A1 Calculation of mean population fitness

W, the mean population fitness, is the sum of the right sides of Eqns. (5), i.e.,

$$W = 1 + r(\tilde{u}_r + \tilde{x}_r) + R(\tilde{u}_R + \tilde{x}_R),$$

and from (2), (3), and (4),

$$W = 1 + r\left[uL_u(p_r, r) + x\left(L_u(p_r, r) + \delta_K p_r + \delta_{\pi_C}\frac{r}{r+R}\right)\right] + R\left[u(1 - L_u(p_r, r)) + x\left(1 - L_u(p_r, r) - \delta_K p_r - \delta_{\pi_C}\frac{r}{r+R}\right)\right].$$

Since $u + x = 1$,

$$\begin{aligned} W &= 1 + r\left[L_u(p_r, r) + x\left(\delta_K p_r + \delta_{\pi_C}\frac{r}{r+R}\right)\right] \\ &\quad + R\left[1 - L_u(p_r, r) - x\left(\delta_K p_r + \delta_{\pi_C}\frac{r}{r+R}\right)\right], \qquad \text{(A1.1)} \\ &= 1 + R + (r - R)\left[L_u(p_r, r) + x\left(\delta_K p_r + \delta_{\pi_C}\frac{r}{r+R}\right)\right]. \end{aligned}$$

A2 Proof of Result 3.2

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We first prove part (i). From (14a), $W - \delta = 1$ because $\hat{N}_p > 0$. Then from Eqn. (18), $\delta = R$ is only possible if $L(\hat{u}_r, \hat{r}) = 0$ or $\hat{r} = R$. If $L(\hat{u}_r, \hat{r}) = 0$, assuming $\pi_C > 0$, then from (2), $\hat{r} = K\hat{u}_r = 0$.

Parts (ii) and (iii) follow from Eq. (17) and the requirement that $0 \leq L(\hat{u}_r, \hat{r}) \leq 1$ because $L(\hat{u}_r, \hat{r})$ is a probability.

A2.1 Note on the Rare Case of $\hat{r} = R = \delta$

Note that in the rare case of $\hat{r} = R = \delta$, from (14)b

$$(1 + \delta)\hat{u}_r = L(\hat{u}_r, \hat{r})(1 + \delta)$$

so

$$\hat{u}_r = L(\hat{u}_r, \hat{r}) = K\hat{u}_r + \pi_C/2$$

and thus

$$\hat{u}_r = \frac{1}{2}\left(\frac{\pi_C}{1 - K}\right). \qquad \text{(A2.1)}$$

If there is no time delay, i.e. the predation term is $P(r, u_r) = L_{total}$ as defined in (12), and $\hat{r} = \delta$, then the equilibrium population size is

$$\hat{N}_p = \frac{1-\delta}{\beta L(\hat{u}_r, \hat{r})}$$

which simplifies to

$$\hat{N}_p = \frac{2(1-\delta)(1-K)}{\beta \pi_C}.$$

If there is a time delay, i.e. the predation term is $p_r = u_r$, and $\hat{r} = \delta$, then because $\hat{r} = 1 - \beta \hat{N}_p \hat{u}_r$, the equilibrium forager population size is

$$\hat{N}_p = \frac{1-\delta}{\beta \hat{u}_r},$$

which, from (A2.1), simplifies to

$$\hat{N}_p = \frac{2(1-\delta)(1-K)}{\beta \pi_C}. \tag{A2.2}$$

A3 Proof of Result 3.3

If $\hat{r} > 0$, then since $0 < L_u(\hat{u}_r, \hat{r}) \leq 1$ and $R < \delta$, from (17) we must have $\hat{r} \geq \delta$. Substituting (17) for $L_u(\hat{u}_r, \hat{r})$ and $W = 1 + \delta$ into (14b),

$$\hat{u}_r = L_u(\hat{u}_r, \hat{r}) \frac{1+\hat{r}}{1+\delta} = \frac{(\delta - R)(1+\hat{r})}{(\hat{r} - R)(1+\delta)}. \tag{A3.1}$$

Since $\hat{r} \geq \delta > R$, (A3.1) is legitimate, i.e. $0 < \hat{u}_r \leq 1$, if

$$(\delta - \hat{r})(1 + R) \leq 0$$

which is true because $\hat{r} \geq \delta$.

