2018 QM: Three spins in Hilbertland.

Notes: $\hat{S} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$

$\hat{S}^2 = (\hat{S}_{1x} + \hat{S}_{2x} + \hat{S}_{3x})^2 + (\hat{S}_{1y} + \hat{S}_{2y} + \hat{S}_{3y})^2 + (\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z})^2$

$\hat{S}^2 |s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$

$\hat{S}_z |s,m\rangle = (\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z})|s,m\rangle = \hbar m |s,m\rangle$

$|s_1 - s_2 - s_3| \leq s \leq s_1 + s_2 + s_3$

$m = -s, -s+1, \dots, s-1, s$

a) Basis:

$|1\rangle = |+\rangle_1 |+\rangle_2 |+\rangle_3$
$|2\rangle = |+\rangle|+\rangle|-\rangle$
$|3\rangle = |+\rangle|-\rangle|+\rangle$
$|4\rangle = |+\rangle|-\rangle|-\rangle$
$|5\rangle = |-\rangle|+\rangle|+\rangle$
$|6\rangle = |-\rangle|+\rangle|-\rangle$
$|7\rangle = |-\rangle|-\rangle|+\rangle$
$|8\rangle = |-\rangle|-\rangle|-\rangle$

Determine state $|3/2, -1/2\rangle \rightarrow s = 3/2,\ m = -1/2$.
Recall that for 2 spin-1/2 particles, we had:

triplet: $s=1$
- $|+\rangle|+\rangle$
- $|+\rangle|-\rangle + |-\rangle|+\rangle$
- $|-\rangle|-\rangle$

singlet $\longrightarrow |+\rangle|-\rangle - |-\rangle|+\rangle$

We can treat the 3 spin-1/2 as 1 spin-1/2 + 2 spin-1/2:
The triplet state then becomes:

"triplet" $s = 3/2$
- $|+\rangle(|+\rangle|+\rangle) = |+\rangle|+\rangle|+\rangle \quad m = 3/2$
- $|-\rangle(|+\rangle|+\rangle) = |-\rangle|+\rangle|+\rangle \quad m = 1/2$
- $|+\rangle(|+\rangle|-\rangle + |-\rangle|+\rangle) = |+\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|+\rangle \quad m = \frac{1}{2}$
- $|-\rangle(|+\rangle|-\rangle + |-\rangle|+\rangle) = |-\rangle|+\rangle|-\rangle + |-\rangle|-\rangle|+\rangle \quad m = -\frac{1}{2}$
- $|+\rangle(|-\rangle|-\rangle) = |+\rangle|-\rangle|-\rangle \quad m = -1/2$
- $|-\rangle(|-\rangle|-\rangle) = |-\rangle|-\rangle|-\rangle \quad m = -3/2$.

and the singlet becomes (?)

$m=\frac{1}{2}$

$S=\frac{1}{2} \rightarrow |+\rangle(|+\rangle|-\rangle - |-\rangle|+\rangle) = |+\rangle|+\rangle|-\rangle - |+\rangle|-\rangle|+\rangle$

$|-\rangle(|+\rangle|-\rangle - |-\rangle|+\rangle) = |-\rangle|+\rangle|-\rangle - |-\rangle|+\rangle|+\rangle$

$m=-\frac{1}{2}$

Again, $|3/2, -1/2\rangle \rightarrow m=-1/2$: there are 2 states derived from the triplet that satisfies that condition:

$|-\rangle|+\rangle|-\rangle + |-\rangle|-\rangle|+\rangle$ and $|+\rangle|-\rangle|-\rangle$.

$$\therefore \boxed{|3/2, -1/2\rangle = \frac{1}{\sqrt{3}}\left(\underbrace{|-\rangle|+\rangle|-\rangle}_{|6\rangle} + \underbrace{|-\rangle|-\rangle|+\rangle}_{|7\rangle} + \underbrace{|+\rangle|-\rangle|-\rangle}_{|4\rangle}\right)}$$

b) $\hat{H} = \frac{2A}{\hbar^2}\left(\hat{S}_1\cdot\hat{S}_2 + \hat{S}_2\cdot\hat{S}_3 + \hat{S}_3\cdot\hat{S}_1\right)$

