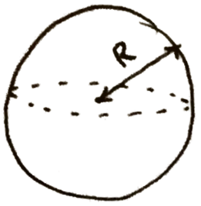
SM Statistical Mechanics in the Sky | 2018 long Qu 2

Degenerate Fermi Gas of N neutrons of mass m_n

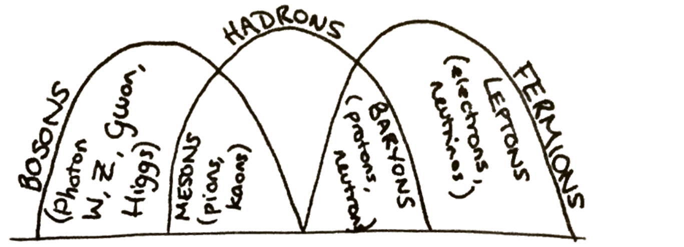


a) Fermi Energy E_F, assuming neutrons are non-relativistic:

$$E = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$$

$$E = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

$n = 1$ fermions



Fermi energy: energy difference between the highest & lowest occupied single particle states in a quantum system of non-interacting fermions at absolute zero temperature → Fermi gas: the lowest occupied state is taken to have zero K.E. (but in a metal the lowest occupied state is typically the bottom of the conduction band).

$$E_F = \frac{\hbar^2}{2m_0}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$ for non-interacting ensemble of identical fermions spin-½ in a 3D non-relativistic system.

Fermi temperature $T_F = E_F / k_B$ Can easily derive!

$$V = \frac{M}{\rho} \qquad V = \frac{4}{3}\pi R^3$$

$$E_F = \frac{\hbar^2}{2m_n}\left(\frac{\pi N}{4R^3}\right)^{2/3}$$ Derive - long - CBA to do again.

b) Total Kinetic Energy of Star: $U = \frac{3}{5} N E_F$

$\cancel{\frac{1}{2}mv^2 = \frac{3}{2}k_B T}$

$\cancel{U = \frac{3}{2} N k_B T}$ $\cancel{T_F = \frac{E_F}{k_B}}$ $\cancel{U = \frac{3}{2} N k_B T_F = \frac{3 N E_F}{2}}$ No!

$$U_k = 2\iiint E_F \, d^3 n = 2\int_0^{n_F}\int_0^{\pi/2}\int_0^{\pi/2} \frac{\hbar^2}{2m_n}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = 2\iiint \frac{h^2}{8mV^{2/3}} n^2 n^2 \sin\theta \, dn \, d\theta \, d\phi$$

to account for spin ↑ & ↓ for each state

limits due to angle of octant

$$= \frac{\pi h^2}{8mV^{2/3}}\frac{n_F^5}{5} = \left(\frac{h^2}{8mV^{2/3}} n_F^2\right)\left(\frac{\pi}{3} n_F^3\right)\left(\frac{3}{5}\right)$$

$$U_K = \frac{3}{5} N E_F$$

c) $U_{grav} = -\frac{3}{5}\frac{GM^2}{R}$

$$U_T = \frac{3}{5}\left(NE_F - \frac{GM^2}{R}\right) = \frac{3}{5} N \frac{\hbar^2}{2m}\left(\frac{\pi N}{4R^3}\right)^{2/3} - \frac{GM}{R}$$

$$\frac{\partial U}{\partial R} = 0 = \frac{3}{5}\frac{N\hbar^2}{2m}\left(\frac{\pi N}{4}\right)^{2/3}\frac{\partial (R^3)^{-2/3}}{\partial R} - GM\frac{\partial}{\partial R}\left(\frac{1}{R}\right)$$

$$\frac{\partial U}{\partial R} = \frac{3}{5}\frac{N\hbar^2}{2m}\left(\frac{N\pi}{4}\right)^{2/3}(-2)R^{-3} + \frac{GM}{R^2} = 0$$

$$\frac{GM}{R^2} = 2\cdot\frac{3}{5}\frac{N\hbar^2}{2m}\left(\frac{N\pi}{4}\right)^{2/3}\frac{1}{R^3}$$

$$R = \frac{3}{5}\frac{N\hbar^2}{GMm}\left(\frac{N\pi}{4}\right)^{2/3}$$

$$M = Nm_n \rightarrow N = \frac{M}{m_n}$$

$$= \frac{3}{5}\frac{\hbar^2}{Gm_n^2}\left(\frac{N\pi}{4}\right)^{2/3}$$

$$= \frac{3}{5}\frac{\hbar^2}{Gm_n^2}\left(\frac{M\pi}{m_n 4}\right)^{2/3}$$