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Results in Mathematics

Korovkin-Type Approximation Theorem for Double Sequences of Positive Linear Operators via Statistical *A*-Summability

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Abstract. In this paper, using the concept of statistical A-summability which is stronger than the A-statistical convergence we provide a Korovkin-type approximation theorem on the space of all continuous real valued functions defined on any compact subset of the real two-dimensional space. We also study the rates of statistical A-summability of positive linear operators.

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1. Introduction

Approximation theory has important applications in theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations, etc. [1]. The well-known Korovkin approximation theorem [1,7] for a sequence of positive linear operators, say $\{L_n(f;x)\}$, is mainly based on the existence of the limit $\lim_{n\to\infty} L_n(f;x) = f(x)$. Korovkin [7] first noticed the sufficient conditions for the uniform convergence of $L_n(f;x)$ to a function f by using the test functions $x^i, (i = 0, 1, 2)$. Many researchers have investigated these conditions for various operators defined on different spaces. Recently, some Korovkin type approximation theorems have been studied via statistical convergence and equi-statistical convergence [3,6]. Also, it was proved that those results are stronger than the classical Korovkin theorem. Our primary interest in the present paper is to obtain a

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Korovkin-type approximation theorem for a double sequence of positive linear operators defined on $C(D)$, which is the space of all continuous real valued functions on any compact subset of the real two-dimensional space. Also, we study the rates of statistical A-summability of positive linear operators.

We first recall the concept of A-statistical convergence for double sequences.

A double sequence $x = (x_{m,n})_{m,n\in\mathbb{N}}$ is convergent to a number L in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. In this case L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [11]).

If there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the two-dimensional set of all positive integers, then $x = (x_{m,n})$ is said to be bounded. Recall that if a single sequence is convergent, then it is also bounded. But, this case does not hold for a double sequence, i.e. the convergence in Pringsheim's sense of a double sequence does not imply the boundedness of the double sequence.

Let

$$A = [a_{j,k,m,n}], \quad j, k, m, n \in \mathbb{N},$$

be a four-dimensional infinite matrix. For a given double sequence $x = (x_{m,n})$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j, k) \in \mathbb{N}^2$. We say that a sequence x is A-summable to l if the A-transform of x exists for all $j, k \in \mathbb{N}$ and is convergent in the Pringsheim's sense i.e.,

$$P - \lim_{p,q} \sum_{m\in\mathbb{N}}^{p} \sum_{n\in\mathbb{N}}^{q} a_{j,k,m,n} x_{m,n} = y_{j,k} \quad \text{and} \quad P - \lim_{j,k} y_{j,k} = l.$$

In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix transformations is known as Silverman–Toeplitz conditions (see, for instance, [5]). In Robison [12] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison–Hamilton conditions, or briefly, RH-regularity. (see, [4, 12])

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison–Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P-\lim_{j,k} a_{j,k,m,n}=0$ for each $(m,n)\in\mathbb{N}^2$,
(ii) $P-\lim_{j,k}\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}=1$,
(iii) $P-\lim_{j,k}\sum_{m\in\mathbb{N}}|a_{j,k,m,n}|=0$ for each $n\in\mathbb{N}$,
(iv) $P-\lim_{j,k}\sum_{n\in\mathbb{N}}|a_{j,k,m,n}|=0$ for each $m\in\mathbb{N}$,
(v) $\sum_{(m,n)\in\mathbb{N}^2}|a_{j,k,m,n}|$ is P-convergent for each $(j,k)\in\mathbb{N}^2$,
(vi) there exist finite positive integers A and B such that $\sum_{m,n>B}|a_{j,k,m,n}|<A$ holds for every $(j,k)\in\mathbb{N}^2$.

Now let $A=[a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K\subset\mathbb{N}^2$. Then A-density of K is given by

$$\delta_A^2(K):=P-\lim_{j,k}\sum_{(m,n)\in\mathbb{K}} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x=(x_{m,n})$ is said to be A-statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_A^2\left(\left\{(m,n)\in\mathbb{N}^2:|x_{m,n}-L|\geq\varepsilon\right\}\right)=0.$$

In this case we write $st^2_{(A)}-\lim_{m,n}x_{m,n}=L$. Observe that a P-convergent double sequence is A -statistically convergent to the same value but the converse is not always true.

