Statistical Convergence of Double Sequences on Intuitionistic Fuzzy Normed Spaces

Sevda Karakus and Kamil Demirci

Department of Mathematics, Faculty of Sciences and Arts Sinop, Ondokuz Mayis University, 57000 Sinop-TURKEY
E-mail: skarakus@omu.edu.tr , kamild@omu.edu.tr

Abstract

The concept of statistical convergence was presented by Steinhaus (1951). This concept was extended to the double sequences by Mursaleen and Edely (2003). In this paper, we define and study statistical analogue of convergence and Cauchy for double sequences on intuitionistic fuzzy normed spaces. Then we give a useful characterization for statistically convergent double sequences. Furthermore, we display an example such that our method of convergence is stronger than the usual convergence for double sequences on intuitionistic fuzzy normed spaces.

KEY WORDS: Natural double density, statistical convergence, continuous $t-$norm, continuous $t-$conorm, intuitionistic fuzzy normed space.

1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [29]. Then many authors developed the theory of fuzzy set and applications. The fuzzy logic has been used many fields, like metric and topological spaces [9] , [10] , [16] , [19], theory of functions [4] , [18] , [28] , computer programing [17], econometrics and other fields [1] , [2] , [3] , [8] , [20] , [22] . Also, recently, the concepts of intuitionistic fuzzy metric space has been studied by Park [23], and intuitionistic fuzzy normed space have been studied by Saadati and Park [25] .

In this paper we give statistical analogues of convergence and Cauchy for double sequences which studied in Mursaleen and Osama [21] on intuitionistic fuzzy normed spaces.Also we display an example such that our method of convergence is stronger than the usual convergence for double sequences on intuitionistic fuzzy normed spaces.

Now we recall some notations and definitions which we used in the paper.

Definition 1 [26]*A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* $t-$*norm if it satisfies the following conditions:*

(*a*) $*$ *is associative and commutative,*

(*b*) $*$ *is continuous,*

(*c*) $a * 1 = a$ *for all* $a \in [0,1]$,

(*d*) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t-$*norm are* $a * b = ab$ and $a * b = \min(a,b)$ for all $a, b \in [0,1]$.

Definition 2 [26]*A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* $t-$*conorm if it satisfies the following conditions:*

(*a*) $\diamond$ *is associative and commutative,*

(*b*) $\diamond$ *is continuous,*

(*c*) $a \diamond 0 = a$ *for all* $a \in [0,1]$,

(*d*) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t-$*conorm are* $a \diamond b = \min(a+b, 1)$ and $a \diamond b = \max(a,b)$ for all $a, b \in [0,1]$.

Now we give the concept of intuitionistic fuzzy normed space which has recently introduced by Saadati and Park [25].

Definition 3 [25] *The* $5-$*tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space* (*IFNS*) *if* V *is a vector space,* $*$ *is a continuous* $t-$*norm,* $\diamond$ *is a continuous* $t-$*conorm, and* μ, ν *fuzzy sets on* $V \times (0, \infty)$ *satisfying the following conditions for every* $x, y \in V$ *and* $s, t > 0$:

(*a*) $\mu(x,t) + \nu(x,t) \leq 1$,

(*b*) $\mu(x,t) > 0$,

(*c*) $\mu(x,t) = 1$ *if and only if* $x = 0$,

(*d*) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(*e*) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(*f*) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*

(*g*) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$,

(*h*) $\nu(x,t) < 1$,

(*i*) $\nu(x,t) = 0$ *if and only if* $x = 0$,

(*j*) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,

(l) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous*,

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 0$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm.* We can give an example as follow:

Let $(V, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. If we define

$$\mu_0(x,t) := \frac{t}{t+\|x\|} \text{ and } \nu_0(x,t) := \frac{\|x\|}{t+\|x\|}.$$

for all $x \in V$ and every $t > 0$, then observe that $(V, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Before we present the new definitions and the main theorems, we shall recall some concepts which we need.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [24]. A double sequence $x = (x_{jk})_{jk=0}^{\infty}$ is called convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k \geq N$. L is called the Pringsheim limit of x.