Thus at the equilibrium, from Eqns. (2) and (17),

$$K \frac{(\delta - R)(1+r)}{(r-R)(1+\delta)} + \pi_C \frac{r}{r+R} = \frac{\delta - R}{r - R}. \tag{A3.2}$$

Equation (A3.2) can be rewritten as $Q_r(r) = 0$, where

$$Q_r(r) = r^2 \left[\pi_C + \frac{K(\delta - R)}{(1+\delta)}\right] + r\left[(\delta - R)\left(\frac{K(1+R)}{1+\delta} - 1\right) - R\pi_C\right] - R(\delta - R)\left(1 - \frac{K}{1+\delta}\right), \tag{A3.3}$$

which we write as $Q_r(r) = Ar^2 + Br + C$. Since

$$Q_r(0) = C = -R(\delta - R)\left(1 - \frac{K}{1+\delta}\right) < 0,$$

there is an equilibrium $\hat{r} \geq \delta$ if $Q_r(1) \geq 0$ and $Q_r(\delta) \leq 0$, where

$$Q_r(1) = A + B + C = \pi_C(1 - R) + (\delta - R)(1 + R)\left(\frac{2K}{1 + \delta} - 1\right) \tag{A3.4}$$

and (I showed all my steps so it's easier to check)

$$\begin{aligned}
Q_r(\delta) &= A\delta^2 + B\delta + C & \text{(A3.5)}\\
&= (\delta - R)\left[\frac{K\delta^2}{1+\delta} + \delta\left(\frac{K(1+R)}{1+\delta} - 1\right) - R\left(1 - \frac{K}{1+\delta}\right)\right] + \delta^2\pi_C - R\pi_C\delta\\
&= (\delta - R)\left[\frac{K\delta^2}{1+\delta} - (R+\delta)\left(1 - \frac{K}{1+\delta}\right) + \frac{KR\delta}{1+\delta} + \pi_C\delta\right]\\
&= (\delta - R)\left[(\delta + R)\left(\frac{K\delta}{1+\delta} - 1 + \frac{K}{1+\delta}\right) + \pi_C\delta\right]\\
&= (\delta - R)\left[(\delta + R)\left(\frac{K\delta + K}{1+\delta} - 1\right) + \pi_C\delta\right]\\
&= (\delta - R)\left[(\delta + R)(K - 1) + \pi_C\delta\right]\\
&= (\delta - R)\left[\delta(K + \pi_C - 1) + R(K - 1)\right]\\
&= (\delta - R)\left[-\pi_W\delta + R(K - 1)\right]. & \text{(A3.6)}
\end{aligned}$$

However, $Q_r(\delta) \leq 0$ because $(\delta - R) > 0$ and $-\pi_W\delta + R(K-1) \leq 0$. Thus there is an equilibrium $\delta \leq \hat{r} \leq 1$ if $Q_r(1) \geq 0$.

To complete the proof, we must show there is a legitimate equilibrium predator population size $\hat{N}_p$. No time delay entails $\hat{r} = 1 - \beta\hat{N}_p L_u(\hat{u}_r, \hat{r})$ and a time delay entails $\hat{r} = 1 - \beta\hat{N}_p\hat{u}_r$. If there is no time delay and $\beta > 0$,

$$\hat{N}_p = \frac{(1 - \hat{r})(\hat{r} - R)}{\beta(\delta - R)}, \tag{A3.7}$$

where $L(\hat{u}_r, \hat{r})$ is given by (17) and $\hat{r}$ is the larger root of $Q_r(r)$.

