigms*: Conventional and mRNA vaccine technologies are not just individual products, but sets of techniques used to *produce* and *improve* a variety of products (vaccines). Each technology also required substantial innovation before a viable product could be produced. For example, the practice of conventional vaccination dates back over a millenium, but the production of vaccines for many diseases was not possible before the development of microbiology in the late 19th century (Kinch, 2018). By contrast, mRNA was first discovered only in 1961, and initial experiments suggesting its therapeutic potential were conducted in the early 1990s. The final breakthrough enabling the development of these vaccines came in experiments by Katalin Karikó and Drew Weissman in 2005, for which these researchers were awarded a Nobel Prize in 2023 (Karikó et al., 2005; Hedestam and Sandberg, 2023). As this discussion indicates, the bodies of knowledge on which conventional and mRNA vaccine technologies are built are largely distinct, so that a firm interested in innovating a new vaccine faces the important decision of *which* technology to research.

In the case of mRNA vaccines, incumbent and entrant firms made different choices. Figure

2 displays the number of patents related to mRNA therapies and conventional vaccines that were filed (and eventually granted) in the United States between 1999 and 2018.[9] To more accurately reflect its scientific contribution, each patent is weighted by the number of forward citations from future patents, controlling for the time horizon after publication. The black line in each figure represents the total number of patents filed each year, including patents assigned to individual researchers, firms, universities, public research organizations, and independent research institutes. The blue line represents the subset of patents assigned to the top twenty pharmaceutical companies by revenue in 2023, while the red line represents patents assigned to just four entrant firms whose founders played a key role in the development of mRNA technology: Moderna, BioNTech, the pharmaceutical firm CureVac, and the now-defunct biotech firm RNARx (Hedestam and Sandberg, 2023). All entrants were founded after 2000 specifically to commercialize mRNA therapies.

Figure 2 demonstrates two facts. First, a substantial share of research on mRNA therapies and conventional vaccines is performed outside of the identified incumbent and entrant firms, reflecting the broader decentralization of research in the pharmaceutical industry (Scott Morton and Kyle, 2011). Early contributions to mRNA technology were made primarily by researchers at universities and independent research institutes; Figure 2(a) shows that researchers outside of large pharmaceutical firms or the leading mRNA therapy firms remain influential. Second, the small set of entrant firms played a much bigger role in the innovation of mRNA therapies than the larger set of incumbent pharmaceutical firms. A recent historical account by Dolgin (2021) emphasizes that this was an active decision by many of these incumbents: "In the 1990s and for most of the 2000s, nearly every vaccine company that considered working on mRNA opted to invest its resources elswhere." This remained true even after the final technical barrier to mRNA therapy was resolved in 2005.[10] The entrant firms noted above were responsible not only for much of the foundational basic research on mRNA technology, but also the initial therapy development: CureVac conducted the first clinical trial for an mRNA vaccine in 2017, and Moderna followed quickly to test mRNA vaccines against the Zika virus and two strains of the avian influenza virus.[11] Throughout the time horizon, incumbent pharmaceutical firms remained more active in conventional vaccine innovation than in mRNA therapy innovation (Figure 2(b)).

[9] I consider a broader set of patents than just those for mRNA vaccines, because the underlying technology can also be applied to treat a variety of genetic conditions and cancers. See Appendix D for a full description of the patent data, citation weights, and procedure for identifying mRNA therapy and conventional vaccine patents.

[10] In a recent interview, Weissman recalled that after their 2005 breakthrough he "told [Karikó] our phones are going to ring off the hook. But nothing happened. We didn't get a single call." (Yu, 2021)

[11] See Alberer et al. (2017) and clinical trials NCT03014089, NCT03076385, and NCT03345043 at ClinicalTrials.gov.

Why were incumbents so reluctant to invest in mRNA technology, while entrants pushed it from the laboratory to pharmacy shelves? I argue that competition-based theories of innovation incentives cannot explain this pattern, because the mRNA and conventional vaccine technologies are often *not competing* for uses. Each type of vaccine has its own advantages and disadvantages that make it suitable for different applications. For example, mRNA vaccines are often faster to develop and easier to produce than conventional vaccines, but to maintain efficacy they must be transported and stored at below-freezing temperatures. This limits their use in many contexts, where conventional vaccines may be more suitable (Gote et al., 2023). Firms developing mRNA therapies have also focused on conditions without existing vaccines and/or treatments, including infectious diseases like HIV and avian flu, genetic diseases like cystic fibrosis, and a variety of cancers.[12] These considerations limit the extent to which mRNA therapies cannibalize or "steal business" from existing therapies, the standard explanation for why incumbents might be reluctant to innovate relative to entrants (Arrow, 1962; Reinganum, 1983). Moreover, in the wake of the technology's success during the COVID-19 pandemic, several incumbent pharmaceutical firms have begun developing mRNA therapies.[13] This observation sheds doubt on organizational theories that suggests incumbents did not innovate because they *could not* innovate (Henderson, 1993).

My theory instead offers a simple explanation in line with the quote from Dolgin (2021) above: Incumbent pharmaceutical firms had already developed expertise in other areas of research (including conventional vaccines), and they found it more valuable to continue innovating in those areas than funding early work on mRNA technology. With no such expertise, the entrants instead pursued research for a less developed technology with greater long-run promise. Only after learning from years' worth of innovation by entrants did the incumbent pharmaceutical firms begin seriously contributing to research on mRNA technology.

This pattern of innovation arises naturally in the model described in Section 3, but only if entrants have enough R&D resources to push the new technology forward. Otherwise, the entrants may be unable to make sufficient progress before it becomes more valuable for them to abandon the new technology and join the incumbents in innovating the old one. This observation highlights a key advantage to decentralized R&D ecosystems like that in the pharmaceutical industry: Although incumbents almost universally decided against funding mRNA therapy research, entrants were able to develop the technology to the point that they could distribute highly effective vaccines for COVID-19 within a year of the initial outbreak. This

[12] See the product pipelines for Moderna (modernatx.com/en-US/research/product-pipeline) and BioNTech (biontech.com/int/en/home/pipeline-and-products/pipeline.html).

[13] See the product pipelines for Merck (merck.com/research/product-pipeline) and Roche (roche.com/solutions/pipeline).

decentralization contrasts sharply with current trends in AI, where the “Big Five” technology firms from Figure 1 deploy the vast majority of R&D resources while hiring researchers from academia and startups at a growing rate (Ahmed, Wahed, and Thompson, 2023).[14] As this case study suggests, the associated risks may be substantial: Incumbents may leave socially valuable technologies undeveloped, and entrants may simply choose to join them instead of independently developing other technologies. The model developed in the next two sections describes why and when this might occur.

3 Model

This section describes the baseline model of directed innovation and firm dynamics. I set up the model and define an equilibrium in Section 3.1, I solve the “static block” of the model in Section 3.2, and I define and characterize balanced growth in Section 3.3.

3.1 Setup

Consumption. The economy is deterministic and exists in continuous time, populated by a mass $L > 0$ of *workers*, a mass $S > 0$ of *scientists*, and a mass $N \in (0, 1]$ of *entrepreneurs*. To focus attention on firm innovation decisions, I keep the demand side of the economy as simple as possible: All agents have linear preferences over a unique consumption good, with common discount rate $\rho > 0$. The economy admits a representative consumer who evaluates consumption streams $[C(t)]_t$ with the utility function

$$\int_0^\infty \exp(-\rho t)\, C(t) dt. \tag{1}$$

Workers inelastically supply one unit of labor at wage $w_L(t)$, while entrepreneurs and scientists own all firms and earn profits as described below. All agents can risklessly save and borrow against the total value of firms $\mathcal{A}(t)$ at the equilibrium interest rate $r(t)$. Given the path of interest rates $[r(t)]_t$, the representative consumer solves a standard consumption-savings problem to maximize her utility (1) subject to her budget constraint and transversality condition:

$$\dot{\mathcal{A}}(t) \leq w_L(t)L + r(t)\mathcal{A}(t) - C(t), \tag{2}$$

$$0 \leq \lim_{t\to\infty} \mathcal{A}(t) \exp\left(-\int_0^t r(s) ds\right). \tag{3}$$

[14] See also https://www.nytimes.com/2024/06/13/opinion/big-tech-ftc-ai.html.

I take the final (consumption) good as the numeraire. The main simplifying assumption here is that preferences are linear, so that in equilibrium the interest rate is fixed at $r(t)=\rho$. With less elastic intertemporal preferences, the interest rate would vary along a technological transition and introduce additional dynamics into firm innovation decisions. This general equilibrium effect is potentially interesting but obscures the more fundamental innovation incentives that are the focus of my analysis.

Production. A competitive firm produces the final good by combining labor supplied by workers with intermediates. Intermediates come in two types $\theta \in \{A,B\}$ and are supplied by a unit measure of monopolistic *incumbent* firms; I refer to each type θ as a *technology*. Each incumbent owns one intermediate for each technology θ with an endogenous and firm-specific *quality* $q_\theta(t)$. An incumbent is fully characterized by its vector of qualities $q \equiv (q_A, q_B)$ at each time, and I let $F(q,t)$ denote the distribution of qualities across incumbents at time t.

Given labor input $L(t)$ and intermediate inputs $x_A(q,t)$ and $x_B(q,t)$ from each firm with qualities q, final output is

$$Y(t) \equiv \frac{1}{1-\beta}\left(\int \sum_{\theta\in\{A,B\}} q_\theta^\beta x_\theta(q,t)^{1-\beta} dF(q,t)\right) L(t)^\beta. \tag{4}$$

The final producer chooses the inputs $L(t)$ and $[x_\theta(q,t)]_{\theta,q}$ at each time to maximize its profits, taking as given the wage for workers $w_L(t)$ and the intermediate prices $[p_\theta(q,t)]_{\theta,q}$. The wage $w_L(t)$ is set competitively to clear the market for production labor:

$$L(t) = L. \tag{5}$$

The intermediate prices $[p_\theta(q,t)]_{\theta,q}$ are chosen by firms to maximize profits as described below. Each unit of an intermediate is produced using $\gamma > 0$ units of final output. All remaining output is used for consumption, yielding the market-clearing condition

$$Y(t) = C(t) + \gamma \int \sum_{\theta\in\{A,B\}} x_\theta(q,t) dF(q,t). \tag{6}$$

This production structure is used frequently in models of endogenous growth with quality upgrading, though I introduce the distinction between two sets of intermediates $\theta \in \{A,B\}$.[15] Its key features are additive separability across intermediates in the final production function

[15] For example, see Howitt (1999) and Acemoglu, Aghion, and Zilibotti (2006).

(4) and the absence of any factor reallocation between intermediates. As a result, the final producer's demand curve for each intermediate is independent of the qualities and prices of all others, $x_\theta(q,t) = q_\theta p_\theta(q,t)^{-\frac{1}{\beta}} L$. The flow profits for firms inherit these properties, as each firm sets the prices of its intermediates $p_A(q,t)$ and $p_B(q,t)$ to maximize profits π given its qualities q:[16]

$$\pi(q) \equiv (q_A + q_B)\,\bar{\pi}, \quad \text{where} \quad \bar{\pi} \equiv \max_p (p-\gamma)\, p^{-\frac{1}{\beta}} L. \tag{7}$$

This expression demonstrates that the economy features no competition between firms or substitution between technologies that could affect innovation incentives. Competition is ruled out because each firm's profits do not depend on other firms' qualities, so firms will not innovate to steal business or preempt competition from rival firms.[17] Similarly, the profits earned by technology A intermediates do not depend on the qualities of technology B intermediates. This would be natural if technologies A and B instead produced distinct goods that were imperfectly substitutable in demand or production, as in the existing literature on directed innovation. Improvement in one technology would then trigger relative price and market size adjustments that affect the profits for the other.[18] I exclude these competition and "demand-pull" forces to focus instead on how the innovation process itself shapes firms' incentives.

Innovation and Firm Dynamics. Intermediate qualities are determined endogenously through innovation, entry, and exit. Each firm can raise the quality of its intermediates by employing scientists to conduct research. If a firm with qualities $q(t)$ employs $s_\theta(q(t),t)$ scientists to research technology θ at t, the quality of its intermediate evolves according to

$$\dot{q}_\theta(t) = [\lambda q_\theta(t) + \sigma_I K_\theta(t)]\, \eta_\theta s_\theta(q(t),t). \tag{8}$$

Here $\eta_\theta > 0$ denotes the basic productivity of research for technology θ. This productivity is augmented by the accumulation of knowledge that raises research productivity, represented by the term in brackets. A central feature of my theory is that knowledge accumulates through two channels: First, following an extensive literature on endogenous growth initiated by Romer

[16]Flow profits are also linear and symmetric in the qualities q. Linearity is essentially a normalization from the definition of q_θ, while symmetry ensures that firms have no reason to favor either technology based on assumed differences in the goods market.

[17]These incentives are discussed extensively in the literature on market structure, competition, and innovation. Arrow (1962) and Reinganum (1983) show that business stealing can reduce the incentives of high-quality firms to innovate relative to low-quality firms (the "replacement effect"), while Gilbert and Newbery (1982) and Aghion et al. (2005) demonstrate conditions under which the opposite holds (the "escape competition effect").

[18]These price and market size effects are described extensively in Acemoglu (1998, 2002), and they are applied to study the direction of innovation between clean and dirty technologies in Acemoglu et al. (2012).

(1990), I suppose that each firm can learn from innovations made by all others. This public knowledge is technology-specific, and it is embodied in each technology's *knowledge stock* $K_\theta(t) > 0$. The knowledge stock reflects, for example, all information about technology θ found in the patent data and scientific publications, or shared among scientists at conferences. Its initial value $K_\theta(0) > 0$ is exogenous, and it increases as firms innovate for θ:

$$\dot{K}_\theta(t) = \int \dot{q}_\theta(t) dF(q(t), t). \tag{9}$$

Let $K(t) \equiv (K_A(t), K_B(t))$ denote the vector of knowledge stocks at time t. Second, I make the novel assumption that knowledge also accumulates *within each firm*, embodied in its quality $q_\theta(t)$. This internal knowledge is again technology-specific, and it represents any information produced by the firm that disproportionately improves its own future research efforts. The parameters $\lambda, \sigma_I \geq 0$ control the extent to which firms draw on internal and public knowledge when innovating, respectively.

Several mechanisms could induce knowledge accumulation within firms. For example, intermediates for a technology θ could represent distinct research lines, and the knowledge generated within a research line may be more useful for future innovation than knowledge generated by others. A firm may work as a mechanism to coordinate scientists on a single research line and accelerate innovation. The case study of Section 2 reflects this interpretation: mRNA technology can be used to treat or prevent a variety of different diseases, including genetic conditions, cancer, and viral infections like COVID-19. Each application builds on and contributes to our general understanding of mRNA technology. But the technology must also be tailored to each case, and the leading mRNA firms have indeed specialized within particular applications.[19] Alternatively, work dating back to Marshall (1890) documents that innovation is spatially concentrated, suggesting that knowledge spillovers between researchers happen through direct communication.[20] A firm may employ researchers to facilitate communication or ensure physical proximity in a common office or lab, again catalyzing "internal" knowledge spillovers. Both of these mechanisms imply a role for the firm *per se* in the innovation process, rooted in the insights of Coase (1937) and Alchian and Demsetz (1972) that firms form to internalize externalities and coordinate complementary activities. This role for the firm is essential to the model, and I discuss its implications throughout the analysis below.

Consistent with these microfoundations, I assume that scientists are organized into teams man-

[19] Moderna has primarily focused on developing mRNA vaccines for contageous diseases, while BioNTech has instead pursued mRNA vaccines for cancer.

[20] For example, see Jaffe, Trajtenberg, and Henderson (1993) and

aged by entrepreneurs. A firm innovates by employing one entrepreneur and her team to research its intermediates. For simplicity, I assume that each entrepreneur manages the same number $s \equiv S/N$ of scientists, so all innovating firms have equally sized "R&D departments." However, a fraction $1-N$ of firms are unable to innovate at each time because entrepreneurs are in scarce supply. The team size s is a natural measure of the *concentration of scientists* across firms: Holding the total mass of scientists S fixed, an increase in s reduces the number of firms N that can innovate while raising the R&D resources available to any firm that still can. To reduce notation without any essential changes to the model, I assume that any firm unable to hire an entrepreneur must exit. The distribution $F(q,t)$ describes the qualities for active firms and is scaled by their total measure $N>0$.

In this baseline model, firm entry and exit are exogenous: Each incumbent receives an independent exit shock at rate $\delta > 0$, at which time it ceases production and is replaced by an entrant with initial qualities

$$q^E_\theta(t) \equiv \sigma_E K_\theta(t). \tag{10}$$

Here $\sigma_E \geq 0$ determines the strength of knowledge spillovers to entrants. The entrant immediately employs the entrepreneur and scientists corresponding to the exiting firm.

Before exit, I assume that all profits generated by an innovating firm accrue to its entrepreneur and scientists. The scientists are then allocated at each time to maximize the firm's value $V(q,t)$, taking as given its initial qualities q and the trajectories of the knowledge stocks $[K(t)]_t$ and the interest rate $[r(t)]_t$:

$$V(q,t) = \max_{[s_\theta(q(\tau),\tau)]_{\theta,\tau}} \int_t^\infty \exp\left[-\int_0^\tau (r(t')+\delta)dt'\right]\pi(q(\tau))d\tau, \tag{11}$$

where maximization is subject to the resource constraint $s_A(q,\tau)+s_B(q,\tau) \leq s$ and the quality evolution equation (8). Asset market clearing requires that the total demand for assets from the consumer equal the total value of incumbents at each time:

$$\mathcal{A}(t) = \int V(q,t)dF(q,t). \tag{12}$$

This specification of innovation, entry, and exit implies that the distribution $F(q,t)$ evolves according to the Kolmogorov forward equation (KFE)

$$\frac{\partial F(q,t)}{\partial t} = -\int \dot{q}_A(t) F\left(q_A, dq_B', t\right) - \int \dot{q}_B(t) F\left(dq_A', q_B, t\right) + \delta N \mathbb{1}_{q \geq q^E(t)} - \delta F(q,t). \tag{13}$$

The mass of incumbents with qualities below q declines as incumbents with technology A qualities q_A improve their A intermediates. The first term on the right side of (13) captures the corresponding loss of mass per unit of time, or *flux*, through the boundary $\left\{q' : q_A' = q_A\right\}$. The second term similarly captures the flux through the boundary $\left\{q' : q_B' = q_B\right\}$. The third term reflects the increase in mass from entry, while the last term gives the fall in mass from exit.

Definition 1. An *equilibrium* is a set of trajectories for total output $[Y(t)]_t$, consumption $[C(t)]_t$, assets $[\mathcal{A}(t)]_t$, labor demand $[L(t)]_t$, intermediate quantities $[x_\theta(q,t)]_{\theta,q,t}$, wages $[w_L(t)]_t$, intermediate prices $[p_\theta(q,t)]_{\theta,q,t}$, knowledge stocks $[K(t)]_t$, allocations of scientists $[s_\theta(q,t)]_{\theta,q,t}$, incumbent values $[V(q,t)]_{q,t}$, and the distribution of incumbents $[F(q,t)]_{q,t}$ such that

(i) the representative consumer chooses her consumption and asset holdings to maximize her utility (1) subject to her budget constraint (2) and transversality condition (3);

(ii) labor demand and intermediate quantities are chosen by the final producer to maximize profits, given input prices;

(iii) intermediate prices are chosen to maximize incumbent flow profits (7);

(iv) the markets for labor, goods, and assets clear (6, 5, 12);

(v) the knowledge stocks satisfy the evolution equation (9);

(vi) the incumbent value function satisfies (11), while the scientist allocation solves the corresponding maximization problem; and

(vii) the quality dsitribution satisfies the KFE (13).

I maintain several parameter restrictions throughout the analysis of equilibrium. I assume $\lambda + \sigma_I > 0$ so that innovation is possible, though either of the parameters λ and σ_I may equal zero. I also assume that spillovers to entrants are positive $\sigma_E > 0$, which ensures that the economy always generates long-run growth. Finally, I assume that the exit rate δ is sufficiently large that an initial cohort of incumbents cannot generate aggregate growth in the long run:

$$\delta > (\lambda + \sigma_I N) \max\{\eta_A, \eta_B\} s. \tag{14}$$

3.2 Equilibrium: Static Block

To simplify the characterization of equilibrium, I note that the quality distribution F and the knowledge stocks $K \equiv (K_A, K_B)$ are the state variables in this economy. With the exception of the incumbent value function V and the allocation of scientists $[s_\theta(q,t)]_{\theta,q,t}$, all remaining equilibrium variables are statically determined as a function of the state (F, K). In fact, only the *aggregate qualities* $Q \equiv (Q_A, Q_B)$ are needed to determine this "static block" of the economy, where the aggregate quality of technology θ is

$$Q_\theta(t) \equiv \int q_\theta dF(q,t).$$

Integration by parts reveals that the aggregate quality of technology θ increases with incumbent innovation but potentially declines as incumbents exit:[21]

$$\dot{Q}_\theta(t) = \dot{K}_\theta(t) + \delta \left[N q_\theta^E(t) - Q_\theta(t) \right]. \tag{15}$$

The following lemma characterizes the static block of the economy along with the equilibrium interest rate $r(t)$:

Lemma 1. *In equilibrium,*

(i) the interest rate is $r(t) = \rho$;

(ii) intermediate prices and the wage for workers are

$$p_\theta(q,t) = \frac{\gamma}{1-\beta},$$
$$w_L(t) = [Q_A(t) + Q_B(t)] \frac{\beta \bar{x}^{1-\beta} L^{-(1-\beta)}}{1-\beta};$$

(iii) production labor demand is $L(t) = L$;

[21] The convention in the endogenous growth literature is to identify the knowledge stock K_θ with the aggregate quality Q_θ. This literature generally does not consider entrants that may arrive with lower qualities than the average incumbent ($\sigma_E < 1$), which directly reduces aggregate quality. I introduce the distinction between K_θ and Q_θ so that public knowledge is not artificially reduced by firm entry and exit.

(iv) intermediate quantities and the corresponding flow profis are

$$x_\theta(q,t) = q_\theta \bar{x}, \quad \textit{where} \quad \bar{x} \equiv L\left(\frac{\gamma}{1-\beta}\right)^{-\frac{1}{\beta}},$$
$$\pi(q_\theta) = q_\theta \bar{\pi}, \quad \textit{where} \quad \bar{\pi} \equiv \beta L\left(\frac{\gamma}{1-\beta}\right)^{-\frac{1-\beta}{\beta}};$$

(v) total output and consumption are

$$Y(t) = [Q_A(t) + Q_B(t)]\,\bar{Y}, \quad \textit{where} \quad \bar{Y} \equiv \frac{\bar{x}^{1-\beta}L^\beta}{1-\beta},$$
$$C(t) = [Q_A(t) + Q_B(t)]\,\bar{C}, \quad \textit{where} \quad \bar{C} \equiv \bar{Y} - \gamma\bar{x}.$$

All proofs are found in Appendix A. The wage $w_L(t)$, total output $Y(t)$, and consumption $C(t)$ are all linear in the sum of the aggregate qualities $Q_A(t) + Q_B(t)$, again reflecting the absence of substitution between technologies A and B. As noted above, the fixed interest rate $r(t) = \rho$ aids tractability. In the remaining analysis, I focus on the "dynamic block" of the economy: firm innovation decisions and the resulting dynamics of the state (F, K).

3.3 Balanced Growth

Before studying technological transitions, I characterize the economy's "steady states," or *balanced growth paths*. Toward a definition, note first that the distribution of qualities F is never stationary in equilibrium: As firms innovate and contribute to the knowledge stocks K_θ, they improve the initial qualities for entrants $q_\theta^E(t) = \sigma_E K_\theta(t)$. This process repeats as entrants begin innovating, and it produces an upward shift in the distribution of qualities over time. We can attempt to stabilize the distribution by normalizing the qualities q_θ by the quality of an entrant $q_\theta^E(t)$. Define the *relative quality* of an intermediate by

$$z_\theta(t) \equiv \frac{q_\theta(t)}{q_\theta^E(t)}, \tag{16}$$

and denote a firm's vector of relative qualities by $z = (z_A, z_B)$. Let $H(z,t) \equiv F(zq^E(t), t)$ denote the corresponding distribution, where $zq^E(t)$ denotes the pointwise product $(z_A q_A^E(t), z_B q_B^E(t))$.

Definition 2. A *balanced growth path (BGP)* is an equilibrium in which

(i) K_θ and Q_θ grow at a constant rate $g_\theta^* \geq 0$ for each technology $\theta \in \{A, B\}$; and

(ii) the relative quality distribution is stationary: $H(z,t) = H^*(z)$ for all $t \geq 0$.

We can observe immediately that the economy admits at most three BGPs, only two of which are locally stable. There exists one stable BGP corresponding to each technology θ in which all firms exclusively innovate for θ. As a result, technology θ grows asymptotically while the other technology θ' permanently stagnates. Multiple BGPs arise because innovation for each technology features dynamic increasing returns: Innovation for technology θ produces knowledge externally (through K_θ) and within the firm (through q_θ) that raises the productivity of future research for θ. With positive spillovers to entrants $\sigma_E > 0$, any technology θ that develops a large enough lead in accumulated knowledge attracts all innovation by entrants, reinforcing its advantage. Note that without substitution between technologies A and B in demand or production, there are no relative price adjustments that could redirect innovation toward the lagging technology and ensure a unique "interior" BGP.

Whenever the two stable BGPs exist, there also exists a third BGP that features equal and positive growth rates for both technologies. All scientists must be evenly split across the technologies to sustain equal growth rates. However, this BGP is fragile to perturbations in which a majority of scientists are temporarily directed to just one of the two technologies, which can push the economy toward one of the stable "corner" BGPs. Finally, we can rule out BGPs in which the growth rates g_A^* and g_B^* are both positive but unequal: Eventually all entrants would choose to research the technology with the faster growth rate, and the growth rate of the other would be driven to zero.

I restrict attention to the economy's two stable BGPs. To characterize, suppose all firms permanently direct their scientists toward technology θ. The evolution of intermediate qualities is then completely mechanical:

$$\dot{q}_\theta(t) = [\lambda q_\theta(t) + \sigma_I K_\theta(t)]\,\eta_\theta s \quad \text{and} \quad \dot{q}_{\theta'}(t) = 0.$$

With no innovation directed toward θ', all relative qualities $z_{\theta'}$ remain fixed at one. The BGP is then summarized by the growth rate g_θ^* and the marginal distribution $H_\theta^*(z_\theta)$ of relative qualities for θ. We can use the evolution equation (9) for the knowledge stock K_θ to write the growth rate g_θ^* as a function of the distribution H_θ^*:

$$g_\theta^* = \frac{\dot{K}_\theta}{K_\theta} = \int [\lambda \sigma_E z_\theta + \sigma_I]\,\eta_\theta s\, dH_\theta^*(z_\theta). \tag{17}$$

The distribution H_θ^* is determined by the evolution of qualities through the KFE (13). To see this, let $F_\theta(q_\theta, t)$ denote the non-stationary marginal distribution of qualities for technology θ. This distribution satisfies the following one-dimensional version of the KFE:

$$\frac{\partial F_\theta(q_\theta, t))}{\partial t} = -[\lambda q_\theta + \sigma_I K_\theta(t)]\,\eta_\theta s f_\theta(q_\theta, t) + \delta\left[N\mathbb{1}_{q_\theta \geq q_\theta^E(t)} - F_\theta(q_\theta, t)\right].$$

Here $f_\theta(q_\theta, t) \equiv \partial F_\theta(q_\theta, t)/\partial q_\theta$ denotes the density of F_θ. Differentiating the identity $H_\theta^*(z_\theta) = F_\theta(z_\theta q_\theta^E(t), t)$ then yields a time-invariant differential equation for H_θ^*:

$$0 = -\left[(\lambda - g_\theta^*)z_\theta + \frac{\sigma_I}{\sigma_E}\right]h_\theta^*(z_\theta) + \delta N - H_\theta^*(z_\theta). \tag{18}$$

The solution to this differential equation gives the distribution H_θ^* as a function of the growth rate g_θ^*. Candidates for the BGP growth rate g_θ^* and stationary distribution H_θ^* must solve the system (17, 18).