A double sequence $x = (x_{jk})$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{pq} - x_{jk}| < \varepsilon$ for all $p \geq j \geq N$, $q \geq k \geq N$.

A double sequence x is bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j and k.

So we can give the (μ,ν) analogue of above two definitions as follow:

Definition 4 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{jk})$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that $\mu(x_{jk} - L, t) > 1 - \varepsilon$ and $\nu(x_{jk} - L, t) < \varepsilon$ for all $j, k \geq N$. It is denoted by $(\mu, v)_2 - \lim x = L$ or $x_{jk} \overset{(\mu,\nu)_2}{\to} L$ as $j, k \to \infty$.*

Definition 5 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{jk})$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν) provided that, for every $\varepsilon > 0$ and $t > 0$, there exists $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that $\mu(x_{jk} - x_{pq}, t) > 1 - \varepsilon$ and $\nu(x_{jk} - x_{pq}, t) < \varepsilon$ for all $j, p \geq N$, $k, q \geq M$*

Now we first recall statistical convergence and then in new section, we introduce basic definitions and properties which we mention above .

2 Statistical Convergence of Double Sequence on IFNS

Steinhaus [27] introduced the idea of statistical convergence (see also Fast [11]). If K is a subset of $\mathbb{N}$, the set of natural numbers, then the asymptotic density of K denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, when $|A|$ denotes the cardinality of the set A. A sequence $x = (x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$.

Statistical convergence has been investigated in a number of paper [6], [7], [12], [13], [14], [15].

Now we recall the concept of statistical convergence of double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}.$$

In case the sequence $(K(n,m)\diagup nm)$ has a limit in Pringsheim's sense [24] then we say that K has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

If we consider the set of $K = \{(i,j) : i,j \in \mathbb{N}\}$, then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0.$$

Also, if we consider the set of $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density $1/2$.

If we set $n = m$, we have a two-dimensional natural density considered by Christopher [5].

Now we recall the concepts of statistically convergent and statistically Cauchy for double sequence as follows:

Definition 6 [21] *A real double sequence $x = (x_{jk})$ is said to be statistically convergent the number ℓ provided that, for each $\varepsilon > 0$, the set*

$$\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk} - \ell| \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2 - \lim_{j,k} x_{jk} = \ell$.

Definition 7 [21]*A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy provided that, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j,p \geq N$, $k,q \geq M$, the set*

$$\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk} - x_{pq}| \geq \varepsilon\}$$

has double natural density zero.

Now we give the analogues of these with respect to the intuitionistic fuzzy norm (μ,ν).

Definition 8 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. A real double sequence $x=(x_{jk})$ is statistically convergent to $L\in V$ with respect to the intuitionistic fuzzy norm (μ,ν) provided that, for every $\varepsilon>0$ and $t>0$,*

$$K=\{(j,k)\,,\ j\le n \text{ and } k\le m:\mu(x_{jk}-L,t)\le 1-\varepsilon \text{ or } \nu(x_{jk}-L,t)\ge\varepsilon\} \quad (1)$$

has double natural density zero, i.e., if $K(n,m)$ be the numbers of (j,k) in K

$$\lim_{n,m}\frac{K(n,m)}{nm}=0. \quad (2)$$

In this case we write $st_{(\mu,\nu)_2}-\lim\limits_{j,k} x_{jk}=L$, where L is said to be $st_{(\mu,\nu)_2}-$limit. Also we denote the set of all statistically convergent double sequences with respect to the intuitionistic fuzzy norm (μ,ν) by $st_{(\mu,\nu)_2}$.

By using (2) and the well-known properties of the double natural density, we easily get the following lemma.

