Computational Finance with C++
Tutorial: Constrained Optimisation – Optimality Conditions

Exercise 1. Write down the optimality conditions for the following problem.

$$\begin{aligned} \min\ & f(x) \\ & Ax \geq b \\ & Gx \leq c \\ & a \leq x \leq \widehat{b} \end{aligned} \tag{0.1}$$

Where f is a convex function, A is an $m_1 \times n$ matrix, G is an $m_2 \times n$ matrix, $b \in \mathbb{R}^{m_1}$ and $c \in \mathbb{R}^{m_1}$ and x, a and $\widehat{b}$ are n dimensional vectors.

Exercise 2. Suppose that there are n possible investment opportunities (assets). The return distribution for each asset is bivariate Normal with mean $\bar{r}$ and covariance matrix $\frac{1}{2}\Sigma$ (the $\frac{1}{2}$ in front of the covariance matrix is just for convenience). Suppose that short selling is allowed, a budget of $b = 1$ is given and a target r_m of the returns is to be achieved over a single time period.

1. Write the mean variance optimization problem for this case.
2. Show that the optimal solution in this case is given by,

   $$w^* = \Sigma^{-1}\left(\frac{C - Br_m}{AC - B^2}\mathbf{1} + \frac{Ar_m - B}{AC - B^2}\bar{r}\right),$$

   where,

   $$\begin{aligned} A &= \mathbf{1}^T\Sigma^{-1}\mathbf{1} \\ B &= \mathbf{1}^T\Sigma^{-1}\bar{r} \\ C &= \bar{r}^T\Sigma^{-1}\bar{r} \end{aligned}$$

   You may assume that $B^2 - AC \neq 0$.
3. Show that $B^2 - AC < 0$.