To ensure that a solution exists and delivers finite values for firms and the consumer, I maintain the following parameter restrictions:

$$s > \sigma_E S, \tag{19}$$

$$\rho > -\frac{\delta - \lambda\eta_\theta s - \sigma_I \eta_\theta S}{2} + \sqrt{\left(\frac{\delta - \lambda\eta_\theta s - \sigma_I\eta_\theta S}{2}\right)^2 + (\lambda\sigma_E + \sigma_I)\,\delta\eta_\theta S}. \tag{20}$$

The first condition (19) ensures that spillovers to entrants are sufficiently small that an incumbent's quality $q_\theta(t)$ grows faster than the entrant quality $q_\theta^E(t)$. This assumption implies that relative qualities $z_\theta(t)$ are weakly above one, which is essential to obtain a non-degenerate stationary distribution H_θ^*. The second condition (20) ensures that the discount rate ρ is large enough that the consumer's discounted utility is finite in equilibrium. Together with the lower bound on the exit rate (14), it is also sufficient to ensure that firm values are finite.

Given these assumptions, the following proposition explicitly characterizes the economy's stable BGPs:

Proposition 1. *The economy has two locally stable BGPs, one for each technology θ. In the BGP for technology θ:*

(i) All scientists research technology θ.

(ii) The knowledge stock K_θ and aggregate quality Q_θ grow at rate

$$g_\theta^* = -\frac{\delta - \lambda\eta_\theta s - \sigma_I \eta_\theta S}{2} + \sqrt{\left(\frac{\delta - \lambda\eta_\theta s - \sigma_I \eta_\theta S}{2}\right)^2 + (\lambda\sigma_E + \sigma_I)\,\delta\eta_\theta S}.$$

Their ratio satisfies

$$\frac{Q_\theta(t)}{K_\theta(t)} = \frac{g_\theta^* + \delta N \sigma_E}{g_\theta^* + \delta} < 1.$$

(iii) The stationary distribution H_θ^* is a generalized Pareto disribution with location parameter 1, shape parameter $\varphi_\theta^* > 0$, and tail parameter $\xi_\theta^* \in (-\infty, 1)$, where

$$\varphi_\theta^* = \frac{\left(\lambda + \frac{\sigma_I}{\sigma_E}\right)\eta_\theta s - g_\theta^*}{\delta} \quad \textit{and} \quad \xi_\theta^* = \frac{\lambda\eta_\theta s - g_\theta^*}{\delta}.$$

Along the BGP for technology θ, all macroeconomic aggregates including the knowledge stock K_θ, aggregate quality Q_θ, total output Y, and consumption C grow at the common rate $g_\theta^* > 0$. This growth rate is naturally increasing in the extent of knowledge spillovers (σ_E, σ_I) and the productivity of research (λ, η_θ), and with fully endogenous growth, the model also features a scale effect whereby the growth rate is increasing in the total mass of scientists S. Since all innovation is undertaken by incumbents, the growth rate is declining in the exit rate δ.

The most interesting comparative static for g_θ^* concerns the concentration of scientists across incumbents, s. Provided that incumbents build to some extent on their own past advances ($\lambda > 0$), the growth rate strictly increases with the concentration of scientists. This observation showcases a foundational role for the firm *per se* in the growth process: If scientists build on the knowledge generated by others, and if these spillovers are particularly intense between scientists at the same firm, then larger firms can better catalyze spillovers and accelerate innovation. Any of the microfoundations discussed in Section 3.1 could account for this "theory of the firm," and Proposition 1 draws a sharp implication for balanced growth.

The second part of Proposition 1 shows that the BGP relative quality distribution falls into the familiar generalized Pareto class, with an explicit formula for the distribution function:

$$H_\theta(z_\theta)^* = \begin{cases} N\left[1 - \left(1 + \xi_\theta^* \frac{z_\theta - 1}{\varphi_\theta^*}\right)^{-\frac{1}{\xi_\theta^*}}\right] & \text{if } \xi_\theta^* \neq 0, \\ N\left[1 - \exp\left(-\frac{z_\theta - 1}{\varphi_\theta^*}\right)\right] & \text{if } \xi_\theta^* = 0. \end{cases}$$

This distribution has unbounded support $[1,\infty)$ if $\xi_\theta^* \geq 0$ and bounded support $[1, 1-\varphi_\theta^*/\xi_\theta^*]$ when $\xi_\theta^* < 0$. The shape and tail parameters of the distribution depend endogenously on the growth rate g_θ^* and the parameters of the innovation process. To understand this relationship, note that an intermediate's relative quality z_θ increases as the growth rate of its quality outpaces the aggregate growth rate g_θ^*:

$$\begin{aligned}\frac{\dot{z}_\theta}{z_\theta} &= \frac{\dot{q}_\theta}{q_\theta} - \frac{\dot{K}_\theta}{K_\theta} \\ &= \left[\lambda + \frac{\sigma_I}{\sigma_E} z_\theta^{-1}\right]\eta_\theta s - g_\theta^*.\end{aligned}$$

The numerator of the shape parameter φ_θ^* is equal to the growth rate of relative quality for an entrant ($z_\theta = 1$), while the numerator of the tail parameter ξ_θ^* is equal to the limiting growth rate as an incumbent's relative quality becomes large ($z_\theta \to \infty$). An increase in either parameter tends to reallocate mass toward larger relative qualities; the shape parameter primarily affects the "body" of the distribution (low z_θ), while the tail parameter naturally affects the tail (high z_θ). Hence any change in the model primitives that raises the growth rate of individual firms relative to the growth rate of the economy will produce a more skewed distribution. Holding fixed these growth rates, both parameters shrink (in absolute value) as the exit rate δ increases, which lowers the expected lifetime of firms. Because it plays a key role in the analysis of transitions to follow, the following corollary formally states the comparative static for the tail parameter ξ_θ^* with respect to the concentration of scientists s.

Corollary 1. *The tail parameter ξ_θ^* is strictly increasing in s if and only if $\lambda > 0$.*

I prove additional comparative statics for ξ_θ^* in Appendix A.

4 Equilibrium with Technology Choice

In this section, I analyze the economy's equilibrium when firms can choose to innovate for technology A or technology B. I suppose that initially only technology A is available, and that the economy is following the unique BGP for A when B arrives at $t = 0$. These technologies pose a simple trade-off: Technology B has the higher basic research productivity $\eta_B > \eta_A$, so it can support faster aggregate growth in the long run. But its initial knowledge stock $K_B(0) > 0$ may be lower than that for technology A, allowing slower growth in the short run. Initial incumbents at $t = 0$ are endowed with the entrant quality $q_B^E(0)$, so that no firm has any absolute advantage in innovating technology B to start.

I begin by characterizing the firm's problem in Section 4.1. Firm innovation decisions are generally *path-dependent* and *forward-looking*, and knowledge spillovers generate two kinds of strategic complementarities that critically shape (and complicate) these decisions. To resolve the resulting technical challenges, I study the class of *monotone equilibria* in Section 4.2. These equilibria exist under weak conditions, and they display the pattern of innovation observed in the case study of Section 2 and the empirical analysis of Section 5: Initial incumbents are reluctant to innovate for the new technology B, but they may begin doing so after entrants have made substantial progress. In Section 4.3 I discuss the benchmark case with no knowledge spillovers across incumbents ($\sigma_I = 0$), which features a unique, closed-form equilibrium. The concentration of scientists s has an ambiguous effect on the equilibrium direction of innovation, but it tends to slow or prevent a transition to technology B provided the discount rate ρ is sufficiently high. The social planner may not always transition to technology B *more often* than in equilibrium, but when this is optimal, the social planner always transitions *more quickly*.

4.1 The Firm's Problem

Consider a firm with initial qualities $q(t_0)$ at an some time $t_0 \geq 0$ — for example, an initial incumbent at $t_0 = 0$ or an entrant that arrives after the introduction of technology B. Given a trajectory for the knowledge stocks $[K(t)]_t$, the firm's problem (11) is to allocate its scientists across its intermediates at all times $t \geq t_0$ to maximize the present value of its profits. To simplify this infinite-horizon optimal control problem, I note first that it is without loss of generality to assume that the firm chooses a corner allocation $s_\theta(t) \in \{0, s\}$ at each time. This holds because the evolution equation (8) for $q_\theta(t)$ is linear in the mass of scientists $s_\theta(t)$, so at each time the firm optimally allocates all scientists to the technology with the higher marginal value of research effort. As a result, I can alternatively cast the firm's problem as the choice of an initial innovation direction $\theta_0 \in \{A, B\}$ and a sequence of stopping times $t_0 < T_1 \leq T_2 \leq \ldots$ at which the firm completely reverses its direction. The next proposition records this observation and provides a first-order necessary condition – the *smooth-pasting condition* – for interior stopping times.[22] To state the result, define the values

$$\begin{aligned} k_\theta(t) &\equiv \lambda q_\theta(t) + \sigma_I K_\theta(t), \\ \Psi_\theta(T) &\equiv \int_T^\infty \exp\left(-\int_T^t [\rho + \delta - \lambda \eta_\theta s_\theta(q(\tau), \tau)] \, d\tau \right) dt. \end{aligned}$$

[22]I adopt the terminology of Dixit (1993) for this optimality condition, though I do not emphasize its connection to the differentiability of the firm's value function at the stopping times.

Here $k_\theta(t)$ is the total knowledge of technology θ available to a firm with quality $q_\theta(t)$, including both internal knowledge embodied by $q_\theta(t)$ and external knowledge $K_\theta(t)$. $\Psi_\theta(T)$ gives the value of an incremental innovation for technology θ at time T. This value includes all discounted profits generated by the innovation, including those resulting from an increase in the productivity of future research for technology θ.

Proposition 2. *Given initial qualities $q(t_0)$ and a trajectory for the knowledge stocks $[K(t)]_t$, for any solution to the firm's problem (11) there exists an initial innovation direction $\theta_0 \in \{A, B\}$ and a sequence of stopping times $t_0 < T_1 \leq T_2 \leq \ldots$ such that the firm exclusively innovates for θ_0 at $t \in [t_0, T_1)$ and reverses its innovation direction at each stopping time T_k. Every interior stopping time $T_k \in (t_0, \infty)$ must satisfy the smooth-pasting condition*

$$k_B(T_k)\eta_B\Psi_B(T_k) = k_A(T_k)\eta_A\Psi_A(T_k). \tag{21}$$

This proposition allows us to isolate several new forces that influence firm innovation decisions, and it is worth discussing in detail. I first describe the firm's path-dependent and forward-looking motives before turning to a discussion of spillovers.

To build intuition, suppose first that the solution to the firm's problem features no direction changes (or equivalently that all stopping times T_k are infinite). This holds in equilibrium for σ_I sufficiently small relative to λ, and it implies that the firm's problem reduces to the choice of a permanent innovation direction at time t_0. Let $V^\theta(q(t_0), t_0)$ denote the value from innovating permanently for technology θ. Integrating the quality evolution equation (8) yields the explicit formula

$$\begin{aligned}\frac{V^\theta(q(t_0), t_0)}{\bar{\pi}} = &\ \frac{q_\theta(t_0) + q_{\theta'}(t_0)}{\rho + \delta} \\ &+ \frac{1}{\rho+\delta}\frac{\lambda\eta_\theta s}{\rho + \delta - \lambda\eta_\theta s} q_\theta(t_0) + \int_{t_0}^{\infty} \exp\left(-(\rho+\delta)t\right) \frac{\sigma_I \eta_\theta s K_\theta(t)}{\rho + \delta - \lambda\eta_\theta s} dt.\end{aligned} \tag{22}$$

The first term is the value of the firm's initial qualities $q(t_0)$, while the last two terms capture the additional value generated by innovation for technology θ. The second term gives the value of all innovation by the firm that builds on its own past advances, and it is naturally increasing in the intensity of internal knowledge accumulation λ. The third term gives the value of innovation that builds on external knowledge, and it is similarly increasing in σ_I.[23]

[23]These terms are additive because the quality evolution equation (8) is additively separable between internal and external knowledge.

The firm chooses to innovate for technology B instead of technology A if and only if B yields a higher value, $V^B(q_B(t_0), t_0) \geq V^A(q_A(t_0), t_0)$. The equation above above shows that when knowledge accumulates within firms ($\lambda > 0$), this decision is both *path-dependent* and *forward-looking* at the firm level. Path dependence arises because a firm's past innovation for technology θ raises its current research productivity for θ, incentivizing continued innovation in that direction. However, this mechanism also implies that the firm's choice of direction must be meaningfully forward-looking, because its future research productivities are similarly contingent on its current innovation decision. This incentivizes the firm to innovate for technology B specifically, because it allows for faster quality *growth* through cumulative innovation ($\eta_B > \eta_A$). These dual incentives push in the same direction when the firm has innovated for technology B in the past, but an initial incumbent with past innovation for A must trade them off when deciding its research direction. A key insight from the model is that the path dependence and forward-looking forces arise from the same assumption that knowledge is technology-specific and accumulates within the firm.

With stochastic entry and exit these features of the innovation process naturally generate heterogeneity in innovation incentives across firms. Initial incumbents have an *endogenous comparative advantage in innovating* technology A relative to entrants: By definition these firms have higher initial qualities $q_A(0)$ for technology A than entrants, but they share the same initial qualities $q_B^E(0)$ for technology B. The path dependence effect implies that the difference $V^B(q(0), 0) - V^A(q(0), 0)$ is decreasing in $q_A(0)$, so incumbents have an intrinsic incentive to continue innovating for technology A after B arrives. These incentives readily explain the initial innovation decisions made by firms in the mRNA vaccine case study of Section 2. Entrants opted to explore the potentially promising mRNA technology, while incumbent pharmaceutical firms with existing expertise "opted to invest [their] resources elsewhere" (Dolgin, 2021).

When firms instead draw only on the aggregate knowledge stocks K when innovating ($\lambda = 0$), firm heterogeneity has no impact on firm innovation incentives. I discuss this case in detail in Appendix C, but the intuition is straightforward: With $\lambda = 0$, the quality evolution equation (8) implies that each firm's research productivities are determined only by the aggregate knowledge stocks K. Since flow profits $\pi(q)$ are linear in qualities, the incremental value generated by innovation is also independent of the firm's qualities q. As a result, every firm simply innovates for the technology with the highest overall research productivity $\sigma_I K_\theta(t)\eta_\theta$ at each time t. The assumption of purely external knowledge accumulation is pervasive in the endogenous growth and directed innovation literatures, precluding any meaningful role for the firm (or the concentration of scientists s) in the growth process.[24]

[24]On a technical note, this is also the unique case of the model in which the firm's problem (11) is weakly

The expression for the firm's value (22) also clarifies how the two sources of spillovers impact the choice of direction. For an entrant at time t_0, its initial qualities are determined entirely by the aggregate knowledge stocks, $q^E_\theta(t_0) = \sigma_E K_\theta(t_0)$. When firm innovation decisions are path-dependent ($\lambda > 0$), past innovation for technology θ then raises the entrant's value from choosing technology θ. These "backward-looking" spillovers on entrants generate a strategic complementarity between past and current firms, translating path dependence at the firm level into path dependence economy-wide.[25] A second source of spillovers arises when $\sigma_I > 0$, so that firms draw on external knowledge when innovating. In this case, the firm has incentives to innovate for the technology with the larger current knowledge stock $K_\theta(t)$, generating a strategic complementarity between past and current firms and across all current firms. If in addition $\lambda > 0$, so that the firm's choice of direction is forward-looking, then the firm must account for the future *evolution* of the knowledge stocks: The expectation of high growth in the knowledge stock $K_\theta(t)$ may induce the firm to innovate for technology θ, even if its current research productivity is lower.

This second source of knowledge spillovers can severely complicate equilibrium in two ways. First, it may induce a firm innovating for technology θ to reverse its research direction if the other technology's knowledge stock $K_{\theta'}(t)$ is growing (or expected to grow) quickly enough. In this case, firms do not simply choose an initial research direction according to the value functions (22), but they must also determine the optimal stopping times T_k at which to reverse their research direction. The smooth-pasting condition (21) is a local optimality condition that must be satisfied at each stopping time. It ensures that the marginal values of research $k_\theta(T_k)\eta_\theta\Psi_\theta(T_k)$ are equalized across technologies A and B, so that the firm is indifferent to marginal changes in the stopping time T_k. As the Proposition indicates, for an arbitrary trajectory for the knowledge stocks $[K(t)]_t$ firms may reverse their innovation directions repeatedly.

A more serious issue relates to the role of expectations: With knowledge spillovers across firms ($\sigma_I > 0$) and knowledge accumulation within firms ($\lambda > 0$), firms are more likely to innovate for a technology θ if they expect its knowledge stock $K_\theta(t)$ to grow in the future. Since the knowledge stock grows more rapidly as more firms innovate for θ, these spillovers across firms can produce multiple equilibria, each supported by a self-fulfilling expectation that other firms will (or will not) innovate for a given technology θ in the future. This issue does not arise with spillovers to entrants, because the implied complementarities are purely

convex and can be solved using standard methods. With $\lambda > 0$, the dynamic increasing returns at the firm level render the firm's problem non-convex.

[25] This mechanism is responsible for the appearance of multiple BGPs in the model.

backward-looking.

4.2 Monotone Equilibria

When innovation decisions are both inherently forward-looking ($\lambda > 0$) and subject to knowledge spillovers across incumbents ($\sigma_I > 0$), the model features a dynamic strategic complementarity that renders a full characterization of all equilibria intractable. To simplify while maintaining the key economics, I make two restrictions. First, I consider only equilibria that converge to one of the economy's stable BGPs. This restriction excludes cyclical equilibria as well as equilibria that happen to converge to the economy's unstable BGP. Second, I consider only equilibria in which innovation decisions are appropriately monotone over time:

Definition 3. An equilibrium is *monotone* if $s_\theta(q(t), t)$ is monotone in t for every initial time $t_0 \geq 0$, where $q(t) = q^E(t)$ for $t \leq t_0$ and $q(t)$ evolves according to (8) for $t > t_0$.

To understand the definition, first set $t_0 = 0$. Monotonicity then implies that an initial incumbent reverses its innovation direction at most once. By Proposition 2, this reduces the incumbent's problem to a choice of an initial research direction and a *single* stopping time T. Instead setting $t_0 > 0$, monotonicity implies the same restriction on entrants, with one additional requirement: If at any time $t > 0$ the initial innovation direction for entrants reverses from technology θ to θ', all subsequent entrants permanently innovate for θ'. In this sense, the definition requires that innovation directions are monotone both within firms and across entrants over time.

I focus on monotone equilibria to maintain analytical tractability while allowing for knowledge spillovers across incumbents ($\sigma_I > 0$), which can generate realistic innovation dynamics. For example, we will see in Section 4.3 that in the absence of these spillovers firms never reverse their initial innovation directions. Spillovers to entrants still generate linkages across firms, but they cannot rationalize why, for example, incumbent pharmaceutical firms are now researching and developing new mRNA therapies. By contrast, spillovers across incumbents can induce a firm to alter its direction of innovation. Monotonicity simply ensures that this occurs at most once for each firm, and so works as a joint restriction on σ_I and the trajectories of the knowledge stocks $[K(t)]_t$. The empirical analysis in Section 5 also suggests that these spillovers are important to explain firm patenting behavior, and it shows that incumbent innovation for many new technologies is generally monotone in the sense of Definition 3.

The next proposition characterizes all monotone equilibria converging to the BGP for technology B, while I provide the analogous result for equilibria converging to the BGP for technology

A in Appendix B. To state the proposition, let

$$\kappa(t) \equiv \frac{K_B(t)}{K_A(t)} \tag{23}$$

denote the *aggregate knowledge ratio*, which plays a key role in the analysis below.

Proposition 3. *In any monotone equilibrium converging to the BGP for technology B, there exists a cutoff $\chi(t)$ such that a firm innovates for B if and only if*

$$\frac{k_B(t)}{k_A(t)} > \chi(t).$$

There exists a time $T_E \geq 0$ such that $\chi(t) = \kappa(t)$ for $t \in [0, T_E)$. For $t \in [T_E, \infty)$ the cutoff solves the differential equation

$$\frac{\dot{\chi}(t)}{\chi(t)} = \left(\sigma_I \frac{\dot{K}_B(t) - \chi(t)\dot{K}_A(t)}{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)} - \lambda\eta_A s\right) \mathbb{1}\left[\chi(t) \geq \frac{\eta_A}{\eta_B}\frac{\rho + \delta - \lambda\eta_B s}{\rho + \delta}\right]. \tag{24}$$

The equilibrium is unique (in the class of monotone equilibria) up to the parameters T_E and $\chi(T_E)$, which satisfy the complementary slackness condition

$$T_E\,[\chi(T_E) - \kappa(T_E)] = 0. \tag{25}$$

The knowledge stocks $K(t)$ are the solutions to the dynamical system (B9).

Any monotone equilibrium can be characterized by a simple cutoff function $\chi(t)$ for the *firm knowledge ratio* $\frac{k_B(t)}{k_A(t)}$, reflecting the dynamic sorting of firms to technologies based on comparative advantage in innovation. Determining this cutoff is not straightforward because firms do not make myopic innovation decisions based only on their current knowledge ratios $\frac{k_B(t)}{k_A(t)}$. But monotonicity and convergence to B imply that all entrants after some time T_E exclusively innovate for technology B. The cutoff $\chi(t)$ then matches the aggregate knowledge ratio before T_E but decreases with the knowledge ratio $\frac{k_B(t)}{k_A(t)}$ of the last firm to innovate for A. It remains constant when it reaches the level identified by the smooth pasting condition (21) as the value of the knowledge ratio $\frac{k_B(t)}{k_A(t)}$ at which a firm innovating for A reverses to B.

Proposition 3 also demonstrates that both the cutoff $\chi(t)$ and the knowledge stocks $K(t)$ can be recovered by integrating a second-order dynamical system, dramatically simplifying the computation of equilibria. Moreover, monotone equilibria are unique up to either the initial condition $\chi(0)$ or the transition time for entrants T_E. A monotone equilibrium with $T_E > 0$ is essentially a "translation" of a monotone equilibrium with $T_E = 0$, so I restrict to $T_E = 0$ and

$\chi(0) \geq \kappa(0)$ in what follows.

As the discussion before Proposition 3 clarifies, monotonicity disciplines equilibrium by ensuring that firms do not reverse their research directions more than once. This may not hold for an arbitrary equilibrium converging to the BGP for technology B, because the knowledge stock $K_B(t)$ for technology B may initially decline relative to the knowledge stock $K_A(t)$ for technology A. With strong enough spillovers across incumbents, these dynamics could induce an incumbent to start innovating for B, reverse to A as $\kappa(t)$ falls, and reverse back to B as $\kappa(t)$ diverges asymptotically. To rule out this effect, in Appendix B I derive a condition on primitives and the trajectory of the knowledge stocks $[K(t)]_t$ to ensure that any firm innovating for B can never satisfy the optimality conditions necessary for a reversal to A. The following proposition provides a weaker sufficient condition, exclusively involving primitives, under which monotone equilibria converging to each technology's BGP exist.

Proposition 4. *Suppose*

$$\lambda \eta_A s \geq \frac{\sigma_I}{\lambda \sigma_E + \sigma_I} g_B^*.$$

Then there exist thresholds $\kappa_B^* \leq \kappa_A^*$ such that:

(i) A monotone equilibrium converging to B exists if and only if $\kappa(0) \geq \kappa_B^*$.

(ii) A monotone equilibrium converging to A exists if and only if $\kappa(0) \leq \kappa_A^*$.

The thresholds satisfy $\kappa_B^* < \kappa_A^*$ if and only if $\lambda > 0$ and $\sigma_I > 0$.

This result shows that an equilibrium converging to the BGP for technology θ can only exist provided that the initial advantage for θ' is not too large. Moreover, multiple equilibria can arise for intermediate values of the aggregate knowledge ratio $\kappa(t)$ when the firm's problem (11) is dynamic ($\lambda > 0$) and there are knowledge spillovers across incumbents ($\sigma_I > 0$). As the discussion above indicates, these multiple equilibria are supported by different expectations about other firms' future innovation decisions.

4.3 Benchmark: $\sigma_I = 0$

To develop additional intuition, I consider the benchmark case with no spillovers across incumbents ($\sigma_I = 0$). This case provides a simple illustration of equilibrium dynamics, and it provides several new insights about how market structure and the concentration of R&D affect the aggregate direction of innovation. I sketch the equilibrium derivations below, describing firm innovation decisions, their aggregation, comparative dynamics for the concentration of scientists s, and potential inefficiencies. I provide full details in Appendix C.

Firm Innovation. The case with $\sigma_I = 0$ is particularly tractable because all spillovers between firms are "backward-looking," so that each firm is unaffected by other firms' current and future innovation decisions. With dynamic increasing returns to innovation at the firm level, this conveniently implies that each firm makes an initial decision of its innovation direction with no reversals.

Recall the value $V^\theta(q(t_0), t_0)$ of innovating permanently for technology θ from (22). Rearranging the inequality $V^B(q(t_0), t_0) \geq V^A(q(t_0), t_0)$, we find that a firm with initial qualities $q(t_0)$ innovates for technology B if and only if

$$\frac{\eta_B}{\rho + \delta - \lambda \eta_B s} q_B(t_0) \geq \frac{\eta_A}{\rho + \delta - \lambda \eta_A s} q_A(t_0). \tag{26}$$

We can apply this observation to conveniently characterize entrant and incumbent innovation decisions. First consider an entrant at $t_0 \geq 0$. Substituting the entrant qualities $q^E_\theta(t_0) = \sigma_E K_\theta(t_0)$ into the inequality (26), we can rearrange to find that it innovates for technology B when the aggregate knowledge ratio $\kappa(t)$ is above the *entry threshold*

$$\kappa^E \equiv \frac{\eta_A}{\rho + \delta - \lambda \eta_A s} \left(\frac{\eta_B}{\rho + \delta - \lambda \eta_B s} \right)^{-1} \in (0, 1).$$

Now consider an initial incumbent at $t_0 = 0$ with qualities $q_A(0) = z_A(0) q^E_A(0)$ and $q_B(0) = q^E_B(0)$. Substituting these qualities into (26), we similarly find that the incumbent permanently innovates for technology B when its relative quality $z_A(0)$ is below the cutoff

$$z^*_{A0} \equiv \frac{\kappa(0)}{\kappa^E}$$

This cutoff is inversely related to the initial value of the cutoff function $\chi(t)$ described by Proposition 3, which in this case satisfies $\chi(0) = \kappa^E$. That is, the equilibrium in this case is monotone, and with $\chi(0)$ pinned down it must be unique. Since firms never reverse their innovation directions in equilibrium, there is no need to track the full path of the cutoff function $\chi(t)$, and I can focus on z^*_{A0} in the analysis below.

Aggregation. Using this characterization of innovation decisions, we can directly derive the dynamical system described by Proposition 3 for the evolution of the knowledge stocks K. Suppose $\kappa(0) > \kappa^E$, and consider technology A. For t near zero, all innovation for A is conducted by initial incumbents that chose not to innovate for B. The quality q_A for any such incumbent grows at rate $\lambda\eta_A s$, but these incumbents exit at rate δ. Differentiating the evolution equation (9) for K_A then yields

$$\ddot{K}_A(t) = -(\delta - \lambda\eta_A s)\dot{K}_A(t). \tag{27}$$

With the assumed lower bound on the exit rate (14), this equation implies that the knowledge stock K_A increases more slowly over time as initial incumbents exit and entrants instead choose to innovate for technology B. The initial conditions for this differential equation are the initial knowledge stock $K_A(0)$ and the initial growth rate $\dot{K}_A(0)/K_A(0)$, which is decreasing in the cutoff z^*_{A0}:

$$\frac{\dot{K}_A(0)}{K_A(0)} = \lambda\eta_A s\sigma_E \int_{z^*_{A0}}^{\infty} z_A dH^*_A(z_A). \tag{28}$$

Now consider technology B. For t near zero, entrants as well as initial incumbents with $z_A(0) \leq z^*_{A0}$ innovate for technology B. The quality q_B for any such firm grows at rate $\lambda\eta_B s$, but firms exit at rate δ. Exiting firms are replaced by a mass δN of entrants with initial quality $\sigma_E K_B(t)$, so differentiating the evolution equation (9) for K_B yields

$$\ddot{K}_B(t) = -(\delta - \lambda\eta_B s)\dot{K}_B(t) + \lambda\eta_B s\delta N\sigma_E K_B(t). \tag{29}$$

The initial conditions for this differential equation are the initial knowledge stock $K_B(0)$ and the initial growth rate $\dot{K}_B(0)$, which is increasing in the cutoff z^*_{A0}:

$$\frac{\dot{K}_B(0)}{K_B(0)} = \lambda\eta_B s\sigma_E H^*_A(z^*_{A0}). \tag{30}$$

The equations (27, 29) form a second-order, autonomous, linear system of differential equations for the evolution of the key state variables K (while entrants innovate for B). This is a natural kind of dynamical system to describe the evolution of an economy undergoing an endogenous technological transition: As Jones (1995) discusses in detail, all models of endogenous *balanced* growth rely on a *first*-order, autonomous, linear differential equation of the form $\dot{X}(t) = gX(t)$ to generate exponential growth, where X is an appropriately-defined knowledge stock or productivity variable.[26] But along an endogenous technological *transition*, the growth rates of the old and new technologies must adjust to accommodate the rise of

[26]For example, the present model satisfies this relation for $X = K_\theta$ along each stable BGP.

the new technology. A *second*-order linear system is perhaps the simplest way to describe the evolution of state variables with smoothly changing growth rates.