Lemma 9 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. Then, for every $\varepsilon>0$ and $t>0$, the following statements are equivalent:*

(i) $st_{(\mu,\nu)_2}-\lim\limits_{j,k} x_{jk}=L$

(ii) $\delta_2\{(j,k)\,,\ j\le n \text{ and } k\le m:\mu(x_{jk}-L,t)\le 1-\varepsilon\}=\delta_2\{(j,k)\,,\ j\le n \text{ and } k\le m:\nu(x_{jk}-L,t)\ge\varepsilon\}=0.$

(iii) $\delta_2\{(j,k)\,,\ j\le n \text{ and } k\le m:\mu(x_{jk}-L,t)>1-\varepsilon \text{ and } \nu(x_{jk}-L,t)<\varepsilon\}=1.$

(iv) $\delta_2\{(j,k)\,,\ j\le n \text{ and } k\le m:\mu(x_{jk}-L,t)>1-\varepsilon\}=\delta_2\{(j,k)\,,\ j\le n \text{ and } k\le m:\nu(x_{jk}-L,t)<\varepsilon\}=1.$

(v) $st_2-\lim\mu(x_{jk}-L,t)=1$ and $st_2-\lim\nu(x_{jk}-L,t)=0.$

Theorem 10 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If a double sequence $x=(x_{jk})$ is statistically convergent with respect to the intuitionistic fuzzy norms (μ,ν), then the $st_{(\mu,\nu)_2}-$limit is unique.*

Proof. Let $x=(x_{jk})$ be a double sequence. Suppose that $st_{(\mu,\nu)_2}-\lim x=L_1$ and $st_{(\mu,\nu)_2}-\lim x=L_2$. Let $t>0$ and $\varepsilon>0$. Choose $r\in(0,1)$ such that $(1-r)*(1-r)\ge 1-\varepsilon$ and $r\diamond r\le\varepsilon$. Then, we define the following sets:

$$\begin{aligned}
K_{\mu,1}(r,t) &:= \{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L_1,t)\le 1-r\}\,,\\
K_{\mu,2}(r,t) &:= \{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L_2,t)\le 1-r\}\,,\\
K_{\nu,1}(r,t) &:= \{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk}-L_1,t)\ge r\}\,,\\
K_{\nu,2}(r,t) &:= \{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk}-L_2,t)\ge r\}\,.
\end{aligned}$$

Since $st_{(\mu,\nu)_2} - \lim x = L_1$, we have

$$\delta_2\{K_{\mu,1}(\varepsilon,t)\} = \delta_2\{K_{\nu,1}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Furthermore, using $st_{(\mu,\nu)_2} - \lim x = L_2$, we get

$$\delta_2\{K_{\mu,2}(\varepsilon,t)\} = \delta_2\{K_{\nu,2}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\varepsilon,t) := \{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. Then observe that $\delta_2\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta_2\{\mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)$, then we have two possible cases. The former is the case of $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}$; and the letter is $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. We first consider that

$$(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}.$$

Then we have

$$\begin{aligned} \mu(L_1 - L_2, t) &\geq \mu(x_{jk} - L_1, \frac{t}{2}) * \mu(x_{jk} - L_2, \frac{t}{2}) \\ &> (1-r) * (1-r) \geq 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\mu(L_1 - L_2, t) = 1$ for all $t > 0$,which yields $L_1 = L_2$. On the other hand, if $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$, then we may write that

$$\begin{aligned} \nu(L_1 - L_2, t) &\leq \nu(x_{jk} - L_1, \frac{t}{2}) \diamond \nu(x_{jk} - L_2, \frac{t}{2}) \\ &< r \diamond r \leq \varepsilon. \end{aligned}$$

Again, since $\varepsilon > 0$ was arbitrary, we have $\nu(L_1 - L_2, t) = 0$ for all $t > 0$, which implies $L_1 = L_2$. Therefore, in all cases, we conclude that the $st_{(\mu,\nu)_2}-$limit is unique. ■

Theorem 11 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $(\mu,\nu)_2 - \lim x = L$ for a double sequence $x = (x_{jk})$, then $st_{(\mu,\nu)_2} - \lim x = L$.*

Proof. By hypothesis, for every $\varepsilon > 0$ and $t > 0$, there is a number $N \in \mathbb{N}$ such that

$$\mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon$$

for all $j \geq N$ and $k \geq N$. This guarantees that the set

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\}$$

has at most finitely many terms. Since every finite subset of the natural numbers has double density zero, we immediately see that

$$\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\} = 0,$$

whence the result. ■

The following example shows that the converse of Theorem 11 is not hold in general.

