PRELIMINARIES:Finite–dimensional product Hilbert spaces, denoted $H_{d_1,d_2}........_{,d_N} = C^{d_1} \otimes ....C^{d_n}$ or $H_{d^n} = C^d \otimes C^d....... \otimes C^d$. Subsystems are denoted $A_1, A_2, ..., A_n$=A in the general multipartite case or A, B, forsmaller systems. For pure states we use the traditional denotations: $|\Psi\rangle, |\Psi\rangle$ , often adding subscripts corresponding to respective (groups of) parties, e.g., $|\Psi\rangle_{ABC}$. We will use the standard basis for all the parties $|i\rangle_{i=0}^{d}$ and the kets will be written as row vectors.

Entanglement. An n–partite pure state$|\Psi\rangle_{A_1A_2.........A_n}$ is said to be fully product if it can be written as $|\Psi\rangle_{A_1A_2.........A_n} = |\Psi\rangle_{A_1} \otimes |\kappa\rangle_{A_2} ....... \otimes |\vartheta\rangle_n$ Otherwise it is called entangled. Among entangled states a particularly interesting class is constituted by genuinely multiparty entangled (GME)

states, i.e., those which cannot be written as$|\Psi\rangle_{A_1A_2.........A_n} = |\Psi\rangle_S \otimes |\kappa\rangle_{\bar{S}}$

for any bipartite cut (biaprtition)$S|\bar{S}$ where S is a subset of the parties

and $\bar{S} = A\backslash S$.In other words a GME state is not biproduct with respect to any bipartite cut of the parties. A canonical example of a GME state is the famous GHZ state$|GHZ\rangle = 1/\sqrt{2}(|000.....00\rangle + |111.......11\rangle)$. A state $|\Psi\rangle$ is called k– product if it is of the form $|\Psi_{\otimes^k}\rangle = |\Psi_1\rangle_{s_1} \otimes |\Psi_2\rangle_{s_2} \otimes$ ............... $\otimes |\Psi_k\rangle_{s_k}$ where $S_1 \cup S_2 \cup S_3........ \cup S_K = A$ is a k–partition. In the particular case k = n, the vector is fully product; when k = 2 it is biproduct.

Completely and genuinely entangled subspaces. It is a subspaces containing only entangled states, so called completely entagled subspaces (CESs). It has been shown that their maximal achievable dimension for $H_{d^n}$ is

$$D_{max}^{CES} = .d^n + nd + n - 1 = (d^{n-1} + d^{n-2} + ...... + 1 - n)(d - 1)$$

] Definition 1. A subspace $\varrho \subset H_{d_1,.........,d_n}$ is called a genuinely entangled subspace (GES) of $H_{d_1,.........,d_n}$ if any $|\Psi\rangle \subset \varrho$ is genuinely multiparty entangled (GME).To obtain the maximal available dimension of a GES one needs to consider maximal dimensions of all bipartite GESs and take the smallest among them. It is then easy to see that for

$$D_{max}^{GES} = (d^n - 1)(d - 1)$$

An example of a two dimensional GES of $H_{2^n}$ is given by the span of the already mentiond GHZ state and the W state,$|W\rangle = \frac{1}{\sqrt{n}}(|000.....01\rangle + |000.....010\rangle + |100.......00\rangle)$.

we have given few other constructions of GESs working in general multiparty scenarios attaining larger dimensions. In particular, one of these constructions gives a GES of dimension $d^{n-2}(d-1)$ . Let us recall it here, for simplicity considering$H_{3^3}$. Given is the set of vectors $(a \in C) : (1, \alpha + \alpha^3, \alpha^2 + \alpha^6) \otimes (1, \alpha^3, \alpha^6) \otimes (1, \alpha, \alpha^2)$.The subspace orthogonal to the span of these vectors is a twelve–dimensional GES. Choosing a set of twelve linearly independent vectors of the form above one obtains an example of a tripartite non–orthogonal unextendbile product basis.

**our consideration** In our case we consider multiple qubit Hilbert spaces, i.e.,$H_{2^3} := (C^2)^{\otimes 3}$

$B = \{|\Psi\rangle \,|\, |\Psi\rangle = (1, a + b\alpha + c\alpha^2)_A \otimes (1, A + B\alpha + C^2)_B \otimes (1, x + y\alpha + Z\alpha^2)_C \,|$

In our cas, the condition that B is a GES is equivalent to saying that it is void of any biproduct vectors, i.e., we require vectors of the for $|\Psi\rangle_S \otimes |\kappa\rangle_{\bar{S}}$, for any bipartition $S|\bar{S}$, not to belong to B. In other words, there can be no such vectors orthogonal to the subspace spanned by the vector in B (in any bipartate cut)

First of all,we move main detail of discusstion,give some direct calculation on it. B=$\{|\Psi\rangle \,|\, |\Psi\rangle = (1, a + b\alpha + c\alpha^2)_A \otimes (1, A + B\alpha + C^2)_B \otimes (1, x + y\alpha + Z\alpha^2$

