Stochastic Processes

Assignment #04

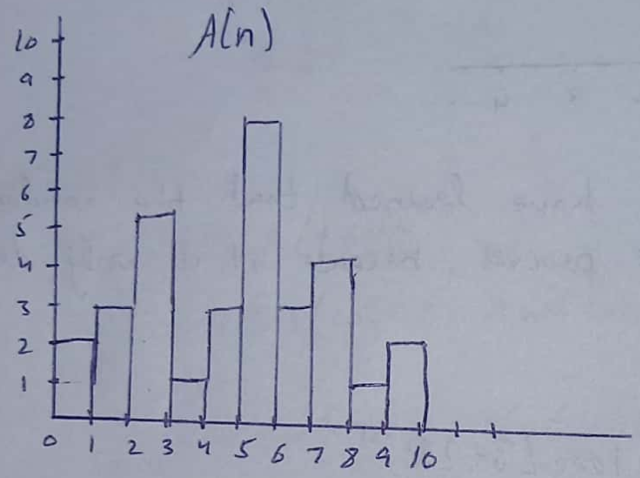
Question #01:

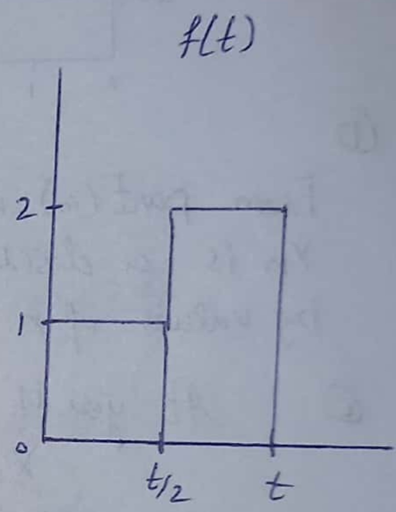
$$x(t) = \sum_{n=-\infty}^{\infty} A_n f(t - nt_1)$$

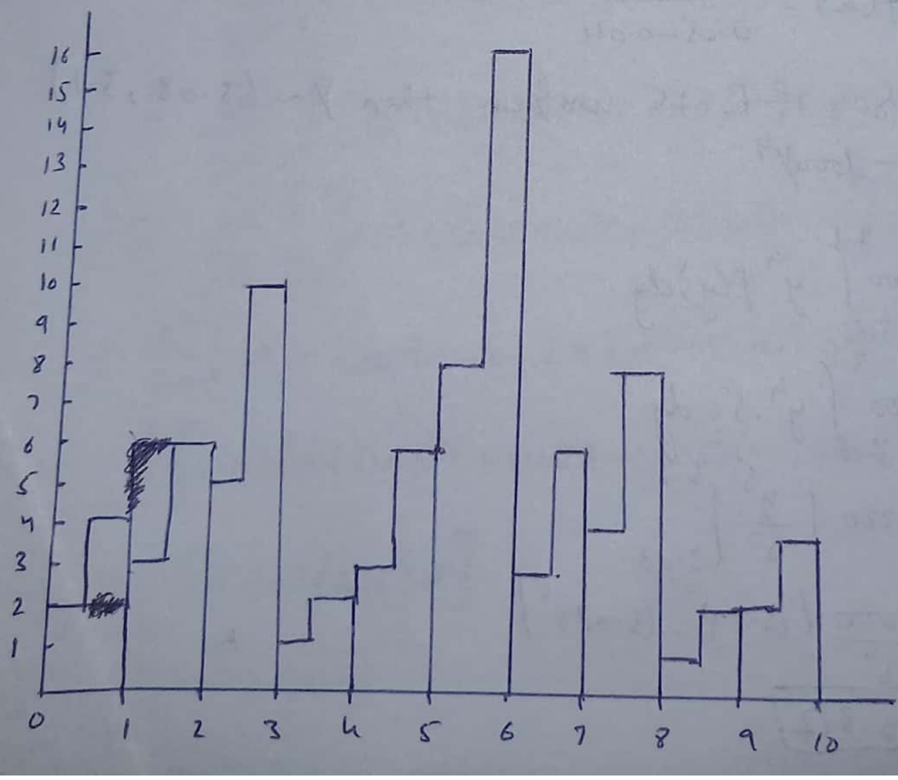
$$f(t) = \begin{cases} 1 \; ; 0 \le t \le t_1/2 \\ 2 \; ; t_1/2 \le t \le t_1 \end{cases}$$

ⓐ Since the amplitude is not entirely specified as a function of time, it is non-deterministic. Here $X(t)$ depends on A_n.

