14.452 Recitation 4: OLG, Romer

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Recitation Plan: Solve the canonical OLG model (Q3 on Problem Set 3) and the Romer (1990) "lab equipment" model with knowledge spillovers

1 OLG: Problem Set 3 Question 3

[Posted after the problem set due date]

2 Romer (1990)

2.1 Setup

This model exists in continuous time $t \in [0, \infty)$ and consists of a representative household with labor endowment $L(t) = L\exp(nt)$, discount rate $\rho > 0$, and consumption utility $u(c) = c^{1-\theta}/(1-\theta)$. A unique final good (and numeraire) is produced at each time t using the modified Cobb-Douglas production technology

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu\right] L_E(t)^{\beta},$$

where $L_E(t)$ denotes the quantity of labor employed in final good production, $x(\nu,t)$ denotes the quantity of intermediate good ν used in final good production, and $N(t)$ denotes the number of intermediate varieties discovered up to time t. Each intermediate is produced using the final good at marginal cost $\psi > 0$, and intermediates are assumed to depreciate completely at each time.

Labor can also be used to conduct research and development (R&D) for the discovery of new intermediate varieties. Given a quantity of labor input $L_R(t)$, the number of varieties increases according to the evolution equation

$$\dot{N}(t) = N(t)^{\phi}\eta L_R(t).$$

Here $\phi \leq 1$ controls the strength of knowledge spillovers across time: With $\phi > 0$, greater existing knowledge makes current researchers more productive in the discovery of new varieties, and this effect is stronger when ϕ is larger. I restrict $\phi \leq 1$ so that we do not obtain "explosive" growth even when $L_R(t)$ is constant over time. I refer to the case with $\phi = 1$ as exhibiting *log-linear spillovers* because $\dot{\log}(N(t))$ is constant in $N(t)$, while the case with $\phi < 1$ exhibits *log-sublinear spillovers* because $\dot{\log}(N(t))$ is instead declining in $N(t)$. We will see below that these two cases have drastically different implications for long-run growth.[1]

Labor is allocated between final good production and R&D according to profit-maximizing behavior by two different kinds of firms. A representative final good producer chooses the quantities of all inputs ($x(\nu,t)$ for $\nu \in [0, N(t)]$ and $L_E(t)$) to maximize profits, taking the price of each intermediate $p(\nu,t)$ and the wage $w(t)$ as given. A large mass of firms also employ labor to discover new intermediate varieties. Each of these "potential monopolists" can employ

[1]This observation is due to Jones (1995), and it was a key point of contention in the endogenous growth literature of the late 1990s.

one unit of labor to discover a new variety at rate $N(t)^{\phi}\eta$.[2] Aggregating across all potential monopolists that employ labor, the total flow rate of new ideas is then $\dot{N}(t) = N(t)^{\phi}\eta L_R(t)$. Potential monopolists find it optimal to employ labor for R&D provided that the value $V(t)$ of discovering a new variety at t dominates the cost of discovery. Equivalently, this holds when the value of employing an additional unit of labor at wage $w(t)$ is weakly smaller than the value generated by that labor, which equals the flow rate of discovery $N(t)^{\phi}\eta$ times the value $V(t)$. In equilibrium, potential monopolists continue to enter until the wage $w(t)$ is driven up to this flow value $N(t)^{\phi}\eta V(t)$, so that we satisfy

$$N(t)^{\phi}\eta V(t) \leq w(t) \quad \text{and} \quad L_R(t) \geq 0,$$

with complementary slackness.

To complete the description of the model, we must determine the value $V(t)$. I assume that each monopolist that successfully invents a new intermediate variety ν receives a perpetual patent on that variety. As a result, it can set its price $p(\nu, t)$ at each time t to maximize profits, taking all remaining equilibrium objects except for the quantity $x(\nu, t)$ as given. Letting $\pi(t)$ denote the profits at each time t, and noting that π does not depend on ν because all existing intermediates $\nu \in [0, N(t)]$ enter final production symmetrically and have the same marginal cost ψ, the value $V(t)$ must satisfy

$$V(t) = \int_t^{\infty} \exp\left(-\int_t^s r(u)du\right)\pi(s)ds.$$

Here $r(t)$ denotes the equilibrium interest rate at time t. The value of ownership of an intermediate is then the present discounted value of all future profit flows, discounted to present using the "market" discount rate $r(t)$. Differentiating with respect to t implies that this value also satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$r(t)V(t) = \pi(t) + \dot{V}(t).$$

This equation expresses the "arbitrage condition" that the instantaneous return to owning an intermediate $r(t)V(t)$ must equal the flow dividend $\pi(t)$ plus any "capital gains" $\dot{V}(t)$.