Note $\hat{S}_1 = (\hat{S}_{1x}, \hat{S}_{1y}, \hat{S}_{1z})$, $\hat{S}_{1x} = \frac{1}{2}(\hat{S}_{1+} + \hat{S}_{1-})$, $\hat{S}_{1y} = \frac{-i}{2}(\hat{S}_{1+} - \hat{S}_{1-})$

Also,

$$2\hat{S}_1\cdot\hat{S}_2 = 2\hat{S}_{1z}\cdot\hat{S}_{2z} + 2(\hat{S}_{1x}\cdot\hat{S}_{2x} + \hat{S}_{1y}\cdot\hat{S}_{2y})$$
$$= 2\hat{S}_{1z}\cdot\hat{S}_{2z} + 2\left[\left(\frac{1}{4}\right)(\hat{S}_{1+}+\hat{S}_{1-})\cdot(\hat{S}_{2+}+\hat{S}_{2-}) - \left(\frac{1}{4}\right)(\hat{S}_{1+}-\hat{S}_{1-})(\hat{S}_{2+}-\hat{S}_{2-})\right]$$
$$= 2\hat{S}_{1z}\hat{S}_{2z} + \frac{1}{2}\left[2\hat{S}_{1+}\hat{S}_{2-} + 2\hat{S}_{1-}\hat{S}_{2+}\right]$$

$$2\hat{S}_1\cdot\hat{S}_2 = 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

similarly, $2\hat{S}_2\cdot\hat{S}_3 = 2\hat{S}_{2z}\hat{S}_{3z} + \hat{S}_{2+}\hat{S}_{3-} + \hat{S}_{2-}\hat{S}_{3+}$

and $2\hat{S}_3\cdot\hat{S}_1 = 2\hat{S}_{3z}\hat{S}_{1z} + \hat{S}_{3+}\hat{S}_{1-} + \hat{S}_{3-}\hat{S}_{1+}$

Let's break the Hamiltonian in 3 Hamiltonians.

$$\hat{H} = \frac{A}{\hbar^2}\left[\hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}\right]$$

(3)

$$\hat{H}_{12} = 2\hat{S}_1 \cdot \hat{S}_2.$$

$$\hat{H}_{12} = \begin{array}{c} \begin{array}{cccccccc} |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle & |6\rangle & |7\rangle & |8\rangle \end{array} \\ \begin{pmatrix} \langle 1|\hat{H}_{12}|1\rangle & & & & & & & \\ \langle 2|\hat{H}_{12}|1\rangle & \ddots & & & & & & \\ \vdots & & \ddots & & & & & \\ \langle 8|\hat{H}_{12}|1\rangle & & & & & & & \langle 8|\hat{H}_{12}|8\rangle \end{pmatrix} \end{array} \begin{array}{c} \langle 1| \\ \langle 2| \\ \langle 3| \\ \langle 4| \\ \langle 5| \\ \langle 6| \\ \langle 7| \\ \langle 8| \end{array}$$

$$\hat{H}_{12}|1\rangle = \left(2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)|+\rangle_1|+\rangle_2|+\rangle_3$$

$$= \left[2\cdot\frac{1}{2}\cdot\frac{1}{2}\hbar^2\right]|1\rangle = \frac{1}{2}|1\rangle$$

$$\boxed{\to \hat{H}_{12}|1\rangle = \frac{1}{2}|1\rangle\hbar^2}$$

$$\hat{H}_{12}|2\rangle = \left(2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)|+\rangle|+\rangle|-\rangle \quad *$$

$$= \left(2\cdot\frac{1}{2}\cdot\frac{1}{2}\hbar^2\right)|+\rangle|+\rangle|-\rangle = \frac{1}{2}|2\rangle$$

$$\boxed{\to \hat{H}|2\rangle = \frac{1}{2}|2\rangle\hbar^2}$$

$$\hat{H}_{12}|3\rangle = \left(\quad * \quad \right)|+\rangle|-\rangle|+\rangle.$$

$$= \hbar^2\left(2\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\right)|3\rangle + \hat{S}_{1-}\hat{S}_{2+}\underbrace{|+\rangle}_{A}\underbrace{|-\rangle}_{B}|+\rangle$$