ⓑ







(c)

Since the Random process is uniformally distributed at half time and discrete at other half. It is mixed.

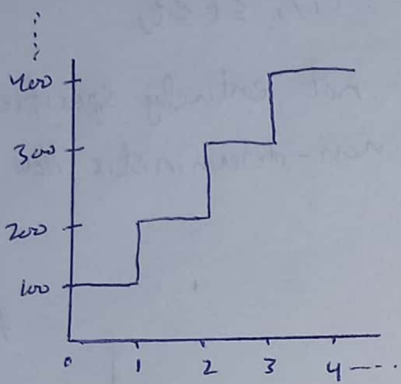
Question #02 :

$X_n = 1000 (3+2R)^n$ for $n = 0, 1, 2, \cdots$

R is R.V that $R \sim U(0.04, 0.05)$

(a)

Since interest rate compound anually.



(b)

From part (a), we have learned that the random process X_n is a discrete process, because it is only defined by values of n.

(c) At year 4

$$X_4 = 1000 (3+2R)^4$$

$$f(x) = \frac{1}{0.05-0.04} = 100$$

Let $Y = 3+2R$ so, if R is uniform then $Y \sim (3.08, 3.1)$

$$X_4 = 1000 Y^4$$

$$E[X_4] = 1000 \int_{3.08}^{3.1} y^4 f(y) dy$$

$$= 1000 \int_{3.08}^{3.1} y^4 . 50 \, dy$$

$$= 50000 \left| \frac{y^5}{5} \right|_{3.08}^{3.1}$$

$$= \frac{50000}{5} [(3.1)^5 - (3.08)^5]$$

$$\boxed{= 9116.812}$$

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Question #03:

$$x(t) = A\cos(\omega t+\theta) \; ; \; -\infty < t < +\infty$$

ω & A constant and $\theta \sim U(0,\pi)$

ⓐ Mean:

$$E[x(t)] = \int_0^{\pi} x\, f(x)\, dx$$

$$= \int_0^{\pi} \frac{1}{\pi} A\cos(\omega t+\theta)\, d\theta$$

$$= \frac{A}{\pi}\int_0^{\pi} \cos(\omega t+\theta)\, d\theta$$

$$= \frac{A}{\pi} \left| \sin(\omega t+\theta) \right|_0^{\pi}$$

$$= \frac{A}{\pi}[\sin \omega t + \sin\pi - \sin\omega t - \sin 0]$$

$$= \frac{A}{\pi}[0-0] = 0$$

Autocorelation:

$$R_{xx} = E[x(t)\,x(t+\tau)]$$

$$= \int_0^{\pi} \frac{1}{\pi} A\cos(\omega t+\theta) \cdot A\cos(\omega(t+\tau)+\theta)\, d\theta$$

$$= \frac{A^2}{\pi}\int_0^{\pi} \cos(\omega t+\theta)\cdot \cos(\omega t+\omega\tau+\theta)\, d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{\pi} 2\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta)\, d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{\pi} [\cos(2\omega t+\omega\tau+2\theta) - \cos(-\omega\tau)]\, d\theta$$

$$= \frac{A^2}{2\pi}\left[\int_0^{\pi} \cos(2\omega t+\omega\tau+2\theta)\, d\theta - \int_0^{\pi} \cos(\omega\tau)\, d\theta\right]$$

$$= \frac{A^2}{2\pi}\left[\left| \sin(2\omega t+\omega\tau+2\pi) - \sin(2\omega t+\omega\tau) \right| - \cos(\omega\tau)\pi \right]$$