If there is a time delay and $\beta > 0$,

$$\hat{N}_p = \frac{1 - \hat{r}}{\beta\hat{u}_r}, \tag{A3.8}$$

or from (A3.1)

$$\hat{N}_p = \frac{(1 - \hat{r}^2)(\delta - R)}{\beta(\hat{r} - R)(1 + \delta)}. \tag{A3.9}$$

A4 Proof of Result 3.4

If $R > \delta$, then we know $\hat{r} \leq \delta < R$, so $\hat{r} > 0$ exists if $Q_r(r)$ has at least one root between $r = 0$ and $r = \delta$. Note that

$$Q_r(0) = C = -R(\delta - R)\left(1 - \frac{K}{1+\delta}\right) \geq 0,$$

and from (A3.5), $Q_r(\delta) \geq 0$ because $R > 0$. Thus if $A < 0$, i.e. the parabola of $Q_r(r)$ opens down, there is no equilibrium with $\hat{r} > 0$.

On the other hand, if the parabola $Q_r(r)$ points upward, i.e. $A > 0$, then $Q_r(r)$ can only have roots within the range $0 < \hat{r} \leq \delta$ if $Q_r'(0) = B < 0$ and $Q_r'(\delta) = 2A\delta + B > 0$. The roots of $Q_r(r)$ are

$$\hat{r} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

where, if the discriminant $B^2 - 4AC > 0$, then there are two $\hat{r} > 0$ equilibria. If the discriminant is instead nonnegative, then there are no real roots of $Q_r(r)$.

A5 Proof of Result 3.5

Given $\pi_C > 0$, if $\hat{r} = 0$ then at equilibrium $Wu_r = Ku_r$, so either $W = K$ or $\hat{u}_r = 0$. Say $W = K$ and $\hat{u}_r \neq 0$. Then from (15),

$$K = 1 + R(1 - Ku_r)$$

which is a contradiction because $0 \leq K < 1$, leaving us with $\hat{u}_r = \hat{r} = 0$.

Next, we aim to show that if $\hat{u}_r = \hat{r} = 0$, then $\hat{N}_p = 0$ if $R < \delta$. The mean population fitness is $W = 1 + R$ because $\hat{u}_r = \hat{r} = 0$. Since $R < \delta$, $W \neq 1 + \delta$, and thus the only possible equilibrium is $\hat{N}_p = 0$.

For the last part of the proof, if $R > \delta$, then N_p would increase infinitely because $W > 1 + \delta$, so $N_p' > N_p$.

A6 Proof of Result 3.6

From (14b), at this equilibrium $\hat{W}\hat{u}_r = 2L(\hat{u}_r, \hat{r})$ where $L(\hat{u}_r, \hat{r}) = K\hat{u}_r + \frac{\pi_C}{1+R}$ and, from (15), $\hat{W} = 1 + R + (1 - R)L(\hat{u}_r, \hat{r})$. Then

$$u_r\left[1 + R + (1 - R)\left(Ku_r + \frac{\pi_C}{1+R}\right)\right] = 2\left(Ku_r + \frac{\pi_C}{1+R}\right)$$

or

$$Q_u(u_r) = u_r^2 K(1 - R) + u_r\left[1 + R + (1 - R)\frac{\pi_C}{1+R} - 2K\right] - 2\frac{\pi_C}{1+R} = 0. \qquad \text{(A6.1)}$$

Thus $\hat{u}_r$ is the larger root of $Q_u(u_r)$ because $K(1-R) > 0$, $Q_u(0) < 0$, and

$$Q_u(1) = -(1-R)\left(K + \frac{\pi_C}{1+R}\right)(1+R) > 0.$$

A7 Derivation of Internal Stability Jacobian

Near the equilibrium $(\hat{N}_p, \hat{u}_r, \hat{r})$,

$$\begin{aligned} L_u(\hat{u}_r + \Delta_{u_r}, \hat{r} + \Delta_r) &= K(\hat{u}_r + \Delta_{u_r}) + \frac{\hat{r} + \Delta_r}{\hat{r} + \Delta_r + R}\pi_C \\ &= L_u(\hat{u}_r, \hat{r}) + \Delta_L \end{aligned}$$