(A) $\hat{S}_{1-}|+\rangle = \sqrt{s(s+1) - m(m-1)}\,|-\rangle$
$= \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\,|-\rangle = |-\rangle$

(B) $\hat{S}_{2+}|-\rangle = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}\,|+\rangle = |+\rangle$

General: $\hat{S}_+|-\rangle = |+\rangle \quad ; \quad \hat{S}_-|+\rangle = |-\rangle$

$$\hat{H}_{12}|3\rangle = -\frac{1}{2}|3\rangle + |-\rangle|+\rangle|+\rangle$$

$$\boxed{\hat{H}_{12}|3\rangle = -\frac{1}{2}|3\rangle + |5\rangle}$$

$$\star = 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

$$\hat{H}_{12}|4\rangle = (\quad \star \quad)|+\rangle|-\rangle|-\rangle$$
$$= \left(2\cdot\frac{1}{2}\left(\frac{-1}{2}\right)\right)|+\rangle|-\rangle|-\rangle + \hat{S}_{1-}\hat{S}_{2+}|+\rangle|-\rangle|-\rangle$$
$$= -\frac{1}{2}|4\rangle + |-\rangle|+\rangle|-\rangle$$
$$\boxed{\hat{H}_{12}|4\rangle = -\frac{1}{2}|4\rangle + |6\rangle}$$

$$\hat{H}_{12}|5\rangle = \left(2\cdot\frac{-1}{2}\cdot\frac{1}{2}\right)|-\rangle|+\rangle|+\rangle + |+\rangle|-\rangle|+\rangle$$
$$\boxed{\hat{H}_{12}|5\rangle = -\frac{1}{2}|5\rangle + |3\rangle}$$

$$\hat{H}_{12}|6\rangle = \left(2\cdot\frac{-1}{2}\cdot\frac{1}{2}\right)|-\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|-\rangle$$
$$\boxed{\hat{H}_{12}|6\rangle = -\frac{1}{2}|6\rangle + |4\rangle}$$

$$\hat{H}_{12}|7\rangle = \left(2\cdot\frac{-1}{2}\cdot\frac{-1}{2}\right)|-\rangle|-\rangle|+\rangle + 0$$
$$\boxed{\hat{H}_{12}|7\rangle = \frac{1}{2}|7\rangle}$$
$$\boxed{\hat{H}_{12}|8\rangle = \frac{1}{2}|8\rangle}$$

Therefore,

$$\hat{H}_{12} = \frac{1}{2}\begin{array}{c} \\ \left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \end{array}\right. \end{array}$$

$$\hat{H}_{12} = \frac{1}{2}\begin{array}{cc} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

Summary for $\hat{H}_{12}$:

$\hat{H}_{12}|1\rangle = \frac{1}{2}|1\rangle \cdot \hbar^2$
$\hat{H}_{12}|2\rangle = \frac{1}{2}|2\rangle \cdot \hbar^2$
$\hat{H}_{12}|3\rangle = (-\frac{1}{2}|3\rangle + |5\rangle) \cdot \hbar^2$
$\hat{H}_{12}|4\rangle = (-\frac{1}{2}|4\rangle + |6\rangle) \hbar^2$
$\hat{H}_{12}|5\rangle = (-\frac{1}{2}|5\rangle + |3\rangle) \hbar^2$
$\hat{H}_{12}|6\rangle = (-\frac{1}{2}|6\rangle + |4\rangle) \hbar^2$
$\hat{H}_{12}|7\rangle = (\frac{1}{2}|7\rangle) \cdot \hbar^2$
$\hat{H}_{12}|8\rangle = \frac{1}{2}|8\rangle \cdot \hbar^2$

Now, let's look at $\hat{H}_{23}$.

$$\hat{H}_{23} = 2\hat{S}_2 \cdot \hat{S}_3 = 2\hat{S}_{2z}\hat{S}_{3z} + \hat{S}_{2+}\hat{S}_{3-} + \hat{S}_{2-}\hat{S}_{3+}$$