Example 12 *Let* $(\mathbb{R}, |\cdot|)$ *denote the space of real numbers with the usual norm, and let* $a * b = ab$ *and* $a \diamond b = \min\{a+b, 1\}$ *for all* $a, b \in [0,1]$. *For all* $x \in \mathbb{R}$ *and every* $t > 0$, *consider*

$$\mu_0(x,t) := \frac{t}{t+|x|} \text{ and } \nu_0(x,t) := \frac{|x|}{t+|x|}.$$

In this case observe that $(\mathbb{R}, \mu, \nu, *, \diamond)$ *is an IFNS. Now define a double sequence* $x = (x_{jk})$ *whose terms are given by*

$$x_{jk} := \begin{cases} 1, & \text{if } j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then, for every $0 < \varepsilon < 1$ *and for any* $t > 0$, *let* $K_n(\varepsilon,t) := \{(j,k),\ j \leq n \text{ and } k \leq m : \mu_0(x_{jk},t) \leq 1-\varepsilon \text{ or } \nu_0(x_{jk},t) \geq \varepsilon\}$. *Since*

$$\begin{aligned} K_n(\varepsilon,t) &= \left\{(j,k),\ j \leq n \text{ and } k \leq m : \frac{t}{t+|x_{jk}|} \leq 1-\varepsilon \text{ or } \frac{|x_{jk}|}{t+|x_{jk}|} \geq \varepsilon\right\} \\ &= \left\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk}| \geq \frac{\varepsilon t}{1-\varepsilon} > 0\right\} \\ &= \{(j,k),\ j \leq n \text{ and } k \leq m : x_{jk} = 1\} \\ &= \{(j,k),\ j \leq n \text{ and } k \leq m : j \text{ and } k \text{ are squares}\} \end{aligned}$$

we have

$$\delta_2\left(K_n(\varepsilon,t)\right) \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0.$$

Hence, we get $st_{(\mu_0,\nu_0)_2} - \lim x = 0$. *However, since the sequence* $x = (x_{jk})$ *given by (3) is not convergent in the space* $(\mathbb{R}, |\cdot|)$, *by Lemma 4.10 of* [25], *we also see that* x *is not convergent with respect to the intuitionistic fuzzy norm* (μ_0, ν_0).

Theorem 13 *Let* $(V, \mu, \nu, *, \diamond)$ *be an IFNS. Then* $st_{(\mu,\nu)_2} - \lim x = L$ *if and only if there exists a subset* $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$, $j, k = 1, 2, ...$, *such that* $\delta_2(K) = 1$ *and* $(\mu,\nu)_2 - \lim\limits_{\substack{j,k\to\infty \\ (j,k)\in K}} x_{jk} = L$.

Proof. We first assume that $st_{(\mu,\nu)_2} - \lim x = L$. Now, for any $t > 0$ and $j \in \mathbb{N}$, let

$$K_r := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-L,t) \leq 1 - \frac{1}{r} \text{ or } \nu(x_{jk}-L,t) \geq \frac{1}{r}\right\}$$

and

$$\begin{aligned} M_r &= \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-L,t) > 1 - \frac{1}{r} \text{ and } \nu(x_{jk}-L,t) < \frac{1}{r}\right\}, \\ &\quad (r = 1, 2, ...). \end{aligned}$$

Then $\delta_2(K_r) = 0$ and

(1) $M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ...$

and

(2) $\delta_2(M_r) = 1, \quad r = 1, 2, ...$

Now we have to show that for $(j,k) \in M_r$, (x_{jk}) is convergent to L. Suppose that (x_{jk}) is not convergent to L. Therefore there is $\varepsilon > 0$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\}$$

for infinitely many terms.

Let

$$M_\varepsilon = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon\}$$

and $\varepsilon > \frac{1}{r} \quad (r = 1, 2, ...)$.

Then

(3) $\delta_2(M_\varepsilon) = 0$,

and by (1), $M_r \subset M_\varepsilon$. Hence $\delta_2(M_r) = 0$ which contradicts (2). Therefore (x_{jk}) is convergent to L.