We should note that if we take $A=C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [8,10]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

Definition 1.1. Let $A=[a_{j,k,m,n}]$ be a nonnegative RH-regular summability matrix and $x=(x_{m,n})$ be a double sequence. We say that x is statistically A-summable to L if for every $\varepsilon>0$,

$$\delta^2_{C(1,1)}\left(\left\{(j,k)\in\mathbb{N}^2:|(Ax)_{j,k}-L|\geq\varepsilon\right\}\right)=0,$$

i.e.,

$$\lim_{p,q}\frac{1}{pq}\left|\{j\leq p,k\leq q:|(Ax)_{j,k}-L|\geq\varepsilon\}\right|=0$$

where $|B|$ denotes the cardinality of the set B. Thus $x=(x_{m,n})$ is statistically A-summable to L if and only if Ax is statistically convergent to L. In this case we write $(A)^2_{st}-\lim_{m,n}x_{m,n}=L$ or $st^2_{C(1,1)}-\lim_{j,k}(Ax)_{j,k}=L$.

We note that if we take $A=C(1,1)$ then statistical A-summability is reduced to the statistical $C(1,1)$-summability due to Moricz [9].

Now we prove the following relation between statistical A-summability and A-statistical convergence.

Theorem 1.2. *If a double sequence is bounded and A-statistically convergent to L then it is A-summable to L and hence statistically A-summable to L but not conversely.*

Proof. Let $x = (x_{m,n})$ be bounded and A-statistically convergent to L, and

$$K(\varepsilon) = \{m \le p, n \le q : |x_{m,n} - L| \ge \varepsilon\}.$$

Then

$$\begin{aligned}
|(Ax)_{j,k} - L| &\le \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}(x_{m,n} - L) \right| + |L| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} - 1 \right| \\
&\le \left| \sum_{(m,n)\notin K(\varepsilon)} a_{j,k,m,n}(x_{m,n} - L) \right| + \left| \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n}(x_{m,n} - L) \right| \\
&\quad + |L| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} - 1 \right| \\
&\le \varepsilon \sum_{(m,n)\notin K(\varepsilon)} a_{j,k,m,n} + \sup_{(m,n)\in\mathbb{N}^2} |x_{m,n} - L| \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} \\
&\quad + |L| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} - 1 \right|.
\end{aligned}$$

By using the definition of A-statistical convergence and the conditions of RH-regularity of A, we get

$$P - \lim_{j,k} |(Ax)_{j,k} - L| = 0$$

since $\varepsilon > 0$ was arbitrary. Hence $st^2_{C(1,1)} - \lim_{j,k}(Ax)_{j,k} = L$. □

To see that the converse does not hold, we construct the following example:

Example 1.1. Let A be double Cesáro matrix, i.e.

$$a_{j,k,m,n} = \begin{cases} \frac{1}{jk}, & 1 \le m \le j, 1 \le n \le k, \\ 0, & \text{otherwise.} \end{cases}$$

and $\alpha = (\alpha_{m,n})$ is a double sequence defined by $\alpha_{m,n} = (-1)^{m+n}$. Observe now that, $A = [a_{j,k,m,n}]$ is a nonnegative RH-regular summability matrix and for the sequence $\alpha = (\alpha_{m,n})$

$$st^2_{C(1,1)} - \lim_{j,k}(A\alpha)_{j,k} = 0.$$

However, the sequence $(\alpha_{m,n})$ does not convergent in the Pringshem sense.

2. A Korovkin-Type Theorem

By $C(D)$, we denote the space of all continuous real valued functions on any compact subset of the real two-dimensional space. This space is equipped with the supremum norm

$$\|f\|_{C(D)} = \sup_{(x,y)\in D} |f(x,y)|, \quad (f \in C(D)).$$

Let L be a linear operator from $C(D)$ into $C(D)$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of Lf a point $(x,y) \in D$ by $L(f;x,y)$.

Throughout the paper, we also use the following test functions

$$f_0(x,y) = 1, \quad f_1(x,y) = x, \quad f_2(x,y) = y, \quad f_3(x,y) = x^2 + y^2.$$

First we recall the statistical and classical cases of the Korovkin-type results introduced in [2, 14], respectively.

Theorem 2.1. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into $C(D)$. Then, for all $f \in C(D)$,*

$$P - \lim_{m,n} \|L_{m,n}(f) - f\|_{C(D)} = 0$$

if and only if

$$P - \lim_{m,n} \|L_{m,n}(f_i) - f_i\|_{C(D)} = 0, \quad (i = 0,1,2,3).$$

Theorem 2.2. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into $C(D)$. Then, for all $f \in C(D)$,*

$$st^2_{C(1,1)} - \lim_{m,n} \|L_{m,n}(f) - f\|_{C(D)} = 0$$

if and only if

$$st^2_{C(1,1)} - \lim_{m,n} \|L_{m,n}(f_i) - f_i\|_{C(D)} = 0, \quad (i = 0,1,2,3).$$

Now we have the following main result.