I show in Appendix C that the system (27, 29) can be integrated in closed form. The solution has two key implications: The aggregate knowledge ratio $\kappa(t)$ depends on initial conditions only through its initial value $\kappa(0)$, in which it is strictly increasing; and $\kappa(t)$ is "U-shaped" over time.[27] These properties are useful because the system (27, 29) only describes the dynamics of the knowledge stocks $K(t)$ while entrants continue to research technology B, $\kappa(t) \geq \kappa^E$. If this condition is ever violated, the economy fails to transition in aggregate to technology B and instead converges back to the BGP for technology A. Thus there exists a threshold κ^* such that the economy converges to the BGP for technology B if and only if $\kappa(0) \geq \kappa^*$. Note that this threshold characterization of the aggregate direction of innovation reflects Proposition 4 for the case in which the equilibrium is unique.

Equilibrium Transition. The threshold κ^* determines the economy's propensity to transition to technology B in equilibrium, and it depends richly on model primitives. To gain intuition for the key forces, note that the analysis above implies that a simple sufficient condition for the economy to transition is for the initial growth rate of K_B to dominate the initial growth rate of K_A. By (28, 30), this holds if and only if

$$\eta_B H_A^*\left(z_{A0}^*\right) \geq \eta_A \int_{z_{A0}^*}^{\infty} z_A dH_A^*(z_A). \tag{31}$$

This inequality depends on the initial aggregate knowledge ratio $\kappa(0)$ only through the cutoff z_{A0}^*, which is strictly increasing in $\kappa(0)$. As $\kappa(0)$ rises, the left side increases as a larger mass of initial incumbents innovate for B, raising the initial growth rate of K_B. The right side instead decreases as fewer initial incumbents innovate for A, lowering the initial growth rate of K_A. There exists a unique value $\bar{\kappa}$ at which the inequality (31) binds:

$$1 = \frac{\eta_A}{\eta_B} \frac{1}{1-\xi_A^*} \frac{\frac{\bar{\kappa}}{\kappa^E}}{\left(\frac{\bar{\kappa}}{\kappa^E}\right)^{1/\xi_A^*} - 1}. \tag{32}$$

Here I have substituted $z_{A0}^* = \frac{\kappa(0)}{\kappa^E}$ and the BGP relative quality distribution H_A^* from Proposition 1, which is a standard Pareto distribution when $\sigma_I = 0$.

The right side of equation (32) is strictly increasing in ξ_A^* and κ^E and strictly decreasing in $\tilde{\kappa}$,

[27] The latter holds because exit and entry are sluggish, so that technology A can continue improving relative to technology B until enough entrants have begun innovating for B.

and it delivers a simple but powerful intuition about the drivers of a technological transition: Any change that thickens the tail of the old technology's firm-quality distribution slows the transition, because it raises both the relative mass of incumbents who choose not to transition and their initial innovation rates. Both effects increase their collective influence over the aggregate direction of innovation, which may be decisive if it induces entrants to switch back to innovating for technology A. However, any change that raises incentives for new firms to innovate for the new technology instead accelerates the transition by raising the relative mass of incumbents who choose to transition. The tail parameter ξ_A^* and the entry threshold κ^E respectively capture these "composition" and "growth" forces, but they depend on many of the same model primitives. The following proposition provides an explicit comparative static for $\bar{\kappa}$ with respect to the concentration of scientists s:

Proposition 5. *There exists a discount rate $\bar{\rho} \geq 0$ that depends on model primitives such that $\bar{\kappa}$ is strictly increasing (decreasing) in s locally if and only if ρ is larger (smaller) than $\bar{\rho}$.*

The trade-off between the composition and growth effects hinges on the discount rate ρ. When ρ is relatively high, firms are relatively myopic, and the increase in the growth rate of technology B with s has little effect on firm incentives. The composition effect then dominates, and the increase in the concentration of scientists can delay or prevent a transition to technology B. In Section 6, I provide a simple calibrated example to show that an increase in the concentration of scientists can raise the thresholds $\bar{\kappa}$ and κ^* for reasonable values of ρ.

To show that this may be inefficient, I suppose that a social planner instead chooses the initial cutoff z_{A0} to maximize the representative consumer's utility (1) given the initial knowledge stocks $K(0)$, with all innovation decisions by entrants as in equilibrium. The following proposition characterizes properties of the solution $\hat{z}_{A0}$:

Proposition 6. *A solution $\hat{z}_{A0}$ to the social planner's problem exists and depends on $K(0)$ only through $\kappa(0)$. There exists a threshold $\hat{\kappa}$ such that*

(i) the solution $\hat{z}_{A0}$ yields a transition to technology B if and only if $\kappa(0) \geq \hat{\kappa}$;

(ii) $\hat{z}_{A0} > z_{A0}^*$ if $\kappa(0) \geq \hat{\kappa}$; and

(iii) $\hat{z}_{A0} \leq z_{A0}^*$ if $\kappa(0) < \hat{\kappa}$, with equality only if $z_{A0}^* = 1$.

In general, the transition thresholds for the social planner $\hat{\kappa}$ and the equilibrium κ^* cannot be ranked. This holds because the social planner internalizes knowledge spillovers on future entrants when choosing the long-run direction of innovation, but these spillovers are not necessarily always larger for a given technology: Technology B spillovers are larger in the long-run given $\eta_B > \eta_A$, but technology A spillovers may be larger in the short-run given incumbents' initial expertise for technology A (i.e., the initial distribution H_A^*). However, Proposition C.5 shows that for a given long-run innovation direction, the social planner always prefers to direct greater initial innovation in that direction than in equilibrium.

5 Empirical Analysis

In this section, I present an empirical analysis of firm innovation decisions as reflected in patent publications. I document three facts consistent with the theory: First, a firm's current patenting is highly correlated with its previous patenting, controlling for other determinants of innovation outcomes (Section 5.2). Second, for the collection of new technologies identified by Kalyani et al. (2023), a firm's current patenting for a given technology is better predicted by past patenting *within that technology* than patenting in general (Section 5.3). Third, incumbents with greater patenting experience patent *less* for a new technology than less-experienced firms, though this gap shrinks as the technology matures (Section 5.4). The first two facts support the theory's key assumptions that knowledge is both cumulative within firms and technology-specific, generalizing existing evidence from the auto industry (Aghion et al., 2016). The third supports the key prediction that experienced incumbents are reluctant to innovate for new technologies given their expertise in old ones. I discuss these results in Section 5.5.

5.1 Data

The main dataset for the analysis is a panel of US public firms over 1980-2021, which includes measures of (i) each firm's patenting, both overall and within specific technologies; (ii) the aggregate stock of knowledge (patents) available to each firm, both overall and within specific technologies; and (iii) each firm's R&D expenditures. Below I summarize the principal data sources, with additional details in Appendix D.

Patents. I use data on the set of all utility patents filed at the US Patent and Trademark Office (USPTO) after 1980 and granted through 2023. This dataset covers almost seven million patents and is made available through the USPTO's PatentsView platform. It includes information about each patent's title, abstract, assignees (initial owners), inventors, technology area, and citations made to other USPTO patents.

Patent-Firm Matching. Accurately grouping patents by firm is difficult, because assignee names are not standardized and are not adjusted to reflect changes in firm ownership through mergers and acquisitions.[28] I address these issues using the DISCERN 2.0 dataset, which employs an extensive matching process to identify all patents granted to US public firms from 1980 to 2021 (Arora et al., 2024). The resulting 1,865,633 patents are matched to 5,680 firms, and for tractability I restrict the analysis to the top 10% of firms by total number of patents. The final subset includes 1,659,998 patents matched to 568 firms. Restricting to public firms reduces the scope of the analysis, but it ensures an accurate matching of patents to firms and is common throughout the empirical literature on firm innovation (Hall, Mairesse, and Mohnen, 2010).

R&D Expenditures. I use data on yearly R&D expenditures from Compustat North America. I normalize by US GDP per capita each year to measure inflation-adjusted R&D expenditures in "scientist equivalent" units.

New Technologies. To group patents into technologies in a theory-consistent way, I start with the list of new technologies produced by Kalyani et al. (2023). They identify new technologies by extracting all two-word combinations that appear in the text of US patents over 1976-2014, excluding any combination found in a representative sample of pre-1970 American English text. Of the remaining "novel" combinations, the authors retain only the ones that appear in a sufficiently large number of citation-weighted patents. Finally, to ensure that these combinations refer to technologies instead of scientific concepts or problems to solve, the authors search for a Wikipedia page corresponding to each combination and verify that it describes a technology. Each of the resulting 1,148 two-word combinations is a *new technology*. Kalyani et al. (2023) also estimate the "emergence year" of a new technology, equal to the first year in which it is mentioned in a minimal number of citation-weighted patents and achieves ten percent growth in this number over each of the next five years.

I identify the set of patents related to each new technology by searching for the two-word combination in the title and abstract of each patent. I include any patent that mentions the combination, as well as any patent that cites such a patent. Table 1 lists the top ten new technologies by total number of patents, along with their emergence years. The list primarily includes information and communication technologies ("data store," "code-division multiple access") along with biomedical technologies ("thermometer," "ion channel") and industrial

[28] PatentsView provides disambiguated assignee and inventor names meant to resolve the first issue, but my experience matching patents to pharmaceutical companies for the case study in Section 2 suggests that the PatentsView disambiguation protocol is still a work-in-progress.

Technology	Patents	Emergence Year
data store	133,992	1990
code-division multiple access	107,638	1986
memory address	107,358	1992
thermometer	106,336	1991
heat treating	104,587	1983
microsoft access	100,097	1987
error detection correction	96,819	1986
holographic optical element	95,231	1992
text-based user interface	94,379	1994
ion channel	94,097	1992

Table 1: Top New Technologies by Total Patents

Notes: Each technology's name corresponds to the title of its Wikipedia page, not the associated two-word combination.

technologies ("heat treating"), representative of the broader set of 1,148 technologies.

How well do these groups of patents capture the notion of a technology in the theory? The latter corresponds to (i) a shared set of production techniques and (ii) a shared base of knowledge used to improve those techniques. As Table 1 suggests, the new technologies identified in the patent data generally meet the first condition: Each new technology spans patents with similar applications, from data storage ("data store") to biological targets for drug treatment ("ion channel"). They also meet the second condition essentially by design: Any patent associated with a new technology must either mention it *or cite* one that does, so that it contributes directly to or builds on the technology's base of knowledge. To see that patents for a new technology are generally focused on similar topics, we can analyze how they are classified into technology areas by patent examiners. Each patent is assigned to potentially several of 132 "classes" and 672 "subclasses" defined in the Cooperative Patent Classification (CPC). These groupings are meant to identify all patents with similar technological content, with greater specificity for subclasses than classes. On average, I find that a share 0.58 of patents in a new technology belong to a *single class*, while a share 0.45 belong to a *single subclass*, so that patents associated with a new technology are highly concentrated within technology areas.[29]

[29] I do not use the CPC hierarchy to directly define technologies for two reasons. First, classes and subclasses are still too broad to provide a compelling definition: Examples include class A61 "Medical or Veterinary Science; Hygiene" and subclass A61K "Preparations for Medical, Dental, or Toiletry Purposes", which include patents with a

This method to group patents into technologies has several limitations. First, identifying an initial set of patents by keywords may include patents that simply mention the keywords but actually address other applications. I mitigate this concern by searching only in the title and abstract of each patent, which are more tightly focused on the patent's core applications. Second, Kalyani et al. (2023) explicitly identify *new* technologies that arrive over 1976-2014, but my theory focuses on the contrast between new and old technologies. In my empirical analysis, I proxy for old technologies with the set of all patents not associated with new technologies. Finally, some of the new technologies identified by Kalyani et al. (2023) have substantial overlap, such as "data store" and "microsoft access" (a database management system). This will attenuate my results about the technology specificity of knowledge to the extent that I misclassify patents as unrelated to a given new technology. However, it may also artificially increase precision by inflating the number of new technologies above its true value.

5.2 Fact 1: Knowledge Accumulation within Firms

I first provide evidence consistent with knowledge accumulation within firms: A firm's current patenting is highly correlated with its previous patenting, holding fixed R&D expenditures, previous patenting by all firms, and the firm's latent propensity to patent. The estimating equation is

$$Pat_{it} = \exp\left(\beta_1 \log\left(K_{it}^{\text{Firm}}\right) + \beta_2 \log\left(K_{it}^{\text{Agg}}\right) + \beta_3 \log\left(s_{it}\right) + \alpha' X_{it} + \varepsilon_{it}\right). \tag{33}$$

Here Pat_{it} denotes the number of patents filed by firm i in year t. I weight each patent by the number of forward citations it receives to better reflect the value of the underlying innovations.[30] The firm-level knowledge stock K_{it}^{Firm} measures firm i's accumulated internal knowledge at time t. I construct this stock from past patent flows by the perpetual inventory method, setting the depreciation rate ν to the standard value of 0.15:[31]

$$K_{it}^{\text{Firm}} = (1-\nu)K_{it-1}^{\text{Firm}} + Pat_{it-1}.$$

The aggregate knowledge stock K_{it}^{Agg} measures the aggregate knowledge available to firm i at time t. Since each firm draws on different knowledge depending on its area of focus, I construct K_{it}^{Agg} in two steps. I first compute the knowledge stocks for each CPC class by the perpetual

variety of applications. Second, many reasonably-defined technologies include patents assigned to many different classes and subclasses. For example, the two most common subclasses among mRNA therapy vaccines identified in Section 2 are A61K and C12N "Microorganisms or Enzymes; Compositions Thereof; Propogating, Preserving, or Maintaining Microorganisms; Mutation or Genetic Engineering; Culture Media."

[30] See Appendix D for additional details about the variable constructions.

[31] See Hall, Jaffe, and Trajtenberg (2005).

inventory method. I then compute K_{it}^{Agg} as a weighted average of these stocks at each time t, with weights given by the distribution of firm i's total patents across all CPC classes. Finally, I control for R&D expenditures s_{it} and a vector of controls X_{it} that includes fixed effects by year, firm, and the number of years for which the firm has been publicly listed.

The estimating equation (33) is an empirical analogue to the quality evolution equation (8). I use patent flows Pat_{it} in place of the increase in quality $\dot{q}_\theta(t)$, the firm-level knowledge stock K_{it}^{Firm} in place of the quality $q_\theta(t)$, and K_{it}^{Agg} in place of the aggregate knowledge stock $K_\theta(t)$. However, in this section the estimating equation (33) is not technology-specific; it instead captures determinants of overall firm-level innovation. I also adopt a log-linear functional form so that the coefficients of interest β_1, β_2, and β_3 may be interpreted as unit-free elasticities. The hypothesis that knowledge accumulates within firms implies $\beta_1 > 0$, so that a firm's past patenting has a larger impact on its current patenting than predicted by its effect on the aggregate knowledge stock K_{it}^{Agg}.

I estimate the model (33) on the panel of public firms described in Section 5.1. To mitigate truncation issues in the patent data and in the unbalanced panel of firms, I restrict to the years 1985-2016, and I drop any observations for the first year when a firm becomes publicly listed.[32] I estimate the model directly as a negative binomial regression and as a linear regression (OLS) with dependent variable $\log(1+Pat_{it})$. The corresponding estimates are found in columns (1) and (2) of Table 2.

In each specification, I find that a firm's patenting is positively correlated with its past patenting K_{it}^{Firm} conditional on the other covariates. The estimates of the corresponding elasticity suggest that a 10% increase in a firm's internal knowledge stock K_{it}^{Firm} is associated with a 7% increase in the firm's contemporaneous patenting Pat_{it}. This is consistent with the assumption in the theory that knowledge accumulates within firms ($\lambda > 0$), so that the productivity of a firm's R&D expenditures s_{it} is increasing in its own past patenting K_{it}^{Firm}. The model controls for many alternative explanations: For example, firm fixed effects ensure that this relationship is not driven by variation in firms' latent propensity to patent at all times, which would yield a positive correlation between past and current patenting. Controlling for R&D expenditures similarly ensures that the relationship is not driven by variation in the scale of each firm's R&D program. To the extent that R&D expenditures are elastic to *persistent* innovation opportunities, this control also rules out the explanation that firms simply patent more when they discover a

[32]Truncation issues arise in the patent data because I only observe *granted* patents, and there is often a multi-year lag between when a patent is filed and when it is granted. A similar problem arises for observed citations, which I address by normalizing forward citations both within year and across CPC classes (see Appendix D). A left truncation issue appears for firms because I only observe them after they become publicly listed, biasing my initial estimates of their internal knowledge stocks K_{it}^{Firm} downward if they filed for patents before then.

	Overall Patenting		Technology-Specific Patenting	
	(1) Neg. Binomial	(2) OLS	(3) Neg. Binomial	(4) OLS
$\log\left(K_{it}^{\text{Firm}}\right)$	0.696***	0.716***	0.286***	0.132***
	(0.0329)	(0.0335)	(0.0617)	(0.0268)
$\log\left(K_{it}^{\text{Agg}}\right)$	0.297*	0.248*	0.554	-0.0997
	(0.125)	(0.119)	(0.554)	(0.147)
$\log\left(s_{it}\right)$	0.0778***	0.0951***	0.0961*	0.0450**
	(0.0185)	(0.0203)	(0.0473)	(0.0162)
$\log\left(K_{i\theta t}^{\text{Firm}}\right)$			0.504***	0.397***
			(0.0214)	(0.0285)
$\log\left(K_{\theta t}^{\text{ext}}\right)$			0.337***	0.0919***
			(0.0423)	(0.0122)
Observations	12237	12228	79093	79092
Firms	555	546	257	256

*Significance: *** 0.01, ** 0.05, * 0.1*

Table 2: Regression Results: Firm Patenting

Notes: The overall patenting regressions include fixed effects by year, firm, and the number of years for which the firm has been publicly listed. They also include dummy variables for zero internal knowledge stock K_{it}^{Firm}, zero aggregate knowledge stock K_{it}^{Agg}, and zero R&D expenditures s_{it}. The technology-specific patenting regressions additionally include fixed effects by technology and the time since emergence. They also add dummy variables for zero internal and aggregate technology-specific knowledge stocks $K_{i\theta t}^{\text{Firm}}$ and $K_{\theta t}^{\text{Agg}}$. All standard errors are clustered at the firm level.

valuable line of research.

Table 2 also shows that a firm's patenting is positively correlated with the measure of the aggregate knowledge K_{it}^{Agg} available to the firm, which reflects positive spillovers across incumbents in the theory ($\sigma_I > 0$). The estimates of the corresponding elasticity imply that a 10% increase in the firm's external knowledge K_{it}^{Agg} is associated with a 2.5-3% increase in contemporaneous patenting. I also find a positive elasticity of patenting to R&D expenditures of 8-9%, in line with previous estimates in the literature (e.g., Bloom, Schankerman, and Van Reenen, 2013).

5.3 Fact 2: Technology-Specific Knowledge

The estimates described above support the claim that knowledge accumulates within firms, but they say nothing about the extent to which this knowledge is *technology-specific*. To assess this second assumption of the theory, I consider a related estimating equation for a firm's patenting in a given technology θ:

$$Pat_{i\theta t} = \exp\Big(\beta_1 \log\left(K_{it}^{\text{Firm}}\right) + \beta_2 \log\left(K_{it}^{\text{Agg}}\right) + \beta_3 \log\left(s_{it}\right) \\ + \beta_4 \log\left(K_{i\theta t}^{\text{Firm}}\right) + \beta_5 \log\left(K_{\theta t}^{\text{Agg}}\right) + \alpha' X_{i\theta t} + \varepsilon_{i\theta t}\Big). \tag{34}$$

Here the dependent variable $Pat_{i\theta t}$ is the number of citation-weighted patents filed by firm i for technology θ in year t. Regressors again include the firm's internal knowledge stock K_{it}^{Firm}, aggregate knowledge stock K_{it}^{Agg}, and R&D expenditures s_{it}, none of which are technology-specific. But I also add the technology-specific knowledge stocks $K_{i\theta t}^{\text{Firm}}$ and $K_{\theta t}^{\text{Agg}}$. Here $K_{i\theta t}^{\text{Firm}}$ is defined analogously to K_{it}^{Firm} as the discounted sum of past citation-weighted patents by firm i for technology θ. The aggregate knowledge stock $K_{\theta t}^{\text{Agg}}$ is simply the discounted sum of all past patents for technology θ and does not vary across firms. The vector of controls $X_{i\theta t}$ includes the same fixed effects as in the first model (33), in addition to fixed effects by technology θ and the number of years after the emergence of the technology as defined by Kalyani et al. (2023). The hypothesis that knowledge is technology-specific implies $\beta_4 > \beta_1$ and $\beta_5 > \beta_2$, so that knowledge related to a given technology θ has a larger impact on a firm's current patenting for θ than knowledge in general.

For tractability, I consider only the top 10% of technologies θ by total number of citation-weighted patents with emergence years after 1990. I make a final restriction to ensure that the estimates of the model (34) apply to pairs of firms i with technologies θ that they could plausibly innovate: For each technology θ, I compute the distribution of its associated patents across CPC classes. I then exclude any firm-technology pairs for which the cosine similarity between the firm and technology patent distributions falls below 0.8. This excludes, for example, the pair of Microsoft and "antenna radio" (cosine similarity 0.43) while including the pair of Microsoft and "network virtualization" (cosine similarity 0.82). Including more firm-technology pairs attenuates the coefficient estimates, as firms patent little for technologies outside their areas of focus.

The negative binomial and OLS estimates for the technology-specific model (34) are found in columns (3) and (4) of Table 2. These estimates broadly demonstrate that knowledge is technology-specific: The estimated elasticities corresponding to the technology-specific knowledge stocks $K_{i\theta t}^{\text{Firm}}$ and $K_{\theta t}^{\text{Agg}}$ are significantly positive and generally larger than the elasticities corresponding to the generic knowledge stocks K_{it}^{Firm} and K_{it}^{Agg}. For example, a 10% increase in a firm's past patenting for a technology θ correlates with a 4-5% increase in the firm's contemporaneous patenting for θ, while the corresponding elasticity for the firm's generic past patenting is only 1.3-3%. Finally, note that the elasticity of patenting for technology θ with respect to R&D expenditures naturally attenuates relative to the firm-level specification (33): s_{it} is at best an imprecise proxy for the R&D expenditures specifically allocated by the firm to technology θ, which I cannot observe.

5.4 Fact 3: Incumbents and New Technologies

When knowledge accumulates within firms and is technology-specific, the theory predicts that incumbent firms with extensive experience in existing technologies should be reluctant to innovate new ones, relative to less-experienced incumbents or entrants (Proposition 2). To test this prediction, I estimate several models that relate a firm's patenting for a new technology θ to the characteristics of the firm at the time of emergence T_θ:

$$\begin{aligned}\log(1+Pat_{i\theta}^{10}) = \beta_1 \log\left(K_{iT_\theta}^{\text{Firm}}\right) + \beta_2 \log\left(K_{iT_\theta}^{\text{Agg}}\right) + \beta_3 \log\left(s_{iT_\theta}\right) \\ + \beta_4 \log\left(K_{i\theta T_\theta}^{\text{Firm}}\right) + \beta_5 \log\left(K_{\theta T_\theta}^{\text{Agg}}\right) + \alpha' X_{i\theta} + \varepsilon_{i\theta}.\end{aligned} \tag{35}$$

Here $Pat_{i\theta}^{10}$ denotes the total number of citation-weighted patents filed by firm i for technology θ in the ten years after the technology's emergence. The regressors include all firm-specific and aggregate knowledge stocks at the emergence time T_θ, along with the firm's R&D expenditures s_{iT_θ}. The vector of controls $X_{i\theta}$ includes fixed effects by firm i, emergence year T_θ, and the number of years for which the firm has been publicly listed at the emergence time.[33] I also consider the alternative model in which the outcome variable is the *share* of technology θ in all patents filed by firm i in the ten years after emergence:

$$TechShare_{i\theta}^{10} \equiv \frac{Pat_{i\theta}^{10}}{\sum_{t=T_\theta}^{T_\theta+9} Pat_{it}}.$$

The estimating equation (35) is similar to the previous model of technology-specific patenting (34), but with no panel dimension. It simply relates a firm's medium-run innovation for a new technology $Pat_{i\theta}^{10}$ to the firm's characteristics when that technology emerges. The coefficient of interest is β_1, which measures how the firm's initial knowledge stock at the time of emergence $K_{iT_\theta}^{\text{Firm}}$ predicts its subsequent patenting in the new technology θ. When the dependent variable is $TechShare_{i\theta}^{10}$, the theory predicts that this coefficients should be negative: The empirical results documented thus far suggest that firm innovation decisions should be to some extent

[33] In this baseline specification, I exclude technology fixed effects so as to estimate the effect of the initial aggregate knowledge stock $K_{\theta T_\theta}^{\text{Agg}}$. The results in Table 3 are robust to the inclusion of technology fixed effects (see Appendix E).

	Technology Patents		Technology Patent Share	
	(1) Full Sample	(2) No Early Patents	(3) Full Sample	(4) No Early Patents
$\log(K^{\text{Firm}}_{iT_\theta})$	-0.0833***	-0.0843***	-0.0023***	-0.0013**
	(0.0306)	(0.0270)	(0.0008)	(0.0006)
Observations	13,190	11,024	13,190	11,024

*Significance: *** 0.01, ** 0.05, * 0.1*

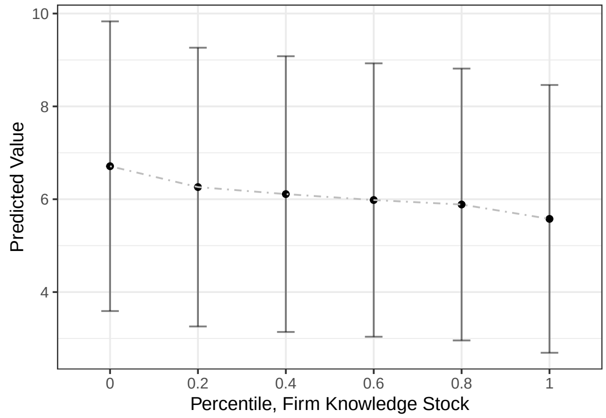
Table 3: Regression Results: Technology Patenting after Emergence

Notes: All regressions include fixed effects by firm, emergence year, and the number of years for which the firm has been publicly listed. They also include dummy variables for zero values of each of the knowledge stocks and R&D expenditures s_{it}. All standard errors are clustered at the firm level.

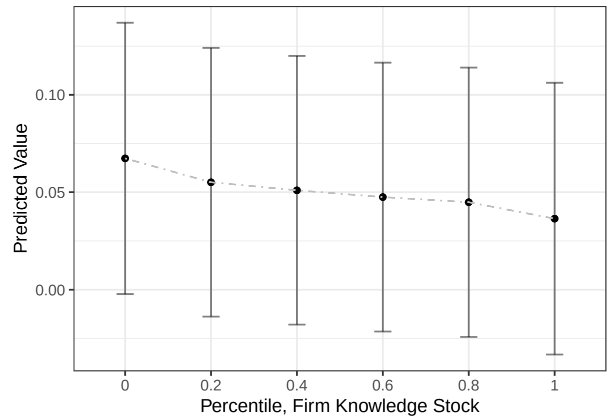
path-dependent. As a result, incumbents with greater internal knowledge stocks when a new technology θ emerges should devote a smaller share of their R&D resources to technology θ, yielding in a smaller share for technology θ in the firm's total patents $TechShare^{10}_{i\theta}$. This need not imply that these firms patent *absolutely* less for technology θ, because their greater research productivity may allow them to patent more in many different technologies.[34] Only when the path dependence force is particularly strong does the theory predict that β_1 should be negative when the dependent variable in the estimating equation (35) is the transformed patent count $\log(1 + Pat^{10}_{i\theta})$.

