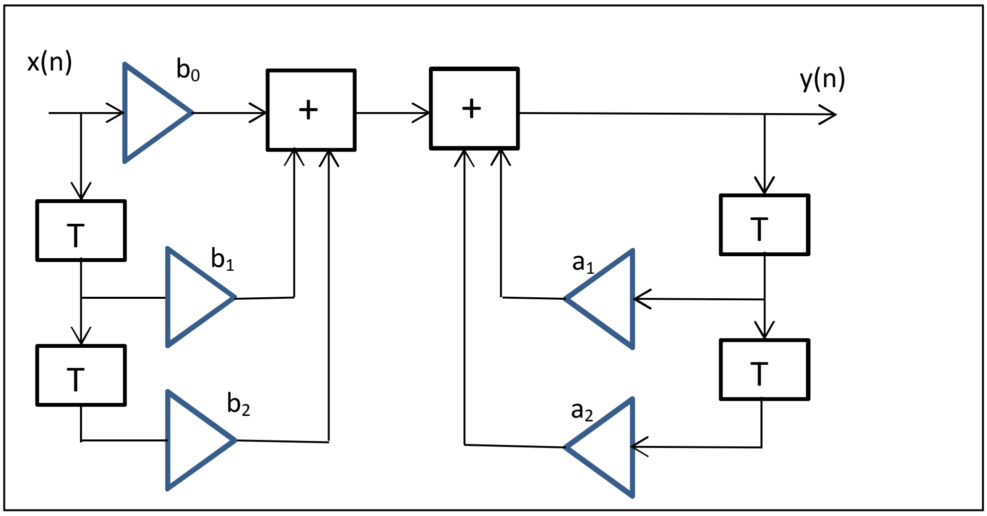
Filter types

A digital filter can be described using the following general equations:

$$y(n)=\sum_{k=0}^{M}b_k x(n-k)+\sum_{k=1}^{L}a_k y(n-k)$$
$$Y(z)=\sum_{k=0}^{M}b_k X(z)z^{-k}+\sum_{k=1}^{L}a_k Y(z)z^{-k}$$
$$Y(z)\left(1-\sum_{k=0}^{L}a_k Y(z)z^{-k}\right)=\sum_{k=1}^{M}b_k X(z)z^{-k}$$
$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M}b_k z^{-k}}{\left(1-\sum_{k=1}^{L}a_k z^{-k}\right)}$$
$$\text{For example, if } M=L=2,\ 2^{nd}\ \text{order}:$$
$$y(n)=\sum_{k=0}^{2}b_k x(n-k)+\sum_{k=1}^{2}a_k y(n-k)$$
$$=y(n)=b_0x(n)+b_1x(n-1)+b_2x(n-2)+a_1y(n-1)+a_2y(n-2)$$
$$Y(z)=b_0X(z)+b_1X(z)z^{-1}+b_2X(z)z^{-2}+a_1Y(z)z^{-1}+a_2Y(z)z^{-2}$$
$$H(z)=\frac{\sum_{k=0}^{2}b_k z^{-k}}{\left(1-\sum_{k=1}^{2}a_k z^{-k}\right)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}$$

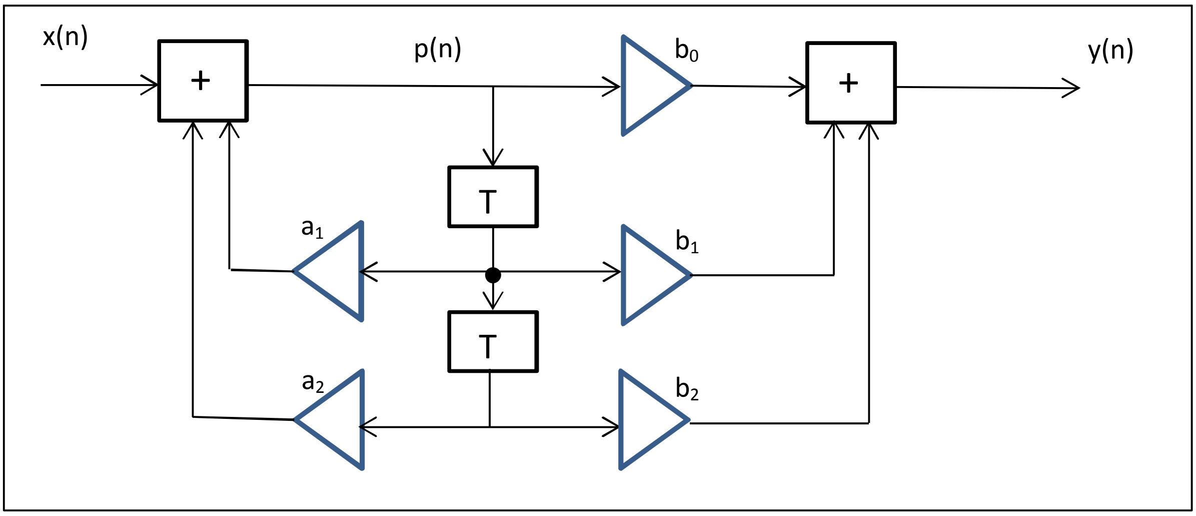
For the sake of brevity, we assume 2nd order systems from this point on, although the analysis equally applies to any order desired.