Conversely, suppose that there exists a subset $K = \{(j,k)\} \subset \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $(\mu,\nu)_2 - \lim_{j,k} x_{jk} = L$, i.e. there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ and $t > 0$

$$\mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon, \quad \forall j, k \geq N.$$

Now

$$\begin{aligned} K_\varepsilon &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\} \\ &\subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), \ (j_{N+2}, k_{N+2}), ...\}. \end{aligned}$$

Therefore $\delta_2(K_\varepsilon) \leq 1 - 1 = 0$. Hence x is statistically convergent to L. ■

Definition 14 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. We say that a double sequence $x = (x_{jk})$ is statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) provided that, for every $\varepsilon > 0$ and $t > 0$, there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j, p \geq N$, $k, q \geq M$, the set*

$$\{(j,k), \ j \leq n, \ k \leq m : \mu(x_{jk} - x_{pq}, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - x_{pq}, t) \geq \varepsilon\}$$

has double natural density zero.

Now using a similar technique in the proof of Theorem 13 one can get the following result at once.

Theorem 15 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS, and let $x = (x_{jk})$ be a double sequence whose terms are in the vector space V. Then, the following conditions are equivalent:*

(i) x is a statistically Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν).

(ii) There exists an increasing index sequence $K = \{(j,k)\} \subseteq \mathbb{N}\times\mathbb{N}$, $j,k = 1,2,...$ such that $\delta_2(K) = 1$ and the subsequence $\{x_{jk}\}_{(j,k)\in K}$ is a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν).

Now we show that statistically convergence of double sequences on IFNS has some arithmetical properties similar to properties of the usual convergence on $\mathbb{R}$.

Lemma 16 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$ and $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$ then $st_{(\mu,\nu)_2} - \lim (x_{jk}+y_{jk}) = \xi+\eta$.*

Proof. Let $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$, $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$, $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r)*(1-r) \geq 1-\varepsilon$ and $r \diamond r \leq \varepsilon$. Then we define the following sets:

$$\begin{aligned}
K_{\mu,1}(r,t) &: = \{(j,k)\in\mathbb{N}\times\mathbb{N} : \mu(x_{jk}-\xi,t) \leq 1-r\}, \\
K_{\mu,2}(r,t) &: = \{(j,k)\in\mathbb{N}\times\mathbb{N} : \mu(y_{jk}-\eta,t) \leq 1-r\}, \\
K_{\nu,1}(r,t) &: = \{(j,k)\in\mathbb{N}\times\mathbb{N} : \nu(x_{jk}-\xi,t) \geq r\}, \\
K_{\nu,2}(r,t) &: = \{(j,k)\in\mathbb{N}\times\mathbb{N} : \nu(y_{jk}-\eta,t) \geq r\}.
\end{aligned}$$

Since $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$, we have

$$\delta_2\{K_{\mu,1}(\varepsilon,t)\} = \delta_2\{K_{\nu,1}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Similarly, since $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$, we get

$$\delta_2\{K_{\mu,2}(\varepsilon,t)\} = \delta_2\{K_{\nu,2}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\varepsilon,t) := \{K_{\mu,1}(\varepsilon,t)\cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t)\cup K_{\nu,2}(\varepsilon,t)\}$. Then observe that $\delta_2\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta_2\{\mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)$, then we have two possible cases. The former is the case of $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t)\cup K_{\mu,2}(\varepsilon,t)\}$; and the letter is $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t)\cup K_{\nu,2}(\varepsilon,t)\}$. We first consider that

$$(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t)\cup K_{\mu,2}(\varepsilon,t)\}.$$

Then we have

$$\begin{aligned}
\mu((x_{jk}-\xi)+(y_{jk}-\eta),t) &\geq \mu(x_{jk}-\xi,\frac{t}{2}) * \mu(y_{jk}-\eta,\frac{t}{2}) \\
&> (1-r)*(1-r) \geq 1-\varepsilon.
\end{aligned}$$

On the other hand, if $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t)\cup K_{\nu,2}(\varepsilon,t)\}$, then we can write that

$$\begin{aligned}
\nu((x_{jk}-\xi)+(y_{jk}-\eta),t) &\leq \nu(x_{jk}-\xi,\frac{t}{2}) \diamond \nu(x_{jk}-\eta,\frac{t}{2}) \\
&< r \diamond r \leq \varepsilon.
\end{aligned}$$