I estimate the model (35) by OLS for both dependent variables. For brevity, I display the estimates for β_1 in Table 3 and present the estimates for the remaining coefficients in Appendix E. Columns (1) and (3) provide the estimate $\hat{\beta}_1$ for the full sample of firms, while columns (2) and (4) exclude firms with any patenting for technology θ before the emergence year T_θ. Consistent with firm-level path dependence, the estimates suggest a sizable negative effect of a firm's initial patenting experience on its subsequent patenting for new technologies: A 10% increase in a firm's initial internal knowledge stock $K^{\text{Firm}}_{iT_\theta}$ is associated with a 0.8% decrease in the firm's subsequent patents for the new technology. It also corresponds to a decrease of 0.1-0.2 percentage points in the technology's share of the firm's subsequent patents. To provide a sense of the effect size, Figure 3 plots predicted values from the models in columns (1) and (3) at different percentiles of the distribution of initial knowledge stocks $K^{\text{Firm}}_{iT_\theta}$, holding other covariates fixed at their mean values. Moving from the 10th percentile to the 90th percentile of the initial knowledge distribution reduces the firm's subsequent patents for a new technology

[34]This force is absent in the theoretical model because all knowledge is technology-specific and firms cannot deploy more R&D resources as their research becomes more productive.



(a) Patents: $\log\left(1+Pat^{10}_{i\theta}\right)$



(b) Patent Share: $TechShare^{10}_{i\theta}$

Figure 3: Predicted Values: Technology Patenting after Emergence

Notes: Figures 4(a) and 4(b) respectively display predicted values from the models in columns (1) and (3) of Table 3, with 95% confidence intervals. The firm-specific internal knowledge stock K^{Firm}_{it} ranges from -3.55 to 10.08, with median 4.50 and mean 4.18.

by nearly 45%; it also reduces the technology's share of the firm's subsequent patents by 1.6 percentage points, or approximately 27%. These effects suggest substantial path dependence in incumbent innovation, consistent with a fundamental prediction of the theory.

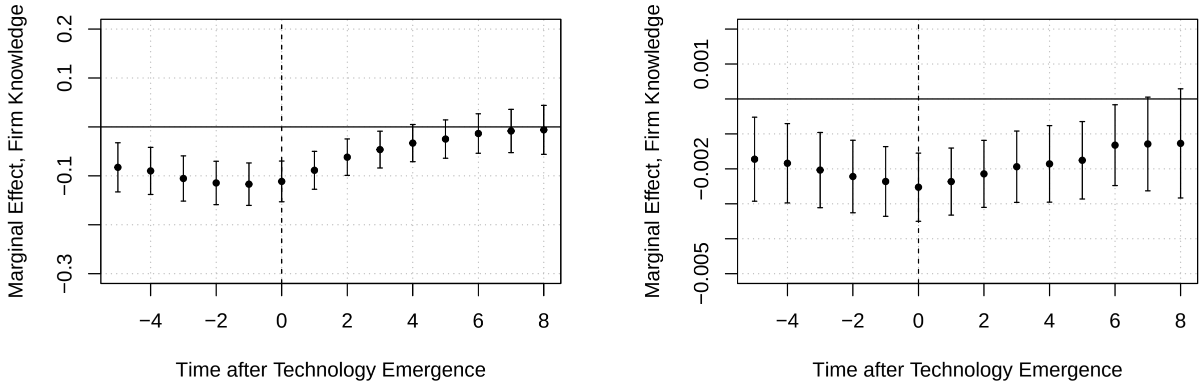
To assess how path dependence varies as a new technology matures, I estimate a dynamic version of the estimating equation (35), allowing the effect of the initial internal knowledge stock $K^{\text{Firm}}_{iT_\theta}$ to vary over time:

$$\begin{aligned}\log(1+Pat_{i\theta w}) = &\sum_{\tilde{w}=-2}^{5} \beta_{1\tilde{w}} \log\left(K^{\text{Firm}}_{iT_\theta}\right) \mathbb{1}_{w=\tilde{w}} + \beta_2 \log\left(K^{\text{Agg}}_{iT_\theta}\right) + \beta_3 \log\left(s_{iT_\theta}\right) \\ &+ \beta_4 \log\left(K^{\text{Firm}}_{i\theta T_\theta}\right) + \beta_5 \log\left(K^{\text{Agg}}_{\theta T_\theta}\right) + \alpha' X_{i\theta w} + \varepsilon_{i\theta w}.\end{aligned} \tag{36}$$

Here I group all years from 1980-2022 into three-year windows, and for each technology I denote the window containing the emergence time T_θ by $w=0$. I let $Pat_{i\theta w}$ denote the firm's total citation-weighted patents for technology θ in window w. I group years into windows so that I can consider the alternative model in which the outcome variable is the share of technology θ in all patents filed by firm i in window w:

$$TechShare_{i\theta w} \equiv \frac{Pat_{i\theta w}}{Pat_{iw}}.$$

The timing of patent filings is noisy from year to year, so grouping years into windows reduces



(a) Patents: $\log(1 + Pat_{i\theta w})$

(b) Patent Share: $TechShare_{i\theta w}$

Figure 4: Regression Results: Technology Patenting Over Time

*Notes:*All regressions include fixed effects by firm-year, technology-window, and the number of years for which the firm has been publicly listed. They also include dummy variables for zero values of each of the knowledge stocks and R&D expenditures s_{it}. All standard errors are clustered at the firm level.

the number of observations dropped when the denominator of $TechShare_{i\theta w}$ is zero.

The model (36) relates the firm's patenting after the emergence of a technology to its various knowledge stocks at the time of emergence, allowing for heterogeneous effects across time for the firm's generic internal knowledge stock $K^{\text{Firm}}_{iT_\theta}$. The vector of controls $X_{i\theta w}$ now includes fixed effects for each firm-year it, each technology-window θw, and the number of years for which the firm has been publicly listed at the time of emergence T_θ. Note that I do *not* include firm-technology fixed effects, so the estimating equation (36) does *not* correspond to an event study. Rather, the coefficients of interest β_{1w} capture the *level differences* in patenting across firms with different internal knowledge stocks $K^{\text{Firm}}_{iT_\theta}$ at the emergence time. The key prediction of the theory is that these coefficients should be negative $\beta_{1w} < 0$ for windows w near zero (i.e., near emergence).

I estimate the model (36) by OLS for the dependent variables $\log(1+Pat_{i\theta w})$ and $TechShare_{i\theta w}$, and the corresponding estimates ("marginal effects") $\hat{\beta}_{1w}$ are plotted in Figure 4. These estimates provide clear evidence for firm-level path dependence: A firm with greater initial patenting experience patents strictly less for an emerging technology, both absolutely and relative to its total flow of patents.[35] Figure 4(a) shows that this negative effect on total patenting attenuates both before and after the emergence time. Experienced incumbents are particularly

[35] As expected, the coefficient magnitudes are similar to those from the cross-sectional regressions in Table 3.

unlikely to innovate for an emerging technology, but after 10 years a firm's initial experience has no effect on its total patenting for the new technology. Figure 4(b) replicates the initial negative effect of the firm's internal knowledge stock for the share of the new technology in the firm's total patents. This negative effect displays the same "V shape" around emergence, but it is more persistent. On average, an experienced incumbent always has a lower share of patents for a new technology than less-experienced firms.

5.5 Discussion

The empirical results presented in this section support the key assumptions and implications of my theory. The results in Section 5.2 provide evidence that firms accumulate knowledge through innovation that raises the productivity of their future R&D. Most microeconomic evidence on the return to R&D at the firm level ignores this dynamic channel (Hall, Mairesse, and Mohnen, 2010; Bloom, Schankerman, and Van Reenen, 2013), which my results suggest is quantitatively important. Klette (1996) provides an early discussion of this issue, which I extend with an alternative model for firm-level innovation and more comprehensive evidence about the knowledge accumulation channel.

The results in Section 5.3 additionally demonstrate that knowledge produced through innovation is highly technology-specific, indicating that firm innovation decisions are subject to the path-dependent and forward-looking forces highlighted in the theory (Proposition 2). These findings generalize existing work in environmental economics that show knowledge from dirty and clean innovation is technology-specific (Dechezleprêtre, Märtin, and Mohnen, 2014) and may generate firm-level path dependence (Aghion et al., 2016). I show that these properties are pervasive, extending to a variety of technologies in many industries. I also explicitly control for other firm-level determinants of innovation outcomes (like R&D expenditures) that could offer an alternative explanation for these findings.

Finally, the analysis in Section 5.4 tests the theory's basic implication that firms with substantial expertise in existing technologies should be reluctant to innovate for new ones. The findings are consistent with the theory, and they provide novel evidence that experienced incumbents generally play a limited role in innovating emerging technologies. Moreover, the pattern of innovation uncovered by this analysis largely reflects the history of innovation in mRNA vaccines described in Section 2: Incumbents initially innovate less for a new technology, but their efforts intensify after younger firms develop the technology over several years. Lessons from the case study appear to generalize to the emergence of a variety of new technologies.

Parameter Value	Target/Source
$S = 1$	Normalization
$\delta = 0.075$	Hopenhayn, Neira, and Singhania (2022)
$\lambda = 1$	Normalization
$\sigma_I = 0$	Table 2 estimates
$\sigma_E = 0.4$	$\frac{1}{\xi_A^*} = 2.12$ (Benhabib, Perla, and Tonetti, 2021)
$\eta_A = 0.026$, $\eta_B = 0.03$	$g_A^* = 0.02$, $g_B^* = 0.025$
$s = 2.1$	$\lambda \eta_A s = 0.055$ (Haltiwanger, Jarmin, and Miranda (2013))
$\rho = 0.075$	Gormsen and Huber (2023)

Table 4: Model Calibration

6 Quantitative Example

In this section, I provide a calibrated example of the model to assess how an increase in the concentration of scientists might affect equilibrium innovation and social welfare. I describe the calibration in Section 6.1, and I provide simulation results in Section 6.2.

6.1 Calibration

I calibrate all parameters of the model necessary to compute the equilibrium path of the knowledge stocks K and the aggregate qualities Q. I first normalize the total mass of scientists to one, $S = 1$. I set the exit rate to $\delta = 0.075$, in line with recent estimates of the aggregate exit rate for US firms reported by Hopenhayn, Neira, and Singhania (2022). I choose the discount rate $\rho = 0.075$ to ensure that the firms' total discount rate $\rho + \delta = 0.15$ falls in the middle of the range reported by Gormsen and Huber (2023).

To calibrate the parameters of the innovation process, I note that the estimates of the firm patenting equation (33) reported in Table 2 indicate that firms build substantially on their own past knowledge when innovating, with a smaller contribution from external knowledge. To keep the equilibrium dynamics simple, I consider the limiting case described in Section 4.3 and set the contribution of spillovers across incumbents to zero $\sigma_I = 0$. I then normalize $\lambda = 1$. I choose the parameters $\sigma_E = 0.4$ and $\eta_A = 0.026$ jointly to ensure that the initial BGP growth rate is $g_A^* = 0.02$ and that the tail parameter of the BGP firm-quality distribution ξ_A^* matches the corresponding value for the US firm-productivity distribution as in Benhabib, Perla, and Tonetti (2021). Finally, I choose technology B's productivity parameter $\eta_B = 0.03$ to ensure that this

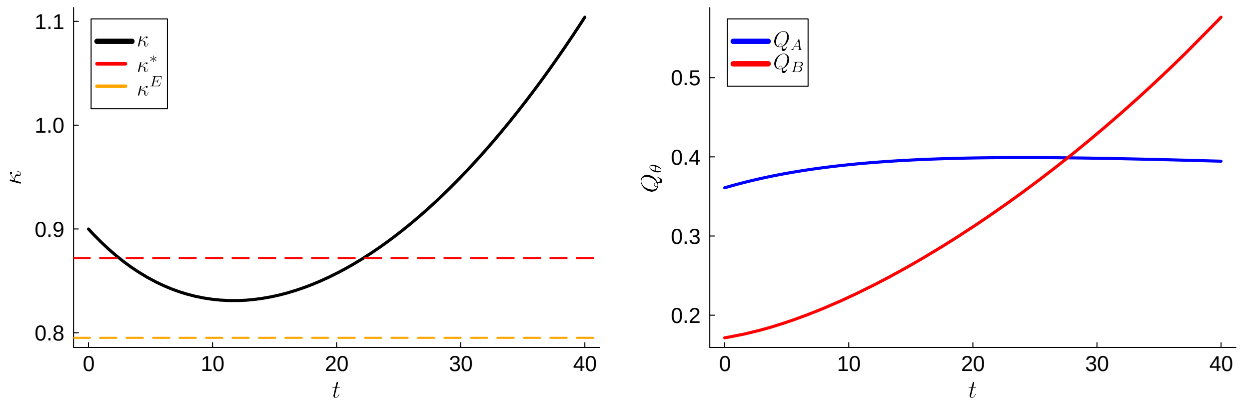


Figure 5: Equilibrium Trajectories

technology allows for 0.5% faster growth than technology A along its BGP, $g_B^* = 0.025$.

The calibration of the model is summarized in Table 4.

6.2 Simulation Results

The calibrated model has a unique equilibrium, and it features a transition from technology A to technology B if and only if the aggregate knowledge ratio $\kappa(t)$ is initially above the threshold $\kappa^* \approx 0.872$. Figure 5 displays the trajectories of the aggregate knowledge ratio $\kappa(t)$ and the aggregate qualities $Q(t)$ for an equilibrium with a transition. The aggregate knowledge ratio $\kappa(t)$ is "U-shaped," initially decreasing before increasing asymptotically. But it remains above the threshold κ^E at which entrants would instead innovate for technology A, ensuring that the economy converges to the BGP for technology B in the long run. The aggregate quality Q_B for technology B increases with innovation, while the aggregate quality Q_A for technology A initially increases before declining to a positive limiting value due to firm reallocation.

To explore how the concentration of scientists s can affect the equilibrium's propensity to transition, in Figure 6 I plot the equilibrium threshold κ^* for different values of s and ρ, keeping all other parameters of the model fixed. The figure indicates that an increase in the concentration of scientists increases the threshold κ^* under the baseline calibration of the model, implying a *lower* propensity to develop the high-growth technology B in equilibrium. The threshold κ^* increases faster with s for higher values of the discount rate ρ, while it can instead decline with s for lower values of ρ. These observations are exactly consistent with the trade-off between the composition and growth effects discussed in Section 4.3.

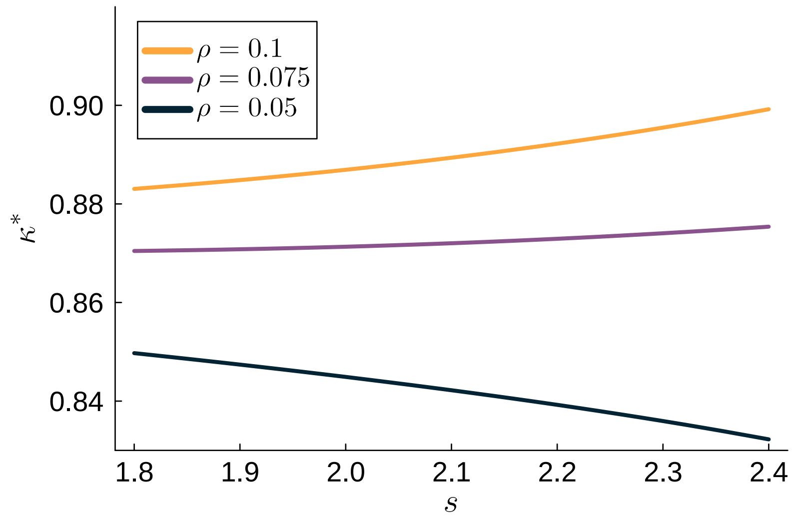


Figure 6: Equilibium Threshold Comparative Statics

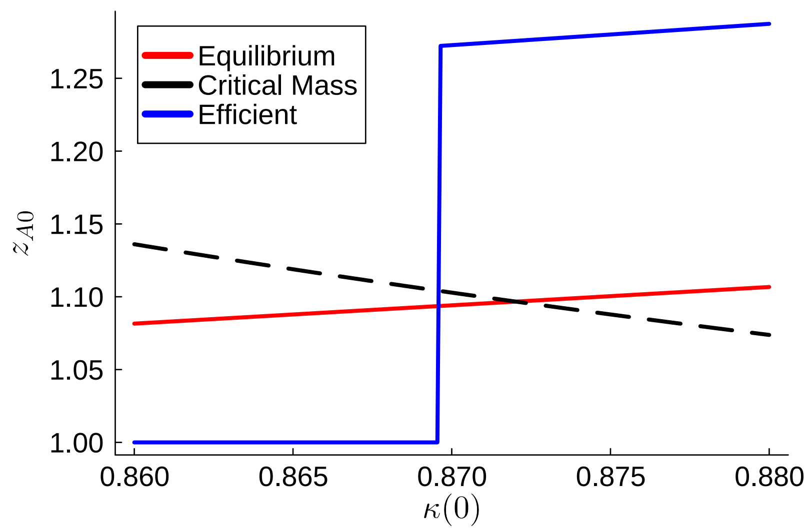


Figure 7: Equilibrium and Efficient Cutoffs z_{A0}

Finally, it is straightforward to see that an increase in the concentration of scientists s can actually reduce social welfare. In Figure 7, I plot the initial incumbent cutoff z_{A0} chosen in equilibrium (red line) and by the social planner (blue line) as a function of the initial aggregate knowledge stock ratio $\kappa(0)$. This cutoff determines which initial incumbents innovate for technology B ($z_A(0) \leq z_{A0}$), and it must be greater than the dashed "critical mass" line to ensure a transition to the BGP for techonlogy B. Under the baseline calibration, the social planner always prefers to transition more often than in equilibrium, and consistent with Proposition 6 the efficient cutoff $\hat{z}_{A0}$ is always greater than the equilibrium cutoff z^*_{A0} when a transition to technology B is efficient. As noted above, an increase in the concentration of scientists s raises the equilibrium transition threshold κ^*, represented in Figure 7 by the intersection between the red and black lines. A marginal increase in s can strictly reduce welfare by precluding a transition to technology B when this is socially efficient.

7 Conclusion

In this paper, I presented a new model of directed innovation and firm dynamics to clarify a novel connection between market structure, the direction of innovation, and economic growth. Knowledge accumulation within firms generates heterogeneous innovation incentives, and spillovers between firms generate complementarities that imply an industry's initial market structure can be decisive for long-run growth. These observations have immediate relevance for the growing concentration of R&D in AI: Increasing concentration risks leaving valuable alternative innovation directions unexplored, as large incumbent firms develop this technology according to their existing expertise. Empirical evidence suggests that these concerns are substantive. Innovation policy can help address any potential inefficiencies by ensuring entrant firms have sufficient resources to pursue their own research directions.

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Appendix

Contents

A Proofs

This appendix provides proofs for results in the main text.

A.1 Proofs for Section 3

Proof of Lemma 1: With linear, additively separable preferences over consumption sequences, the representative consumer's Euler equation and asset market clearing jointly imply $r(t) = \rho$. The intermediate price $p_\theta(q,t)$ must solve the maximization problem in the definition of flow profits (7), which implies a markup of $1/(1-\beta)$ over marginal cost: $p_\theta(q,t) = \gamma/(1-\beta)$. Given the final producer's intermediate demand curve, this implies the stated equations for intermediate quantities $x_\theta(q,t)$ and profits $\pi(q_\theta)$. Market-clearing for workers requires $L(t) = L$, and substituting the derived input quantities into the production function yields the state equation for output $Y(t)$. Market-clearing for goods then implies the corresponding equation for consumption $C(t)$. Finally, the wage for workers is recovered from the marginal product condition $w_L(t)L = \beta Y(t)$. ∎

Proof of Proposition 1: The equations (17, 18) derived in the discussion before the Proposition characterize the growth rate g_θ^* and the relative quality distribution H_θ^*. I reproduce these equations here:

$$g_\theta^* = \int (\lambda \sigma_E z_\theta + \sigma_I)\, \eta_\theta s dH_\theta^*(z_\theta), \tag{A1}$$

$$0 = -\left[(\lambda - g_\theta^*) z_\theta + \frac{\sigma_I}{\sigma_E}\right] h_\theta^*(z_\theta) + \delta N - H_\theta^*(z_\theta). \tag{A2}$$

The evolution equation (15) for Q_θ additionally implies

$$g_\theta^* = \frac{K_\theta(t)}{Q_\theta(t)}\left(g_\theta^* + \delta N \sigma_E\right) - \delta,$$

which can be rearranged to give the expression for $Q_\theta(t)/K_\theta(t)$ stated in the Proposition. The inequality $Q_\theta(t)/K_\theta(t) < 1$ follows immediately from this expression and the assumed upper bound on entrant spillovers (19), recalling that market-clearing for scientists requires $S = Ns$.

To solve the differential equation (A2) for H_θ^*, define the values

$$a \equiv \lambda \eta_\theta s - g_\theta^*,$$
$$b \equiv \frac{\sigma_I}{\sigma_E} \eta_\theta s.$$

Then (A2) has the general solution

$$H_\theta^*(z_\theta) = \begin{cases} c\left((az_\theta + b)^{-\frac{\delta}{a}}\right) + N & a \neq 0, \\ c\exp\left(-\frac{\delta}{b} z_\theta\right) + N & a = 0. \end{cases}$$

The integration constant c is determined by the condition that H_θ^* equals zero at the lower bound of its support. To determine the support, consider the quality of an intermediate introduced at $t = 0$. In a BGP with $K_\theta(t) = K_\theta(0)\exp\left(g_\theta^* t\right)$, we can integrate the evolution equation (8) for q_θ to find

$$q_\theta(t) = \begin{cases} \left[-\frac{b}{a} + \left(1 + \frac{b}{a}\right)\exp(at)\right]\sigma_E K_\theta(t) & a \neq 0, \\ (1 + bt)\,\sigma_E K_\theta(t) & a = 0. \end{cases}$$

Hence

$$z_\theta(t) = \begin{cases} -\frac{b}{a} + \left(1 + \frac{b}{a}\right)\exp(at) & a \neq 0, \\ 1 + bt & a = 0. \end{cases}$$

With $a > 0$ or $b > 0$, z_θ is strictly increasing over time and limits to infinity. Since firms exit at a finite rate, the BGP distribution H_θ^* must then have support $[1, \infty)$. In fact, I show below that $a = 0$ implies $b > 0$, so that H_θ^* has support $[1, \infty)$ when $a \geq 0$. If instead $a < 0$, I conjecture and verify below that $b > -a$. The expression above then implies that z_θ is strictly increasing over time and limits to the finite value $-b/a > 1$, so the distribution H_θ^* has bounded support $[1, -b/a]$.

In all cases, the integration constant c is determined by the initial condition $H_\theta^*(1) = 0$. Solving this equation for c and substituting yields

$$H_\theta^*(z_\theta) = \begin{cases} N\left[1 - \left(1 + \frac{a}{\delta}\frac{z_\theta - 1}{\frac{a+b}{\delta}}\right)^{-\frac{\delta}{a}}\right] & a \neq 0, \\ N\left[1 - \exp\left(-\frac{\delta}{b}(z_\theta - 1)\right)\right] & a = 0. \end{cases}$$

Hence H_θ^* is a generalized Pareto distribution with location parameter 1, shape parameter $\varphi_\theta^* = \frac{a+b}{\delta}$, and tail parameter $\xi_\theta^* = \frac{a}{\delta}$. Provided that $\xi_\theta^* < 1$, the mean of this distribution (scaled by N) is

$$\int z_\theta dH_\theta^*(z_\theta) = N\frac{\delta + b}{\delta - a} = N\frac{\delta + \frac{\sigma_I}{\sigma_E}\eta_\theta s}{\delta - (\lambda\eta_\theta s - g_\theta^*)}$$

Substituting into (A1), the growth rate g_θ^* must solve the fixed-point equation

$$g_\theta^* = \left[\lambda\frac{\sigma_E\delta + \sigma_I\eta_\theta s}{\delta - \left(\lambda\eta_\theta s - g_\theta^*\right)} + \sigma_I\right]\eta_\theta S.$$

Multiplying through by the denominator on the right side yields a quadratic equation in g_θ^*, and the solution is given by the expression for g_θ^* in the Proposition.[36] With this expression, we immediately observe that the assumption of positive spillovers from innovation $\lambda\sigma_E + \sigma_I > 0$ ensures that g_θ^* is positive, and in fact large enough to guarantee $\xi_\theta^* < 1$: This inequality holds if and only if $g_\theta^* > \lambda\eta_\theta s - \delta$. This is immediate if $\lambda\eta_\theta s < \delta$, and otherwise it follows from $\lambda\sigma_E + \sigma_I > 0$ after substituting the expression for g_θ^* from the Proposition.

It remains to verify that $a = 0$ implies $b > 0$ and $b > -a$. For the former, note that $b \geq 0$, with equality if and only if $\sigma_I = 0$. If both $a = 0$ and $\sigma_I = 0$, the fixed-point equation for g_θ^* above simplifies to $s = \sigma_E S$, contradicting the assumed upper bound on entrant spillovers (19). To see that $b > -a$, note that this holds if and only if $\lambda + \frac{\sigma_I}{\sigma_E}\eta_\theta s > g_\theta^*$. After substituting the expression for g_θ^* from the Proposition, direct calculation shows that this inequality is also implied by the assumed upper bound on entrant spillovers (19). Note that this implies that the shape parameter φ_θ^* is positive.

To conclude, note that in all cases the assumed lower bound for ρ (20) ensures that the consumer's transversality condition (3) holds. Moreover, the expression for $q_\theta(t)$ derived avove implies an asymptotic growth rate of $\lambda\eta_\theta s$ for flow profits, so the assumed lower bound (14) on the exit rate δ ensures that the value function $V(q,t)$ remains finite. ■

Proof of Corollary 1: I first state the complete set of comparative statics: The tail parameter ξ_θ^* is strictly decreasing in σ_E, σ_I, and S, and it is strictly increasing in λ and s. If σ_I is sufficiently close to zero, ξ_θ is strictly increasing in η_θ and strictly decreasing in δ, while the opposite holds for σ_I sufficiently large.

The comparative statics with respect to σ_E, σ_I, and S follow immediately by the formula for ξ_θ

[36] The lower root of the quadratic equation is negative and hence extraneous.

in Proposition 1. For the comparative static with respect to s, we can differentiate the equation for g_θ to find

$$\frac{\partial g_\theta}{\partial s} = \frac{\lambda\eta_\theta}{2}\left[1 - \frac{\frac{\delta-\lambda\eta_\theta s-\sigma_I\eta_\theta S}{2}}{\sqrt{\left(\frac{\delta-\lambda\eta_\theta s-\sigma_I\eta_\theta S}{2}\right)^2 + (\lambda\sigma_E+\sigma_I)\,\delta\eta_\theta S}}\right].$$

With $\lambda\sigma_E + \sigma_I > 0$ by assumption, the term in brackets is bounded strictly below 2. Thus $\partial g_\theta/\partial s < \lambda\eta_\theta$, and ξ_θ is strictly increasing in s. A similar calculation implies $\partial g_\theta/\partial\lambda < \eta_\theta s$, making use of the assumed upper bound to spillovers on entrants (19). This implies that ξ_θ is strictly increasing in λ. Finally, for η_θ,

$$\frac{\partial g_\theta}{\partial \eta_\theta} = \frac{\lambda s+\sigma_I S}{2} + \frac{-\frac{\lambda s+\sigma_I S}{2}\frac{\delta-\lambda\eta_\theta s-\sigma_I\eta_\theta S}{2} + (\lambda\sigma_E+\sigma_I)\,\delta S}{\sqrt{\left(\frac{\delta-\lambda\eta_\theta s-\sigma_I\eta_\theta S}{2}\right)^2 + (\lambda\sigma_E+\sigma_I)\,\delta\eta_\theta S}}.$$

If $\sigma_I = 0$, this value is below λs if and only if $s > \sigma_E S$, which holds by assumption (19). However, this value also limits to infinity as $\sigma_I \to \infty$. Thus ξ_θ is increasing in η_θ for σ_I small, while it is decreasing in η_θ for σ_I large. Since g_θ is linearly homogeneous in (δ, η_θ), the opposite comparative statics hold for δ: ξ_θ is decreasing in δ for σ_I small, while it is increasing in δ for σ_I large. ■

A.2 Proofs for Section 4

Proof of Proposition 3: See Appendix B.2. ■

Proof of Proposition 4: See Appendix B.3. ■

Proof of Proposition 5: See the proof of Proposition C.4 in Appendix C. ■

Proof of Proposition 6: See the proof of Proposition C.5 in Appendix C. ■

B Monotone Equilibria

This appendix characterizes and proves the existence of monotone equilibria.

