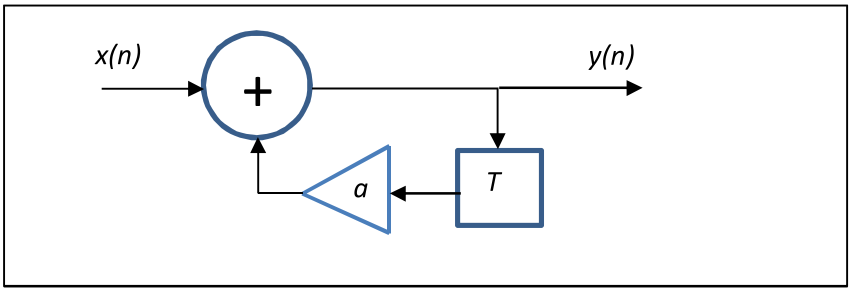
Cut-off frequency (LPF IIR)



$$y(n) = x(n) + ay(n-1) \Rightarrow Y(z) = X(z) + aY(z)z^{-1}$$
$$\Rightarrow Y(z)\left(1 - az^{-1}\right) = X(z)$$
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$
$$H(\theta) = \frac{1}{1 - ae^{-j\theta}} = \frac{1}{1 - a\left\{\cos(\theta) - j\sin(\theta)\right\}}$$
$$|H(\theta)|^2 = \frac{1}{\left\{1 - a\cos(\theta)\right\}^2 + \left\{a\sin(\theta)\right\}^2} = \frac{1}{1 - 2a\cos(\theta) + a^2\left\{\cos^2(\theta) + \sin^2(\theta)\right\}}$$
$$|H(\theta)|^2 = \frac{1}{1 - 2a\cos(\theta) + a^2}$$
$$\frac{|H(0)|^2}{2} = |H(\theta_c)|^2 \Rightarrow \frac{1}{2\left(1 - 2a + a^2\right)} = \frac{1}{1 - 2a\cos(\theta_c) + a^2}$$
$$2 - 4a + 2a^2 = 1 - 2a\cos(\theta_c) + a^2 \Rightarrow 2a\cos(\theta_c) = -1 + 4a - a^2$$
$$\theta_c = \cos^{-1}\left\{\frac{4a - 1 - a^2}{2a}\right\}$$

Alternatively

$$2 - 4a + 2a^2 = 1 - 2a\cos(\theta_c) + a^2$$
$$a^2 - 2a\left(2 - \cos\theta_c\right) + 1 = 0$$
$$a = \left(2 - \cos\theta_c\right) - \sqrt{\left(2 - \cos\theta_c\right)^2 - 1}$$
$$\textit{Since solution of quadratic } Ax^2 + Bx + C \textit{ is:}$$
$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We select the minus option as the plus option would result in a value of 'a' that lies outside the unit circle resulting in instability.

For example, if the sampling frequency is 10 kHz and the filter cut-off frequency is 600 Hz, then the digital frequency

$$\theta = \frac{2\pi f_a}{f_s} = \frac{2\pi(600)}{10000} = \frac{12\pi}{100} = \frac{3\pi}{25}$$

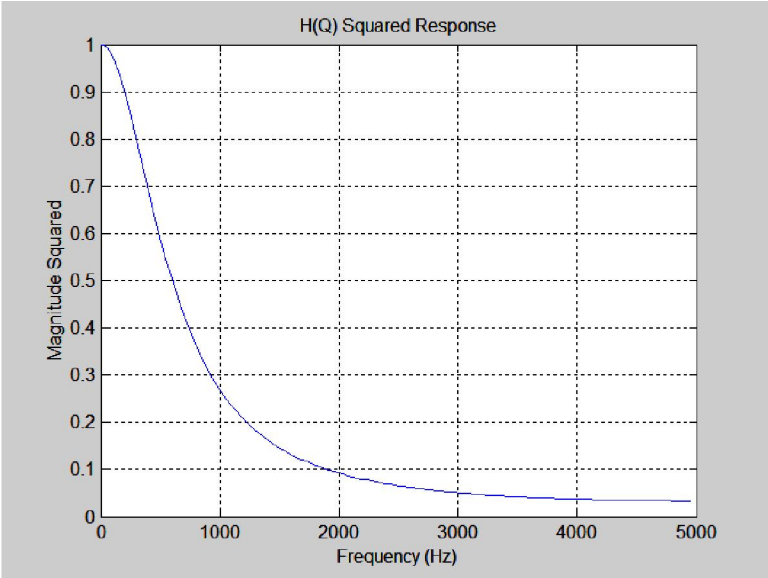
$$\begin{aligned} a &= (2-\cos\theta) - \sqrt{(2-\cos\theta)^2 - 1} \\ &= \left(2-\cos\frac{3\pi}{25}\right) - \sqrt{\left(2-\cos\frac{3\pi}{25}\right)^2 - 1} \\ &= 0.6889 \end{aligned}$$

If a filter with a dc gain of unity is required the filter transfer function is altered to:

$$H(z) = \frac{1-a}{1-az^{-1}}$$

This can be seen by setting z=1, corresponding to θ=0. We can analyze the filter in Matlab as follows:

```
fs=10000;
fc=600;
q=2*pi*fc/fs;
temp=2-cos(q);
a=temp-sqrt(temp^2-1);
bb=1-a;
aa=[1 -a];
[h,f]=freqz(bb,aa,128,fs);
plot(f,(abs(h)).^2);
grid;
xlabel 'Frequency (Hz)'
ylabel 'Magnitude Squared'
title 'H(Q) Squared Response'
```



Problem

The output of a digital IIR LPF is:

$$y(n) = x(n) + ay(n-1)$$

Show that the transfer function can be defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

Whence show that the cut-off frequency (Θ_c) is:

$$\theta_c = \cos^{-1}\left\{\frac{4a - 1 - a^2}{2a}\right\}$$

Or that the filter coefficient 'a' is:

$$a = (2 - \cos\theta_c) - \sqrt{(2 - \cos\theta_c)^2 - 1}$$

This digital filter requires a 1.8 kHz cut-off frequency. Determine Θ_c and 'a'. It may be assumed that the sampling frequency of the filter is 64 KHz.