Study Guide: Foundations of Operations Research and Linear Programming

1.0 Introduction to Operations Research (OR)

Operations Research (OR) is a scientific discipline dedicated to improving decision-making in complex systems. It involves the application of scientific methods, techniques, and tools to problems concerning the operations of a system, with the ultimate goal of providing decision-makers with optimal solutions. By modeling and analyzing intricate real-world problems, OR provides a quantitative basis for making strategic choices that enhance efficiency and effectiveness across an entire organization.

The field of Operations Research can be defined in several ways, each highlighting a core aspect of its function:

- "Operation Research is also called optimization technique because operation research model consist of maximizing or minimizing objective function under the given conditions."
- "Operation research deals with optimization of problems and help in dealing men, power, machinery, money and material to their most/maximum utilization."
- "It is the application of scientific method, technique and tools to problems involving the operations of the system so as to provide those in control of the system with optimum solution to the problem."

The primary objective of Operations Research is to assist decision-makers in solving problems that involve the interaction of various components across different disciplines. The aim is to find a solution that is in the best interest of the organization as a whole. The best solution obtained through this process is known as the "optimal decision/solution."

Key Application Areas of Operations Research

OR techniques are applied across a wide range of fields to solve complex problems. Key areas include:

1. Defence Operations (Military)
2. Industry
3. Agriculture
4. Planning for Economic Growth
5. Traffic Control
6. Hospital Management

Key Operations Research Models

To solve problems in these areas, Operations Research utilizes various models. Common models include:

1. Linear Programming Model
2. Non-linear Programming Model
3. Game Theory Model

4. Networking Model
5. Replacement Model
6. Transportation Model

This guide will now focus on the most fundamental of these: the Linear Programming model.

2.0 Core Concepts of Linear Programming (LP)

Linear Programming (LP) is a cornerstone technique within Operations Research used to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Understanding its fundamental components is essential for formulating and solving optimization problems effectively.

2.1 Objective Function

- The Objective Function is a mathematical expression that represents the goal of the problem, which is to be either maximized or minimized.
- **Example:** $Z = 4x_1 + 2x_2$ is an objective function where Z is the value to be optimized.

2.2 Decision Variable

A Decision Variable is a variable quantity (e.g., x_1, x_2) that represents a specific choice to be made by the decision-maker. The set of values determined for these variables constitutes the solution to the problem.

2.3 Constraints / Restrictions / Conditions

- Constraints account for the physical limitations of the system being modeled, such as limited resources, time, or capacity.
- These are expressed as a mathematical function (typically an inequality or equality) that limits the decision variables to their feasible or allowable values.

2.4 Feasible Solution

- A Feasible Solution is any set of values for the decision variables that satisfies all the constraints of the problem. Two definitions are provided:
  - It is any point within the "Feasible Region," which is the set of all points satisfying the system of linear inequalities (constraints).
  - It is any solution that also satisfies the non-negativity condition (e.g., $x_1 \geq 0, x_2 \geq 0$).

2.5 Optimal Solution

- The Optimal Solution is the specific feasible solution that yields the best possible value for the objective function (i.e., the maximum or minimum value).
- This is also referred to as the "Optimal Feasible Solution."

These core concepts provide the framework for building and interpreting LP models, which can then be solved using specific methods to find the optimal solution.

3.0 Solving LP Problems: The Graphical Method

The Graphical Method is a straightforward and visual technique for solving Linear Programming problems that involve only two decision variables. Its strategic value lies in its ability to clearly illustrate the feasible region, which contains all possible solutions, and the corner points, among which the optimal solution is always found.

The process for solving an LP problem using the Graphical Method involves three key steps:

1. Graph the constraints to determine the solution space, which is the common region that satisfies all constraints simultaneously. This is also known as the feasible region.
2. Determine the extreme points (or corner points) of the feasible region.
3. Evaluate the objective function at each of these extreme points to identify the optimal solution—the point that maximizes or minimizes the objective function.

Example: Graphical Method in Practice

Let's analyze a worked example to demonstrate this method.

Problem: $\text{Max } Z = 4x_1 + 2x_2$

Subject to the constraints:

1. $x_1 + 2x_2 \le 5$
2. $x_1 + x_2 \le 4$
3. $x_1, x_2 \ge 0$

Step 1: Find Intercepts for Each Constraint

- For $x_1 + 2x_2 = 5$:
    - If $x_2 = 0$, then $x_1 = 5$. Point: **(5, 0)**
    - If $x_1 = 0$, then $x_2 = 2.5$. Point: **(0, 2.5)**
- For $x_1 + x_2 = 4$:
    - If $x_2 = 0$, then $x_1 = 4$. Point: **(4, 0)**
    - If $x_1 = 0$, then $x_2 = 4$. Point: **(0, 4)**

Step 2: Identify Feasible Corner Points

By graphing these lines and identifying the common shaded region, the following corner points of the feasible region are found: **O(0,0), C(4,0), E(3,1), and B(0, 2.5)**.

Step 3: Evaluate the Objective Function

The value of the objective function Z is calculated at each corner point.