where

$$\Delta_L \approx K\Delta_{u_r} + \frac{R\pi_C\Delta_r}{(\hat{r}+R)^2} \tag{A7.1}$$

because

$$\frac{\hat{r}+\Delta_r}{\hat{r}+\Delta_r+R} \approx (\hat{r}+\Delta_r)\left(1 - \frac{\Delta_r}{\hat{r}+R}\right)\left(\frac{1}{\hat{r}+R}\right) \approx \frac{\hat{r}}{\hat{r}+R} + \frac{\Delta_r}{\hat{r}+R}\left(1 - \frac{\hat{r}}{\hat{r}+R}\right)$$

and $1 - \frac{\hat{r}}{\hat{r}+R} = \frac{R}{\hat{r}+R}$. The mean population fitness near the equilibrium is

$$\begin{aligned} W + \Delta_W &= 1 + R + (\hat{r} + \Delta_r - R)L_u(\hat{u}_r + \Delta_{u_r}, \hat{r} + dr) \\ \Delta_W &\approx \Delta_L(\hat{r} - R) + \Delta_r L_u(\hat{u}_r, \hat{r}) \end{aligned} \tag{A7.2}$$

which simplifies to

$$\Delta_W = K(\hat{r}-R)\Delta_{u_r} + \Delta_r\left[L(\hat{u}_r, \hat{r}) + \frac{R\pi_C(\hat{r}-R)}{(\hat{r}+R)^2}\right]. \tag{A7.3}$$

The predator population size near the equilibrium $\hat{N}_p$ is

$$\hat{N}_p + \Delta'_{N_p} = (\hat{N}_p + \Delta_{N_p})(\hat{W} + \Delta_W - \delta)$$

so

$$\begin{aligned} \Delta'_{N_p} &\approx \Delta_W \hat{N}_p + \Delta_{N_p}(W - \delta) \\ &= \Delta_{N_p}(\hat{W} - \delta) + \hat{N}_p\left\{K(\hat{r}-R)\Delta_{u_r} + \Delta_r\left[L(\hat{u}_r, \hat{r}) + \frac{R\pi_C(\hat{r}-R)}{(\hat{r}+R)^2}\right]\right\}. \end{aligned} \tag{A7.4}$$

The frequency of predators exploiting the CP near the equilibrium is

$$\hat{u}_r + \Delta'_{u_r} = \frac{1}{\hat{W} + \Delta_W}(L(\hat{u}_r, \hat{r}) + \Delta_L)(1 + \hat{r} + \Delta_r)$$

so

$$\hat{u}_r + \Delta'_{u_r} \approx \frac{1}{\hat{W}}\left(1 - \frac{\Delta_W}{\hat{W}}\right)[L(\hat{u}_r,\hat{r})(1+\hat{r}) + \Delta_r L(\hat{u}_r,\hat{r}) + \Delta_L(1+\hat{r})]$$
$$= \left(1 - \frac{\Delta_W}{\hat{W}}\right)\left[\hat{u} + \Delta_r \frac{L(\hat{u}_r,\hat{r})}{\hat{W}} + \Delta_L \frac{(1+\hat{r})}{\hat{W}}\right],$$

and thus the perturbation from equilibrium is

$$\Delta'_{u_r} \approx \Delta_r \frac{L(\hat{u}_r,\hat{r})}{\hat{W}} + \Delta_L \frac{(1+\hat{r})}{\hat{W}} - \frac{\hat{u}_r}{\hat{W}}\Delta_W,$$

which, after substituting (A7.1) for Δ_L and $\hat{W} = 1+\delta$, becomes

$$\Delta'_{u_r} = \Delta_{u_r}\frac{K}{\hat{W}}(1 + \hat{r} - \hat{u}_r(\hat{r} - R))$$
$$+ \Delta_r \left\{\frac{L(\hat{u}_r,\hat{r})}{\hat{W}}(1 - \hat{u}_r) + \frac{\pi_C R}{\hat{W}(\hat{r}+R)^2}[1 + \hat{r} - \hat{u}_r(\hat{r} - R)]\right\} \quad \text{(A7.5)}$$