Finally, note that in this version of the Romer (1990) model, the household can "save" only in a fairly implicit way. Just as in the neoclassical growth model, we allow the household

[2] I write this as if each potential monopolist can only employ one unit of labor for R&D, but since the "production technology for knowledge" $\dot{N} = N^{\phi}\eta L_R$ exhibits constant returns to scale in L_R, it's all the same if each potential monopolist can employ any quantity of labor it wishes.

access to an asset $\mathcal{A}(t)$ that pays an instantaneous return $r(t)$ at each time t and, from the household's perspective, allows it to transfer consumption across time. The household's optimal consumption stream can again be summarized by the Euler equation and the transversality condition

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho),$$
$$0 = \lim_{t\to\infty} \exp\left(-\int_0^t r(s)ds\right)\mathcal{A}(t).$$

But how does "saving" actually happen, and what is the asset $\mathcal{A}(t)$ since this model does not have physical capital? In equilibrium, the household's assets at each time t must be equal to the value of all intermediate monopolists: $\mathcal{A}(t) = N(t)V(t)$. Intuitively, when the household wants to transfer consumption into the future, the economy responds by reducing the quantity of labor $L_E(t)$ employed in final good production and raising the quantity of labor $L_R(t)$ employed in R&D. This raises the rate at which new intermediates are discovered and hence the "supply" of assets $N(t)V(t)$. As we will see below, this works to raise consumption in the future by making labor more productive in producing the final good, which increases consumption (holding the labor input fixed).

The way this works in equilibrium is as follows: Fix a path for per capita consumption $[c(t)]_{t\geq 0}$, and note that the interest rate $r(t)$ is pinned down at each time by the household's Euler equation. Suppose we temporarily increase the household's desire for saving at time t (say, by reducing ρ temporarily). This leads the household to demand more assets $\mathcal{A}(t)$, which places downward pressure on the interest rate $r(t)$. But this raises the discounted present value $V(t)$ of future profits earned by an intermediate monopolist, stimulating additional entry and increasing the rate of production of new assets $\dot{N}(t)$.[3] This can only happen if labor is reallocated away from final good production and toward R&D, which reduces present consumption in favor of future consumption.

2.2 Static Equilibrium Conditions

Before studying the dynamic equilibrium in this model, we can make some progress by studying the static equilibrium conditions of the final good producer and the monopolists of existing intermediates $\nu \in [0, N(t)]$. Given the wage $w(t)$ and the intermediate prices $[p(\nu, t)]_{\nu=0}^{N(t)}$,

[3] This explanation assumes that initially $\dot{N}(t) > 0$. If we are instead in an equilibrium with no R&D, then the interest rate is the only part of the equilibrium that adjusts to the increased propensity to save, ensuring that the household finds it optimal to consume according to the original consumption path at each time. This adjustment is just as in the asset pricing model of Lucas (1978), and I'm happy to explain further if helpful.

the final good producer chooses $L_E(t)$ and $[x(\nu,t)]_{\nu=0}^{N(t)}$ to maximize profits. The first-order optimality conditions are

$$
\begin{aligned}
w(t) &= \beta \frac{Y(t)}{L_E(t)}, \\
p(\nu,t) &= L_E(t)^{\beta} x(\nu,t)^{-\beta}.
\end{aligned}
$$

We will eventually use the first condition to determine the wage $w(t)$. The second condition defines the (inverse) demand curve observed by each intermediate monopolist ν. Given this demand curve, the monopolist chooses the price $p(\nu,t)$ to maxmize its own profits at t:

$$
\max_{p} (p-\psi) L_E(t) p^{-1/\beta}.
$$

The solution to this problem is

$$
p(\nu,t) = \frac{1}{1-\beta}\psi,
$$

with corresponding quantity and profits

$$
\begin{aligned}
x(\nu,t) &= \bar{x} L_E(t), \quad \text{where} \quad \bar{x} = \left(\frac{\psi}{1-\beta}\right)^{-\frac{1}{\beta}} \\
\pi(t) &= \bar{\pi} L_E(t), \quad \text{where} \quad \bar{\pi} = \beta \left(\frac{\psi}{1-\beta}\right)^{-\frac{1-\beta}{\beta}}.
\end{aligned}
$$