Corner Point	Coordinates (x_1, x_2)	Calculation ($Z = 4x_1 + 2x_2$)	Result (Z)
O	(0, 0)	Z = 4(0) + 2(0)	0
C	(4, 0)	Z = 4(4) + 2(0)	16
E	(3, 1)	Z = 4(3) + 2(1)	14

B	(0, 2.5)	Z = 4(0) + 2(2.5)	5

The final optimal solution is the one that yields the highest value for Z. In this case, the **optimal point is C(4,0)** and the **optimal value is 16**.

While the Graphical Method is highly intuitive for two-variable problems, more complex problems with numerous variables require more advanced algebraic methods like the Simplex Method.

4.0 Advanced Solution Concepts for Algebraic Methods

To move beyond the limitations of graphical solutions and tackle more complex Linear Programming problems, it is necessary to understand more formal algebraic concepts. Concepts such as the 'basic solution' and 'standard form' provide the essential foundation for algorithmic approaches like the Simplex method, which can handle problems with any number of variables.

Basic Solution

For a system of linear equations with m equations and n variables (where $n > m$), an associated **Basic Solution** is determined by setting $n-m$ of the variables equal to zero and then solving the m equations for the remaining m variables. This assumes the resulting solution exists and is unique.

Basic Feasible vs. Infeasible Solutions

From the set of basic solutions, we can distinguish between those that are feasible and those that are not.

- **Basic Feasible Solution:** A basic solution is considered feasible if all its solution values are non-negative.
- **Infeasible Solution:** A solution that contains a negative value is called an infeasible solution.

Standard Form

To apply algebraic methods like the Simplex algorithm, an LP problem must first be converted into **Standard Form**. This conversion is facilitated by introducing new variables, but is defined by two fundamental requirements:

1. All constraints must be expressed as equations with non-negative right-hand sides.
2. All variables must be non-negative.

To meet the first requirement, we use slack and surplus variables:

- **Slack Variables:** For ≤ (less-than-or-equal-to) constraints, a non-negative **slack variable** is added to the left side to convert it into an equation. This variable represents the unused amount of a resource.
  - **Example:** $2x_1 + 3x_2 + 5x_3 \le 10$ becomes $2x_1 + 3x_2 + 5x_3 + S_2 = 10$.
- **Surplus Variables:** For ≥ (greater-than-or-equal-to) constraints, a non-negative **surplus variable** is subtracted from the left side. This variable represents the amount by which the minimum requirement is exceeded.

- **Example:** $x_1 + x_2 \geq 5$ becomes $x_1 + x_2 - S_3 = 5$.

These advanced concepts are the necessary prerequisites for understanding the systematic mechanics of the Simplex Method.

5.0 The Primal Simplex Method

The Simplex Method is a powerful and widely used iterative algebraic procedure that systematically finds the optimal solution to a linear programming problem. The method starts from an initial feasible basic solution and moves through a series of successive feasible solutions, each time improving the value of the objective function, until an optimal solution is reached.

The progression of the Simplex Method is governed by two primary conditions:

1. **Optimality Condition:** This rule determines which variable should enter the basis to improve the solution. The entering variable is the non-basic variable with the most negative coefficient in the objective Z-equation for a maximization problem, or the most positive coefficient for a minimization problem. An optimum is reached when all non-basic coefficients in the Z-equation are non-negative (for maximization).
2. **Feasibility Condition:** This rule determines which current basic variable must leave the basis to maintain feasibility. The leaving variable is the one associated with the smallest positive ratio, calculated by dividing the solution values by the corresponding coefficients in the entering variable's column.

Steps of the Simplex Method

The algorithm can be summarized in the following steps:

1. **Step I:** Select an entering variable from the current non-basic variables using the optimality condition.
2. **Step II:** Select a leaving variable from the current basic variables by using the feasibility condition.
3. **Step III:** Determine the value of the new basic variables by making the entering variable basic and the leaving variable non-basic. This is done through a pivot operation, and the process is repeated until the optimality condition is met.

Example: Simplex Method in Practice

Let's trace the process for the following maximization problem.

Problem: $\text{Max } Z = 6x_1 + 5x_2$

Subject to:

- $x_1 + x_2 \leq 5$
- $3x_1 + x_2 \leq 12$
- $x_1, x_2 \geq 0$

1. Conversion to Standard Form: The problem is converted to standard form by adding slack variables S_1 and S_2:

- $\text{Max } Z - 6x_1 - 5x_2 = 0$

- $x_1 + x_2 + S_1 = 5$
- $3x_1 + x_2 + S_2 = 12$

2. Iterative Process:

- **Iteration 1:** The process begins with S_1 and S_2 as the basic variables. The entering variable is x_1 because it has the most negative coefficient (-6) in the Z-row. To find the leaving variable, we calculate the ratios: for the S_1 row, the ratio is $5/1 = 5$; for the S_2 row, the ratio is $12/3 = 4$. Since 4 is the smallest positive ratio, S_2 is the leaving variable. A pivot operation is performed.
- **Iteration 2:** After the first pivot, x_1 is basic, but the Z-row still contains a negative coefficient, so the solution is not yet optimal. The next entering variable is x_2. A new ratio test is performed, which identifies S_1 as the leaving variable. Another pivot operation is performed.
- **Optimality:** After the second pivot, the algorithm terminates because all coefficients in the Z-row are non-negative, satisfying the optimality condition.