A7.0.1 Internal Stability, no time delay

The CP relative density near equilibrium is

$$\hat{r} + \Delta'_r = \frac{(\hat{r} + \Delta_r)\left[2 - \beta(\hat{N}_p + \Delta_{N_p})(L(\hat{u}_r,\hat{r}) + \Delta_L)\right]}{1 + \hat{r} + \Delta_r}$$
$$\approx \frac{(\hat{r} + \Delta_r)\left(2 - \beta\hat{N}_p L(\hat{u}_r,\hat{r}) - \beta\Delta_{N_p} L(\hat{u}_r,\hat{r}) - \beta\hat{N}_p\Delta_L\right)}{1 + \hat{r} + \Delta_r}.$$

To simplify, note that $\frac{1}{1+\hat{r}+\Delta_r} \approx \frac{1}{1+\hat{r}}\left(1 - \frac{\Delta_r}{1+\hat{r}}\right)$. Then

$$\hat{r} + \Delta'_r = \frac{\hat{r}(2 - \beta\hat{N}_p L(\hat{u}_r,\hat{r}) - \hat{r}\beta\left(L(\hat{u}_r,\hat{r})\Delta_{N_p} + \hat{N}_p\Delta_L\right) + \Delta_r\left(2 - \beta\hat{N}_p L(\hat{u}_r,\hat{r})\right)}{1+\hat{r}}\left(1 - \frac{\Delta_r}{1+\hat{r}}\right).$$

Additional steps are shown in red:

$$\hat{r} + \Delta'_r = \left(\hat{r} - \frac{\hat{r}\beta(\Delta_{N_p} L(\hat{u}_r,\hat{r}) + \hat{N}_p\Delta_L)}{1+\hat{r}} + \Delta_r\frac{2 - \beta\hat{N}_p L(\hat{u}_r,\hat{r})}{1+\hat{r}}\right)\left(1 - \frac{\Delta_r}{1+\hat{r}}\right)$$
$$\Delta'_r \approx -\Delta_r\frac{\hat{r}}{1+\hat{r}} - \frac{\hat{r}\beta(\Delta_{N_p} L(\hat{u}_r,\hat{r}) + \hat{N}_p\Delta_L)}{1+\hat{r}} + \Delta_r\frac{2 - \beta\hat{N}_p L(\hat{u}_r,\hat{r})}{1+\hat{r}},$$

and substituting (A7.1) for Δ_L gives

$$\Delta_r' \approx -\Delta_{N_p}\frac{\hat{r}\beta L(\hat{u}_r,\hat{r})}{1+\hat{r}} - \Delta_{u_r}\frac{K\hat{N}_p\hat{r}\beta}{1+\hat{r}} + \left(\frac{2-\beta\hat{N}_pL(\hat{u}_r,\hat{r})-\hat{r}}{1+\hat{r}} - \frac{\hat{N}_p\hat{r}\beta R\pi_C}{(1+\hat{r})(\hat{r}+R)^2}\right)\Delta_r. \quad \text{(A7.6)}$$

Using Eqs (A7.4), (A7.5), and (A7.6), the local stability matrix for the equilibrium $\hat{N}_p, \hat{u}_r, \hat{r}$ is of the form

$$J^* = \begin{pmatrix} \hat{W}-\delta & a & b \\ 0 & c & d \\ e & f & g \end{pmatrix}, \quad \text{(A7.7)}$$

where a, b are the coefficients of Δ_{u_r} and Δ_r, respectively, from (A7.4), c and d are the coefficients of Δ_{u_r} and Δ_r, respectively, from (A7.5), and e, f, and g are the coefficients of $\Delta_{N_p}, \Delta_{u_r}$, and Δ_r, respectively, from (A7.6).