Theorem 2.3. *Let $A = [a_{j,k,m,n}]$ be a nonnegative RH-regular summability matrix and let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into $C(D)$. Then, for all $f \in C(D)$,*

$$st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} = 0 \tag{2.1}$$

if and only if

$$st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_i) - f_i \right\|_{C(D)} = 0, \quad (i = 0,1,2,3). \tag{2.2}$$

Proof. Since each $f_i \in C(D), (i = 0, 1, 2, 3)$, the implication (2.1)⇒(2.2) is obvious. Now, to prove the implication (2.2)⇒(2.1) assume that (2.2) holds. Let $f \in C(D)$. By the continuity of f on compact set D, we can write

$$|f(x,y)| \leq M$$

where $M := \|f\|_{C(D)}$. Also, since f is continuous on D, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ for all $(u,v) \in D$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, we get

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\}. \tag{2.3}$$

Using the linearity and the positivity of the operators $\{L_{m,n}\}$ and (2.3), we obtain

$$\begin{aligned}
&\left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f;x,y) - f(x,y)\right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(|f(u,v) - f(x,y)|;x,y) \\
&\qquad + |f(x,y)| \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\};x,y\right) \\
&\qquad + M \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\quad \leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\left(E^2 + F^2\right)\right) \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0;x,y) - f_0(x,y)\right| \\
&\qquad + \frac{4M}{\delta^2} E \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_1;x,y) - f_1(x,y)\right| \\
&\qquad + \frac{4M}{\delta^2} F \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_2;x,y) - f_2(x,y)\right| \\
&\qquad + \frac{2M}{\delta^2} \left|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_3;x,y) - f_3(x,y)\right|
\end{aligned}$$

where $E := \max|x|, F := \max|y|$. Taking supremum over $(x,y) \in D$, we get

$$\begin{aligned}
&\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} \\
&\leq \varepsilon + K \left\{ \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} \right. \\
&\quad + \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_1) - f_1 \right\|_{C(D)} \\
&\quad + \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_2) - f_2 \right\|_{C(D)} \\
&\quad \left. + \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_3) - f_3 \right\|_{C(D)} \right\}
\end{aligned}$$

where $K := \max\left\{\varepsilon + M + \frac{2M}{\delta^2}\left(E^2+F^2\right), \frac{4M}{\delta^2}E, \frac{4M}{\delta^2}F, \frac{2M}{\delta^2}\right\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon > r$. Then, setting

$$D := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} \geq r \right\},$$

$$D_1 := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} \geq \frac{r-\varepsilon}{4K} \right\},$$

$$D_2 := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_1) - f_1 \right\|_{C(D)} \geq \frac{r-\varepsilon}{4K} \right\},$$

$$D_3 := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_2) - f_2 \right\|_{C(D)} \geq \frac{r-\varepsilon}{4K} \right\},$$

$$D_4 := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_3) - f_3 \right\|_{C(D)} \geq \frac{r-\varepsilon}{4K} \right\},$$

it is easy to see that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$. Then, we obtain

$$\lim_{p,q} \frac{1}{pq} |D| \leq \lim_{p,q} \frac{1}{pq} |D_1| + \lim_{p,q} \frac{1}{pq} |D_2| + \lim_{p,q} \frac{1}{pq} |D_3| + \lim_{p,q} \frac{1}{pq} |D_4| .$$

Then using the hypothesis (2.2) we get

$$\lim_{p,q} \frac{1}{pq} \left| \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} \geq r \right\} \right| = 0.$$

So the proof is complete. □

Remark 2.4. If we replace the matrix A in Theorem 2.3 by the identity double matrix, then we immediately get the Theorem 2.1.

Remark 2.5. Now, we exhibit an example of a sequence of positive linear operators of two variables satisfying the conditions of Theorem 2.3 but that does not satisfy the conditions of Theorems 2.1 and 2.2.