3. Final Solution: The optimal solution is reached when the Z-row becomes non-negative. The final values are identified as $\mathbf{x_1 = 7/2}$, $\mathbf{x_2 = 3/2}$, and the maximum objective value $\mathbf{Z = 57/2}$.

While highly effective, the Simplex Method has a counterpart, the Dual Simplex Method, which provides an alternative approach for solving certain types of LP problems.

6.0 The Dual Simplex Method

In the theory of linear programming, **Duality** implies that every LP problem (referred to as the **Primal** problem) has a corresponding associated problem called its **Dual**. The Dual Simplex Method is an algorithm that leverages this principle. Unlike the standard Simplex method, it operates by first selecting a leaving variable and then an entering variable. It begins with a solution that is dual feasible (optimal) but primal infeasible, and iteratively moves toward primal feasibility while maintaining optimality. The method selects the leaving variable first based on the most negative solution value, and then selects the entering variable based on a ratio test with negative coefficients in the pivot row.

The key difference in the starting conditions for the two simplex methods is fundamental:

- **Simplex Method:** Starts from a condition that is *feasible but not optimal*. It maintains feasibility at every step while working towards optimality.
- **Dual Simplex Method:** Starts from a condition that is *optimal but infeasible*. It maintains optimality at every step while working towards feasibility.

Simplex vs. Dual Simplex Method Comparison

The operational steps of the two methods are related but distinct.

Simplex Method	Dual Simplex Method
1. First, we find the **entering variable**.	1. First, we find the **leaving variable**.

2. Calculate ratios by dividing each value in the Solution column by the corresponding positive value in the Entering variable's column.	2. Calculate ratios by dividing each value in the Z-row by the corresponding negative value in the Leaving variable's row.
3. While calculating the ratio, **only positive entries** in the entering column are considered.	3. While calculating the ratio, **only negative entries** in the leaving row are considered.
4. Then we find the Leaving Variable with the **least ratio**.	4. Here we find the Entering Variable with the **least ratio**.
5. Then the table is revised, and the process continues until **optimality** is reached.	5. Then the table is revised, and the process continues until **feasibility** is reached.

Advantages and Disadvantages of the Dual Simplex Method

Advantages of Dual Simplex Algorithm

- Suitable for problems where it is easier to find a dual feasible solution than a primal feasible one.
- Efficient in re-optimizing problems after small changes to constraints.
- If you have a feasible dual solution but an infeasible primal, the dual simplex method can restore feasibility without restarting the entire optimization.
- This is often useful after adding new constraints to an already-solved problem.

Disadvantages of the Dual Simplex

- Finding a dual-feasible starting point can be difficult compared to finding a primal-feasible one, which is what the regular Simplex method uses.
- The pivot rules and their interpretation are more complex than in the Primal Simplex method.
- The algorithm can be more sensitive to rounding errors due to handling infeasible primal solutions.
- Choosing the correct leaving and entering variables requires careful consideration to maintain dual feasibility.

The Dual Simplex method is particularly valuable for sensitivity analysis and for solving problems where adding a new constraint makes a previously optimal solution infeasible.

7.0 Comprehensive Glossary of Key Terms

Term	Definition
Basic Solution	For a system with `m` equations and `n` variables, it is a solution obtained by setting `n-m` variables to zero and solving for the remaining `m` variables.
Basic Feasible Solution	A basic solution where all of its solution values are non-negative.
Constraints	Mathematical functions that account for the physical limitations of a system

	and limit decision variables to their feasible or allowable values.
Decision Variable	A variable quantity that represents a decision to be made by the decision-maker (e.g., x_1, x_2).
Duality	The principle that every linear programming problem (the Primal) is associated with another linear programming problem called its Dual.
Dual Simplex Method	A method that starts from a condition that is optimal but infeasible and iteratively works towards feasibility.
Feasible Region	The common shaded solution space representing the set of all points that satisfy a system of linear inequalities (constraints).
Feasible Solution	Any solution that satisfies the problem's constraints, including the non-negativity condition.
Infeasible Solution	A solution that contains a negative value.
Linear Programming	An optimization model used to solve problems involving maximizing or minimizing a linear objective function subject to linear constraints.
Objective Function	A function which is to be maximized or minimized. For example: $Z = 4x_1 + 2x_2$.
Operations Research	The application of scientific methods, techniques, and tools to problems involving the operations of a system to provide an optimum solution.
Optimal Solution	The feasible solution which maximizes or minimizes the objective function. Also known as the Optimal Feasible Solution.
Primal Simplex Method	An iterative method that starts from a feasible basic solution and moves through successive feasible solutions until the optimal solution is reached.
Slack Variable	A non-negative variable added to a $\leq$ constraint to convert it into an equality.
Standard Form	A format for an LP problem where all constraints are expressed as equalities with non-negative right-hand sides and all variables are non-negative.
Surplus Variable	A non-negative variable subtracted from a $\geq$ constraint to convert it into an equality.