$$= \frac{A^2}{2\pi}\left[\sin 2\omega t + \sin\omega\tau + \sin 2\pi - \sin 2\omega t - \sin\omega\tau - \cos(\omega\tau)\pi\right]$$

$$= \frac{A^2}{2\pi}[-\cos(\omega\tau)\pi]$$

$$\boxed{= -\frac{A^2}{2}\cos(\omega\tau)}$$

(b) A random process $X(t)$ is wide sense stationary (WSS) if its mean and autocorrelation function are time-invariant, they do not depend on the specific time instance t. In the case of $X(t)$, we have shown that the mean is zero and the autocorrelation function depends on τ, but not on t. Therefore, $X(t)$ is WSS random process.

Question #04:

$$W(t) = A + 2Bt \qquad t \in [0, +\infty)$$

A & B independent RV $\quad A \sim N(1,1)$, $B \sim N(2,2)$

(a) $X = W(1)$

PDF of X

$X = W(1) = A + 2B$ then X is also normal

If A & B are Normal

$E(x) = E(A+2B) = E(A) + 2E(B) = 1 + 2(2) = 5$

$Var(x) = var(A+2B) = Var(A) + 4 var B = 1 + 4(2) = 9$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5}{3}\right)^2}$$

(b) $Y = W(2)$ calculate pdf of Y

$Y = W(2) = A + 4B$

$E[Y] = E(A+4B) = E(A) + 4E(B) = 1 + 4(2) = 9$

$Var(Y) = var(A+4B) = var(A) + 16 var(B) = 1 + 16(2) = 33$

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi \times 33}} e^{-\frac{1}{2}\left(\frac{x-9}{\sqrt{33}}\right)^2}$$

(c) $Z = W(3)$ Pdf of Z

$Z = W(3) = A + 6B$

$E(Z) = E(A+6B) = E(A) + 6E(B) = 1 + 6(2) = 13$

$Var(Z) = var(A+6B) = var(A) + 36 var(B) = 1 + 36(2) = 73$

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\, 73}} e^{-\frac{1}{2}\left(\frac{x-13}{\sqrt{73}}\right)^2}$$

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$$E[XYZ] = E[(A+2B)(A+4B)(A+6B)]$$
$$= E[(A^2+4AB+2AB+8B^2)(A+6B)]$$
$$= E[A^3+6A^2B+6A^2B+36AB^2+8AB^2+48B^3]$$
$$= E[A^3+12A^2B+44AB^2+48B^3]$$
$$= E[A^3]+12E[A^2B]+44E[AB^2]+48E[B^3]$$
$$= E[A^2 \cdot A]+12E[A^2 \cdot B]+44E[A \cdot B^2]+48E[B^2 \cdot B]$$
$$= E(A^2) \cdot E(A)+12E(A^2)E(B)+44E(A)E(B^2)\cancel{+48E(B^2)E(B)}$$
$$+48E(B^2)E(B)$$

$$E(A^2) = Var(A)+[E(A)]^2$$
$$= 1+1 = 2$$
$$E(B^2) = var(B)+[E(B)]^2$$
$$= 2+4 = 6$$
$$E[XYZ] = 2 \cdot 1+12 \cdot 2 \cdot 2+44 \cdot 1 \cdot 6+48 \cdot 6 \cdot 2$$
$$= 2+48+264+576$$
$$\boxed{= 890}$$

Question #05:

$$X_n = 1000(1+R)^n \text{ for } n = 0,1,2\cdots$$

R is uniform RV $(0.04, 0.05)$;

Mean: $$f(r) = \frac{1}{0.05-0.04} = 100$$

$$Y \sim U(1.04, 1.05)$$
$$X(n) = 1000Y^n$$
$$E[X_n] = 1000\int_{1.04}^{1.05} y^n f(y)dy$$
$$= 1000\int_{1.04}^{1.05} y^n 100 dy$$
$$= 10^5 \left| \frac{y^{n+1}}{n+1} \right|_{1.04}^{1.05}$$
$$\boxed{= \frac{10^5}{n+1}\left((1.05)^{n+1} - (1.04)^{n+1}\right)}$$