Total output then satisfies

$$
Y(t) = \frac{\bar{x}^{1-\beta}}{1-\beta} N(t) L_E(t),
$$

so that the wage becomes

$$
w(t) = \beta \frac{Y(t)}{L_E(t)} = \beta \frac{\bar{x}^{1-\beta}}{1-\beta} N(t).
$$

Finally, total consumption is

$$
\begin{aligned}
C(t) &= Y(t) - \psi \int_0^{N(t)} x(\nu,t) d\nu \\
&= Y(t) - \psi \bar{x} N(t) L_E(t) \\
&= \left(1-(1-\beta)^{1+\beta}\right) Y(t).
\end{aligned}
$$

Note that the crucial feature of this model is that final output $Y(t)$ is proportional to $N(t)$ and $L_E(t)$: $N(t)$ acts like labor-augmenting technological progress, so provided that $L_E(t)$ eventually settles to a constant value, we expect to achieve constant growth in output per capita if $N(t)$ increases at a constant rate.

2.3 Log-Linear Spillovers: $\phi = 1$, $n = 0$

To characterize the equilibrium with $\phi = 1$ and $n = 0$, I begin as usual by studying the balanced growth path. Suppose an equilibrium in which output and consumption grow at the constant rate $g \geq 0$. The household's Euler equation then implies that the interest rate is constant and satisfies the standard "Ramsey formula"

$$r^* = \rho + \theta g.$$

There are two cases to consider: Either $\dot{N}(t) = 0$ always, or $\dot{N}(t) > 0$ at some time t. I consider these cases in turn.

Case 1: $\dot{N}(t) \equiv 0$. In this case, we must have $L_E(t) = L$ at each time t, so that the economy permanently stagnates with output $Y(t) = \frac{\bar{x}^{1-\beta}}{1-\beta} N(0)L$, wage $w(t) = \beta \frac{\bar{x}^{1-\beta}}{1-\beta} N(0)$, and interest rate $r^* = \rho$. To ensure that this is a valid equilibrium, we must only check that potential monopolists find it weakly optimal not to conduct R&D. The value $V(t)$ of an intermediate is $V(t) = \bar{\pi} L/\rho$, so that free entry with $L_R(t) = 0$ requires

$$N(0)\eta V(t) \leq w(0) \iff \rho \geq \eta(1-\beta)L.$$

Under this parameter restriction, the economy has a balanced growth path with $\dot{N}(t) \equiv 0$.

Case 2: $\dot{N}(t) > 0$ at some t. In this case, we must have $L_R(t) > 0$, so that the free-entry condition implies

$$N(t)\eta V(t) = w(t) \Rightarrow \eta V(t) = \beta \frac{\bar{x}^{1-\beta}}{1-\beta}.$$

The implication follows from the characterization of the wage $w(t)$ above. Hence $V(t) = V^*$ is constant,[4] and the HJB equation for $V(t)$ implies

$$V^* = \frac{\bar{\pi} L_E(t)}{r^*}.$$

[4] Here I'm really assuming that $\dot{N}(t) > 0$ on an interval or that $\dot{N}(t)$ is continuous, both of which are innocuous along a balanced growth path.

But then $L_E(t)$ must be constant, $L_E(t) \equiv L_E^*$. This constant quantity of labor employed in final good production and the growth rate g must satisfy the system of equations

$$
\begin{aligned}
L_E^* &= r^* \frac{V^*}{\bar{\pi}} = \frac{\rho + \theta g}{\eta(1-\beta)}, \\
g &= \frac{\dot{N}(t)}{N(t)} = \eta\left(L - L_E^*\right).
\end{aligned}
$$

The solution is

$$
\begin{aligned}
L_E^* &= \frac{1}{\eta} \frac{\theta \eta L + \rho}{1 - \beta + \theta}, \\
g &= \frac{\eta(1-\beta)L - \rho}{1 - \beta + \theta}.
\end{aligned}
$$

