Question 1 (30%, each of the 6 sub-questions below is worth 5%).

(a) What is the output of the following code?

```
#include <cstdlib>
#include <iostream>
using namespace std;
int main()
{
        int ran = rand();
        cout << ran << endl;
}
```

(b) Will the following code compile correctly?

```
class Test {
private:
        int x=5;
};
int main()
{
        Test t;
        cout << t.x;
        return 0;
}
```

Briefly justify your answer.

(c) Is the following statement true or false:
A member function can always access the data in the class of which it is member.

(d) Consider the following definition of the function f.

```
int f(int &x, int c) {
        c  = c - 1;
        if (c == 0) return 1;
        x = x + 1;
        return f(x, c) * x;
}
```

Briefly explain why the return value of q defined below will be 9^4.

```
int main()
{
        int p;
        p=5;
        int q=f(p,p);
        ...
}
```

Note that the first parameter is passed by reference, whereas the second parameter is passed by value.

(e) What is the default constructor for a C++ class?

(f) What is the output of the following code?

```
#include<iostream>

using namespace std;
class Base1 {
        public:
        Base1()
        { cout << " Base1's constructor called" << endl;
}
};
class Base2 {
        public:
        Base2()
        { cout << "Base2's constructor called" << endl;
}
};
class Derived: public Base1, public Base2 {
        public:
        Derived()
        {  cout << "Derived's constructor called" << endl; }};
int main()
{
        Derived d;
        return 0;
}
```

Briefly explain your answer.

Question 2 (30%, each of the 3 sub-questions below is worth 10%).
Suppose that $r_1, r_2, \dots, r_n$ are independent and identically distributed observations of a random variable X.

(a) Show that,

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i,$$

is an unbiased estimator for the mean of X and compute its variance.

(b) Show that,

$$\bar{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2$$

is an unbiased estimator for the variance of X.

(c) Consider the following Stochastic Differential Equation (SDE):

$$dS_t = \mu dt + \sigma^2 dW_t.$$

Describe how the Euler-Maruyama method can be used to estimate,

$$E(F(S_T)),$$

for some function $F : \mathbb{R} \to \mathbb{R}$ and with S_0 given. What are the sources of error for the Euler-Maruyama method?

Question 3 (40%, each of the 3 sub-questions below is worth 10%,20% and 10%). Consider a market with n risky assets with,

- mean returns $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$
- covariances σ_{ij} for $i, j = 1, 2, \ldots, n$.

(a) Formulate the Markowitz model for the market above when short-selling is allowed and with a required target return of $\bar{r}_P$.

(b) Write down the Lagrangian associated with the optimization model in (a) and derive the linear system of equations that the optimal solution satisfies.

(c) Consider the following optimization problem,

$$\min_x \; f(x)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function. A minimiser is to be approximated with the Newton-Raphson method as defined below:

$$x_{k+1} = x_k - \left(\frac{d^2 f(x_k)}{dx^2}\right)^{-1} \frac{df(x_k)}{dx}$$

What are the possible shortcomings of the iterative method defined above.

Answer 1 (30%, each of the 6 sub-questions below is worth 5%).

(a) 012012

(b) No x is a private member that is accessed outside the class.

(c) True.

(d) Since c is passed by value and x is passed by reference, all functions will have same copy of x, but different copies of c. f(5, 5) = f(x, 4)*x = f(x, 3)*x*x = f(x, 2)*x*x*x = f(x, 1)*x*x*x*x = 1*x*x*x*x = x^4 Since x is incremented in every function call, it becomes 9 after f(x, 2) call. So the value of expression x^4 becomes 9^4 which is 6561.

(e) A constructor without any arguments.

(f)

```
Base1's constructor called
Base2's constructor called
Derived's constructor called
```

When a class inherits from multiple classes, constructors of base classes are called in the same order as they are specified in inheritance.

Answer 2 (30%, each of the 3 sub-questions below is worth 3%).

(a) (sketch) Textbook, σ^2/n where σ^2 is the variance of X.

(b) Textbook

(c) (sketch) Calculate N paths as follows,

$$S^i(t+\delta t) = \mu S^i(t) + \sigma S^i(t)\sqrt{\delta t}Z, \quad i = 1, \ldots, N$$

Where $Z\ N(0,1)$ and δt is the step-size. The the expectation is calculated as,

$$E(F(S_T)) \approx \frac{1}{N}\sum F(S_T^i)$$

Error is $O(\max(\delta t, \sqrt{N})$ (a justification of how the error is estimated should be provided)

Answer 3 (40%, each of the 3 sub-questions below is worth 10%,20% and 10%).

(a)

$$\begin{array}{lll} \text{minimize} & \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j & = \tfrac{1}{2}\sigma_{\mathrm{P}}^2 \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_{\mathrm{P}} & = \text{ exp. return target} \\ & \displaystyle\sum_{i=1}^{n} w_i = 1 & = \text{ weights sum to 1} \end{array}$$

(b) (sketch) The associated Lagrangian function L is given by

$$L(\boldsymbol{w}, \lambda, \mu) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda \left(\boldsymbol{w}^\top \bar{\boldsymbol{r}} - \bar{r}_{\mathrm{P}}\right) - \mu \left(\boldsymbol{w}^\top \boldsymbol{e} - 1\right) ,$$

while the optimality conditions become

$$\Sigma \boldsymbol{w} - \lambda \bar{\boldsymbol{r}} - \mu \boldsymbol{e} = \boldsymbol{0} , \quad \bar{\boldsymbol{r}}^\top \boldsymbol{w} = \bar{r}_{\mathrm{P}} \quad \text{and} \quad \boldsymbol{e}^\top \boldsymbol{w} = 1 .$$

(c) Possible problems are: converge to stationary point (not local min or max) or the algorithm may cycle.