Autocorrelation:

$$R_{xx} = E[x(t)x(t+\tau)]$$

$$= \int_{1.04}^{1.05} 100(1000y^n)(1000y^{n+\tau})\,dy$$

$$= 10^8 \int_{1.04}^{1.05} y^n y^{n+\tau}\,dy$$

$$= 10^8 \int_{1.04}^{1.05} y^{2n+\tau}\,dy$$

$$= 10^8 \left| \frac{y^{2n+\tau+1}}{2n+\tau+1} \right|_{1.04}^{1.05}$$

$$\boxed{= \frac{10^8}{2n+\tau+1}\left[1.05^{2n+\tau+1} - 1.04^{2n+\tau+1}\right]}$$

Question #06:

$$x(t) = A + Bt$$

A & B are independent normal $N(1,1)$

Mean:

$$E[x(t)] = E(A+Bt)$$

$$= E(A) + t\,E(B)$$

$$\boxed{= 1+t}$$

Autocorrelation:

$$R_{xx} = E[x(t)x(t+\tau)]$$

$$= E[(A+Bt)(A+B(t+\tau))]$$

$$= E[(A+Bt)(A+Bt+B\tau)]$$

$$= E[A^2 + ABt + AB\tau + ABt + B^2t^2 + B^2t\tau]$$

$$= E(A^2) + t\,E(AB) + \tau E(AB) + t\,E(AB) + t^2E(B^2) + t\tau(B^2)$$

$$E(A^2) = var(A) + \{E(A)\}^2 = 1+1 = 2$$

$$E(B^2) = var(B) + \{E(B)\}^2 = 1+1 = 2$$

$$= 2 + 2t + \tau + 2t^2 + 2t\tau$$

$$\boxed{= 2t^2 + 2t + 2t\tau + \tau + 2}$$

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Question # 07:

$$X(t) = A + Bt$$

$$Y(t) = A + Ct$$

A, B & C independent normal $N(1,1)$

Calculate $R_{xy}(t_1, t_2)$

$$R_{xy}(t_1, t_2) = E[(A+Bt)(A+Ct)]$$

$$= E[A^2 + ACt + ABt + BCt^2]$$

$$= E[A^2 + ACt + ABt + BCt^2)(A^2 + AC(t+\tau) + AB(t+\tau) + BC(t+\tau)^2)]$$

$$R_{xy}(t, t+\tau) = E[(A^2 + ACt + ABt + BCt^2)(A^2 + ACt + ABt + BCt^2 + BC\tau^2 + 2BC(t\tau))]$$

$$= E[(A^2 + ACt + \cancel{Bt}^{ABt} + BCt^2)(A^2 + ACt + AC\tau + ABt + AB\tau + BCt^2 + BC\tau^2 + 2BC(t\tau)]$$

$$= E[A^4 + A^3Ct + A^3C\tau + A^3Bt + A^3B\tau + A^2BCt^2 + A^2BC\tau^2 + 2A^2BCt\tau + A^3Ct + A^2C^2t^2$$
$$+ A^2C^2t\tau + A^2BCt^2 + A^2BC\tau t + ABC^2t^3 + ABC^2\tau^2t + 2ABC^2t^2\tau + A^3Bt + A^2BCt^2$$
$$+ A^2BCt\tau + A^2B^2t^2 + A^2B^2t\tau + AB^2Ct^3 + AB^2Ct\tau^2 + 2AB^2Ct^2\tau + A^2BCt^2$$
$$+ ABC^2t^3 + ABC^2t^2\tau + AB^2Ct^3 + AB^2Ct^2\tau + B^2C^2t^4 + B^2C^2t^2\tau^2 + 2B^2C^2t^3\tau]$$