To ensure that we have characterized a valid equilibrium, we must check that (i) the free-entry condition is satisfied with $g > 0$ and (ii) $r^* > g$, so that the equilibrium features finite expected discounted output. The first condition holds provided that $\rho < \eta(1-\beta)L$, and the second condition holds provided that

$$
\rho > (1-\theta) g \iff \rho > \frac{\eta(1-\theta)(1-\beta)L}{2-\beta}
$$

The analysis above provides a full characterization of the unique balanced growth path in this economy. What about transitional dynamics? We shouldn't expect any in this model, because the economy does not have any "concave" features (like diminishing marginal returns to an accumulating factor) that would yield sluggish adjustment to the balanced growth path. To see this, suppose the balanced growth path features positive growth, and note that the free-entry condition and Euler equations imply the following characterizations of the interest rate $r(t)$:

$$
\begin{aligned}
r(t) &= \rho + \theta \left[\frac{\dot{N}(t)}{N(t)} + \frac{\dot{L}_E(t)}{L_E(t)} \right] \\
&= \rho + \theta \left[\eta(L - L_E(t)) + \frac{\dot{L}_E(t)}{L_E(t)} \right], \\
r(t) &= \eta(1-\beta) L_E(t).
\end{aligned}
$$

These equations imply

$$L_E(t) - L_E^* = \frac{1}{\eta}\frac{\theta}{1-\beta+\theta}\frac{\dot{L}_E(t)}{L_E(t)}$$

When $L_E(t)$ is above its BGP value L_E^*, this equation implies that the growth rate of $L_E(t)$ is positive. If ever $L_E(t) > L_E^*$, the unique solution to this differential equation would feature $L_E(t) \to \infty$, violating the labor market clearing condition $L_E(t) + L_R(t) \leq L$. Similarly, if ever $L_E(t) < L_E^*$, the solution would feature $L_E(t) \to 0$, implying no consumption as $t \to \infty$ and violating the household's transversality condition. We conclude that any equilibrium must feature $L_E(t) = L_E^*$ at all times t, so that we immediately follow the balanced growth path.

2.4 Log-Sublinear Spillovers: $\phi < 1$, $n > 0$

Weakening knowledge spillovers by reducing ϕ below 1 has (surprisingly!) strong implications for growth in this model. For example, if the number of workers allocated to R&D is held fixed, it is easy to see that the growth rate of labor productivity will tend to zero over time:

$$\frac{\dot{N}(t)}{N(t)} = \eta N(t)^{\phi-1} L_R \to 0,$$

where the limit holds since $\phi < 1$.[5] Conversely, when $\phi = 1$, the growth rate of labor productivity *diverges* if the number of workers allocated to R&D increases at a constant rate (e.g., because of population growth):

$$\frac{\dot{N}(t)}{N(t)} = \eta L_R(t) \to \infty.$$

This is the essence of the *scale effect* in the model with $\phi = 1$: The growth rate on the balanced growth path is increasing in the *quantity* of labor L. As Jones (1995) discusses, this scale effect is counterfactual across many countries and time periods, and that paper proposes a variation of the Romer (1990) model with $\phi < 1$ but $n > 0$ to remove it.

In a balanced growth path with a constant *share* of workers s allocated to R&D, it is straightforward to determine the growth rate g of labor productivity (or output per worker). Since

[5] You can think of this case as encapsulating a kind of "dynamic decreasing returns": The marginal improvement in R&D productivity from additional knowledge accumulation decays over time.

output per worker is proportional to $N(t)$, g must satisfy

$$g = \frac{\dot{N}(t)}{N(t)} = \eta N(t)^{\phi-1} s L(t).$$

Log differentiating both sides implies

$$0 = \frac{\dot{L}(t)}{L(t)} - (1-\phi)\frac{\dot{N}(t)}{N(t)} = n - (1-\phi)g \iff g = \frac{n}{1-\phi}.$$

Hence the long-run growth rate of output per worker is entirely determined by the growth rate of the population and the elasticity of R&D productivity to existing knowledge ϕ. Using similar arguments to those as in the $\phi = 1$ case, it is straightforward to characterize the remaining equilibrium objects along the balanced growth path when $\phi < 1$.