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a) $f(z) = \log(1+z)$

where

$$z = x^T x \qquad x \in \mathbb{R}^d$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \qquad x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \end{bmatrix}$$

$$x^T x = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$$

$$= \left[x_1^2 + x_2^2 + \ldots + x_d^2 \right]$$

using chain rule

$$\frac{df}{dx} = \frac{df}{dz} \times \frac{dz}{dx}$$

$$= \frac{d}{dz} \log(1+z) \times \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \frac{d}{dx}\left(\{x_1^2 + x_2^2 + \ldots + x_d^2 \right)$$

$$= \left(\frac{1}{1+x^T x} \right) \cdot 2 \sum_{i=1}^{d} x_i$$

So the gradient of the first eq is

$$\left(\frac{2}{1+x^T x} \right) \sum_{i=1}^{d} (x_i)$$

b) $f(z) = e^{-z/2}$

where $z = g(y) = y^T S^{-1} y$

$y = h(x) = x - \mu$

$x, \mu \in R^d, \quad S \in R^{d \times d}$

using the chain rule

$$\frac{df}{dx} = \frac{df}{dz} \times \frac{dz}{dy} \times \frac{dy}{dx}.$$

$$\Rightarrow \frac{df}{dx} = \frac{d}{dz}\left(e^{-z/2}\right) \times \frac{d}{dy}\left(y^T S^{-1} y\right) \times \frac{d}{dx}(x-\mu)$$

So,

$$\Rightarrow \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{1}{2} e^{-z/2}$$

$$\Rightarrow \frac{d}{dy}\left(y^T S^{-1} y\right) = \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h S^{-1} h - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h\left(y^T S^{-1} + S^{-1} y + S^{-1} h\right)}{h}$$

$$= \lim_{x \to 0} \left(y^T S^{-1} + S^{-1} y + S^{-1} h \right)$$

$$= \lim_{h \to 0} \left(y^T S^{-1} + S^{-1} y + S^{-1} h \right)$$

$$= y^T S^{-1} + S^{-1} y$$

$$= \frac{y^T}{S} + \frac{y}{S}.$$

$\rightarrow \frac{d}{dx}(x - \mu) = 1$

So, $\frac{df}{dx} = \frac{df}{dz} \times \frac{dt}{dy} \times \frac{dy}{dx}$.

$$= \left(-\frac{1}{2} e^{-z/2}\right)\left(y^T S^{-1} + S^{-1} y\right) \times 1$$

So the derivative is, $-\frac{1}{2} e^{-z/2} \left(y^T S^{-1} + S^{-1} y\right)$.