Direct form I



Direct form II

In this case we place the poles 1st in the transfer function:



$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}=\frac{1}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}\left(b_0+b_1z^{-1}+b_2z^{-2}\right)$$

$$P(z)=a_1P(z)z^{-1}+a_2P(z)z^{-2}+X(z)$$

$$P(z)\left(1-a_1z^{-1}-a_2z^{-2}\right)=X(z)$$

$$P(z)=\frac{X(z)}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}$$

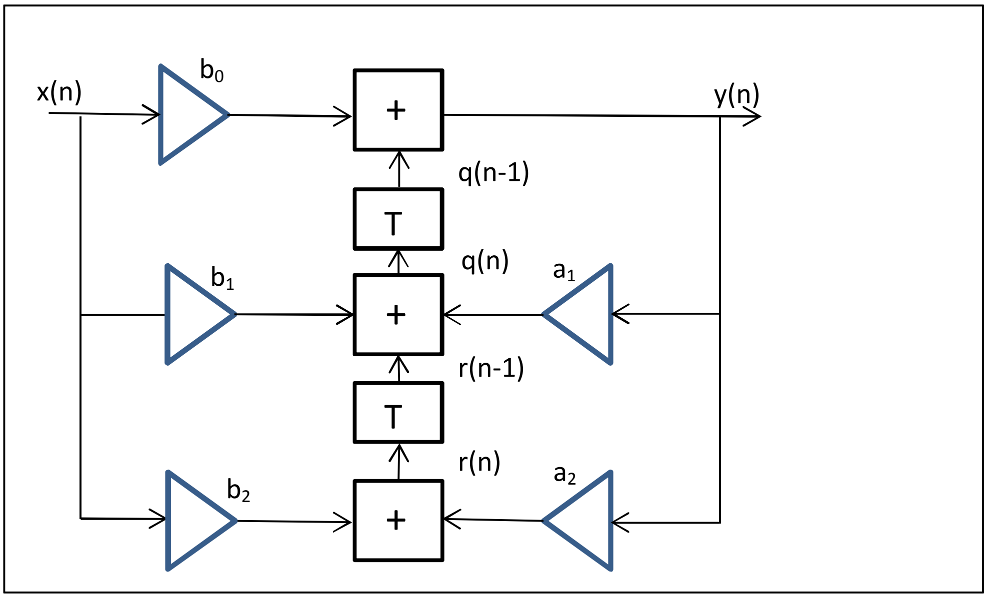
$$Y(z)=b_0P(z)+b_1P(z)z^{-1}+b_2P(z)z^{-2}$$

$$=\frac{X(z)}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}\left(b_0+b_1z^{-1}+b_2z^{-2}\right)$$

$$\Rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{1}{\left(1-a_1z^{-1}-a_2z^{-2}\right)}\left(b_0+b_1z^{-1}+b_2z^{-2}\right)$$

Note that the number of delays are reduced in this form, which is an advantage.

Transpose



$$R(z) = b_2 X(z) + a_2 Y(z)$$

$$Q(z) = b_1 X(z) + a_1 Y(z) + R(z) z^{-1}$$

$$= b_1 X(z) + a_1 Y(z) + \left(b_2 X(z) + a_2 Y(z)\right) z^{-1}$$

$$= b_1 X(z) + a_1 Y(z) + b_2 X(z) z^{-1} + a_2 Y(z) z^{-1}$$

$$= X(z)\left(b_1 + b_2 z^{-1}\right) + Y(z)\left(a_1 + a_2 z^{-1}\right)$$

$$Y(z) = b_0 X(z) + Q(z) z^{-1}$$

$$= b_0 X(z) + \left(X(z)\left(b_1 + b_2 z^{-1}\right) + Y(z)\left(a_1 + a_2 z^{-1}\right)\right) z^{-1}$$

$$= b_0 X(z) + X(z)\left(b_1 z^{-1} + b_2 z^{-2}\right) + Y(z)\left(a_1 z^{-1} + a_2 z^{-2}\right)$$

$$Y(z) - Y(z)\left(a_1 z^{-1} + a_2 z^{-2}\right) = b_0 X(z) + X(z)\left(b_1 z^{-1} + b_2 z^{-2}\right)$$

$$Y(z)\left(1 - a_1 z^{-1} + a_2 z^{-2}\right) = X(z)\left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)}{\left(1 - a_1 z^{-1} + a_2 z^{-2}\right)}$$