$\hat{H}_{23}|1\rangle = \frac{1}{2}|1\rangle \cdot \hbar^2$
$\hat{H}_{23}|2\rangle = (-\frac{1}{2}|2\rangle + |3\rangle) \cdot \hbar^2$
$\hat{H}_{23}|3\rangle = (-\frac{1}{2}|3\rangle + |2\rangle) \cdot \hbar^2$
$\hat{H}_{23}|4\rangle = \frac{1}{2}|4\rangle \cdot \hbar^2$
$\hat{H}_{23}|5\rangle = \frac{1}{2}|5\rangle \cdot \hbar^2$
$\hat{H}_{23}|6\rangle = (-\frac{1}{2}|6\rangle + |7\rangle) \hbar^2$
$\hat{H}_{23}|7\rangle = (-\frac{1}{2}|7\rangle + |6\rangle) \cdot \hbar^2$
$\hat{H}_{23}|8\rangle = \frac{1}{2}|8\rangle \cdot \hbar^2$

And $\hat{H}_{31} = 2\hat{S}_{3z}\hat{S}_{1z} + \hat{S}_{3+}\hat{S}_{1-} + \hat{S}_{3-}\hat{S}_{1+}$

$\hat{H}_{31}|1\rangle = \frac{1}{2}|1\rangle \cdot \hbar^2$
$\hat{H}_{31}|2\rangle = (-\frac{1}{2}|2\rangle + |5\rangle) \hbar^2$
$\hat{H}_{31}|3\rangle = (\frac{1}{2}|3\rangle) \hbar^2$
$\hat{H}_{31}|4\rangle = (-\frac{1}{2}|4\rangle + |7\rangle) \hbar^2$
$\hat{H}_{31}|5\rangle = (-\frac{1}{2}|5\rangle + |2\rangle) \hbar^2$
$\hat{H}_{31}|6\rangle = \frac{1}{2}|6\rangle \hbar^2$
$\hat{H}_{31}|7\rangle = (-\frac{1}{2}|7\rangle + |4\rangle) \hbar^2$
$\hat{H}_{31}|8\rangle = (+\frac{1}{2}|8\rangle) \hbar^2$

Now, we calculate $\hat{H}$:

$$\hat{H} = \frac{A}{\hbar^2} \cdot \left[\hat{H}_{12} + \hat{H}_{23} + \hat{H}_{37} \right]$$

$$\hat{H}|1\rangle = \frac{A}{\hbar^2} \cdot \frac{\hbar^2}{2} \left[3|1\rangle \right] = \frac{3}{2} A\, |1\rangle = \frac{A}{2}\left[3|1\rangle\right]$$

$$\hat{H}|2\rangle = \frac{A}{\hbar^2} \cdot \frac{\hbar^2}{2} \left[|2\rangle + |2\rangle + 2|3\rangle + |2\rangle + 2|5\rangle \right] = \frac{A}{2}\left[3|2\rangle + 2|3\rangle + 2|5\rangle\right]$$

$$\hat{H}|3\rangle = \frac{A}{2}\left[|3\rangle + 2|5\rangle + |3\rangle + 2|2\rangle + |3\rangle \right] = \frac{A}{2}\left[3|3\rangle + 2|2\rangle + 2|5\rangle\right]$$

$$\hat{H}|4\rangle = \frac{A}{2}\left[|4\rangle + 2|6\rangle + |4\rangle + |4\rangle + 2|7\rangle \right] = \frac{A}{2}\left[3|4\rangle + 2|6\rangle + 2|7\rangle\right]$$

$$\hat{H}|5\rangle = \frac{A}{2}\left[3|5\rangle + 2|3\rangle + 2|2\rangle \right]$$

$$\hat{H}|6\rangle = \frac{A}{2}\left[3|6\rangle + 2|4\rangle + 2|7\rangle \right]$$

$$\hat{H}|7\rangle = \frac{A}{2}\left[3|7\rangle + 2|4\rangle + 2|6\rangle \right]$$

$$\hat{H}|8\rangle = \frac{A}{2} \cdot 3|8\rangle$$

Therefore,

$$\hat{H} = \frac{A}{2} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$