B.1 Optimal Stopping: Proof of Proposition 2

To facilitate the analysis of monotone equilibria, I first prove several properties of solutions to the firm's problem (11) given an arbitrary but differentiable knowledge stock trajectory $[K(t)]_t$. Define the notation

$$\begin{aligned} k_\theta(t) &\equiv \lambda q_\theta(t) + \sigma_I K_\theta(t), \\ \Psi_\theta(T) &\equiv \int_T^\infty \exp\left(-\int_T^t [\rho + \delta - \lambda\eta_\theta s_\theta(q(\tau),\tau)]\, d\tau\right) dt, \\ \dot{k}_\theta(T-) &\equiv \lim_{t\uparrow T} \dot{k}(t). \end{aligned}$$

Note that $\Psi_\theta(T)$ satisfies the bounds

$$\Psi_\theta(T) \in \left[\frac{1}{\rho+\delta}, \frac{1}{\rho+\delta-\lambda\eta_\theta s}\right].$$

The following lemma provides a stopping time representation of solutions to the firm's problem (11), implying Proposition 2 in Section 4.1.

Lemma B.1. *Suppose $[K(t)]_t$ is differentiable, and suppose a piecewise-continuous solution $[s_\theta(q(t),t)]_t$ to (11) given initial qualities $q(t_0)$ at time $t_0 \geq 0$. Then without loss of generality $s_\theta(q(t),t) \in \{0,s\}$ for all $t \geq t_0$, and there exists a sequence of stopping times $t_0 < T_1 \leq T_2 \leq \ldots$ such that*

(i) if $s_\theta(q(t),t) = s$ for $t \in [T_m, T_{m+1})$, then $s_\theta(q(t),t) = 0$ for $t \in [T_{m+1}, T_{m+2})$;

(ii) every positive stopping time $T_m > 0$ satisfies the smooth-pasting condition

$$k_B(T_m)\eta_B\Psi_B(T_m) = k_A(T_m)\eta_A\Psi_A(T_m);$$

(iii) every positive stopping time $T_m > 0$ at which the firm transitions from A to B satisfies the necessary second-order condition

$$\frac{\dot{k}_B(T_m-)}{k_B(T_m)} + \rho + \delta - \lambda\eta_B s - \frac{1}{\Psi_B(T_m)} \geq \frac{\dot{k}_A(T_m-)}{k_A(T_m)} + \rho + \delta - \frac{1}{\Psi_A(T_m)}; \text{ and}$$

(iv) *every positive stopping time $T_m > 0$ at which the firm transitions from B to A satisfies the necessary second-order condition*

$$\frac{\dot{k}_B(T_m-)}{k_B(T_m)} + \rho + \delta - \frac{1}{\Psi_B(T_m)} \leq \frac{\dot{k}_A(T_m-)}{k_A(T_m)} + \rho + \delta - \lambda\eta_A s - \frac{1}{\Psi_A(T_m)}.$$

Proof of Lemma B.1: The optimality of corner allocations $s_\theta(q(t),t) \in \{0,s\}$ follows immediately from the linearity of the evolution equation (8) in $s_\theta(q,t)$. Hence any solution can be identified with a sequence of (potentially infinite) stopping times $t_0 < T_1 \leq T_2 \leq \ldots$ that prescribe when the firm should reverse its innovation direction.

Consider any positive stopping time $T_m > 0$, and suppose without loss of generality that the firm transitions from A to B at T_m. We can directly integrate the quality evolution equation (8) to find that for $t \geq T_m$,

$$\begin{aligned} q_\theta(t) = {} & \exp\left(\lambda\eta_\theta \int_{T_m}^{t} s_\theta(q(\tau),\tau)d\tau\right) q_\theta(T_m) \\ & + \sigma_I\eta_\theta \int_{T_m}^{t} s_\theta(q(t'),t')K_\theta(t')\exp\left(\lambda\eta_\theta \int_{t'}^{t} s_\theta(q(\tau),\tau)d\tau\right)dt'. \end{aligned}$$

The incumbent's value at t_0 is

$$V(q(t_0),t_0) = \bar{\pi}\int_{t_0}^{\infty} \exp\left(-(\rho+\delta)t\right)\left[q_A(t) + q_B(t)\right]dt.$$

The stopping time $T_m > 0$ must satisfy the interior first-order condition

$$0 = \frac{\partial}{\partial T_m}\frac{V(q(t_0),t_0)}{\bar{\pi}} = \int_{T_m}^{\infty} \exp\left(-(\rho+\delta)t\right)\left[\frac{\partial q_A(t)}{\partial T_m} + \frac{\partial q_B(t)}{\partial T_m}\right]dt.$$

For $t \geq T_m$, we can directly calculate

$$\frac{\partial q_\theta(t)}{\partial T_m} = \exp\left(\lambda\eta_\theta \int_{T_m}^{t} s_\theta(q(\tau),\tau)d\tau\right)\left[\dot{q}_\theta(T_m-) - \dot{q}_\theta(T_m+)\right].$$

Here $\dot{q}_\theta(T_m-)$ denotes the evolution of q_θ just before T_m, while $\dot{q}_\theta(T_m+)$ denotes the evolution just after T_m. The interior first-order condition for T_m simplifies to

$$0 = \sum_{\theta\in\{A,B\}} \left[\dot{q}_\theta(T_m-) - \dot{q}_\theta(T_m+)\right]\Psi_\theta(T_m).$$

Now given that the firm transitions from A to B at T_m< we have

$$\dot{q}_A(T_m-) - \dot{q}_A(T_m+) = k_A(T_m)\eta_A s,$$
$$\dot{q}_\theta(T_m-) - \dot{q}_\theta(T_m+) = -k_B(T_m)\eta_B s.$$

Hence the interior first-order condition is exactly the smooth-pasting condition stated in the Lemma:

$$k_B(T_m)\eta_B\Psi_B(T_m) = k_A(T_m)\eta_A\Psi_A(T_m).$$

The second-order necessary condition for T_m requires that the left side of this equation be weakly increasing relative to the right side just before T_m. Log differentiating yields the necessary condition stated in the Lemma. The corresponding necessary condition for a transition from B to A is derived analogously. ■

B.2 Equilibrium Characterization: Proof of Proposition 3

In this section, I prove Proposition 3 and characterize all monotone equilibria converging to the BGP for technology B. I then state the analogous result for monotone equilibria converging to the BGP for A. The proof of this result is almost identical to that for Proposition 3, so I only sketch the differences.

B.2.1 Equilibria Converging to B: Proof of Proposition 3

Step 1: T_E existence. The time $T_E \geq 0$ corresponds to the first time at which entrants begin innovating immediately for technology B. The existence of this time is immediate: If the equilibrium converges to B, all entrants must eventually innovate for technology B exclusively. Hence there exists a smallest time $T_E < \infty$ after which this holds, and monotonicity implies that all entrants at $t \in [0, T_E)$ initially innovate for technology A.

Step 2: $\chi(t)$ existence. To prove the existence of the cutoff $\chi(t)$, fix a time $t_0 \geq 0$ and a firm with qualities $q_\theta(t_0)$. Given trajectories for the knowledge stocks $[K(t)]_t$ and the firm's allocation of scientists $[s_\theta(q(t), t)]_{\theta,t}$, we can directly integrate the quality evolution equation

(8) to find

$$q_\theta(t) = \exp\left(\lambda\eta_\theta \int_{t_0}^{t} s_\theta(q(\tau),\tau)d\tau\right) q_\theta(t_0)$$
$$+\sigma_I\eta_\theta \int_{t_0}^{t} s_\theta(q(t'),t')K_\theta(t')\exp\left(\lambda\eta_\theta \int_{t'}^{t} s_\theta(q(\tau),\tau)d\tau\right)dt'.$$

This value is convex and supermodular in $[s_\theta(q(\tau),\tau)]_{\tau\in(t_0,t]}$, with increasing differences in $q_\theta(t_0)$ and $[s_\theta(q(\tau),\tau)]_{\tau\in(t_0,t]}$. Since flow profits are linear in qualities, the objective of the firm's problem (11) must also be supermodular in $[s_A(q(t),t)]_t$, with increasing differences in $q_A(t_0)$ and $[s_A(q(t),t)]_t$. Theorem 4 of Milgrom and Shannon (1994) then implies that the optimal value of $s_A(q(t),t)$ is non-decreasing in $q_A(t_0)$ at each $t \geq t_0$.

This observation immediately implies the existence of the cutoff $\chi(t)$: First setting $t_0 = 0$, we can observe that initial incumbents with higher relative qualities $z_A(0)$ transition later to technology B (if at all). The expression for $q_\theta(t)$ above implies that $q_A(t)$ is strictly increasing in $z_A(0)$ for these firms, so that the ratio

$$\frac{k_B(t)}{k_A(t)} = \frac{\lambda q_B(t) + \sigma_I K_B(t)}{\lambda q_A(t) + \sigma_I K_A(t)}$$

is always strictly decreasing in $z_A(0)$. If $T_E = 0$, we can simply define $\chi(t)$ as the least upper bound of this ratio across all initial incumbents still innovating for technology A. If instead $T_E > 0$, then entrants at each time $t_0 \in [0, T_E)$ may also innovate for technology A. But each such firm always has a lower A quality $q_A(t)$ than an initial incumbent who innovates for A, and moreover this quality is decreasing with the entry time t_0. The monotone comparative static described above implies that entrants always transition earlier to technology B (if at all), with the transition time weakly decreasing in the entry time t_0. Since the ratio $\frac{k_B(t)}{k_A(t)}$ is decreasing in $q_A(t)$, we can again define the cutoff $\chi(t)$ as the least upper bound of this ratio across all firms still innovating for A.

Step 3: $\chi(t)$ characterization. If $T_E > 0$, the construction above shows that we can set

$$\chi(t) = \frac{K_B(t)}{K_A(t)} \quad \text{for } t \in [0, T_E],$$

as all initial incumbents and entrants innovate for technology A before T_E. If instead $T_E = 0$, then the initial value of the threshold $\chi(0)$ is indeterminate but weakly above $\frac{K_B(0)}{K_A(0)}$. This implies the complementary slackness condition (25).

After T_E, new entrants permanently innovate for technology B, while all remaining firms must choose whether to continue innovating for A or transition back to B. Lemma B.1 implies that any firm transitioning to B at $T > T_E$ must satisfy the smooth-pasting condition

$$\frac{k_B(T)}{k_A(T)} = \frac{\eta_A}{\eta_B}\frac{\rho+\delta-\lambda\eta_B s}{\rho+\delta}, \tag{B1}$$

with the left side increasing at T. I claim that these conditions are met at most once for each firm that initially innovates for technology A.

To prove this fact, it is helpful to first derive evolution equations for the knowledge stocks $K(t)$. Given that the economy starts along the BGP for A at $t=0$, for $t \in [0, T_E)$ we trivially have

$$\frac{\dot{K}_A(t)}{K_A(t)} = g_A^* \quad \text{and} \quad \frac{\dot{K}_B(t)}{K_B(t)} = 0.$$

For $t \geq T_E$, we can can explicitly write evolution equations starting from the definition (9). Let $T(z_A) \geq 0$ denote the transition time for a firm with relative quality z_A for technology A at T_E. Let $\tilde{z}_A(t)$ denote the inverse of this function, taking value 1 outside its image. Since the distribution of relative qualities z_A at T_E is just the BGP distribution H_A^*, the evolution equation for K_A can then be written

$$\dot{K}_A(t) = \exp\left(-\delta(t-T_E)\right)\int_{\tilde{z}_A(t)}^{\infty}\left[\lambda q_A(t;z_A)+\sigma_I K_A(t)\right]\eta_A s\, dH_A^*(z_A),$$

where $q_A(t;z_A)$ denotes the quality for a firm with relative quality z_A at T_E. Let $N_\theta(t)$ denote the total mass of firms innovating for technology θ at $t \geq T_E$, where

$$N_A(t) = \exp\left(-\delta(t-T_E)\right)\left[N - H_A(\tilde{z}_A(t))\right] \quad \text{and} \quad N_B(t) = N - N_A(t).$$

Then we can differentiate again to derive the second-order differential equation

$$\begin{aligned}\ddot{K}_A(t) = &-\left[\delta-(\lambda+\sigma_I N_A(t))\,\eta_A s\right]\dot{K}_A(t) \\ &-\frac{\rho+\delta}{\rho+\delta-\lambda\eta_B s}\left[\lambda\sigma_E K_B(0)+\sigma_I K_B(t)\right]\eta_B s \exp\left(-\delta(t-T_E)\right)h_A^*(\tilde{z}_A(t))\dot{\tilde{z}}_A(t).\end{aligned} \tag{B2}$$

Turning to technology B, for $t \geq T_E$ we can similarly write the evolution equation (9) as

$$\begin{aligned}\dot{K}_B(t) &= \delta N \int_{T_E}^{t} \exp(-\delta(t-t_0))\left[\lambda q_B(t;t_0) + \sigma_I K_B(t)\right]\eta_B s\, dt_0 \\ &\quad + \int_{1}^{\tilde{z}_A(t)} \exp(-\delta(t-T_E))\left[\lambda q_B(t;z_A) + \sigma_I K_B(t)\right]\eta_B s\, dH_A^*(z_A),\end{aligned}$$

where I abuse notation by letting $q_B(t;t_0)$ denote the quality for a firm that entered at time $t_0 \geq T_E$ and $q_B(t;z_A)$ the quality of a firm with relative quality z_A for A at T_E. Differentiating again yields

$$\begin{aligned}\ddot{K}_B(t) &= -\left[\delta - (\lambda + \sigma_I N_B(t))\eta_B s\right]\dot{K}_B(t) + \delta N(\lambda\sigma_E + \sigma_I)\eta_B s K_B(t) \\ &\quad + \left[\lambda\sigma_E K_B(0) + \sigma_I K_B(t)\right]\eta_B s \exp(-\delta(t-T_E))\, h_A^*(\tilde{z}_A(t))\dot{\tilde{z}}_A(t).\end{aligned} \tag{B3}$$

The evolution equations (B2, B3) have two important implications for the remainder of the proof. First, (B2) easily implies that $\ddot{K}_A(t) < 0$ because of the assumed lower bound (14) on δ. Second, (B3) implies that the growth rate $g_B(t) \equiv \frac{\dot{K}_B(t)}{K_B(t)}$ is strictly increasing to its BGP value g_B^*. This growth rate evolves according to

$$\begin{aligned}\dot{g}_B(t) &= \frac{\ddot{K}_B(t)}{K_B(t)} - g_B(t)^2 \\ &\geq \delta N(\lambda\sigma_E + \sigma_I)\eta_B s - \left[\delta - (\lambda + \sigma_I N_B(t))\eta_B s + g_B(t)\right]g_B(t).\end{aligned}$$

The right side of this inequality must be strictly positive for $t < \infty$: Define a new function $\check{g}(t)$ such that $\check{g}(T_E) = g_B(T_E)$, but with evolution equation

$$\dot{\check{g}}(t) = \delta N(\lambda\sigma_E + \sigma_I)\eta_B s - \left[\delta - (\lambda + \sigma_I N_B(T_E))\eta_B s + \check{g}_B(t)\right]\check{g}(t).$$

Direct calculation implies $\dot{\check{g}}(T_E) > 0$, making use of the assumed upper bound (19) on spillovers to entrants σ_E. With $\dot{N}_B(t) > 0$, we then have $\dot{\check{g}}(t) > 0$ and hence $\dot{g}_B(t) > 0$.

Returning to the smooth-pasting condition (B1), consider any firm innovating for technology A at $t = T_E$. If $\lambda\eta_A s > g_B^*$, it is easy to see that the firm never innovates for technology B: Lemma B.1 implies that the necessary second-order condition corresponding to the smooth-pasting condition (B1) is

$$\frac{\sigma_I K_B(T)}{k_B(T)} g_B(T) \geq \lambda\eta_A s + \frac{\sigma_I K_A(T)}{k_A(T)} g_A(T). \tag{B4}$$

But this inequality must be violated when $\lambda\eta_A s > g_B^*$, because the growth rate $g_B(T)$ is strictly lower than its BGP value g_B^* and $\sigma_I K_B(T) < k_B(T)$ by definition.[37]

Suppose instead $\lambda\eta_A s < g_B^*$ and $\sigma_I > 0$. I claim that any firm innovating for technology A at $t = T_E$ must have a finite stopping time $T > 0$ after which it permanently innovates for technology B. The conditions $\lambda\eta_A s < g_B^*$ and $\sigma_I > 0$ easily imply that such a stopping time exists: With $g_B(t) \uparrow g_B^*$ and $g_A(t) \downarrow 0$, the technology B research productivity $k_B(t)$ eventually grows faster than the technology A research productivity $k_A(t)$. Moreover, it is straightforward to see that T is the unique solution to the smooth-pasting condition (B1) that satisfies the corresponding second-order condition (B4). While the firm innovates for technology A, the left side of this condition is strictly increasing because both $g_B(t)$ and $\sigma_I K_B(T)/k_B(T)$ are strictly increasing. The right side satisfies

$$\frac{\partial}{\partial T}\frac{\dot{K}_A(T)}{k_A(T)} = \frac{\dot{K}_A(T)}{k_A(T)}\left(\frac{\ddot{K}_A(T)}{\dot{K}_A(T)} - \frac{\lambda\eta_A s + \sigma_I \dot{K}_A(T)}{k_A(T)}\right)$$

This value is negative because $\ddot{K}_A(T) < 0$. Thus for any firm that innovates for technology A at $t = T_E$, the smooth-pasting and necesssary second-order conditions for a permanent transition to technology B are satisfied exactly once, and they provide a complete characterization of the transition time T.

Returning to the characterization of $\chi(t)$: The first firm to transition must have entered just before T_E, so we can define $\chi(t)$ to coincide with this firm's ratio $\frac{k_B(t)}{k_A(t)}$ until it reaches the right side of the smooth-pasting condition (B4), after which χ remains constant. This yields the evolution equation (24) stated in the Proposition, noting that the ratio $\frac{k_B(t)}{k_A(t)}$ satisfies

$$\begin{aligned}\frac{d}{dt}\log\left(\frac{k_B(t)}{k_A(t)}\right) &= \frac{\sigma_I \dot{K}_B(t)}{k_B(t)} - \frac{\sigma_I \dot{K}_A(t)}{k_A(t)} - \lambda\eta_A s \\ &= \sigma_I \frac{\dot{K}_B(t) - \frac{k_B(t)}{k_A(t)}\dot{K}_A(t)}{\lambda K_B(T_E) + \sigma_I K_B(t)} - \lambda\eta_A s.\end{aligned}$$

Moreover, the arguments above imply that the cutoff $\chi(t)$ yields a valid description of firm innovation decisions: For any firm innovating for technology A at $t = T_E$, the ratio $\frac{k_B(t)}{k_A(t)}$ is strictly above $\chi(t)$ until it satisfies the smooth-pasting condition (B4), at which time the firm transitions to B.

[37] Here $g_B(t) < g_B^*$ follows by noting that at each time $t < \infty$, fewer than N firms innovate for B, and the distribution of relative qualities $H_B(z_B, t)$ is first-order stochastically dominated by the BGP distribution $H_B^*(z_B)$.

Step 4: $K(t)$ evolution. To conclude the proof, I provide a self-contained version of the evolution equations (B2, B3) for the knowledge stocks. Given $T_E \geq 0$ and $\chi(T_E)$, let $z_{A0} \geq 1$ denote the time T_E relative quality of the firm with initial ratio $\frac{k_B(T_E)}{k_A(T_E)}$ equal to $\chi(T_E)$:

$$\frac{\lambda\sigma_E + \sigma_I}{\lambda\sigma_E z_{A0} + \sigma_I} \frac{K_B(T_E)}{K_A(T_E)} = \chi(T_E). \tag{B5}$$

Let T_0 denote the time at which this firm transitions to technology B. For $t \in [T_E, T_0)$, the knowledge stocks evolve according to

$$\begin{aligned}
\ddot{K}_A(t) &= -[\delta - (\lambda + \sigma_I N_A(t))\eta_A s]\dot{K}_A(t),\\
\ddot{K}_B(t) &= -[\delta - (\lambda + \sigma_I N_B(t))\eta_B s]\dot{K}_B(t) + \delta N(\lambda\sigma_E + \sigma_I)\eta_B s K_B(t),
\end{aligned}$$

where $N_A(t) = \exp(-\delta(t - T_E))[N - H_A(z_{A0})]$ and $N_B(t) = N - N_A(t)$. The initial conditions for these differential equations are

$$\begin{aligned}
&K_A(T_E), \quad K_B(T_E),\\
&\frac{\dot{K}_A(T_E)}{K_A(T_E)} = \int_{z_{A0}}^{\infty} [\lambda\sigma_E z_A + \sigma_I]\eta_A s dH_A(z_A),\\
&\frac{\dot{K}_B(T_E)}{K_B(T_E)} = [\lambda\sigma_E + \sigma_I]\eta_B s H_A(z_{A0}).
\end{aligned} \tag{B6}$$

To determine the time T_0, for arbitrary $z_A \geq z_{A0}$ let $k_A(t; z_A)$ solve the differential equation

$$\dot{k}_A(t; z_A) = \lambda\eta_A s k_A(t; z_A) + \sigma_I \dot{K}_A(t)$$

with initial condition $k_A(T_E; z_A) = (\lambda\sigma_E z_A + \sigma_I)K_A(T_E)$. Then setting $z_A = z_{A0}$, if $\lambda\eta_A s < g_B^*$ the time T_0 is the unique time at which the ratio

$$\frac{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)}{k_A(t; z_{A0})}$$

is increasing and satisfies the smooth-pasting condition (B1). The inequality $t \geq T_0$ then holds if and only if

$$\begin{aligned}
&\frac{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)}{k_A(t; z_{A0})} \geq \frac{\eta_A}{\eta_B}\frac{\rho + \delta - \lambda\eta_B s}{\rho + \delta} \quad \text{and}\\
&\frac{\sigma_I \dot{K}_B(t)}{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)} \geq \lambda\eta_A s + \frac{\sigma_I \dot{K}_A(t)}{k_A(t; z_{A0})}.
\end{aligned} \tag{B7}$$

If $\lambda\eta_A s > g_B^*$, the second inequality in this condition cannot be satisfied; hence $T_0 = \infty$.

For $t \geq T_0$, we can explicitly identify the time T_E relative quality z_A of firms transitioning to technology B. This function is simply $\tilde{z}_A(t)$ defined above, and for $t \geq T_0$ it satisfies the smooth-pasting condition

$$\frac{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)}{k_A(t;\tilde{z}_A(t))} = \frac{\eta_A}{\eta_B}\frac{\rho+\delta-\lambda\eta_B s}{\rho+\delta}.$$

Integrating the evolution equation for $k_A(t;z_A)$ reveals that this function is linear in z_A:

$$k_A(t;z_A) = k_A(t;z_{A0}) + (z_A - z_{A0})\exp(\lambda\eta_A s t)\,\lambda\sigma_E K_A(T_E).$$

Substituting into the previous equation yields an implicit equation for $\tilde{z}_A(t)$:

$$\frac{\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)}{k_A(t;z_{A0}) + (\tilde{z}_A(t) - z_{A0})\exp(\lambda\eta_A s t)\,\lambda\sigma_E K_A(T_E)} = \frac{\eta_A}{\eta_B}\frac{\rho+\delta-\lambda\eta_B s}{\rho+\delta}.$$

Solving for $\tilde{z}_A(t)$ and recalling the characterization (B7) of T_0, we can write

$$\begin{aligned}&\tilde{z}_A(t) \\ &= z_{A0} + \frac{\exp(-\lambda\eta_A s t)}{\lambda\sigma_E K_A(T_E)}\left\{[\lambda\sigma_E K_B(T_E) + \sigma_I K_B(t)]\frac{\eta_B}{\eta_A}\frac{\rho+\delta}{\rho+\delta-\lambda\eta_B s} - k_A(t;z_{A0})\right\}\mathbb{1}[(\text{B7})].\end{aligned} \tag{B8}$$

To summarize, I find that the knowledge stocks satisfy the dynamical system

$$\begin{cases}\ddot{K}_A(t) & = -[\delta - (\lambda + \sigma_I N_A(t))\,\eta_A s]\,\dot{K}_A(t) \\ & \quad -\frac{\rho+\delta}{\rho+\delta-\lambda\eta_B s}[\lambda\sigma_E K_B(0) + \sigma_I K_B(t)]\,\eta_B s\exp(-\delta(t-T_E))\,h_A^*(\tilde{z}_A(t))\dot{\tilde{z}}_A(t), \\ \ddot{K}_B(t) & = -[\delta - (\lambda + \sigma_I N_B(t))\,\eta_B s]\,\dot{K}_B(t) + \delta N\,(\lambda\sigma_E + \sigma_I)\,\eta_B s K_B(t) \\ & \quad +[\lambda\sigma_E K_B(0) + \sigma_I K_B(t)]\,\eta_B s\exp(-\delta(t-T_E))\,h_A^*(\tilde{z}_A(t))\dot{\tilde{z}}_A(t), \\ \dot{k}_A(t;\tilde{z}_A) & = \lambda\eta_A s k_A(t;z_A) + \sigma_I \dot{K}_A(t),\end{cases} \tag{B9}$$

where $\tilde{z}_A$ is given by (B8) and z_{A0} is given by (B5). The initial conditions are (B6) and $k_A(T_E;z_{A0}) = (\lambda\sigma_E z_{A0} + \sigma_I)K_A(T_E)$. ■

B.2.2 Equilibria Converging to A

The following proposition characterizes all monotone equilibria converging to the BGP for technology A, analogous to Proposition 3 in Section 4.2:

Proposition B.1. *In any monotone equilibrium converging to the BGP for technology A, there exists a cutoff $\chi(t)$ such that a firm innovates for B if and only if*

$$\frac{k_B(t)}{k_A(t)} \geq \chi(t).$$

There exists a time $T_E \geq 0$ such that $\chi(t) = \min\{\chi(0), \kappa(t)\}$ for $t \in [0, T_E]$, with $\chi(T_E) = \kappa(T_E)$. For $t \in (T_E, \infty)$, the cutoff solves the differential equation

$$\frac{\dot{\chi}(t)}{\chi(t)} = \left(\lambda\eta_B s + \sigma_I \frac{\frac{1}{\chi(t)}\dot{K}_B(t) - \dot{K}_A(t)}{\lambda\sigma_E K_A(T_E) + \sigma_I K_A(t)}\right) \mathbb{1}\left[\chi(t) \leq \frac{\eta_A}{\eta_B}\frac{\rho+\delta}{\rho+\delta-\lambda\eta_A s}\right]. \tag{B10}$$

The equilibrium is unique up to the parameters $\chi(0)$ and $T_E \geq 0$. The knowledge stocks $K(t)$ are the solutions to the dynamical system (...).

Note several differences from Proposition 3: First, firms choose B provided that the weak inequality $\frac{k_B(t)}{k_A(t)} \geq \chi(t)$ holds, instead of the strong inequality $\frac{k_B(t)}{k_A(t)} > \chi(t)$. This ensures that entrants begin innovating for technology B for $t \leq T_E$. The evolution equation (B10) ensures that $\chi(t)$ equals the knowledge ratio $\frac{k_B(t)}{k_A(t)}$ for the last entrant to innovate first for technology B, until the following smooth-pasting condition is satisfied:

$$\frac{k_B(t)}{k_A(t)} = \frac{\eta_A}{\eta_B}\frac{\rho+\delta}{\rho+\delta-\lambda\eta_A s}.$$

After this time, $\chi(t)$ remains equal to the expression on the right side.

To derive the dynamical system for the knowledge stocks $K(t)$, note first that

B.3 Equilibrium Existence: Proof of Proposition 4

In this section, I provide a full proof of the existence of monotone equilibria converging to the BGP for technology B, given the assumptions of Proposition 4. I sketch the analogous proof for equilibria converging to the BGP for A, which is almost identical. Finally, I show that multiple equilibria arise whenever $\lambda, \sigma_I > 0$.