A8 Proof of Result 3.7

At E0, i.e. $\hat{N}_p = \hat{r} = \hat{u}_r = 0$, the Jacobian is

$$J^*_{E0} = \begin{pmatrix} \hat{W}-\delta & 0 & 0 \\ 0 & \frac{K}{\hat{W}} & \frac{\pi_C}{\hat{W}R} \\ 0 & 0 & 2 \end{pmatrix} \quad \text{(A8.1)}$$

where from (15) $\hat{W} = 1+R$. The eigenvalues are thus $1+R-\delta$, $\frac{K}{1+R}$, 2, so E0 is unstable.

However, consider the situation in which the predator population has depleted the CP, i.e. the system has $r=0$. Then along this null-cline, if $R<\delta$, then E0 is stable because from (15), $W = 1+R(1-L(u_r,0)) < 1+\delta$ and thus $N' < N$.

Now say we start from a point off the nullcline $r=0$, where N_p, u_r, and r are very small. Then from (16),

$$\frac{r'}{r} = \frac{2-\beta N_p P(r,u_r)}{1+r} \approx 2$$

so $r' > r$.

For the second part of the proof, we look at the Jacobian J^* from (A7.7) at some point alone the nullcline $r=0$ for $R<\delta$. Here,

$$J^*(r=0) = \begin{pmatrix} W-\delta & a & b \\ 0 & c & d \\ 0 & 0 & g \end{pmatrix} \quad \text{(A8.2)}$$

because $e=f=0$. The eigenvalues are $W-\delta$, c, and g, where $W-\delta<1$ because $R<\delta$,

$$c = \frac{K(1+u_rR}{W} = \frac{K(1+u_rR)}{1+R(1-Ku_r)} \quad \text{(A8.3)}$$

is less than one if $R > -\frac{1-K}{1-2Ku_r}$, and

$$g = 2 - \beta N_p K u_r \tag{A8.4}$$

is less than one if $1 - \beta N_p K u_r < 0$.

A9 Proof of Result 3.8

At E1, i.e. $\hat{N}_p = 0$, $\hat{r} = 1$, and $\hat{u}_r$ is the larger root of (A6.1), the Jacobian is

$$J^*_{E1} = \begin{pmatrix} \hat{W} - \delta & 0 & 0 \\ 0 & \frac{K}{\hat{W}}(2 - \hat{u}_r(1-R)) & \frac{L(\hat{u}_r,1)}{\hat{W}}(1-\hat{u}_r) + \frac{\pi_C R(2-\hat{u}_r(1-R))}{\hat{W}(1+R)^2} \\ -\frac{1}{2}\beta L(\hat{u}_r, 1) & 0 & \frac{1}{2} \end{pmatrix}$$

with eigenvalues $\lambda_1 = \hat{W} - \delta, \lambda_2 = \frac{K}{\hat{W}}(2 - \hat{u}_r(1-R))$, and $\lambda_3 = 1/2$. From (15),

$$\hat{W} = 1 + R + (1-R)L(\hat{u}_r, 1)$$

so $\lambda_1 < 1$ if $R - \delta + (1-R)L(\hat{u}_r, 1) < 0$. Furthermore, $\lambda_2 < 1$ if

$$K(2 - \hat{u}_r(1-R)) < 1 + R + (1-R)L(\hat{u}_r, 1).$$

which can be written as

$$2K(1 - \hat{u}_r(1-R)) < 1 + R + (1-R)\frac{\pi_C}{1+R}.$$

A10 Derivation of Internal Stability Jacobian with Time Delay

The equations for Δ'_{u_r} and Δ'_{N_p} are the same, but equation (A7.6) changes. Instead,

$$\hat{r} + \Delta'_r \approx \left(\hat{r} - \frac{\hat{r}\beta(\Delta_{N_p}\hat{u}_r + \hat{N}_p\Delta_{u_r})}{1+\hat{r}} + \Delta_r \frac{2 - \beta\hat{N}_p\hat{u}_r}{1+\hat{r}}\right)\left(1 - \frac{\Delta_r}{1+\hat{r}}\right)$$

