Z-Transform, LTI, Causality, BIBO

As there are quite a number of complex issues in this area, the following summary emphasises the important concepts:

The Z-Transform is defined:

$$x(n) \Leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z = e^{sT} = e^{j\omega T} = e^{j\theta}$$

$$T = \frac{1}{f_s}$$

$$In\ practice\ X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}\ \ (1\ sided\ transform)\ is\ used$$

The Unit Impulse

$$\delta(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$

$$\Delta(z) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \ldots\ldots = 1$$

The Unit Step

$$u(n) = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

$$U(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = 1 \times z^{0} + 1 \times z^{-1} + 1 \times z^{-2} + \ldots\ldots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots = \frac{z}{z-1}$$

$$Let\ S = 1 + r + r^2 + r^3 + \ldots\ldots$$

$$rS = r + r^2 + r^3 + \ldots\ldots$$

$$S - rS = 1 + r + r^2 + r^3 + \ldots\ldots - r + r^2 + r^3 + \ldots\ldots = 1$$

$$S(1 - r) = 1$$

$$\Rightarrow S = \frac{1}{1-r}$$

$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Delay of a Signal

$$\boxed{Let\ y(n) = x(n-k) \Rightarrow Y(z) = z^{-k}X(z)}$$

The Unit Delay

$$\boxed{Let\ x(n) = \delta(n-k) \Rightarrow X(z) = z^{-k}}$$

	Table of Z-Transforms	
	Sequence	**Z-Transform**
1	$\delta(n)$	1
2	$u(n)$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	e^{an}	$\frac{z}{z-a^2}$
5	n	$\frac{z}{(z-1)^2}$
6	n^2	$\frac{z(z+1)}{(z-1)^3}$
7	b^nn	$\frac{bz}{(z-b)^2}$
8	e^{an}n	$\frac{za^2}{(z-a^2)^2}$
9	$\sin(an)$	$\frac{\sin(a)\,z}{z^2-2\,z\cos(a)+1}$
10	$b^n\sin(an)$	$\frac{\sin(a)\,b\,z}{z^2-2\,bz\cos(a)+b^2}$
11	$\cos(an)$	$\frac{z(z-\cos(a))}{z^2-2\,z\cos(a)+1}$
12	$b^n\cos(an)$	$\frac{z(z-b\cos(a))}{z^2-2\,bz\cos(a)+b^2}$

Linear time invariance (LTI)

If an input to a system is $x_1(n)$ and its output is $y_1(n)$ then it is linear if:

$$x_1(n) \rightarrow y_1(n)$$
$$x_2(n) \rightarrow y_2(n)$$
$$a_1 x_1(n) + a_2 x_2(n) \rightarrow a_1 y_1(n) + a_2 y_2(n)$$

The system is time invariant if

$$x(n-k) \rightarrow y(n-k)$$

A system is LTI if both properties are met.

Bounded input bounded output (BIBO)

A system is BIBO if a bounded input (limited) produces a bounded output.

Causality

A system is causal if the output is zero for n<0.

Problems:

Find the z-transform of the following sequences:

$$x(n) = \delta(n)$$
$$x(n) = 4\delta(n-1)$$
$$x(n) = 8\delta(n-2) + 4\delta(n-3) + 2\delta(n-3)$$
$$x(n) = u(n) - u(n-6)$$