$E[A] = 1 \quad E[A^2] = 2$

$E[B] = \cancel{0}1 \quad E[B^2] = 2$

$E[C] = 1 \quad E[C^2] = 2$

After Applying mean properties:

$$R_{xy}(t_1, t_2) = 4 + 2t + 2\tau + 2t + 2\tau + 2t^2 + 2\tau^2 + 4t\tau + 2t + 4t^2 + 4t\tau + 2t^2$$
$$+ 2t\tau + 2t^3 + 2t\tau^2 + 4t^2\tau + 2t + 2t^2 + 2t\tau + 4t^2 + 4t\tau + 2t^3 + 2t\tau^2$$
$$+ 4t^2\tau + 2t^2 + 2t^3 + 2t^2\tau + 2t^3 + 2t^2\tau + 4t^4 + 4t^2\tau^2 + 8t^3\tau$$

$$\boxed{= 4 + 8t + 16t^2 + 6t^3 + 4t^4 + 4\tau + 2\tau^2 + 16t\tau + 4t\tau^2 + 12t^2\tau + 4t^2\tau^3 + 8t^3}$$

Question #08 :

ⓐ Cauchy-Schwarz inequality

$$|R_x(\tau)| \le R_x(0)$$

Cauchy-Schwarz inequality states that

$$|u \cdot v| \le \|u\| \cdot \|v\|$$

where $u \cdot v$ denotes the dot product of u and v, $\|u\|$ & $\|v\|$ denotes the Euclidean norms of u and v.

Let $X(t)$ be a random process and $R_X(t)$ be its autocorrelation

Using Cauchy-Schwarz inequality.

$|R_x(\tau)| = |E[X(t)X(t+\tau)]| \le sqrt(E[X(t)^2 X(t+\tau)^2]) = sqrt(R_x(0).R_x(\tau$

By using the fact $R_x(0) = E[x(t)^2]$ for all t.

Taking square on both side

$$|R_x(\tau)|^2 \le R_x(0) \cdot R_x(\tau)$$

Both $R_x(0)$ & $R_x(\tau)$ are non-negative, divide both side by $R_x(\tau)$

$$\boxed{|R_x(\tau)| \le R_x(0)}$$

ⓑ Use definition of autocorrelation

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) = E[X(t)X(t+\tau)]$$

Using definition of $R_x(\tau)$

$$R_x(-\tau) = E[X(t)X(t-\tau)]$$

Since RP $X(t)$ is stationary, it follows that $X(t-\tau)$ has same statistical properties as $X(t+\tau)$.

$$E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)]$$

$$R_x(-\tau) = E[X(t+\tau)X(t)] = R_x(\tau)$$

$$\boxed{R_x(\tau) = R_x(-\tau)}$$

Question # 07:

$\mu_x = 0$

$R_x(\tau) = \delta(\tau)$

$Y(y) = \int_{t-2}^{t} x(u)du$

ⓐ $\mu_y(t) = E[Y(t)]$

$$\mu_y(t) = E[Y(t)]$$
$$= E\left(\int_{t-2}^{t} x(u)du\right)$$
$$= \int_{t-2}^{t} E[x(u)]du$$
$$= \int_{t-2}^{t} 0\, du$$
$$= 0$$

Therefore, the mean $Y(t)$ is zero for all t.

ⓑ $R_{XY}(t_1, t_2)$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$= E\left[X(t_1)\int_{t_2-2}^{t_2} X(u)du\right]$$
$$= \int_{t_2-2}^{t_2} E[X(t_1)X(u)]du$$

~~In this~~

$R_{XX}(\tau) = E[X(t)X(t+\tau)]$ only depend on time difference:

In this case, $R_{XX}(\tau) = \delta\tau$

$$E[X(t_1)Y(t_2)] = \int_{t_2-2}^{t_2} E[X(t_1)X(u)]du$$
$$= \int_{t_2-2}^{t_2} R_{XX}(t_1-u)du$$
$$= \int_{t_2-2}^{t_2} \delta(t_1-u)du$$

$$E[X(t_1)Y(t_2)] = \begin{cases} 1 & ; t_2-2 \le t_1 \le t_2 \\ 0 & ; \text{otherwise} \end{cases}$$