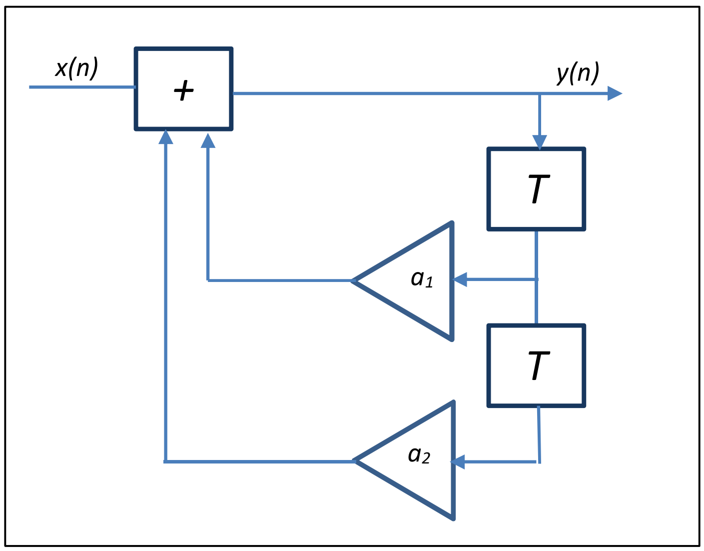
Bandpass IIR pole zero + resonant frequency

The applications of digital bandpass filters include:

- Transmitters and receivers
- Graphic equalizers (Filter banks)
- Speech recognition systems
- Noise rejection
- Mains frequency rejection

We will examine the design of a 2nd order system as shown below. For higher order systems, numerical methods are the normal method of design.



This filter would be classified as a 2nd order IIR type.

$$y(n) = x(n) + a_1 y(n-1) + a_2 y(n-2)$$

$$Y(z) = X(z) + a_1 Y(z) z^{-1} + a_2 Y(z) z^{-2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{z^2}{z^2 - a_1 z - a_2}$$

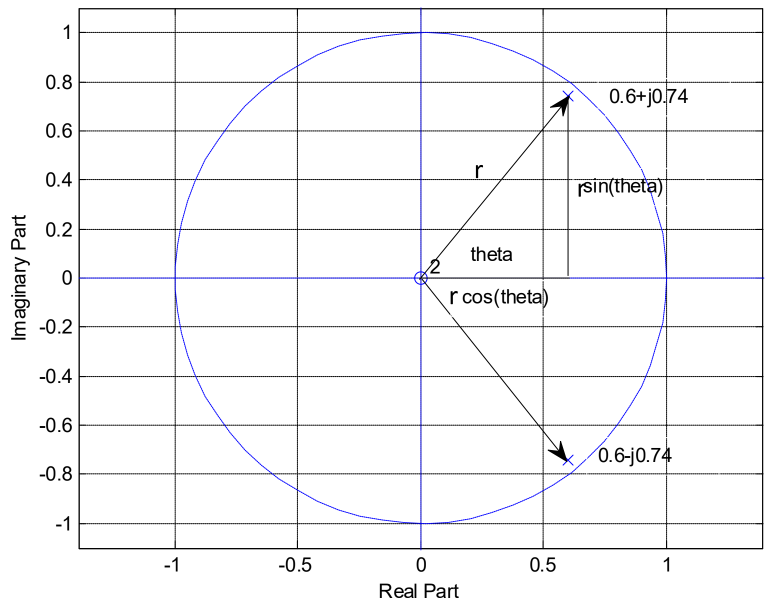
$$Roots\ of\ z^2 - a_1 z - a_2\ are \qquad z = \frac{a_1}{2} \pm \frac{\sqrt{a_1^2 + 4a_2}}{2} = \frac{a_1}{2} \pm j\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{2}$$

$$H(z) = \frac{z^2}{\left(z - \frac{a_1}{2} - j\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{2}\right)\left(z - \frac{a_1}{2} + j\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{2}\right)}$$

$$Let \quad a_1 = 1.2 \quad a_2 = -0.91$$

$$Poles\ are \quad z = \frac{a_1}{2} \pm j\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{2} = \frac{1.2}{2} \pm j\frac{\sqrt{-\left(4\times(-0.91)+(1.2)^2\right)}}{2}$$

$$= 0.6 \pm j\frac{\sqrt{-(-3.64+1.44)}}{2} = 0.6 \pm j\frac{\sqrt{2.2}}{2} = 0.6 \pm j0.74$$