which simplifies to,

$$\Delta'_r \approx \Delta_{N_p}\left(-\frac{\hat{r}\beta\hat{u}_r}{1+\hat{r}}\right) + \Delta_{u_r}\left(-\frac{\hat{r}\beta\hat{N}_p}{1+\hat{r}}\right) + \Delta_r\left(\frac{2 - \beta\hat{N}_p\hat{u}_r - \hat{r}}{1+\hat{r}}\right) \tag{A10.1}$$

The Jacobian formed by Eqs (A7.4), (A7.5), and (A10.1), is of the form

$$J^* = \begin{pmatrix} \hat{W} - \delta & a & b \\ 0 & c & d \\ e & f & g, \end{pmatrix} \tag{A10.2}$$

where we redefine a, b, c, d, e, f, and g to be

$$a = \hat{N}_p K, \qquad b = \hat{N}_p \left[L(\hat{u}_r, \hat{r}) + \frac{R\pi_C(\hat{r} - R)}{(\hat{r} + R)^2} \right], \tag{A10.3}$$

$$c = \frac{K}{\hat{W}} \left(1 + \hat{r} - \hat{u}_r(\hat{r} - R)\right), \tag{A10.4}$$

$$d = \frac{L(\hat{u}_r, \hat{r})}{\hat{W}}(1 - \hat{u}_r) + \frac{\pi_C R \left[1 + \hat{r} - \hat{u}_r(\hat{r} - R)\right]}{\hat{W}(\hat{r} + R)^2}, \tag{A10.5}$$

$$e = \frac{-\hat{r}\beta\hat{u}_r}{1 + \hat{r}} \qquad f = \frac{-\beta\hat{r}\hat{N}_p}{1 + \hat{r}}, \tag{A10.6}$$

$$g = \left(\frac{2 - \beta\hat{N}_p\hat{u}_r - \hat{r}}{1 + \hat{r}} \right). \tag{A10.7}$$

For equilibrium E2, if $\hat{N}_p > 0$, then $\hat{N}_p$ is defined in (A2.2), $\hat{u}_r$ is (A3.1), $\hat{r}$ is the solution to (20), and $\hat{W} = 1 + \delta$. The eigenvalues are defined using equations (27) and (28), where we substitute the new definitions of a, b, c, d, e, f, and g from (A10.3) into (27).

A11 External Stability Jacobian

Near the equilibrium $\hat{N}_p, \hat{u}_r, \hat{r}$,

$$\Delta'_{x_r} \approx \frac{1}{\hat{W}} \Delta_{x_r} L_x(\hat{u}_r, \hat{r})(1 + \hat{r}) \tag{A11.1a}$$

$$\Delta'_{x_R} \approx \frac{1}{\hat{W}} \Delta_{x_R} (1 - L_x(\hat{u}_r, \hat{r}))(1 + R) \tag{A11.1b}$$

so that the Jacobian determining local stability is of the form

$$J_x = \begin{pmatrix} a_x & a_x \\ b_x & b_x \end{pmatrix} \tag{A11.2}$$

for $a_x = \frac{1}{\hat{W}} L_x(\hat{u}_r, \hat{r})(1 + \hat{r})$ and $b_x = \frac{1}{\hat{W}}(1 - L_x(\hat{u}_r, \hat{r}))(1 + R)$. There is one nonzero eigenvalue for this matrix, and it is

$$\lambda_x = a_x + b_x = \frac{1}{\hat{W}} \left[1 + R + L_x(\hat{u}_r, \hat{r})(\hat{r} - R)\right] \tag{A11.3}$$

which simplifies to

$$\lambda_x = 1 + \frac{1}{\hat{W}} \left(\delta_K \hat{u}_r + \delta_{\pi_C} \frac{\hat{r}}{\hat{r} + R} \right) (\hat{r} - R). \tag{A11.4}$$