This show that

$$\delta_2\left(\left\{\begin{array}{r} (j,k)\in\mathbb{N}\times\mathbb{N}:\mu((x_{jk}-\xi)+(y_{jk}-\eta)\,,t)\leq 1-\varepsilon \\ or\quad \nu((x_{jk}-\xi)+(y_{jk}-\eta)\,,t)\geq\varepsilon \end{array}\right\}\right)=0$$

so $st_{(\mu,\nu)_2}-\lim\left(x_{jk}+y_{jk}\right)=\xi+\eta$. ■

Lemma 17 *Let* $(V,\mu,\nu,*,\diamond)$ *be an IFNS. If* $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$ *and* $\alpha\in\mathbb{R}$ *then* $st_{(\mu,\nu)_2}-\lim\alpha x_{jk}=\alpha\xi$.

Proof. Let $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$, $\varepsilon\in(0,1)$ and $t>0$. First of all we consider the case of $\alpha=0$. In this case

$$\mu(0x_{jk}-0\xi,t)=\mu(0,t)=1>1-\varepsilon.$$

Similarly we observe that

$$\nu(0x_{jk}-0\xi,t)=\nu(0,t)=0<\varepsilon$$

for $\alpha=0$. So we obtain $(\mu,\nu)_2-\lim 0x=0$. Then from Theorem 11 we have $st_{(\mu,\nu)_2}-\lim 0x_{jk}=0$.

Now we consider the case of $\alpha\in\mathbb{R}\ (\alpha\neq 0)$. From definition we can write

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi,t\right)\leq 1-\varepsilon\quad\text{or}\quad\nu\left(x_{jk}-\xi,t\right)\geq\varepsilon\right\}\right)=0.$$

So, if we define the sets:

$$\begin{aligned} K_{\mu,1}(\varepsilon,t)\quad &:\quad =\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\xi,t)\leq 1-\varepsilon\} \\ K_{\nu,1}(\varepsilon,t)\quad &:\quad =\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk}-\xi,t)\geq\varepsilon\} \end{aligned}$$

then we can say $\delta_2\left\{K_{\mu,1}\left(\varepsilon,t\right)\right\}=\delta_2\left\{K_{\nu,1}(\varepsilon,t)\right\}=0$ for all $t>0$. Now let $K_{\mu,\nu}(\varepsilon,t)=K_{\mu,1}(\varepsilon,t)\cup K_{\nu,1}(\varepsilon,t)$ then $\delta_2\left\{K_{\mu,\nu}(\varepsilon,t)\right\}=0$ which implies $\delta_2\left\{\mathbb{N}\times\mathbb{N}\diagdown K_{\mu,\nu}(\varepsilon,t)\right\}=1$. If $(j,k)\in\mathbb{N}\times\mathbb{N}\diagdown K_{\mu,\nu}(\varepsilon,t)$ then for $\alpha\in\mathbb{R}\ (\alpha\neq 0)$

$$\begin{aligned} \mu(\alpha x_{jk}-\alpha\xi,t) &= \mu(x_{jk}-\xi,\frac{t}{|\alpha|}) \\ &\geq \mu(x_{jk}-\xi,t)*\mu(0,\frac{t}{|\alpha|}-t) \\ &= \mu(x_{jk}-\xi,t)*1 \\ &= \mu(x_{jk}-\xi,t)>1-\varepsilon. \end{aligned}$$

Similarly, we observe that for $\alpha\in\mathbb{R}\ (\alpha\neq 0)$

$$\begin{aligned} \nu(\alpha x_{jk}-\alpha\xi,t) &= \nu(x_{jk}-\xi,\frac{t}{|\alpha|}) \\ &\leq \nu(x_{jk}-\xi,t)\diamond\nu(0,\frac{t}{|\alpha|}-t) \\ &= \nu(x_{jk}-\xi,t)\diamond 0 \\ &= \nu(x_{jk}-\xi,t)<\varepsilon. \end{aligned}$$