The poles are inside the unit circle. This results in a stable system. If the poles were outside the unit circle, the system becomes unstable and if they are on the unit circle the system is marginally stable.

$$r\cos(\theta_0) = \frac{a_1}{2} \quad r\sin(\theta_0) = \frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{2} \quad \frac{r\sin(\theta_0)}{r\cos(\theta_0)} = \tan(\theta_0) = \frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1}$$

$$\Rightarrow \theta_0 = \tan^{-1}\left(\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1}\right) \quad f_0 = \frac{f_s}{2\pi}\tan^{-1}\left(\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1}\right)$$

$$a_1 = 1.2 \quad a_2 = -0.91 \quad f_0 = 1133\ Hz \quad f_s = 8000\ Hz$$

$$y(n) = x(n) + 1.2y(n-1) - 0.91y(n-2)$$

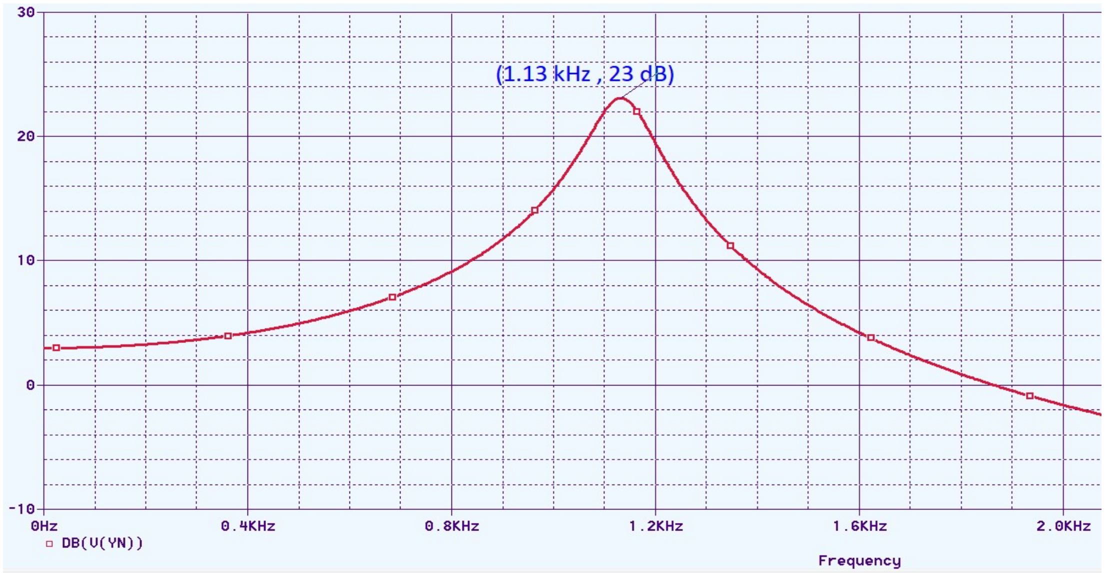
$$H(z) = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} \quad H(\theta_0) = \frac{1}{1 - a_1 e^{-j\theta_0} - a_2 z^{-j2\theta_0}}$$

$$H(\theta_0) = \frac{1}{1 - a_1\{\cos(\theta_0) - j\sin(\theta_0)\} - a_2\{\cos(2\theta_0) - j\sin(2\theta_0)\}}$$

$$\left|H(\theta_0)\right|^2 = \frac{1}{\{1 - a_1\cos(\theta_0) - a_2\cos(2\theta_0)\}^2 + \{a_1\sin(\theta_0) + a_2\sin(2\theta_0)\}^2}$$

$$\left|H(\theta_0)\right|_{dB} = -10\log\left(\{1 - a_1\cos(\theta_0) - a_2\cos(2\theta_0)\}^2 + \{a_1\sin(\theta_0) + a_2\sin(2\theta_0)\}^2\right) = 23\ dB$$

f_0 is the resonant (centre) frequency of the band-pass filter. Simulating and plotting the FFT gives:



Problem

See if you can adjust the values of a_1 and a_2 so that the centre frequency of the BPF is exactly 1 kHz.

$$\theta_0 = \tan^{-1}\left(\frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1}\right) \quad \Rightarrow \tan\left(\theta_0\right) = \frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1}$$

$$\theta_0 = \frac{2\pi f_0}{f_s} = \frac{2\pi(1)}{8} = \frac{\pi}{4} \quad \Rightarrow \tan\left(\frac{\pi}{4}\right) = \frac{\sqrt{-\left(4a_2 + a_1^2\right)}}{a_1} = 1$$

$$\sqrt{-\left(4a_2 + a_1^2\right)} = a_1 \quad \Rightarrow -\left(4a_2 + a_1^2\right) = a_1^2$$

$$-4a_2 = 2a_1^2 \Rightarrow -2a_2 = a_1^2$$

If you look at the pole/zero diagram, any value along the line from 0 to 1 with the 45^0 angle will result in the correct centre frequency. The closer to 1 you put it though will result in a higher Q-factor. So you could arbitrarily choose:

$$r = 0.9$$

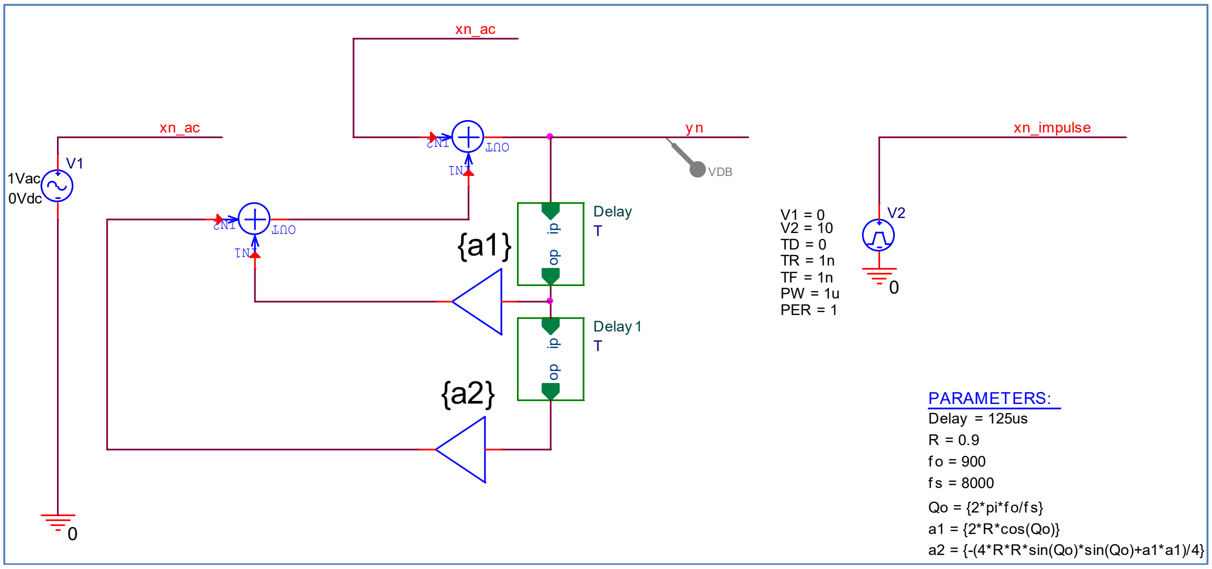
$$r\cos\left(\theta_0\right) = \frac{a_1}{2} \Rightarrow a_1 = 2r\cos\left(\theta_0\right) = \frac{2r}{\sqrt{2}} = r\sqrt{2}$$

$$a_1 = 0.9\sqrt{2} = 1.27$$

$$a_2 = -\frac{a_1^2}{2} = -\frac{(0.9)^2\,2}{2} = -(0.9)^2 = -0.81$$

Draw the block diagram of an IIR BPF with a_2=-1.5 and a_1=1.8. What is the centre frequency of the filter if the sampling frequency is 16 kHz?