Let $I = [0,1]$ *and* $D := I \times I$*. Now we consider the double sequence* $\{L_{m,n}\}$ *of positive linear operators defined by*

$$L_{m,n}(f;x,y) = (1 + \alpha_{m,n}) B_{m,n}(f;x,y) \tag{2.4}$$

where $\{B_{m,n}\}$ *are the Bernstein polynomials of two variables defined on* $C(D)$ *by*

$$B_{m,n}(f;x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m}{j} x^j (1-x)^{m-j} \binom{n}{k} y^k (1-y)^{n-k}$$

(see [13]*) and* $(\alpha_{m,n})$ *is a double sequence defined by* $\alpha_{m,n} = (-1)^{m+n}$*. Then, observe that*

$$\begin{aligned}
L_{m,n}(f_0;x,y) &= (1+\alpha_{m,n}) f_0(x,y),\\
L_{m,n}(f_1;x,y) &= (1+\alpha_{m,n}) f_1(x,y),\\
L_{m,n}(f_2;x,y) &= (1+\alpha_{m,n}) f_2(x,y),\\
L_{m,n}(f_3;x,y) &= (1+\alpha_{m,n}) \left[f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n} \right].
\end{aligned}$$

Now we take $A = C(1,1)$ then, from Example 1.1 we see that

$$st^2_{C(1,1)} - \lim_{j,k} (A\alpha)_{j,k} = 0.$$

So, we conclude that

$$st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_i) - f_i \right\|_{C(D)} = 0, \quad (i = 0,1,2,3).$$

Therefore, by the Theorem 2.3 we see that

$$st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} = 0.$$

Also, since $(\alpha_{m,n})$ is not P-convergent and A-statistical convergent to zero, we can say that the Theorems 2.1 and 2.2 do not work for our operators defined by (2.4).

3. Rates of Convergence

In this section, using the concept of statistical A-summability we study the rate of convergence of positive linear operators with the help of the modulus of continuity. To this end we assume convexity of D.

Let $f \in C(D)$. Then the modulus of continuity of f, defined to be

$$w(f;\delta) = \sup \left\{ |f(u,v) - f(x,y)| : (u,v),(x,y) \in D \text{ and } \sqrt{(u-x)^2+(v-y)^2} \leq \delta \right\}$$

for $\delta > 0$. It is easy to see that, for any $c > 0$ and all $f \in C(D)$

$$w(f;c\delta) \leq (1+[c])\, w(f;\delta)$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Then we have the following result.

Theorem 3.1. *Let $A = [a_{j,k,m,n}]$ be a nonnegative RH-regular summability matrix and let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into $C(D)$. Assume that the following conditions hold:*

(i) $st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} = 0,$

(ii) $st^2_{C(1,1)} - \lim_{j,k} w(f;\delta) = 0,$ *where*

$$\delta := \delta_{(j,k)} := \sqrt{\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(\varphi) \right\|_{C(D)}} \text{ with } \varphi(u,v) = (u-x)^2 + (v-y)^2.$$

Then, we have, for all $f \in C(D)$,

$$st^2_{C(1,1)} - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} = 0.$$

Proof. Let $(x, y) \in D$ and $f \in C(D)$ be fixed. Using the linearity and the positivity of the operators $L_{m,n}$, for all $(m, n) \in \mathbb{N}^2$ and any $\delta > 0$, we have

$$\begin{aligned}
&\left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f; x, y) - f(x, y) \right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(\left|f(u, v) - f(x, y)\right|; x, y) \\
&\qquad + |f(x, y)| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0; x, y) - f_0(x, y) \right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n} \left(w \left(f; \delta \frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} \right); x, y \right) \\
&\qquad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(f_0; x, y\right) - f_0\left(x, y\right) \right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n} \left(\left(1 + \left[\frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} \right] \right) w\left(f; \delta\right); x, y \right) \\
&\qquad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(f_0; x, y\right) - f_0\left(x, y\right) \right| \\
&\quad \leq \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} w\left(f; \delta\right) L_{m,n} \left(\left(1 + \frac{(u-x)^2 + (v-y)^2}{\delta^2} \right); x, y \right) \\
&\qquad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(f_0; x, y\right) - f_0\left(x, y\right) \right| \\
&\quad \leq w\left(f; \delta\right) \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(f_0; x, y\right) - f_0\left(x, y\right) \right| \\
&\qquad + \|f\|_{C(D)} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}\left(f_0; x, y\right) - f_0\left(x, y\right) \right| \\
&\qquad + w(f; \delta) + \frac{w(f; \delta)}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}((u-x)^2 + (v-y)^2; x, y).
\end{aligned}$$