This show that

$$\delta_2\left(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(\alpha x_{jk}-\alpha\xi,t)\le 1-\varepsilon \quad or \quad \nu(\alpha x_{jk}-\alpha\xi,t)\ge\varepsilon\}\right)=0$$

so $st_{(\mu,\nu)_2}-\lim \alpha x_{jk}=\alpha\xi$. ■

Lemma 18 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$ and $st_{(\mu,\nu)_2}-\lim y_{jk}=\eta$ then $st_{(\mu,\nu)_2}-\lim\left(x_{jk}-y_{jk}\right)=\xi-\eta$.*

Proof. The proof is clear from Lemma 16 and Lemma 17. ■

Definition 19 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS . We say that a double sequence $x=(x_{jk})$ is IF-bounded if there exist $t>0$ and $0<r<1$ such that $\mu\left(x_{jk},t\right)>1-r$ and $\nu\left(x_{jk},t\right)<r$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$.*

Definition 20 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS . For $t>0$, we define open ball $B\left(x,r,t\right)$ with center $x\in V$ and radius $0<r<1$, as*

$$B\left(x,r,t\right)=\left\{y\in V:\mu\left(x-y,t\right)>1-r\ ,\ \nu\left(x-y,t\right)<r\right\}.$$

It follows from Lemma 16 Lemma 17 and Lemma 18, that the set of all IF-bounded statistically convergent double sequences on IFNS is a linear subspace of the linear normed space $\ell_\infty^{(\mu,\nu)_2}\left(V\right)$ of all IF-bounded sequences on IFNS.

Theorem 21 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS and the set $st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)$ is closed linear subspace of the set $\ell_\infty^{(\mu,\nu)_2}\left(V\right)$.*

Proof. It is clear that $st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)\subset\overline{st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)}$. Now we show that $\overline{st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)}\subset st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)$. Let $y\in\overline{st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)}$. Since $B\left(y,r,t\right)\cap\left(st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)\right)\ne\varnothing$, there is a $x\in B\left(y,r,t\right)\cap\left(st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)\right)$.

Let $t>0$ and $\varepsilon\in(0,1)$. Choose $r\in(0,1)$ such that $(1-r)*(1-r)\ge 1-\varepsilon$ and $r\diamond r\le\varepsilon$. Since $x\in B\left(y,r,t\right)\cap\left(st_{(\mu,\nu)_2}\left(V\right)\cap\ell_\infty^{(\mu,\nu)_2}\left(V\right)\right)$, there is a set $K\subseteq\mathbb{N}\times\mathbb{N}$ with $\delta\left(K\right)=1$ such that

$$\mu\left(y_{jk}-x_{jk},\frac{t}{2}\right)>1-r \quad\text{and}\quad \nu\left(y_{jk}-x_{jk},\frac{t}{2}\right)<r$$

and

$$\mu\left(x_{jk},\frac{t}{2}\right)>1-r \quad\text{and}\quad \nu\left(x_{jk},\frac{t}{2}\right)<r$$

for all $(j,k)\in K$. Then we have

$$\begin{aligned}\mu\left(y_{jk},t\right) &= \mu\left(y_{jk}-x_{jk}+x_{jk},t\right)\\ &\ge \mu\left(y_{jk}-x_{jk},\frac{t}{2}\right)*\mu\left(x_{jk},\frac{t}{2}\right)\\ &> (1-r)*(1-r)\ge 1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}
\nu\left(y_{jk},t\right) &= \nu\left(y_{jk}-x_{jk}+x_{jk},t\right) \\
&\leq \nu\left(y_{jk}-x_{jk},\frac{t}{2}\right)\diamond\nu\left(x_{jk},\frac{t}{2}\right) \\
&< r\diamond r\leq\varepsilon
\end{aligned}$$

for all $(j,k)\in K$. Hence

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(y_{jk},t\right)>1-\varepsilon \text{ and } \nu\left(y_{jk},t\right)<\varepsilon\right\}\right)=1$$

and thus $y\in st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)$. ■

Conclusion 22 *In this paper, we obtained results on statistical convergence in intuitionistic fuzzy normed spaces. As every ordinary norm induces a intuitionistic fuzzy norm, the results obtained here are more general than the corresponding of normed spaces.*

3 REFERENCES

[1] C. D. Aliprantis and K. C. Border. Infinite dimensional analysis. Springer Verlag, Heidelberg (1994).