B.3.1 Equilibrium Converging to B

Given the characterization of monotone equilibria from Proposition 3, it is clear that if a monotone equilibrium with initiation time $T_E > 0$ exists, there also exists a monotone equilibrium with initiation time $T_E = 0$. To prove existence, I then confine attention to monotone equilibria in which all entrants exclusively innovate for technology B $(T_E = 0)$. Each such equilibrium is uniquely determined by the initial cutoff $\chi(0) \geq \frac{K_B(0)}{K_A(0)}$. I let $\tilde{z}_A$ denote the initial relative quality for an initial incumbent at the cutoff:

$$\frac{\lambda\sigma_E + \sigma_I}{\lambda\sigma_E \tilde{z}_A + \sigma_I}\kappa(0) = \chi(0).$$

The following proposition proves the existence of a monotone equilibrium converging to the BGP for technology B under weaker conditions than assumed in Proposition 4:

Proposition B.2. *There exists a threshold κ_B such that a monotone equilibrium converging to the BGP for technology B exists if the following hold:*

(i) For each $t \geq 0$

$$\lambda\eta_B s + \frac{\sigma_I \dot{K}_B(t)}{k_B(t)} \geq \frac{\sigma_I \dot{K}_A(t)}{k_A(t)}, \tag{B11}$$

where $\dot{k}_A(t) = \sigma_I \dot{K}_A(t)$ and $\dot{k}_B(t) = \lambda\eta_B s k_B(t) + \sigma_I \dot{K}_B(t)$ with $k_\theta(0) = (\lambda\sigma_E + \sigma_I)K_\theta(0)$;

(ii) $\kappa(0) \geq \kappa_B$.

The initial cutoff $\chi(0)$ is such that $\tilde{z}_A$ is a stable solution to the fixed-point equation (B12).

Proof of Proposition B.2: Fix an initial cutoff $\chi(0)$ and the corresponding initial relative quality $\tilde{z}_A$. The purpose of the proof is to demonstrate that the innovation decisions implied by the resulting cutoff $\chi(t)$ are privately optimal for all firms.

First consider an initial incumbent with relative quality $z_A \geq \tilde{z}_A$. This firm initially innovates for technology A, and the arguments in Step 3 of the proof of Proposition 3 demonstrate that the innovation decisions implied by the cutoff $\chi(t)$ are optimal.

Now consider an initial incumbent with relative quality $z_A < \tilde{z}_A$. This firm initially innovates for technology B, and the innovation decisions implied by the cutoff require that it never innovates for technology A. The firm's research productivities then evolve according to

$$\frac{\dot{k}_A(t)}{k_A(t)} = \frac{\sigma_I \dot{K}_A(t)}{k_A(t)} \quad \text{and} \quad \frac{\dot{k}_B(t)}{k_B(t)} = \lambda\eta_B s + \frac{\sigma_I \dot{K}_B(t)}{k_B(t)}.$$

If the incumbent reverted back to technology A at some time T, Lemma B.2 would require that

k_A and k_B satisfy the smooth-pasting condition

$$k_B(T)\eta_B\Psi_B(T) = k_A(T)\eta_A\Psi_A(T),$$

with the corresponding second-order necessary condition

$$\lambda\eta_B s + \frac{\sigma_I \dot{K}_B(t)}{k_B(t)} + \rho + \delta - \frac{1}{\Psi_B(T)} \le \frac{\sigma_I \dot{K}_A(t)}{k_A(t)} + \rho + \delta - \lambda\eta_A s - \frac{1}{\Psi_A(T)}.$$

Given the bounds $\Psi_\theta(T) \in \left[\frac{1}{\rho+\delta}, \frac{1}{\rho+\delta-\lambda\eta_\theta s}\right]$, this inequality is ruled out precisely by (B11).

Finally, note that entrants at $t_0 \ge 0$ innovate for technology B permanently if and only if this yields higher value than innovating initially for technology A and transitioning to technology B at a later stopping time $T(t_0) \ge t_0$. This stopping time is either ∞ (if $\sigma_I = 0$ or $\lambda\eta_A s \ge g_B^*$) or is given by the unique solution to the smooth-pasting condition

$$\frac{k_B(T(t_0);t_0)}{k_A(T(t_0);t_0)} = \frac{\eta_A}{\eta_B}\frac{\rho+\delta-\lambda\eta_B s}{\rho+\delta},$$

where k_A and k_B satisfy the evolution equations

$$\frac{\dot{k}_A(t;t_0)}{k_A(t;t_0)} = \lambda\eta_A s + \frac{\sigma_I \dot{K}_A(t)}{k_A(t;t_0)} \quad \text{and} \quad \frac{\dot{k}_B(t;t_0)}{k_B(t;t_0)} = \frac{\sigma_I \dot{K}_B(t)}{k_B(t;t_0)},$$

with initial conditions $k_\theta(t_0;t_0) = (\lambda\sigma_E + \sigma_I)K_\theta(t_0)$.

If an entrant at t_0 permanently innovates for technology B, let $q_\theta^B(t;t_0)$ denote the corresponding quality for technology θ at time t. These qualities satisfy the evolution equations

$$\begin{aligned}
\dot{q}_A^B(t;t_0) &= 0 \\
\dot{q}_B^B(t;t_0) &= \left[\lambda q_B^B(t;t_0) + \sigma_I K_B(t)\right]\eta_B s,
\end{aligned}$$

with initial conditions $q_\theta^B(t_0,t_0) = \sigma_E K_\theta(t_0)$. If an entrant instead innovates initially for technology A, let $q_\theta^A(t;t_0)$ denote the corresponding quality for technology θ. The evolution equations are now

$$\begin{aligned}
\dot{q}_A^A(t;t_0) &= \mathbb{1}_{t<T(t_0)}\left[\lambda q_A^A(t;t_0) + \sigma_I K_A(t)\right]\eta_A s \\
\dot{q}_B^B(t;t_0) &= \mathbb{1}_{t\ge T(t_0)}\left[\lambda q_B^B(t;t_0) + \sigma_I K_B(t)\right]\eta_B s,
\end{aligned}$$

with the same initial conditions as above. The entrant's value when choosing to innovate technology θ initially is

$$V_E^\theta(t_0) \equiv \bar{\pi} \int_{t_0}^{\infty} \exp\left(-(\rho+\delta)(t-t_0)\right)\left[q_A^\theta(t;t_0)+q_B^\theta(t;t_0)\right]dt.$$

Entrants at t_0 innovate for B if and only if $V_E^B(t_0) \geq V_E^A(t_0)$.

To determine when this condition holds, note first that the dynamical system (B9) that describes the evolution of the knowledge stocks is linearly homogeneous in $K(t)$ conditional on $\tilde{z}_A$. As a result, there exists a function $\tilde{K}(t;\tilde{z}_A,\kappa(0))$ that depends on initial conditions only through $\tilde{z}_A$ and $\kappa(0)$ such that

$$K(t) = \tilde{K}(t;\tilde{z}_A,\kappa(0))K_A(0).$$

This normalization implies $\tilde{K}_A(0;\tilde{z}_A,\kappa(0)) = 1$ and $\tilde{K}_B(0;\tilde{z}_A,\kappa(0)) = \kappa(0)$. To understand how these normalized knowledge stocks depend on $\tilde{z}_A$ and $\kappa(0)$, hold $\kappa(0)$ fixed and consider an increase in $\tilde{z}_A$, or equivalently a decrease in $\chi(0)$. This adjustment raises the initial growth rate of $\tilde{K}_B$ and lowers the initial growth rate of $\tilde{K}_A$, implying that initial incumbents transition more rapidly to B. These effects are mutually reinforcing, and they imply that $\tilde{K}_A$ and $\tilde{K}_B$ are respectively strictly decreasing and strictly increasing in $\tilde{z}_A$ at $t>0$. Now hold $\tilde{z}_A$ fixed, and consider an increase in $\kappa(0)$. This leaves the initial growth rates of $\tilde{K}_A$ and $\tilde{K}_B$ unaffected, and if all initial incumbents with $z_A > \tilde{z}_A$ never transition to B, it only raises $\tilde{K}_B$ for $t \geq 0$ while leaving $\tilde{K}_A$ unchanged. But if initial incumbents eventually transition to B, the increase in $\tilde{K}_B$ induces the incumbents to transition more rapidly. This leads to a further increase in $\tilde{K}_B$ and a decrease in $\tilde{K}_A$ for t sufficiently large.

These linear homogeneity and comparative dynamics observations carry over directly to the qualities $q_\theta^{\theta'}(t;t_0)$ and the values $V_E^\theta(t_0)$. In particular, the difference

$$\frac{V_E^B(t_0)}{K_A(0)} - \frac{V_E^A(t_0)}{K_A(0)}$$

depends on the initial conditions $K(0)$ only through $\kappa(0)$, and the Envelope Theorem implies that it is strictly increasing in $\kappa(0)$ and $\tilde{z}_A$ when $T(t_0)>0$. With $g_B(t)\uparrow 1$ and $g_A(t)\downarrow 0$, there also exists a finite time t_0 after which the difference is (weakly) positive. The Intermediate

Value Theorem implies a function $\kappa^E(t, \tilde{z}_A$ such that $V_E^B(t_0) \geq V_E^A(t_0)$ if and only if

$$\kappa(0) \geq \kappa^E(t_0, \tilde{z}_A),$$

where κ^E is strictly decreasing in $\tilde{z}_A$ and strictly decreasing in t_0 for t_0 sufficiently large.[38]

Now consider optimality for initial incumbents. Given an incumbent with relative quality z_A, denote the value of innovating initially for technology θ by

$$\tilde{V}^\theta(z_A; \tilde{z}_A, \kappa(0)) K_A(0),$$

where the same arguments as above imply linear homogeneity in $[K(t)]_t$. Similarly, the difference

$$\Delta(z_A; \tilde{z}_A, \kappa(0)) \equiv \tilde{V}^B(z_A; \tilde{z}_A, \kappa(0)) - \tilde{V}^A(z_A; \tilde{z}_A, \kappa(0))$$

is also strictly increasing in $\tilde{z}_A$ and $\kappa(0)$ when the firm's transition time is positive, while the Envelope Theorem implies that it is strictly decreasing in z_A (again when the transition time is positive). In equilibrium, $\tilde{z}_A \geq 1$ must be such that this difference is zero:[39]

$$\Delta(\tilde{z}_A; \tilde{z}_A, \kappa(0)) = 0. \tag{B12}$$

Note first that a solution can only exist for $\kappa(0)$ sufficiently large: (B12) is trivially violated when $\kappa(0) = 0$. Fixing any $z_A \geq 1$, we have that $\Delta(z_A; z_A, \kappa(0)) \to \infty$ as $\kappa(0) \to \infty$, so we can take any $\kappa(0)$ such that $\Delta(z_A; z_A, \kappa(0)) > 0$. It is also straightforward to see that $\Delta(z_A; z_A, \kappa(0)) \to -\infty$ as $z_A \to \infty$: Even with all initial incumbents innovating for technology

[38] In the case with $\sigma_I = 0$ or $\lambda\eta_A s \geq g_B$, we can derive an explicit formula:

$$\kappa^E(t, \tilde{z}_A) = \frac{\frac{\lambda\eta_A s}{\rho+\delta-\lambda\eta_A s}\frac{\sigma_E \tilde{K}_A(t;\tilde{z}_A)}{\rho+\delta} + \int_t^\infty \exp\left(-(\rho+\delta)(\tau-t)\right)\frac{\eta_A s\sigma_I \tilde{K}_A(\tau;\tilde{z}_A)}{\rho+\delta-\lambda\eta_A s}d\tau}{\frac{\lambda\eta_B s}{\rho+\delta-\lambda\eta_B s}\frac{\sigma_E \tilde{K}_B(t;\tilde{z}_A)}{\rho+\delta} + \int_t^\infty \exp\left(-(\rho+\delta)(\tau-t)\right)\frac{\eta_B s\sigma_I \tilde{K}_B(t;\tilde{z}_A)}{\rho+\delta-\lambda\eta_B s}d\tau}.$$

Here $\tilde{K}_A(t; \tilde{z}_A) \equiv K_A(t)/K_A(0)$ as in the text of the proof, while for this expression I instead define $\tilde{K}_B(t; \tilde{z}_A) \equiv K_B(t)/K_B(0)$. I can normalize both knowledge stocks independently because the function $\hat{z}_A(t)$ described in the dynamical system (B9) is constant in this case, so the differential equations for K are not interdependent conditional on $\tilde{z}_A$.

[39] Using the same assumptions and notation as in footnote 38, this equation can be written explicitly:

$$\frac{\lambda\eta_A s}{\rho+\delta-\lambda\eta_A s}\frac{\sigma_E}{\rho+\delta}\tilde{z}_A = \left[\frac{\lambda\eta_B s}{\rho+\delta-\lambda\eta_B s}\frac{\sigma_E}{\rho+\delta} + \int_0^\infty \exp\left(-(\rho+\delta)\tau\right)\frac{\eta_B s\sigma_I \tilde{K}_B(\tau;\tilde{z}_A)}{\rho+\delta-\lambda\eta_B s}d\tau\right]\kappa(0)$$
$$-\int_0^\infty \exp\left(-(\rho+\delta)\tau\right)\frac{\eta_A s\sigma_I \tilde{K}_A(\tau;\tilde{z}_A)}{\rho+\delta-\lambda\eta_A s}d\tau.$$

B, an incumbent's value from doing so remains uniformly bounded. By the Intermediate Value Theorem, these arguments imply that for $\kappa(0)$ sufficiently large, there exists a solution $\tilde{z}_A$ to the fixed-point equation (B12). This solution must also be *stable* in the sense that $\Delta(z_A; z_A, \kappa(0))$ is strictly declining in z_A in a neighborhood of the solution $\hat{z}_A(0)$. The Implicit Function Theorem then guarantees that this solution is strictly increasing in $\kappa(0)$.

To conclude, recall that entrant optimality requires $\kappa(0) \geq \kappa^E(t_0, \tilde{z}_A)$ for all $t_0 \geq 0$. With $\tilde{z}_A \geq 1$ and $\kappa(0) > 0$ fixed, there exists a time $\bar{t} \geq 0$ such that this inequality is satisfied for $t_0 \geq \bar{t}$. With κ^E strictly decreasing in $\tilde{z}_A$ and $\tilde{z}_A$ strictly increasing in $\kappa(0)$ for any stable solution to (B12), we also have that the entry condition can is satisfied for all $t_0 \geq 0$ provided that $\kappa(0)$ is sufficiently large. ■

The following lemma verifies that the condition (B11) is implied by the assumptions of Proposition 4.

Lemma B.2. *If $\lambda \eta_B s \geq \frac{\sigma_I}{\lambda \sigma_E + \sigma_I} g_A^*$, then (B11) holds in any equilibrium with $\ddot{K}_A(t) \leq 0$ for all $t \geq 0$ and $g_A(0) \leq g_A^*$.*

Proof of Lemma B.2: Differentiating the right side of (B11) in t yields

$$\frac{\partial}{\partial t} \frac{\sigma_I \dot{K}_A(t)}{k_A(t)} = \frac{\sigma_I \ddot{K}_A(t)}{k_A(t)} - \left(\frac{\sigma_I \dot{K}_A(t)}{k_A(t)} \right)^2 \leq 0.$$

Hence the right side of (B11) is bounded above by

$$\frac{\sigma_I K_A(0)}{k_A(0)} g_A(0) = \frac{\sigma_I}{\lambda \sigma_E + \sigma_I} g_A(0) \leq \frac{\sigma_I}{\lambda \sigma_E + \sigma_I} g_A^*.$$

■

B.3.2 Equilibrium Converging to A

C Equilibrium with Technology Choice: Benchmarks

In this appendix, I provide a full characterization of the economy's equilibria in two benchmark cases. In Appendix C.1, firms build exclusively on the aggregate knowledge stock when innovating ($\lambda = 0$), reflecting the most common way of modeling innovation in existing work on directed innovation and technological transitions. I show that this assumption leaves no role for firm heterogeneity or the concentraton of R&D to affect the economy's equilibrium. In Appendix C.2, I address the other extreme in which firms build exclusively on their own past advances ($\sigma_I = 0$), so that all knowledge spillovers are confined to entrants. As discussed in Section 4.3, in this case the economy's equilibrium depends richly on the initial distribution of incumbents and the concentration of scientists. Both cases feature a unique equilibrium, which is also monotone in the sense of Definition 3.

C.1 External Knowledge Accumulation: $\lambda = 0$

C.1.1 Equilibrium

Suppose firms build exclusively on external knowledge when innovating ($\lambda = 0 < \sigma_I$), and consider the problem of a firm with qualities $q(t_0)$ at time $t_0 \geq 0$ choosing its innovation direction at each time $t \geq t_0$. We can directly integrate the quality evolution equation (8) to find

$$q_\theta(t) = q_\theta(t_0) + \sigma_I \eta_\theta \int_{t_0}^{t} K_\theta(\tau) s_\theta(q(\tau), \tau) d\tau.$$

The firm's objective (11) can then be written

$$\frac{\bar{\pi}(q_A(t_0) + q_B(t_0))}{\rho + \delta} + \bar{\pi} \int_{t_0}^{\infty} \exp\left(-(\rho + \delta)(t - t_0)\right) \sum_{\theta \in \{A,B\}} \frac{\sigma_I K_\theta(t) \eta_\theta s_\theta(q(t), t)}{\rho + \delta} dt.$$

The first term gives the discounted value of profits given the initial qualities $q(t_0)$, and the second term incorporates the additional value generated by innovation at each time $t \geq t_0$. This objective function is linear in the allocation of scientists at each time $t \geq t_0$, so the firm's solution is simply to allocate all scientists toward the technology θ with the largest marginal productivity of research $K_\theta(t)\eta_\theta$ at each time $t \geq t_0$. Equivalently, a firm exclusively innovates for technology B at time t if and only if the knowledge ratio satisfies $\kappa(t) \geq \eta_A/\eta_B$, and otherwise exclusively innovating for technology A. Comparing to the more general analysis of Section 4.1, here we also obtain a corner solution for the allocation of scientists because each technology's innovation rate is linear in the mass of scientists. This case additionally implies that each firm's allocation is identical, because $\lambda = 0$ and flow profits are linear in qualities.

This threshold characterization of firm innovation decisions carries over to the economy's equilibrium. When $\kappa(t) \geq \eta_A/\eta_B$, all innovation is directed toward technology B, so that the knowledge stock for B grows relative to that for A. The ratio $\kappa(t)$ is then strictly increasing, so that innovation is permanently directed toward B. The opposite holds when $\kappa(t) < \eta_A/\eta_B$. Thus the ratio of research productivities η_A/η_B functions as a threshold for $\kappa(0)$ that determines the equilibrium direction of technological change:

Proposition C.1. *The economy with $\lambda = 0$ has a unique equilibrium. All firms exclusively innovate for technology B at $t \geq 0$ if and only if*

$$\kappa(0) \geq \frac{\eta_A}{\eta_B},$$

with all innovation directed toward A otherwise. If innovation is directed toward technology B, the knowledge stock K_B immediately grows at the BGP rate $g_B^* = \sigma_I \eta_B S$, and $Q_B(t)/K_B(t)$ increases monotonically to its BGP value.

Proof: The threshold characterization of equilibrium research follows immediately from the discussion preceding the proposition. If research is directed toward technology B, the evolution equation (9) for K_B immediately implies $\dot{K}_B(t) = \sigma_I \eta_B S K_B(t)$. Integrating the evolution equation (15) for Q_B then yields

$$Q_B(t) = \frac{g_B + \delta N \sigma_E}{g_B + \delta} K_B(t) - \frac{(1 - N\sigma_E)\, g_B}{g_B + \delta} K_B(0) \exp(-\delta t).$$

Hence

$$\frac{Q_B(t)}{K_B(t)} = \frac{g_B + \delta N \sigma_E}{g_B + \delta} - \frac{(1 - N\sigma_E)\, g_B}{g_B + \delta} \exp(-(\delta + g_B) t)$$

By (19), this ratio is strictly increasing in t and limits to the BGP value from Proposition 1. ■

This result is a special case of Propositions 3 and 4. The economy's unique equilibrium features a transition to technology B if and only if the initial ratio of knowledge stocks $\kappa(0)$ is sufficiently high. But with $\eta_B > \eta_A$, the economy may transition even if technology B is initially inferior to technology A ($\kappa(0) < 1$). Along a transition, the knowledge stock K_B features no transitional dynamics, while the aggregate quality Q_B increases relative to K_B as incumbents improve relative to entrants before exit.

C.1.2 Efficiency

A key implication of this analysis is that the equilibrium direction of technological change is essentially myopic: At each time, all firms research the technology with the higher research productivity $\sigma_I K_\theta(t)\eta_\theta$. This holds because firms do not internalize the value of their knowledge spillovers for future innovating firms, and with $\lambda = 0$ they do not benefit directly from their past knowledge production. To understand the resulting inefficiency, consider the problem of a social planner who can choose each firm's allocation of scientists to maximize the consumer's discounted utility (1), but cannot otherwise modify the equilibrium. This social planner solves

$$\max_{[s_\theta(q,t)]_{\theta,q,t}} \int_0^\infty \exp(-\rho t)C(t)dt, \tag{C1}$$

subject to the resource constraint $s_A(q,t)+s_B(q,t) \le s$ and with all remaining quantities determined in equilibrium. The following proposition solves this problem for the case with $\lambda = 0$, demonstrating that the equilibrium transitions to technology B insufficiently often:

Proposition C.2. *With $\lambda = 0$, the social planner allocates all scientists to technology B at $t \ge 0$ if and only if*

$$\kappa(0) \ge \frac{\eta_A}{\eta_B} J,$$

with all innovation directed toward A otherwise. The adjustment factor $J \in (0,1)$ is independent of initial conditions.

Proof: Since incumbents' qualities do not affect their research productivities, it is optimal for the social planner to permanently allocate all scientists to one technology. If the social planner chooses technology A, then Q_A continues to grow at the BGP rate $g_A^* = \sigma_I \eta_A S$, while Q_B remains constant at its initial value $Q_B(0) = N\sigma_E K_B(0)$. With flow consumption equal to $\bar{C}(Q_A + Q_B)$ for a constant $\bar{C} > 0$, this yields social value

$$\begin{aligned}\frac{U^A}{\bar{C}} &= \frac{Q_A(0)}{\rho - g_A^*} + \frac{Q_B(0)}{\rho} \\ &= \frac{1}{\rho - g_A^*} \frac{g_A^* + \delta N \sigma_E}{g_A^* + \delta} K_A(0) + \frac{1}{\rho} N \sigma_E K_B(0).\end{aligned}$$

If instead the social planner permanently allocates all scientists to technology B, then Q_A and Q_B evolve according to

$$\dot{Q}_A(t) = \delta\left[N\sigma_E K_A(0) - Q_A(t)\right],$$
$$\dot{Q}_B(t) = g_B^* K_B(0)\exp\left(g_B^* t\right) + \delta\left[N\sigma_E K_B(0)\exp\left(g_B^* t\right) - Q_B(t)\right].$$

In the second equation I make use of the relation $K_B(t) = K_B(0)\exp\left(g_B^* t\right)$. Integrating these equations yields

$$Q_A(t) = N\sigma_E K_A(0) + \frac{(1-N\sigma_E)\, g_A^*}{g_A^* + \delta} K_A(0)\exp(-\delta t),$$
$$Q_B(t) = \frac{g_B^* + \delta N\sigma_E}{g_B^* + \delta} K_B(0)\exp\left(g_B^* t\right) - \frac{(1-N\sigma_E)\, g_B^*}{g_B^* + \delta} K_B(0)\exp(-\delta t).$$

The social welfare from researching technology B permanently is then

$$\begin{aligned}
\frac{U^B}{\bar{C}} &= \int_0^\infty \exp(-\rho t)\left[Q_A(t) + Q_B(t)\right] dt \\
&= \left[\frac{1}{\rho} N\sigma_E + \frac{1}{\rho+\delta}\frac{(1-N\sigma_E)\, g_A^*}{g_A^* + \delta}\right] K_A(0) \\
&\quad + \left[\frac{1}{\rho - g_B^*}\frac{g_B^* + \delta N\sigma_E}{g_B^* + \delta} - \frac{1}{\rho+\delta}\frac{(1-N\sigma_E)\, g_B^*}{g_B^* + \delta}\right] K_B(0).
\end{aligned}$$

Researching technology B is socially optimal if and only if $U^B \geq U^A$, or equivalently

$$j(g_B^*) g_B^* K_B(0) \geq j(g_A^*) g_A^* K_A(0),$$

where I define

$$j(g) \equiv \left[\frac{1}{\rho - g}\frac{g + \delta N\sigma_E}{g + \delta} - \frac{1}{\rho + \delta}\frac{(1-N\sigma_E)\, g}{g+\delta} - \frac{1}{\rho} N\sigma_E\right]\frac{1}{g}.$$

This function can equivalently be written

$$j(g) = \frac{\frac{g+\delta N\sigma_E}{\rho - g} + \frac{\delta}{\rho+\delta}(1-N\sigma_E)}{g+\delta}\frac{1}{\rho}.$$

Direct calculation implies that this function is strictly increasing in g:

$$j'(g) \propto (g+\delta)\left(1 + \frac{g + \delta N\sigma_E}{\rho - g}\right) - \left[g + \delta N\sigma_E + (\rho - g)\frac{\delta}{\rho+\delta}(1-N\sigma_E)\right].$$

This value is positive if and only if

$$\delta(1-N\sigma_E)+(g+\delta)\frac{g+\delta N\sigma_E}{\rho-g}>-(\rho-g)\frac{\delta}{\rho+\delta}(1-N\sigma_E).$$

This necessarily holds, because the left side is positive and the right side is negative because of the assumed upper bound on entrant spillovers (19) and the assumed lower bound on the discount rate (20). This analysis demonstrates that the social planner exclusively and permanently researches technology B if and only if

$$\kappa(0)\geq\frac{\eta_A}{\eta_B}J,$$

where the adjustment factor $J\equiv j(g_A^*)/j(g_B^*)$ is positive but strictly smaller than one. ■

For intuition, consider the limit $\sigma_E N\uparrow 1$, so that the aggregate qualities Q are unaffected by entry and exit and hence display no transitional dynamics. The factor J then limits to

$$\lim_{\sigma_E N\uparrow 1}J=\frac{\rho-g_B^*}{\rho-g_A^*}.$$

This value is strictly smaller than one precisely because the *long-run* growth rate for technology A is smaller than that for technology B. The social planner internalizes the value of knowledge spillovers for future innovation, so her transition threshold takes into account not only the $t=0$ research productivities for each technology, but also the implied long-run growth rates g_A^* and g_B^*.

Finally, note the distinction between Proposition C.2 and Proposition 6, which is the corresponding efficiency result for the benchmark case with only internal knowledge ($\sigma_I=0$). With only external knowledge, the social planner prefers to transition to technology B more often than in equilibrium; but conditional on the direction of innovation, the equilibrium is efficient (Proposition C.2). With only internal knowledge, the social planner does not necessary prefer to transition to technology B more often; but conditional on the direction of innovation, the equilibrium converges too slowly to the limiting BGP.

C.2 Internal Knowledge Accumulation: $\sigma_I=0$

C.2.1 Equilibrium

Now consider the other extreme case in which firms build exclusively on internal knowledge when innovating ($\sigma_I=0<\lambda$), summarized in Section 4.3. With $\sigma_I=0$, we can again integrate the quality evolution equation (8) to find

$$q_\theta(t) = q_\theta(t_0)\exp\left(\lambda\eta_\theta \int_{t_0}^{\tau} s_\theta(q(\tau),\tau)d\tau\right).$$

In contrast to the case with $\lambda = 0$, $q_\theta(t)$ is not additively separable between the initial quality $q_\theta(t_0)$ and the allocation of scientists. Since firms build on their past advances, an increase in the initial quality $q_\theta(t_0)$ raises the firm's research productivity for θ at each time $\tau \geq t_0$. With profits linear in quality, this mechanism leads to path dependence whereby a firm is more likely to research a technology that it has researched previously. The following lemma precisely characterizes equilibrium innovation decisions:

Lemma C.1. *With $\sigma_I = 0$, a firm with qualities q exclusively innovates for technology B if*

$$\frac{\eta_B}{\rho + \delta - \lambda\eta_B s} q_B \geq \frac{\eta_A}{\rho + \delta - \lambda\eta_A s} q_A,$$

with all innovation directed toward A otherwise.

