Computational Finance with C++
Tutorial: Constrained Optimisation – Optimality Conditions

Answer 1. The Lagrangian is given by,

$$L(x, \lambda) = f(x) + \lambda^T (b - Ax) + \mu^T (Gx - c) + \nu^T (x - \widehat{b}) + \rho^T (a - x).$$

The complementary slackness conditions are,

$$\begin{aligned}
\lambda^T (b - Ax) &= 0 \\
\mu^T (Gx - c) &= 0 \\
\nu^T (x - \widehat{b}) &= 0 \\
\rho^T (a - x) &= 0
\end{aligned}$$

The first order conditions are:

$$\nabla_x f(x) - A^T \lambda + G^T \mu + \nu - \rho = 0$$

and $\lambda \geq 0,\ \ \mu \geq 0,\ \ \nu \geq 0,\ \ \rho \geq 0.$

Answer 2. 1.

$$\begin{aligned}
\min_x \ & \frac{1}{2} x^T \Sigma x \\
& \sum_{i=1} x_i = b \\
& \sum_{i=1} x_i \bar{r}_i = r_m
\end{aligned}$$

2. The Lagrangian is given by:

$$\mathcal{L}(w, \lambda, \gamma) = \frac{1}{2} x^T \Sigma x + \gamma (b - \sum_{i=1}^{n} x_i) + \lambda (r_m - \sum_{i=1}^{n} \bar{r}_i x_i).$$

Using vector notation,

$$\mathcal{L}(w, \lambda, \gamma) = \frac{1}{2} x^T \Sigma x + \gamma(b - \mathbf{1}^T x) + \lambda(r_m - \bar{r}^T x).$$

The first order conditions are:

$$\begin{aligned} \Sigma x &= (\gamma \mathbf{1} + \lambda \bar{r}) \\ \mathbf{1}^T x &= 1 \\ \bar{r}^T x &= r_m \end{aligned} \tag{0.1}$$

By assumption the covariance matrix is positive definite, so we only need to solve the system above, the second order condition is satisfied.

The first equation implies:

$$x = \Sigma^{-1}(\gamma \mathbf{1} + \lambda \bar{r}), \tag{0.2}$$

using this in the other two equations we get:

$$\begin{aligned} \gamma \mathbf{1}^T \Sigma^{-1} \mathbf{1} + \lambda \mathbf{1}^T \Sigma^{-1} \bar{r} &= 1 \\ \gamma \bar{r}^T \Sigma^{-1} \mathbf{1} + \lambda \bar{r}^T \Sigma^{-1} \bar{r} &= r_m. \end{aligned}$$

Using the following definitions:

$$\begin{aligned} A &= \mathbf{1}^T \Sigma^{-1} \mathbf{1} \\ B &= \mathbf{1}^T \Sigma^{-1} \bar{r} \\ C &= \bar{r}^T \Sigma^{-1} \bar{r} \end{aligned}$$

We can write the system:

$$\begin{aligned} A\gamma + B\lambda &= 1 \\ B\gamma + C\lambda &= r_m. \end{aligned}$$

We have two equations and two unknowns with the following solution.

$$\lambda = \frac{A r_m - B}{AC - B^2}$$

$$\gamma = \frac{C - B r_m}{AC - B^2}.$$

Suppose, for the moment, that:

$$AC - B^2 > 0, \tag{0.3}$$

Combining the last two equations with (0.2) we get the optimal solution:

$$w^* = \Sigma^{-1}(\frac{C - Br_m}{AC - B^2}\mathbf{1} + \frac{Ar_m - B}{AC - B^2}\bar{r})$$

All that is left is to justify division with $AC - B^2$. Let $S = \sqrt{\Sigma^{-1}}$, S is guaranteed to be real since Σ is positive definite. Let,

$$x = S\bar{r},\ \ y = S\mathbf{1}. \tag{0.4}$$

Moreover $\bar{r} \neq \kappa\mathbf{1}$, for any κ, i.e. $\bar{r}$ is not proportional to $\mathbf{1}$. This means we can use the Cauchy-Schwartz inequality as a strict inequality:

$$|B| = |x^T y| < \|x\|\|y\| = \sqrt{\bar{r}^T\Sigma^{-1}\bar{r}}\sqrt{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} = \sqrt{C}\sqrt{A},$$

therefore, $B^2 - AC < 0$.