Computational Finance with C++
Tutorial: The Newton-Raphson Method

Answer 1.

a. We have $f'(x) = 4(x - x_0)^3$ and $f''(x) = 12(x - x_0)^2$. Hence Newton's method can be represents as,
$$x_{k+1} = x_k - \frac{x_k - x_0}{3}$$
Therefore,
$$x_{k+1} - x_0 = \frac{2}{3}(x_k - x_0).$$

b. From part (a) $y_k = |x_k - x_0| = \frac{2}{3}|x_{k-1} - x_0| = \frac{2}{3}y_{k-1}$.

c. From part (b), we see that $y^k = (\frac{2}{3})^k y_0$ and therefore $y^k \to 0$. Hence $x_k \to x_0$ for any x_0.

Answer 2.

a. Clearly $f(x) \geq 0$ for all x. We have,
$$f(x) = 0 \Leftrightarrow x_2 - x_1^2 = 0 \text{ and } 1 - x_1 = 0$$
Therefore, $x^* = [1, 1]^\top$ is a global optimum point and since $f(x) > f(x^*)$ for $x \neq x^*$ we conclude that x^* is the unique global minimiser.

b. We compute

$$\nabla f(x) = \begin{bmatrix} 400x_1^3 - 400x_1x_2 + 2x_1 - 2 \\ 200(x_2 - x_1^2) \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

To apply Newton's method we use the inverse of the Hessian, which is

$$\nabla^2 f(x)^{-1} = \frac{1}{80000(x_1^2 - x_2) - 400} \begin{bmatrix} 200 & 400x_1 \\ 400x_1 & 1200x_1^2 - 400x_2 - 2 \end{bmatrix}$$

Applying two iterations of Newton's method we have $x_1 = [1, 0]^\top$, $x_2 = [1, 1]^\top$.

c. Applying the gradient algorithm,

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

with a fixed step-size of $\alpha_k = 0.05$ we obtain, $x_1 = [0.1, 0]^\top$ and $x_2 = [0.17, 0.1]^\top$.