Proof: It is clearly optimal for the firm to innovate permanently for one technology. If the firm innovates permanently for technology A, its value is

$$\frac{V^A}{\bar{\pi}} = \frac{q_A}{\rho + \delta - \lambda\eta_A s} + \frac{q_B}{\rho + \delta}$$

If the instead innovates permanently for technology B, its value is

$$\frac{V^B}{\bar{\pi}} = \frac{q_A}{\rho + \delta} + \frac{q_B}{\rho + \delta - \lambda\eta_B s}.$$

Innovating for technology B is optimal if and only if $V^B \geq V^A$. Substituting the expressions above into this inequality yields the inequality stated in the Lemma. This inequality is self-reinforcing, so an incumbent's innovation direction is perfectly persistent. ■

This result clarifies the two forces that determine the direction of a firm's innovation. First, as discussed above, a firm has a greater propensity to innovate for the technology for which it has a higher quality. Second, firms have a greater propensity to innovate for technology B because of its higher basic research productivity $\eta_B > \eta_A$. This force is stronger with $\lambda > 0$ because the firm's problem is genuinely foward-looking: Innovating for technology θ raises the quality q_θ, making research for θ more productive in the future. The firm internalizes this dynamic effect because its innovation decision has a non-negligible impact on the change in quality q_θ

(in contrast to the knowledge stock K_θ).

We can apply this result to derive a convenient characterization of entrant and incumbent innovation decisions. Substituting the entrant qualities $q^E_\theta(t) = \sigma_E K_\theta(t)$ into the inequality stated in the Lemma, we observe that an entrant at $t \geq 0$ permanently innovates for technology B if and only if the knowledge stock ratio $\kappa(t)$ is larger than the *entry threshold*

$$\kappa^E \equiv \frac{\eta_A}{\rho + \delta - \lambda \eta_A s}\left(\frac{\eta_B}{\rho + \delta - \lambda \eta_B s}\right)^{-1} \in (0,1).$$

An incumbent at $t = 0$ with initial qualities $q_A(0) = z_A(0) q^E_A(0)$ and $q_B(0) = q^E_B(0)$ permanently innovates for technology B if and only if its relative quality $z_A(0)$ is above the cutoff

$$z^*_{A0} \equiv \frac{\kappa(0)}{\kappa^E}$$

Note that path dependence implies that $t = 0$ incumbents always have a lower propensity to innovate for B than entrants at $t = 0$.

Using this characterization of innovation decisions, the following lemma derives a simple dynamical system for the knowledge stocks K, analogous to that of Proposition 3:

Lemma C.2. *With $\sigma_I = 0$, the knowledge stocks K evolve according to*

$$\ddot{K}_A(t) = -(\delta - \lambda \eta_A s)\dot{K}_A(t) + \delta N \lambda \eta_A s \sigma_E K_A(t) \mathbb{1}\left[\kappa(t) < \kappa^E\right], \tag{C2}$$

$$\ddot{K}_B(t) = -(\delta - \lambda \eta_B s)\dot{K}_B(t) + \delta N \lambda \eta_B s \sigma_E K_B(t) \mathbb{1}\left[\kappa(t) \geq \kappa^E\right]. \tag{C3}$$

The corresponding initial conditions are

$$\begin{aligned} & K_A(0), \quad K_B(0), \\ & \frac{\dot{K}_A(0)}{K_A(0)} = \lambda \eta_A s \sigma_E \int_{z^*_{A0}}^{\infty} z_A dH_A(z_A), \\ & \frac{\dot{K}_B(0)}{K_B(0)} = \lambda \eta_B s \sigma_E H_A(z^*_{A0}). \end{aligned} \tag{C4}$$

Proof: For expositional purposes only, suppose $\kappa(0) > \kappa^E$, and fix t small enough that $\kappa(t') > \kappa^E$ for $t' \in [0, t]$. Consider technology A. Using Lemma C.1 and integrating the evolution equation (8) for q_A, we have that for any $t = 0$ incumbent with $z_A(0) > z^*_{A0}$,

$$q_A(t) = z_A(0) \sigma_E K_A(0) \exp(\lambda \eta_A s t).$$

The density of these incumbents at $t = 0$ is $h_A(z_A(0))$. But with exit at rate $\delta > 0$, the density at time $t > 0$ falls to $\exp(-\delta t) h_A(z_A(0))$. Hence the evolution equation (9) for K_A can be written

$$\dot{K}_A(t) = \lambda \eta_A s \sigma_E K_A(0) \left(\int_{z_A > z_{A0}^*} z_A dH_A(z_A) \right) \exp(-(\delta - \lambda \eta_A s) t).$$

Differentiating in t yields (C2).

Turning to technology B, the same argument as above implies that for any $t = 0$ incumbent with $z_A(0) \le z_{A0}^*$,

$$q_B(t) = \sigma_E K_B(0) \exp(\lambda \eta_B s t).$$

By time t, the total mass of $t = 0$ incumbents with $z_A(0) \le z_{A0}^*$ is $\exp(-\delta t) H_A(\hat{z}_A)$. Similarly, for any firm who entered at time $\tau \in [0, t]$,

$$q_B(t) = \sigma_E K_B(\tau) \exp(\lambda \eta_B s (t - \tau)).$$

By time $t \ge \tau$, the density of these firms is $\exp(-\delta(t-\tau)) \delta N$. Hence the evolution equation (9) can be written

$$\begin{aligned} \dot{K}_B(t) &= \lambda \eta_B s \sigma_E K_B(0) H_A(z_{A0}^*) \exp(-(\delta - \lambda \eta_B s) t) \\ &\quad + \lambda \eta_B s \sigma_E \delta N \int_0^t K_B(\tau) \exp(-(\delta - \lambda \eta_B s)(t - \tau)) d\tau. \end{aligned}$$

Differentiating in t yields (C3).

The same arguments yield the evolution equations (C2, C3) regardless of $\kappa(0)$ or t. The initial conditions (C4) follow directly from the evolution equation (9) for the knowledge stocks K, given the assumption that the economy is following the BGP for technology A at $t = 0$. ■

The benchmark case with $\sigma_I = 0$ is convenient precisely because the system (C2, C3) is autonomous and can be integrated in closed form. The following lemma demonstrates this for the case when the economy transitions to the BGP for technology B:

Lemma C.3. *Suppose entrants innovate for technology B at each time, $\kappa(t) \ge \kappa^E$ for $t \ge 0$.*

(i) The solution to the system (C2, C3) is

$$\frac{K_A(t)}{K_A(0)} = c_{A1} - c_{A2}\exp\left(-(\delta - \lambda\eta_A s)\,t\right),$$
$$\frac{K_B(t)}{K_B(0)} = c_{B1}\exp\left(-\left(g_B^* + \delta - \lambda\eta_B s\right)t\right) + c_{B2}\exp\left(g_B^* t\right),$$

where $c_{A1}, c_{A2}, c_{B1}, c_{B2} > 0$ satisfy the initial conditions (C4).

(ii) The knowledge ratio $\kappa(t) = K_B(t)/K_A(t)$ is strictly increasing and strictly convex in the initial condition $\kappa(0)$ for $t > 0$, and there exists a time $\hat{t} \geq 0$ such that $(t - \hat{t})\dot{\kappa}(t) > 0$.

Proof: Integrating the evolution equation (C2) for $\dot{K}_A$ yields

$$\frac{K_A(t)}{K_A(0)} = 1 + \lambda\eta_A s\sigma_E\left(\int_{z_{A0}^*}^{\infty} z_A dH_A(z_A)\right)\frac{1 - \exp\left(-(\delta - \lambda\eta_A s)\,t\right)}{\delta - \lambda\eta_A s}.$$

Hence the integration constants from the Lemma are

$$c_{A1} = 1 + c_{A2},$$
$$c_{A2} = \lambda\eta_A s\sigma_E\left(\int_{z_{A0}^*}^{\infty} z_A dH_A(z_A)\right)\frac{1}{\delta - \lambda\eta_A s}.$$

Integrating the evolution equation (C3) for $\dot{K}_B$ and making use of the formula for g_B^* given in Proposition 1 yields the expression for $K_B(t)/K_B(t)$ given in the Lemma statement. The integration constants jointly satisfy the initial conditions

$$1 = c_{B1} + c_{B2},$$
$$\lambda\eta_B s\sigma_E H_A(z_{A0}^*) = -\left(g_B^* + \delta - \lambda\eta_B s\right)c_{B1} + g_B^* c_{B2}.$$

The solution to this system is

$$c_{B1} = \frac{g_B^* - \lambda\eta_B s\sigma_E H_A(z_{A0}^*)}{2g_B^* + \delta - \lambda\eta_B s},$$
$$c_{B2} = \frac{g_B^* + \delta - \lambda\eta_B s\left(1 - \sigma_E H_A(z_{A0}^*)\right)}{2g_B^* + \delta - \lambda\eta_B s}.$$

Clearly $c_{B2} > 0$ given the assumed lower bound (14) on the exit rate δ. We also have

$$
\begin{aligned}
g_B^* - \lambda\eta_B s\sigma_E H_A(z_{A0}^*) &\geq g_B^* - \lambda\eta_B S\sigma_E \\
&\geq g_B^* - \lambda\eta_B s \\
&= -\frac{\delta - \lambda\eta_B s}{2} - \lambda\eta_B S\sigma_E + \sqrt{\left(\frac{\delta - \lambda\eta_B s}{2}\right)^2 + \lambda\sigma_E\delta\eta_B S}.
\end{aligned}
$$

Direct calculation implies that the right side is positive given the assumed upper bound on spillovers to entrants (19), so $c_{B1} > 0$.

To prove the properties of κ stated in the Lemma, note that the above analysis shows that $K_A(t)/K_A(0)$ depends on the initial knowledge stocks $K(0)$ only through z_{A0}^*. This cutoff is strictly increasing in $\kappa(0)$ while $K_A(t)/K_A(0)$ is strictly decreasing in $\hat{z}_A$ for $t > 0$, so $K_A(t)/K_A(0)$ is strictly decreasing in $\kappa(0)$ for $t > 0$. A similar argument implies that $K_B(t)/K_B(0)$ is strictly increasing in $\kappa(0)$ for $t > 0$. Hence

$$
\kappa(t) = \kappa(0)\frac{K_B(t)/K_B(0)}{K_A(t)/K_A(0)}
$$

depends on $K(0)$ only through $\kappa(0)$, and it is strictly increasing and strictly convex in $\kappa(0)$ for $t > 0$. Finally, note that the growth rate of κ satisfies

$$
\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{\dot{K}_B(t)}{K_B(t)} - \frac{\dot{K}_A(t)}{K_A(t)}.
$$

We can directly calculate

$$
\begin{aligned}
\frac{\dot{K}_A(t)}{K_A(t)} &= \frac{(\delta - \lambda\eta_A s)\,c_{A2}}{c_{A1}\exp\left((\delta - \lambda\eta_A s)\,t\right) - c_{A2}}, \\
\frac{\dot{K}_B(t)}{K_B(t)} &= g_B^* \frac{-\left(1 + \frac{\delta - \lambda\eta_B s}{g_B^*}\right)c_{B1} + c_{B2}\exp\left(\left(2g_B^* + \delta - \lambda\eta_B s\right)t\right)}{c_{B1} + c_{B2}\exp\left(\left(2g_B^* + \delta - \lambda\eta_B s\right)t\right)}.
\end{aligned}
$$

Since all integration coefficients are positive, we immediately have that the growth rate of $K_A(t)$ is declining over time, while the growth rate of $K_B(t)$ is increasing over time. The previous equation then implies that $\dot{\kappa}(t)$ is single-crossing from below, so there exists a time $\hat{t} \geq 0$ such that $\dot{\kappa}(t) < 0$ for $t < \hat{t}$ and $\dot{\kappa}(t) > 0$ for $t > \hat{t}$. ■

The second part of the lemma leverages the solution to the system (C2, C3) to prove two properties of the ratio $\kappa(t)$: It is strictly increasing in its initial value $\kappa(0)$, and it is generally "U-shaped" over time. These properties are useful because the solution in the lemma only describes the dynamics of the knowledge stocks $K(0)$ while entrants continue to research technology B, $\kappa(t) \geq \kappa^E$. If this condition is ever violated, the economy fails to transition to technology B and instead converges back to the BGP for technology A. The properties of $\kappa(t)$ described in Lemma C.3 ensure that a transition takes place exactly when $\kappa(0)$ is sufficiently large.

Proposition C.3. *With $\sigma_I = 0$, the economy has a unique equilibrium. There exists a threshold $\kappa^* > \kappa^E$ such that if $\kappa(0) \geq \kappa^*$, all firms innovate for B as $t \to \infty$, and the economy converges to the BGP for B. Otherwise, all firms innovate for A as $t \to \infty$, and the economy converges to the BGP for A. The economy displays transitional dynamics when $\kappa(0) > \kappa^E$.*

Proof: Clearly if $\kappa(0) \leq \kappa^E$, all innovation is initially directed toward technology A, so that $\kappa(t) < \kappa^E$ for all $t > 0$. Hence all incumbents innovate for technology A, so that the economy continues along the BGP for A.

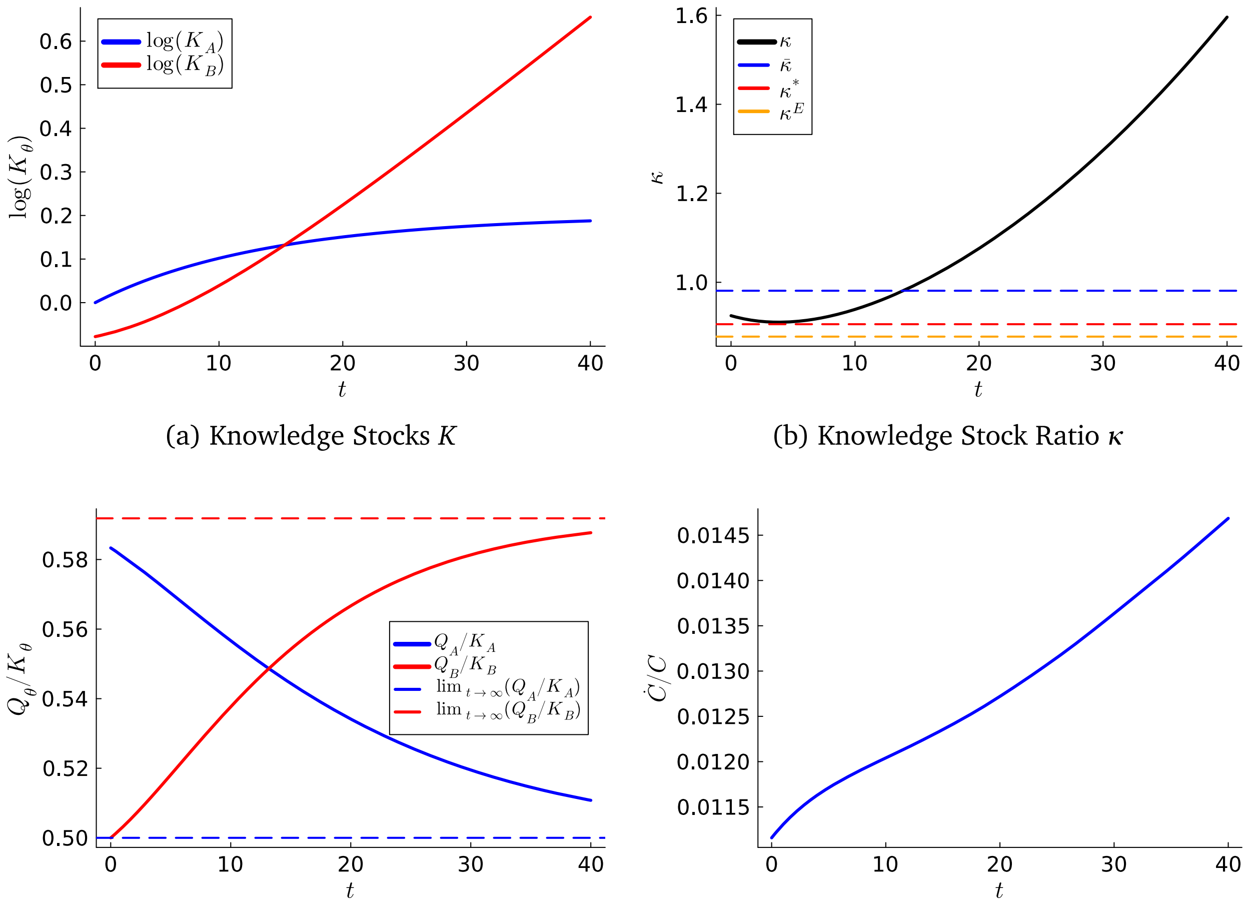
If instead $\kappa(0) > \kappa^E$ but $\kappa(t) < \kappa^E$ for some time $t > 0$, then the economy again converges back to the BGP for A. To see this, let $\underline{t} = \inf\{t : \kappa(t) < \kappa^E\}$. For t in a neighborhood to the right of $\underline{t}$, Lemma C.2 implies that the knowledge stocks K evolve according to

$$\begin{aligned}\ddot{K}_A(t) &= -(\delta - \lambda\eta_A s)\dot{K}_A(t) + \lambda\eta_A s\delta N\sigma_E K_A(t),\\ \ddot{K}_B(t) &= -(\delta - \lambda\eta_B s)\dot{K}_B(t).\end{aligned}$$

As in Lemma C.3, the solution implies that the growth rate $\dot{K}_A(t)/K_A(t)$ is increasing for t in a neighborhood to the right of $\underline{t}$, while the growth rate $\dot{K}_B(t)/K_B(t)$ is decreasing. Since κ must be strictly decreasing for t in a neighborhood to the left of $\underline{t}$, this implies $\dot{K}_A(t)/K_A(t) > \dot{K}_B(t)/K_B(t)$ in a neighborhood to the right of $\underline{t}$. But then the inequality $\kappa(t) < \kappa^E$ is self-reinforcing, and the economy converges back to the BGP for technology A.

The argument above implies that the economy transitions to the BGP for technology B asymptotically if and only if the trajectory of $\kappa(t)$ implied by (C2, C3) with initial conditions (C4) satisfies $\kappa(t) \geq \kappa^E$ for all $t \geq 0$. Lemma C.3 implies that $\kappa(t)$ is asymptotically increasing in t and strictly increasing in $\kappa(0)$, so there must exist a threshold κ^* such that $\kappa(t) \geq \kappa^E$ for all $t \geq 0$ if and only if $\kappa(0) \geq \kappa^*$. Note that this threshold must satisfy $\kappa^* > \kappa^E$, because otherwise $\kappa(t)$ initially falls below κ^E.

The transitional dynamics of the knowledge stocks K and the aggregate qualities Q are described by the system (15, C2, C3). This system is conveniently block diagonal, so that K can be recovered without integrating the evolution equations for Q. ■



(a) Knowledge Stocks K

(b) Knowledge Stock Ratio κ

(c) Quality ratios Q_θ/K_θ

(d) Consumption Growth $\dot{C}/C$

Figure C.1: Example Transition with $\sigma_I = 0$

Notes: To calibrate, I set $\lambda = 1$, $\sigma_E = 0.5$, and $\sigma_I = 0$. I also set $\delta = 0.1$, $S = 1$, and $s = 1$, and I choose η_A and η_B to deliver BGP growth rates $g_A^* = 0.02$ and $g_B^* = 0.0225$. Finally, I set $\rho = 0.075$, and I specify the initial conditions $K_A(0) = 1$ and $K_B(0) = 0.925$ so that $\kappa(0) > \kappa^* \approx 0.91$.

Figure C.1 displays the trajectories of the knowledge stocks K, the growth rates $\dot{K}_\theta/K_\theta$, the knowledge ratio κ, and the ratios Q_θ/K_θ for an example transition. Note that the trajectory of the knowledge stock ratio $\kappa(t)$ is "U-shaped," initially decreasing before increasing asymptotically.

The threshold κ^* determines the economy's propensity to transition in equilibrium, and it depends richly on model primitives. To gain intuition for the key forces, note that Lemma C.3 implies that a simple sufficient condition for the economy to transition is for the initial growth rate of K_B to dominate the initial growth rate of K_A. Given the initial conditions (C4), this holds if and only if

$$\eta_B H_A\left(z_{A0}^*\right) \geq \eta_A \int_{z_{A0}^*}^{\infty} z_A dH_A(z_A). \tag{C5}$$

This inequality depends on $\kappa(0)$ only through the relative quality cutoff $\hat{z}_A$, which is strictly increasing in $\kappa(0)$. As $\kappa(0)$ rises, the left side increases as a larger mass of $t = 0$ incumbents transition to B, raising the initial growth rate of K_B. The right side instead decreases as fewer $t = 0$ incumbents innovate for A, lowering the initial growth rate of K_A. There exists a unique value $\bar{\kappa}$ at which the inequality (C5) binds:

$$1 = \frac{\eta_A}{\eta_B} \frac{1}{1-\xi_A^*} \frac{\frac{\bar{\kappa}}{\kappa^E}}{\left(\frac{\bar{\kappa}}{\kappa^E}\right)^{1/\xi_A^*} - 1}. \tag{C6}$$

Here I have used the expression for z_{A0}^* from Lemma C.1 and the BGP relative quality distribution from Proposition 1, which is a standard Pareto distribution when $\sigma_I = 0$.

The right side of equation (C6) is strictly increasing in ξ_A^* and κ^E and strictly decreasing in $\bar{\kappa}$, and it delivers a simple but powerful intuition about the drivers of a technological transition: Any change that thickens the tail of the old technology's firm-quality distribution slows the transition, because it raises both the relative mass of incumbents who choose not to transition and their initial innovation rates. Both effects increase their collective influence over the aggregate direction of innovation, which may be decisive if it induces entrants to switch back to innovating for technology A. However, any change that raises incentives for new firms to innovate for the new technology instead accelerates the transition by raising the relative mass of incumbents who choose to transition. The tail parameter ξ_A^* and the entry threshold κ^E respectively capture these two forces, but they depend on many of the same model primitives. The following proposition provides explicit comparative statics for $\bar{\kappa}$:

Proposition C.4. *With $\sigma_I = 0$, there exists a threshold $\bar{\kappa} \geq \kappa^*$ such that*

$$\frac{\dot{K}_B(0)}{K_B(0)} \geq \frac{\dot{K}_A(0)}{K_A(0)} \iff \kappa(0) \geq \bar{\kappa}.$$

The threshold $\bar{\kappa}$ is strictly decreasing in σ_E, S, and η_B, and it is strictly increasing in ρ and η_A. For each variable $v \in \{\delta, \lambda, s\}$, there exists a discount rate $\rho^v \geq 0$ that depends on model primitives such that

(i) $\bar{\kappa}$ is strictly increasing (decreasing) in δ locally if and only if ρ is smaller (larger) than ρ^δ;

(ii) $\bar{\kappa}$ *is strictly increasing (decreasing) in* λ *locally if and only if* ρ *is larger (smaller) than* ρ^{λ}*;*

(iii) $\bar{\kappa}$ *is strictly increasing (decreasing) in* s *locally if and only if* ρ *is larger (smaller) than* ρ^{s}*.*

Proof: The existence of the threshold $\bar{\kappa} \geq \kappa^*$ follows immediately from the discussion preceding the Proposition.

Throughout the remainder of the proof, let RHS denote the right side of (C6), and let $\nu \equiv \bar{\kappa}/\kappa^E$. It is immediate that RHS is strictly decreasing in ν and strictly increasing in ξ_A^*:

$$\frac{\partial \text{RHS}}{\partial \nu} = -\frac{\eta_A}{\eta_B}\frac{1}{1-\xi_A^*}\frac{\left(\frac{1}{\xi_A^*}-1\right)\nu^{1/\xi_A^*}+1}{\left(\nu^{1/\xi_A^*}-1\right)^2},$$
$$\frac{\partial \text{RHS}}{\partial \xi_A^*} = \text{RHS}\left[\frac{1}{1-\xi_A^*}+\frac{1}{(\xi_A^*)^2}\frac{\nu^{1/\xi_A^*}\log(\nu)}{\nu^{1/\xi_A^*}-1}\right]$$

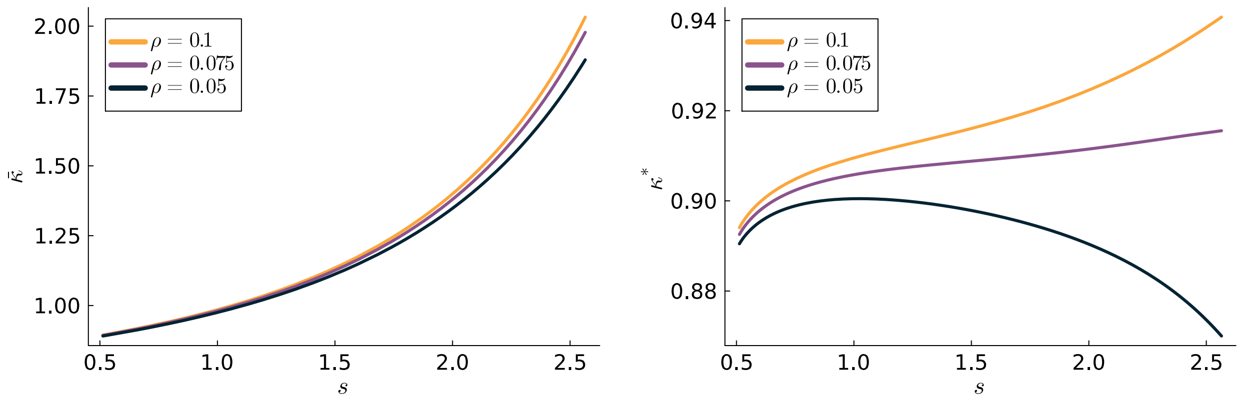
Several comparative statics follow immediately: $\bar{\kappa}$ is strictly decreasing in σ_E and S because these parameters only reduce ξ_A^*. Similarly, $\bar{\kappa}$ is strictly increasing in ρ because an increase in ρ only increases κ^E. Now RHS is directly decreasing in η_B and indirectly decreasing in η_B through κ^E, so $\bar{\kappa}$ is strictly decreasing in η_B. A symmetric argument shows that $\bar{\kappa}$ is strictly increasing in η_A.

For the comparative static with respect to $v \in \{\delta, \lambda, s\}$, we can differentiate to find

$$\frac{d\text{RHS}}{dv} = \frac{\partial \text{RHS}}{\partial \xi_A^*}\frac{\partial \xi_A^*}{\partial v} + \frac{\partial \text{RHS}}{\partial \nu}\nu\kappa^E\frac{\partial(\kappa^E)^{-1}}{\partial v}. \tag{C7}$$

Now (C6) determines ν independently of κ^E, and since RHS depends on ρ only through κ^E, this implies that ν is invariant to ρ. The first summand in (C7) depends on κ^E only through ν, and similarly for the partial derivative $\partial\text{RHS}/\partial\nu$, so these terms are invariant to ρ. With Lemma C.1, we can directly calculate

$$\frac{\partial(\kappa^E)^{-1}}{\partial\delta} = -(\kappa^E)^{-1}\left[\frac{1}{\rho+\delta-\lambda\eta_B s}-\frac{1}{\rho+\delta-\lambda\eta_A s}\right],$$
$$\frac{\partial(\kappa^E)^{-1}}{\partial\lambda} = (\kappa^E)^{-1}\left[\frac{\eta_B}{\rho+\delta-\lambda\eta_B s}-\frac{\eta_A}{\rho+\delta-\lambda\eta_A s}\right]s,$$
$$\frac{\partial(\kappa^E)^{-1}}{\partial s} = (\kappa^E)^{-1}\left[\frac{\eta_B}{\rho+\delta-\lambda\eta_B s}-\frac{\eta_A}{\rho+\delta-\lambda\eta_A s}\right]\lambda.$$



(a) Sufficiency Threshold $\bar{\kappa}$ (b) Equilibrium Threshold κ^*

Figure C.2: Comparative Statics for Thresholds $\bar{\kappa}$ and κ^*

Notes: I vary the discount rate ρ around its baseline value $\rho = 0.075$. The remaining parameters are exactly as in Figure C.1.