Then taking supremum over $(x,y)\in D$, we have

$$\begin{aligned}
&\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}(f)-f\right\|_{C(D)}\\
&\quad\leq \|f\|_{C(D)}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}\\
&\qquad+w\left(f;\delta\right)\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}\\
&\qquad+\frac{w\left(f;\delta\right)}{\delta^2}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(\varphi\right)\right\|_{C(D)}+w\left(f;\delta\right).
\end{aligned}$$

Now, if we take $\delta:=\delta_{(j,k)}:=\sqrt{\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(\varphi\right)\right\|_{C(D)}}$ then we may write that

$$\begin{aligned}
&\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f\right)-f\right\|_{C(D)}\\
&\quad\leq \|f\|_{C(D)}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}\\
&\qquad+w\left(f;\delta_{(j,k)}\right)\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}+2w\left(f;\delta_{(j,k)}\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f\right)-f\right\|_{C(D)}\\
&\quad\leq K\left\{\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}+w\left(f;\delta_{(j,k)}\right)\right.\\
&\qquad\left.+w\left(f;\delta_{(j,k)}\right)\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}L_{m,n}\left(f_0\right)-f_0\right\|_{C(D)}\right\} \qquad (3.1)
\end{aligned}$$

where $K=\max\left\{2,\|f\|_{C(D)}\right\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon > r$. Then, setting

$$D := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} \geq r \right\},$$

$$D_1 := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} \geq \frac{r}{3K} \right\},$$

$$D_2 := \left\{ j \leq p, k \leq q : w\left(f; \delta_{(j,k)}\right) \geq \frac{r}{3K} \right\},$$

$$D_3 := \left\{ j \leq p, k \leq q : w\left(f; \delta_{(j,k)}\right) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} \geq \frac{r}{3K} \right\}.$$

Then, it follows from (3.1) that $D \subset D_1 \cup D_2 \cup D_3$. Also, defining

$$D_3' := \left\{ j \leq p, k \leq q : w\left(f; \delta_{(j,k)}\right) \geq \sqrt{\frac{r}{3K}} \right\},$$

$$D_3'' := \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f_0) - f_0 \right\|_{C(D)} \geq \sqrt{\frac{r}{3K}} \right\},$$

we have $D_3 \subset D_3' \cup D_3''$, which yields

$$D \subset D_1 \cup D_2 \cup D_3' \cup D_3''.$$

Therefore, using (i) and (ii) we get

$$\lim_{p,q} \frac{1}{pq} \left| \left\{ j \leq p, k \leq q : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} L_{m,n}(f) - f \right\|_{C(D)} \geq r \right\} \right| = 0.$$

So the proof is complete. □

References

[1] DeVore, R.A.: The Approximation of Continuous Functions by Positive Linear Operators. Lecture Notes in Mathematics, vol. 293. Springer, Berlin (1972)

[2] Dirik, F., Demirci, K.: Korovkin type approximation theorem for functions of two variables in statistical sense. Turk. J. Math. **34**, 73–83 (2010)

[3] Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mountain J. Math. **32**, 129–138 (2002)

[4] Hamilton, H.J.: Transformations of multiple sequences. Duke Math. J. **2**, 29–60 (1936)

[5] Hardy, G.H.: Divergent Series. Oxford Univ. Press, London (1949)

[6] Karakuş, S., Demirci, K., Duman, O.: Equi-statistical convergence of positive linear operators. J. Math. Anal. Appl. **339**, 1065–1072 (2008)

[7] Korovkin, P.P.: Linear Operators and Approximation Theory. Hindustan Publ. Co., Delhi (1960)

[8] Moricz, F.: Statistical convergence of multiple sequences. Arch. Math. (Basel) **81**, 82–89 (2004)

[9] Moricz, F.: Tauberian theorems for double sequences that are statistically summable $(C, 1, 1)$. J. Math. Anal. Appl. **286**, 340–350 (2003)

[10] Mursaleen, M., Edely, O.H.H.: Statistical convergence of double sequences. J. Math. Anal. Appl. **288**, 223–231 (2003)

[11] Pringsheim, A.: Zur theorie der zweifach unendlichen zahlenfolgen. Math. Ann. **53**, 289–321 (1900)

[12] Robison, G.M.: Divergent double sequences and series. Am. Math. Soc. Transl. **28**, 50–73 (1926)

[13] Stancu, D.D.: A method for obtaining polynomials of Bernstein type of two variables. Am. Math. Monthly **70**(3), 260–264 (1963)

[14] Volkov, V.I.: On the convergence of sequences of linear positive operators in the space of two variables. Dokl. Akad. Nauk. SSSR (N.S.) **115**, 17–19 (1957)

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