[2] C. D. Aliprantis, D. J. Brown and O. Burkinshaw. Edgeworth equilibria, Econometrica, 5, 1109–37 (1997) .

[3] A. Billot. Economic theory of fuzzy equilibria, Springer - Verlag, Berlin (1992).

[4] M. Burgin. Theory of fuzzy limits, Fuzzy Sets Syst.,115, 433-43 (2000).

[5] J. Christoper. The asymptotic density of some k-dimensional sets, Amer. Math. Monthly, 63, 399–401 (1956).

[6] J. Connor. The statistical and strong p-Cesàro convergence of sequences, Analysis, 8, 47–63 (1988).

[7] J. Connor. A topological and functional analytic approach to statistical convergence. Analysis of Divergence. Birkhāuser-Verlag, Boston (1999) 403–13.

[8] G. Debreu. Mathematical Economics: Twenty papers of Gerard Debreu, Cambridge University Press, Cambridge (1983).

[9] Z. Deng. Fuzzy pseudo-metric space, J. Math. Anal. Appl., 86, 74–95 (1982).

[10] M. A. Erceg. Metric spaces in fuzzy set theory, J. Math. Anal. Appl., 69,205–30 (1979).

[11] H. Fast. Sur la convergence statistique, Colloq. Math., 2, 241-44 (1951).

[12] J. A. Fridy. On statistical convergence, Analysis, 5, 301–13 (1985).

[13] J. A. Fridy and C. Orhan. Lacunary statistical convergence, Pacific J.Math., 160, 43–51 (1993).

[14] J. A. Fridy and C. Orhan. Lacunary statistical summability, J. Math.Anal.Appl., 173, 497-504 (1993).

[15] J. A. Fridy and M. K. Khan. Tauberian theorems via statistical convergence, J. Math. Anal. Appl., 228, 73–95 (1998).

[16] A. George, and P. Veeramani. On some result in fuzzy metric space, Fuzzy Sets Syst., 64, 395-9 (1994).

[17] R. Giles. A computer program for fuzzy reasoning, Fuzzy Sets Syst., 4,221-34 (1980).

[18] G. Jäger. Fuzzy uniform convergence and equicontinuity, Fuzzy Sets Syst., 109, 187-98 (2000).

[19] O. Kaleva and S. Seikkala. On fuzzy metric spaces, Fuzzy Sets and Systems,12, 215–29 (1984).

[20] J. Madore. Fuzzy physics, Annals of Physics, 219, 187-98 (1992).

[21] Mursaleen and O. H. H. Edely. Statistical convergence of double sequences, J. Math. Anal. Appl., 288, 223–31 (2003).

[22] S. A. Orlovsky. Decision making with a fuzzy preference relation, Fuzzy Sets and Systems, 1, 155–67 (1978).

[23] J. H. Park. Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals, 22, 1039–46 (2004) .

[24] A. Pringsheim. Zur ttheorie der zweifach unendlichen zahlenfolgen, Math. Ann., 53, 289–321 (1900).

[25] R. Saadati and J. H. Park. On the intuitionistic fuzzy topological spaces, Chaos, Solitons & Fractals, 27, 331–44 (2006).

[26] B. Schweizer and A. Sklar. Statistical metric spaces, Pacific J. Math., 10, 314–44 (1960).

[27] H. Steinhaus. Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2, 73–74 (1951).

[28] K. Wu. Convergences of fuzzy sets based on decomposition theory and fuzzy polynomial function, Fuzzy Sets Syst., 109, 173-85 (2000).

[29] L. A. Zadeh. Fuzzy sets, Inform. Control, 8, 338-53 (1965).

Sevda Karakuş
Ondokuz Mayis University,
Faculty of Sciences and Arts Sinop,
Department of Mathematics
57000 Sinop-TURKEY.
E-Mail: skarakus@omu.edu.trE-Mail: kamild@omu.edu.tr

Kamil Demirci
Ondokuz Mayis University,
Faculty of Sciences and Arts Sinop,
Department of Mathematics
57000 Sinop-TURKEY.
E-Mail: kamild@omu.edu.tr