With $\sigma_I = 0$, Corollary 1 yields

$$\frac{\partial \xi_A^*}{\partial \delta} < 0 \quad \text{and} \quad \frac{\partial \xi_A^*}{\partial \lambda}, \frac{\partial \xi_A^*}{\partial s} > 0.$$

Hence (C7) implies that $d\text{RHS}/d\delta$ is strictly decreasing in ρ, becoming negative in the limit $\rho \to \infty$. There must then exist a value $\rho^\delta \geq 0$ such that $\bar{\kappa}$ is strictly increasing in δ locally if and only if $\rho < \rho^\delta$, while the opposite holds for $\rho > \rho^\delta$. The same argument implies the existence of the values $\rho^\lambda, \rho^s \geq 0$ and the corresponding comparative statics with respect to λ and δ stated in the Proposition. ■

Comparative statics for the thresholds $\bar{\kappa}$ and κ^* with respect to s are illustrated in Figure C.2.

C.2.2 Efficiency

In the case with $\sigma_I = 0$, the solution to the planner's problem (C1) is complex: The planner generally chooses different innovation directions for different firms and may reverse these directions over time. To develop intuition about equilibrium inefficiencies, I consider the simpler problem in which the planner can choose the initial relative quality cutoff $z_{A0} \geq 1$ describing initial incumbents' innovation decisions. The planner then solves

$$\max_{z_{A0} \geq 1} \int_0^\infty \exp(-\rho t) C(t) dt, \tag{C8}$$

with entrant innovation decisions and all remaining quantities determined in equilibrium. The following proposition characterizes properties of the solution $\hat{z}_{A0}$:

Proposition C.5. *A solution $\hat{z}_{A0}$ to the social planner's problem (C8) exists and depends on $K(0)$ only through $\kappa(0)$. There exists a threshold $\hat{\kappa}$ such that*

(i) the solution $\hat{z}_{A0}$ yields a transition to technology B if and only if $\kappa(0) \geq \hat{\kappa}$;

(ii) $\hat{z}_{A0} > z^*_{A0}$ if $\kappa(0) \geq \hat{\kappa}$; and

(iii) $\hat{z}_{A0} \leq z^*_{A0}$ if $\kappa(0) < \hat{\kappa}$, with equality only if $z^*_{A0} = 1$.

Proof: Let U denote the $t = 0$ discounted value of future consumption. Lemma 1 implies that consumption at each time is proportional to output, which is in turn proportional to total flow profits earned by firms. Hence the discounted value of future consumption is proportional to the discounted value of future profits, or equivalently the discounted value of all firms:

$$\begin{aligned}
\frac{\bar{\pi}}{\bar{C}}U = & \int_{z_{A0}}^{\infty}\left[\frac{1}{\rho+\delta-\lambda\eta_A s}z_A\sigma_E K_A(0) + \frac{1}{\rho+\delta}\sigma_E K_B(0)\right]dH_A^*(z_{A0}) \\
& + \int_{1}^{z_{A0}}\left[\frac{1}{\rho+\delta}z_A\sigma_E K_A(0) + \frac{1}{\rho+\delta-\lambda\eta_B s}\sigma_E K_B(0)\right]dH_A^*(z_{A0}) \\
& + \delta N\int_0^{\infty}\exp(-\rho t)\left[\frac{1}{\rho+\delta}\sigma_E K_A(t) + \frac{1}{\rho+\delta-\lambda\eta_B s}\sigma_E K_B(t)\right]\mathbb{1}\left[\kappa(t)\geq\kappa^E\right]dt \\
& + \delta N\int_0^{\infty}\exp(-\rho t)\left[\frac{1}{\rho+\delta-\lambda\eta_A s}\sigma_E K_A(t) + \frac{1}{\rho+\delta}\sigma_E K_B(t)\right]\mathbb{1}\left[\kappa(t)<\kappa^E\right]dt.
\end{aligned}$$

This function is continuous in z_{A0} except at the critical value that separates convergence to the BGP for technology A from convergence to the BGP for technology B. To simplify, I assume that the social planner can choose the asymptotic direction of innovation at this value, so that the planner's problem (C8) is guaranteed to have a solution for $z_{A0} \in [1, \infty]$. Clearly higher values of $K_B(0)$ raise the value of innovating for B relative to A, so the existence of the threshold $\hat{\kappa}$ is immediate given the linearity of the dynamical system (C2, C3) in K and the linearity of the initial conditions (C4) in $K(0)$.

Suppose $\kappa(0) \geq \hat{\kappa}$, so that the social planner's solution $\hat{z}_{A0}$ is such that the economy converges to the BGP for B as $t \uparrow \infty$. If $\kappa(0) < \kappa^*$, so that in equilibrium the economy converges back to the BGP for A, then we immediately have $\hat{z}_{A0} > z^*_{A0}$. Suppose then that $\kappa(0) \geq \hat{\kappa}, \kappa^*$. We must have $\kappa(t) \geq \kappa^E$ for all $t \geq 0$ as entrants innovate for technology B, so given z_{A0} social welfare at $t = 0$ is

$$
\begin{aligned}
\frac{\bar{\pi}}{\bar{C}} U = & \int_{z_{A0}}^{\infty} \left[\frac{1}{\rho + \delta - \lambda \eta_A s} z_A \sigma_E K_A(0) + \frac{1}{\rho + \delta} \sigma_E K_B(0) \right] dH_A^*(z_{A0}) \\
& + \int_{1}^{z_{A0}} \left[\frac{1}{\rho + \delta} z_A \sigma_E K_A(0) + \frac{1}{\rho + \delta - \lambda \eta_B s} \sigma_E K_B(0) \right] dH_A^*(z_{A0}) \\
& + \delta N \int_0^{\infty} \exp(-\rho t) \left[\frac{1}{\rho + \delta} \sigma_E K_A(t) + \frac{1}{\rho + \delta - \lambda \eta_B s} \sigma_E K_B(t) \right] dt.
\end{aligned}
$$

Making use of Lemma C.3, we find

$$
\begin{aligned}
\int_0^{\infty} \exp(-\rho t) K_A(t) dt &= K_A(0) \left(\frac{c_{A1}}{\rho} - \frac{c_{A2}}{\rho + \delta - \lambda \eta_A s} \right), \\
\int_0^{\infty} \exp(-\rho t) K_B(t) dt &= K_B(0) \left(\frac{c_{B1}}{\rho + \delta + g_B^* - \lambda \eta_B s} + \frac{c_{B2}}{\rho - g_B^*} \right).
\end{aligned}
$$

The integration constants c_{A1}, c_{A2}, c_{B1}, and c_{B2} depend on z_{A0} and are described in the proof of Lemma C.3. For our purposes, their derivatives satisfy

$$
\begin{aligned}
\frac{\partial c_{A1}}{\partial z_{A0}} &= -\frac{\lambda \eta_A s \sigma_E}{\delta - \lambda \eta_A s} z_{A0} h_A^*(z_{A0}), \\
\frac{\partial c_{A2}}{\partial z_{A0}} &= \frac{\partial c_{A1}}{\partial z_{A0}}, \\
\frac{\partial c_{B1}}{\partial z_{A0}} &= -\frac{\lambda \eta_B s \sigma_E}{2 g_B^* + \delta - \lambda \eta_B s} h_A^*(z_{A0}), \\
\frac{\partial c_{B2}}{\partial z_{A0}} &= -\frac{\partial c_{B1}}{\partial z_{A0}}.
\end{aligned}
$$

Differentiating social welfare in z_{A0}, we have

$$
\begin{aligned}
\frac{\partial U}{\partial z_{A0}} \propto & \frac{1}{\rho + \delta} z_{A0} \sigma_E K_A(0) + \frac{1}{\rho + \delta - \lambda \eta_B s} \sigma_E K_B(0) - \left[\frac{1}{\rho + \delta - \lambda \eta_A s} z_{A0} \sigma_E K_A(0) + \frac{1}{\rho + \delta} \sigma_E K_B(0) \right] \\
& + \frac{\delta N \sigma_E K_A(0)}{\rho + \delta} \left(\frac{1}{\rho} - \frac{1}{\rho + \delta - \lambda \eta_A s} \right) \frac{1}{h_A^*(z_{A0})} \frac{\partial c_{A1}}{\partial z_{A0}} \\
& + \frac{\delta N \sigma_E K_B(0)}{\rho + \delta - \lambda \eta_B s} \left(\frac{1}{\rho - g_B^*} - \frac{1}{\rho + \delta + g_B^* - \lambda \eta_B s} \right) \frac{1}{h_A^*(z_{A0})} \frac{\partial c_{B2}}{\partial z_{A0}}.
\end{aligned}
$$

Note that the right side is strictly decreasing in z_{A0}, so that the first-order condition $\frac{\partial U}{\partial z_{A0}} = 0$ is both necesasry and sufficient to characterize the planner's solution $\hat{z}_{A0}$. Evaluating the expression above at z_{A0}^*, I observe that the first line collapses to zero by the definition of z_{A0}^*.

Hence

$$\begin{aligned}
\left.\frac{\partial U}{\partial z_{A0}}\right|_{z_{A0}^*} &\propto \frac{1}{\rho+\delta}\left(\frac{1}{\rho}-\frac{1}{\rho+\delta-\lambda\eta_A s}\right)\frac{1}{h_A^*(z_{A0}^*)}\left.\frac{\partial c_{A1}}{\partial z_{A0}}\right|_{z_{A0}^*} \\
&\quad+\frac{\kappa(0)}{\rho+\delta-\lambda\eta_B s}\left(\frac{1}{\rho-g_B^*}-\frac{1}{\rho+\delta+g_B^*-\lambda\eta_B s}\right)\frac{1}{h_A^*(z_{A0}^*)}\left.\frac{\partial c_{B2}}{\partial z_{A0}}\right|_{z_{A0}^*} \\
&=\frac{1}{\rho+\delta}\left(\frac{1}{\rho+\delta-\lambda\eta_A s}-\frac{1}{\rho}\right)\frac{\lambda\eta_A s\sigma_E}{\delta-\lambda\eta_A s}z_{A0}^* \\
&\quad+\frac{\kappa(0)}{\rho+\delta-\lambda\eta_B s}\left(\frac{1}{\rho-g_B^*}-\frac{1}{\rho+\delta+g_B^*-\lambda\eta_B s}\right)\frac{\lambda\eta_B s\sigma_E}{2g_B^*+\delta-\lambda\eta_B s} \\
&\propto -\frac{1}{\rho+\delta}\frac{1}{\rho}\frac{\eta_A}{\rho+\delta-\lambda\eta_A s}z_{A0}^*+\frac{1}{\rho-g_B^*}\frac{1}{\rho+\delta+g_B^*-\lambda\eta_B s}\frac{\eta_B}{\rho+\delta-\lambda\eta_B s}\kappa(0).
\end{aligned}$$

Using the definition $z_{A0}^*=\frac{\kappa(0)}{\kappa^E}$, we have

$$\left.\frac{\partial U}{\partial z_{A0}}\right|_{z_{A0}^*}\propto\frac{1}{\rho-g_B^*}\frac{1}{\rho+\delta+g_B^*-\lambda\eta_B s}-\frac{1}{\rho+\delta}\frac{1}{\rho}.$$

This value is strictly positive if and only if

$$\rho(\rho+\delta)>\left(\rho-g_B^*\right)\left(\rho+\delta+g_B^*-\lambda\eta_B s\right).$$

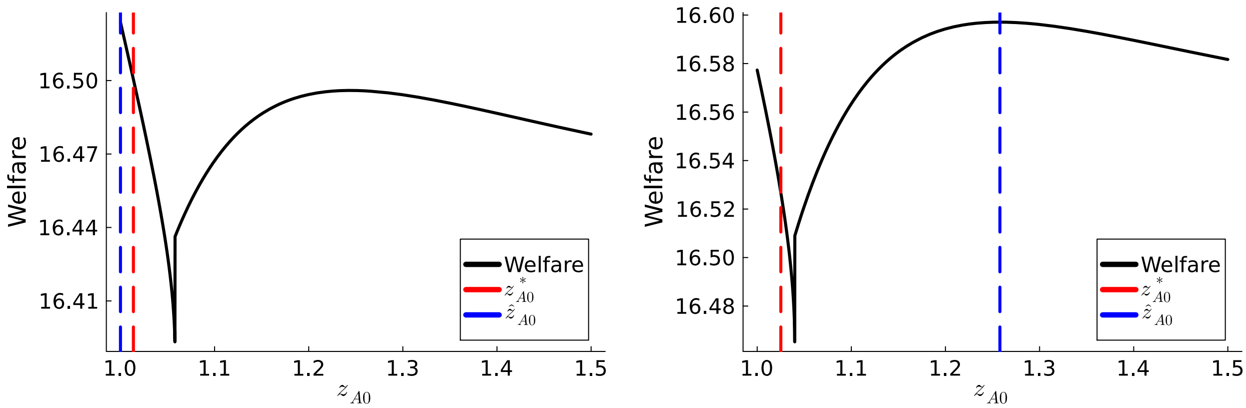
Simplifying yields

$$g_B^*\left(g_B^*+\delta-\lambda\eta_B s\right)>-\rho\lambda\eta_B s.$$

This inequality always holds because of the assumed lower bound (14) on the exit rate δ. We can immediately conclude that $\hat{z}_{A0}>z_{A0}^*$ when $\kappa(0)\geq\hat{\kappa}$.

The remaining statement of the proposition follows by a symmetric argument: Whenever the social planner chooses $\hat{z}_{A0}$ so that the economy converges back to the BGP for technology A, knowledge spillovers lead the social planner to require more initial incumbents to innovate for technology A than in equilibrium. ■

In general, the transition thresholds for the social planner $\hat{\kappa}$ and the equilibrium κ^* cannot be ranked. This holds because the social planner internalizes knowledge spillovers on future entrants when choosing the long-run direction of innovation, but these spillovers are not necessarily always larger for a given technology: Technology B spillovers are larger in the long-run given $\eta_B>\eta_A$, but technology A spillovers may be larger in the short-run given incumbents'



(a) Low $\kappa(0)$

(b) High $\kappa(0)$

Figure C.3: Social Welfare U as a Function of the Cutoff z_{A0}

Notes: In each figure, I plot the social planner's objective (social welfare U) as a function of the initial cutoff z_{A0}. I also note the equilibrium cutoff z^*_{A0} by a red dashed line and the efficient cutoff $\hat{z}_{A0}$ by a blue dashed line. The "Low $\kappa(0)$" figure sets $\kappa(0) = 0.89$, while the "High $\kappa(0)$" figure sets $\kappa(0) = 0.90$. All remaining parameters are exactly as in Figure C.1.

initial expertise for technology A (i.e., the initial distribution H^*_A). However, Proposition C.5 shows that for a given long-run innovation direction, the social planner always prefers to direct greater initial innovation in that direction than in equilibrium.

To provide a sense of the basic non-convexity in the social planner's problem (C8), Figure C.3 displays the objective U as a function of the cutoff z_{A0} for two different values of the initial knowledge stock ratio $\kappa(0)$. In both cases, social welfare attains a global minimum at the "critical mass" value of z_{A0} that divides convergence to the BGP for technology A from convergence to the BGP for technology B. On either side of that point, social welfare attains a global maximum, and the social planner's choice of the long-run innovation direction reduces to comparing the maxima from each side. In both cases, the equilibrium cutoff z^*_{A0} leads to convergence back to the BGP for technology A. When $\kappa(0)$ is low, the social planner also chooses to innovate for technology A in the long run, but consistent with Proposition C.5 the optimal cutoff $\hat{z}_{A0}$ is lower than that in equilibrium. When $\kappa(0)$ is high, the social planner instead chooses to innovate for technology B in the long run.

Figure C.4 plots the equilibrium and efficient cutoffs z^*_{A0} and $\hat{z}_{A0}$ for different values of $\kappa(0)$. In this example, the social planner prefers to transition to technology B more often than in equilibrium ($\hat{\kappa} < \kappa^*$).

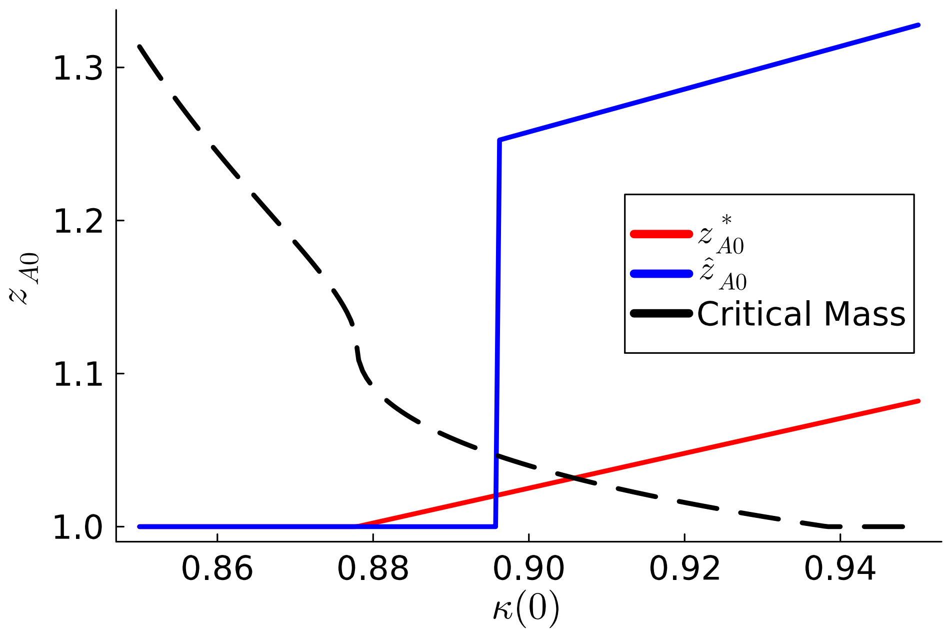


Figure C.4: Equilibrium and Efficient Cutoffs z_{A0}

Notes: The dashed black line denotes the cutoff z_{A0} above which the economy converges to the BGP for technology B. The thresholds κ^* and $\hat{\kappa}$ are the values of $\kappa(0)$ where z^*_{A0} and $\hat{z}_{A0}$ intersect this line, respectively. The remaining parameters are exactly as in Figure C.1.

D Data

This appendix details all data sources and data cleaning procedures for the analyses of Section 2 and Section 5.

D.1 Data Sources

Patents: PatentsView, USPTO-granted patents 1980-2023

- Download date: 08 Oct 2024
- Excluding non-utility, withdrawn, reissued patents
- Excluding all patents assignd to Ethicon, Inc. (anomalous forward citation counts)
- Title and abstract text are processed using the NLTK package in Python to remove standard stop words, punctuation, numbers, and extra white space

US Public Firm Financials: Compustat North America Fundamentals Annual

- Download date: 20 Sept 2024

US Public Firm – Patent Match: DISCERN 2.0 (Arora et al., 2024)

- Download date: 29 Sept 2024

New Technologies: Kalyani et al. (2023)

- Download date: 14 Sept 2024

Aggregate US R&D Expenditures: "Research and Development: US Trends and International Comparisons," National Science Board (NSB-2024-6)

- Download date: 20 Sept 2024

US GDP Deflator/GDP per Capita: Federal Reserve Economic Data (FRED)

D.2 Data Build: mRNA Case Study

mRNA therapy patents: I identify patents related to mRNA therapies by keyword search. I start with all patents that mention at least one term from each of the following lists:

mRNA terms:	“mrna”, “ rna”, “rna ”, “ ribonucleic”
therapy terms:	“therap”, “treat”, “vaccin ”, “innocul”, “immun”

I then exclude patents that mention any terms related to recombinant DNA/RNA or a number of other types of RNA, all of which are related to treatment technologies distinct from mRNA technology:

exclusion terms (RNA): “recombin”, “rna interfer”, “irna ”, “ rnai”, “mirna”, “sirna”, “dsrna”, “trna”, “transfer rna”, “double stranded rna”, “small interfering rna”, “double-stranded rna”, “small-interfering rna”, “micro rna”, “micro-rna”, “microrna”, “reduce expression”, “reducing expression”, “inhibit expression”, “inhibiting expression”, “modulate expression”, “modulating expression”

This procedure identifies 3408 mRNA therapy patents granted between 1980 and 2023 to 1211 unique assignees (as identified by PatentsView). Figure D.1(a) displays the number of patents filed and granted over 1980-2023.

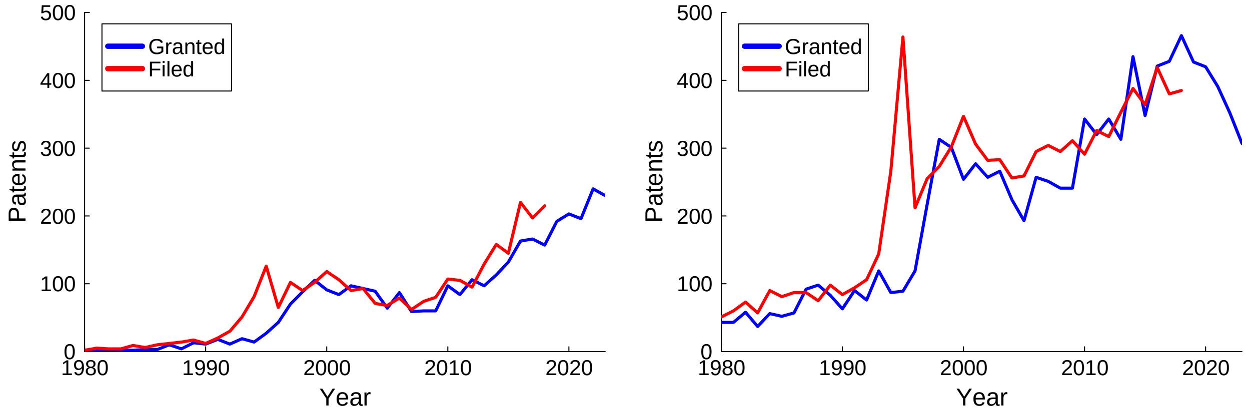
Conventional vaccine patents: I identify patents related to conventional vaccines by keyword search. I start with all patents that mention at least one term from the following list:

vaccine terms: “vaccin”, “innocul”, “immuniz”

I then exclude patents that mention any of the exclusion terms for RNA technologies noted above. I also exclude any patents that mention terms related to cancer, because cancer vaccines constitute a distinct technology from conventional vaccines for infectious diseases:

exclusion terms (cancer): “cancer”, “tumor”, “tumour”, “oncolog”, “oncogen”, “malign”, “mestast”, “neoplas”

Finally, I exclude remaining patents found in the set of mRNA therapy patents constructed above. This procedure identifies 9868 conventional vaccine patents granted between 1980



(a) mRNA Therapy Patents (b) Conventional Vaccine Patents

Figure D.1: Patents Filed and Granted, 1980-2023

Notes: Given the lag between filing year and grant year documented in Figure D.2, I halt the "Filed" series in 2018. The spike in patent filings for conventional vaccines in 1995 is due to the implementation of the TRIPS agreement, which required WTO members to respect pharmaceutical patents.

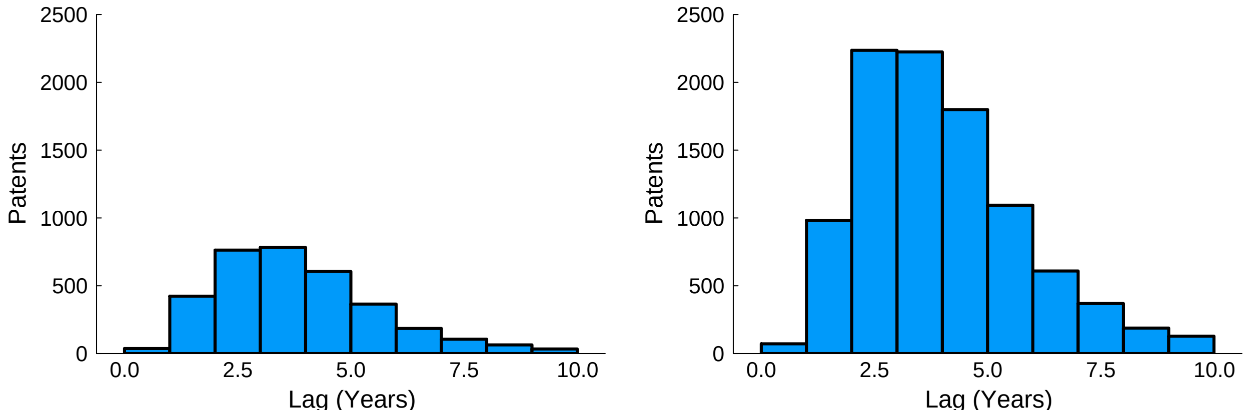
and 2023 to 2683 unique assignees (as identified by PatentsView). Figure D.1(b) displays the number of patents filed and granted over 1980-2023.

Incumbent patents: I determine the set of mRNA therapy and conventional vaccine patents assigned to any of the following top 20 pharmaceutical firms:

Top 20 Pharma: Johnson & Johnson, Sinopharm, Roche, Merck, Pfizer, AbbVie, Bayer, Sanofi, AstraZeneca, Novartis, Bristol-Myers Squibb, GSK, Eli Lilly Novo Nordisk, Shanghai Pharmaceuticals, Takeda, Amgen, Boehringer Ingelheim, Gilead Sciences, Siemens Healthineers

I search the assignees for each patent for keywords related to each of these firms, including the names of any major subsidiaries or recently acquired firms. This procedure identifies 243 mRNA therapy patents and 1385 conventional vaccine patents granted to the firms above.

Entrant patents: I determine the set of mRNA therapy and conventional vaccine patents assigned to Moderna, BioNTech, CureVac, or RNARx by keyword search on patent assignees. To more fully capture these firms' expertise, I also include all patents that list one of their founders as an inventor. This procedure identifies 242 mRNA therapy patents and 47 conventional vaccine patents granted to the four entrant firms or their founders.



(a) mRNA Therapy Patents

(b) Conventional Vaccine Patents

Figure D.2: Lag from Patent Filing to Publication

Notes: For both sets of patents, over 85% of patents have a lag from filing to publication of five years or less.

Forward citations: I directly compute the number of forward citations received by each patent from any other patent in the PatentsView dataset. These counts naturally suffer from truncation bias that becomes severe for more recently-granted patents. To control for this, I normalize each patent's forward citations by the average number of forward citations received by any patent granted in the same year. The normalized citation count of a patent p granted in year t is then

$$\text{Count}_p \equiv \text{FCites}_p \times \frac{|\{p' \text{ granted in } t\}|}{\sum_{p' \text{ granted in } t} \text{FCites}_p}.$$

The average value of Count_p across all mRNA therapy patents is 1.84 , while the average across all conventional vaccine patents is 0.62 .

Sample: On average, mRNA therapy and conventional vaccine patents each have a 3.5-year lag between filing and publication (see Figure D.2). To avoid noise in observed patents filed each year, I consider only granted patents filed through 2018 in Figure 2. Approximately 85% of each of mRNA therapy and conventional vaccine patents have a publication lag of five years or less.

D.3 Data Build: Empirical Analysis

E Empirics: Additional Results and Robustness

	Technology Patents		Technology Patent Share	
	(1) Full Sample	(2) No Early Patents	(3) Full Sample	(4) No Early Patents
$\log(K^{\text{Firm}}_{iT_\theta})$	-0.0833***	-0.0843***	-0.0023***	-0.0013**
	(0.0306)	(0.0270)	(0.0008)	(0.0006)
$\log(K^{\text{Agg}}_{iT_\theta})$	-0.4404***	-0.3957***	-0.0044	-0.0045
	(0.1650)	(0.1498)	(0.0038)	(0.0044)
$\log(s_{iT_\theta})$	0.0179	0.0118	2.14×10^{-5}	8.12×10^{-5}
	(0.0119)	(0.0125)	(0.0003)	(0.0003)
$\log(K^{\text{Firm}}_{i\theta T_\theta})$	0.3305***	0.3729***	0.0069***	0.0093**
	(0.0256)	(0.1018)	(0.0011)	(0.0047)
$\log(K^{\text{Agg}}_{\theta T_\theta})$	0.1880***	0.1839***	0.0032***	0.0030***
	(0.0115)	(0.0119)	(0.0005)	(0.0005)
R^2	0.63910	0.58584	0.27771	0.25158
Observations	13,190	11,024	13,190	11,024
Dep. Var. Mean	2.0682	1.7651	0.01569	0.01373
Dep. Var. SD	2.2352	2.1120	0.04381	0.04161

*Significance: *** 0.01, ** 0.05, * 0.1*

Table 5: Regression Results: Technology Patenting after Emergence (35)

Notes: All regressions include fixed effects by firm, emergence year, and the number of years for which the firm has been publicly listed. They also include dummy variables for zero values of each of the knowledge stocks and R&D expenditures s_{it}. All standard errors are clustered at the firm level.