Using Pspice to Design a Filter



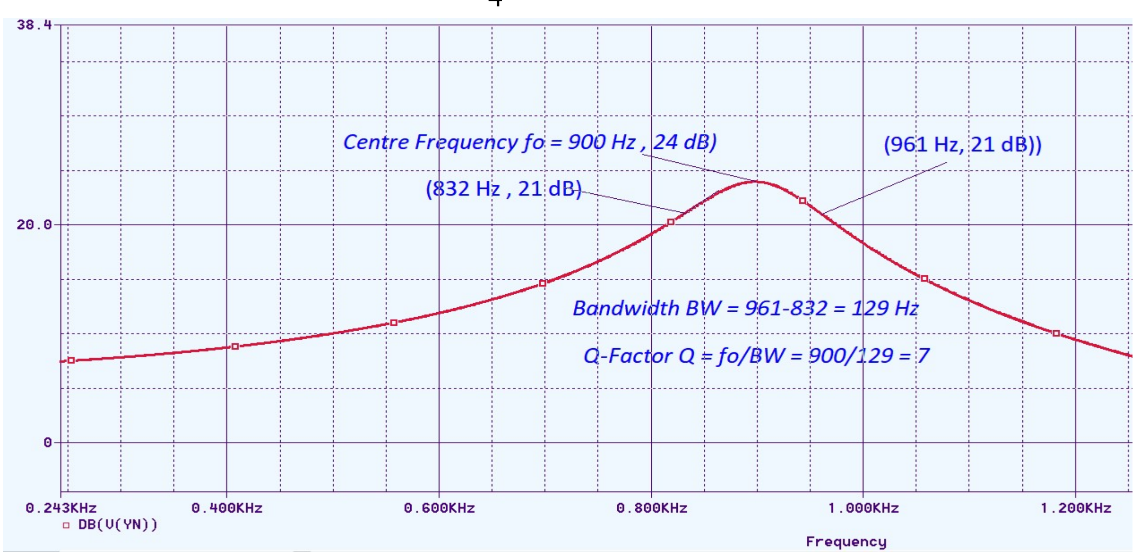
Note:

$$r\cos(\theta_0) = \frac{a_1}{2} \quad \Rightarrow a_1 = 2r\cos(\theta_0) = \frac{a_1}{2}$$

$$r\sin(\theta_0) = \frac{\sqrt{-(4a_2 + a_1^2)}}{2} \Rightarrow 2r\sin(\theta_0) = \sqrt{-(4a_2 + a_1^2)}$$

$$\Rightarrow 4r^2\sin^2(\theta_0) = -(4a_2 + a_1^2) \Rightarrow -4r^2\sin^2(\theta_0) - a_1^2 = 4a_2$$

$$\Rightarrow a_2 = -\frac{4r^2\sin^2(\theta_0) + a_1^2}{4}$$



r IIR BPF Design									
fs	*a2*	*a1*	*fs/2*pi*	*sqrt(-(4a2+a1^2))*	*sqrt(-(4a2+a1^2))/a1*	*invtan()=Q0*	*atan()*fs/2*pi*		
8000	-0.91	1.2	1273.2395	1.483239697	1.236033081	0.89056755	1133.906		
a1cos(Q0)	*a2cos(2Q0)*	*a1sin(Q0)*	*a2sin(Q0)*	*(1-a1cos(Q0)-a2cos(2Q0))^2*	*(a1sin(Q0)+a2sin(2Q0))^2*	*G(f0)*			
0.75476508	0.19	0.9329146	-0.8899438	0.003050896	0.001846489	23.1003575			
fs	*a2*	*a1*	*fs/2*pi*	*sqrt(-(4a2+a1^2))*	*sqrt(-(4a2+a1^2))/a1*	*invtan()=Q0*	*atan()*fs/2*pi*		
16000	-1.5	1.8	2546.4791	1.661324773	0.922958207	0.74535537	1898.032		
a1cos(Q0)	*a2cos(2Q0)*	*a1sin(Q0)*	*a2sin(Q0)*	*(1-a1cos(Q0)-a2cos(2Q0))^2*	*(a1sin(Q0)+a2sin(2Q0))^2*	*G(f0)*			
1.32272446	-0.12	1.2208194	-1.4951923	0.041097207	0.075280487	9.34130251			