$A|BC$

AB: $(1, a + b\alpha + c\alpha^2)_A \otimes (1, A + B\alpha + C^2)_B = (1, a + b\alpha + c\alpha^2, A + B\alpha + C^2, Aa + (Ba + Ab + Bc)\alpha + (Ca + Bb + Ac + Cc)\alpha^2 + (Cb + Bc)\alpha^3 + Cc\alpha^4)$

Assume it is coordinates with respect $\alpha$ linearly independent polynomials have dimension is 4. (depending on spanning property)

C:$(1, x + y\alpha + Z\alpha^2)$ linearly independt polynomial dimestion is 2.(depending on spanning property)

$A|BC$

BC:$(1, A+B\alpha+C^2)\otimes(1, x+y\alpha+Z\alpha^2) = (1, A+B\alpha+C^2, x+y\alpha+Z\alpha^2, XA+(YA+XB+YC)\alpha+(ZA+YB+XC+ZC)\alpha^2+(ZB+YC)\alpha^3+Zc\alpha^4)$ dim is 4

A:$(1, a+b\alpha+c\alpha^2)$ dim is 2.

Second,let $\bar{B}$ be a subspace whose orthogonal to spane of B.and $\bar{B}$ span by $|\xi,\gamma\rangle = |\xi\rangle_S \otimes |\gamma\rangle_{\bar{S}} = (|\xi\rangle_0)_S \otimes (|\gamma\rangle_0, |\gamma\rangle_1)_{\bar{S}}$ we could easly say that ,the tesnodr product of the coorosponding pair vecors in these two set are equal to zero. between two sets have bijective function relation.let say F:$(C^2)^2 \longrightarrow (C^2)^2$,or taking it a step further,we could say that F is a transformation.

Definition: A linear transformation T : v$\longrightarrow v$is said to be non-singular if T(v) = 0 $\Rightarrow v = 0 i.e. N(T) = 0$

Definition: A linear transformation T : V is said to be singular if some $\exists v \epsilon V s.t. v \neq 0$ *and* $T(v) = 0$ i.e. N(T) contains at least one-zero element.

Definition:A linear transformation is an isomorphism if it is one-one and onto. i.e. T : $V \longrightarrow W$is an isomorphism if

(1) T is linear transformation.

(2) T is one-one.

(3) T is onto.

Then V and W are called isomorphic.

We write V $\cong$ W

THEOREM: V $\cong$ W $\Leftrightarrow$dim V = dim W cooroseponding proof in silde in attachment. from the definition of linear transformation ,we could say that

F:$(C^2)^2 \longrightarrow (C^2)^2$ could generate a Matrix F.and matrix F is consist of

$A, B, C$ if we choose a $S|\bar{S}$ bipartition,$S \cup \bar{S} = A$ ,where local dimention of S and $\bar{S}$ are 2,4 respectively. the question will rise in here,which what about the order of bipartition cut ?

the answer coming from Galois theory by Harlod .M.Edwards.page 47

-55.content:Basic Galois Theory: The Galois Group (permutaion of the roots of polynomilas)

for example we suppose S=$B$ and$\bar{S} = AC$ ,which mean we have a permutaion p(123)=213.it is easy to see that this order of permutaion is represent a unique individual bipartition. our main operation on our case staring from new.if we assume A biproduct vector $|\xi,\gamma\rangle = |\xi\rangle_S \otimes |\gamma\rangle_{\bar{S}} = (|\xi\rangle_0)_S \otimes (|\gamma\rangle_0, |\gamma\rangle_1)_{\bar{S}}$ belong to $\bar{B}$ ,it should satsify following eqaution: $\langle\zeta,\gamma|\,|\Psi\rangle = 0 \;\; \forall\alpha$ where $|\Psi\rangle \in B$

$A|BCcut$

$\langle\xi|_0\,|(1, a+b\alpha+c\alpha^2)\rangle\otimes\langle\gamma_0,\gamma|_1\,|(1, A+B\alpha+C\alpha^2),(1, X+Y\alpha+Z\alpha^2)\rangle = 0$

$\langle\zeta|_0\otimes(1, a+b\alpha+c\alpha^2)\otimes\langle\gamma_0,\gamma|_1\,((1, A+B\alpha+C\alpha^2),(1, X+Y\alpha+Z\alpha^2), AX+(BaX+AY+BZ)\alpha+(AZ+YB+XC+ZC)\alpha^2+(ZB+YC)\alpha^3+ZC\alpha^4) = 0$

After sorting:

$\langle\gamma_0,\gamma|_1 * (1+A+XA)+\langle\gamma_0,\gamma|_1 * (B+YA+XB+YC)\alpha+(\langle\gamma_0,\gamma|_1 * (Z+ZA+YB+XC+ZC)\alpha^2+\langle\gamma_0,\gamma|_1\,(ZB+YC)\alpha^3+\langle\gamma_0,\gamma|_1\,(ZC\alpha^4)) = 0$

since left side of all above equations is a ploynomila degree 4 in variable $\alpha$.if we consider must be hold ,then the coefficients are qual to zero.which means :

$\langle\gamma_0,\gamma|_1 * (1+A+XA) = 0$

$\langle\gamma_0,\gamma|_1 * (B+YA+XB+YC) = 0$

$(\langle\gamma_0,\gamma|_1 * (Z+ZA+YB+XC+ZC) = 0$

$\langle\gamma_0,\gamma|_1\,(ZB+YC) = 0$

$\langle\gamma_0,\gamma|_1\,(ZC) = 0$

$$E = \langle\gamma_1| \begin{bmatrix} 1+A+A+XA \\ B+YA+XB+Yc \\ Z+ZA+YB+XC+ZC \\ ZB+YC \\ ZC \end{bmatrix} \tag{1}$$

if we assume $|\gamma_0\rangle$ is a constant parameter,then every bipartition have 5 homogenous linear equations with 2 unknowns .and we could write these linear homogenous equation in matrix form.

$B|ACcut$

$\langle\xi|_0 |(1, A + B\alpha + C\alpha^2)\rangle\otimes\langle\gamma_0, \gamma|_1 |(1, a + b\alpha + c\alpha^2), (1, X + Y\alpha + Z\alpha^2)\rangle = 0$

$\langle\zeta|_0\otimes(1, A+B\alpha+C\alpha^2)\otimes\langle\gamma_0, \gamma|_1 ((1, a+b\alpha+c\alpha^2), (1, X+Y\alpha+Z\alpha^2), Xa+(Ya + Xb + Yc)\alpha + (Za + Yb + Xc + Zc)\alpha^2 + (Zb + Yc)\alpha^3 + Zc\alpha^4) = 0$

After sorting:

$\langle\gamma_0, \gamma|_1 * (1 + a + Xa) + \langle\gamma_0, \gamma|_1 * (b + Ya + Xb + Yc)\alpha + (\langle\gamma_0, \gamma|_1 * (Z + Za + Yb + Xc + Zc)\alpha^2 + \langle\gamma_0, \gamma|_1 (Zb + Yc)\alpha^3 + \langle\gamma_0, \gamma|_1 (Zc\alpha^4)) = 0$ since left side of all above equations is a ploynomila degree 4 in variable $\alpha$.if we consider must be hold ,then the coefficients are qual to zero.which means :

$\langle\gamma_0, \gamma|_1 * (1 + a + Xa) = 0$

$\langle\gamma_0, \gamma|_1 * (b + Ya + Xb + Yc) = 0$

$(\langle\gamma_0, \gamma|_1 * (Z + Za + Yb + Xc + Zc) = 0$

$\langle\gamma_0, \gamma|_1 (Zb + Yc) = 0$

$\langle\gamma_0, \gamma|_1 (Zc) = 0$

$$E = \langle\gamma_1| \begin{bmatrix} 1 + a + Xa \\ b + Ya + Xb + Yc \\ Z + Za + Yb + Xc + Zc \\ Zb + Cc \\ Zc \end{bmatrix} \quad (2)$$

if we assume $|\gamma_0\rangle$ is a constant parameter,then every bipartition have 5 homogenous linear equations with 2 unknowns .and we could write these linear homogenous equation in matrix form.

$C|ABcut$

$\langle\xi|_0 |(1, X + Y\alpha + Z\alpha^2)\rangle\otimes\langle\gamma_0, \gamma|_1 |(1, a + b\alpha + c\alpha^2), (1, A + B\alpha + C\alpha^2)\rangle = 0$

$\langle\zeta|_0\otimes(1, X+Y\alpha+Z\alpha^2)\otimes\langle\gamma_0, \gamma|_1 ((1, a+b\alpha+c\alpha^2), (1, A+B\alpha+C\alpha^2), Aa+(Ba + Ab + Bc)\alpha + (Ca + Bb + Ac + Cc)\alpha^2 + (Cb + Bc)\alpha^3 + Cc\alpha^4) = 0$

After sorting:

$\langle\gamma_0, \gamma|_1 * (1 + a + Aa) + \langle\gamma_0, \gamma|_1 * (b + Ba + Ab + Bc)\alpha + (\langle\gamma_0, \gamma|_1 * (C + Ca + BB + Ac + Cc)\alpha^2 + \langle\gamma_0, \gamma|_1 (Cbb + Bc)\alpha^3 + \langle\gamma_0, \gamma|_1 (Cc\alpha^4)) = 0$

since left side of all above equations is a ploynomila degree 4 in variable $\alpha$.if we consider must be hold ,then the coefficients are qual to zero.which means :

$\langle\gamma_0, \gamma|_1 * (1 + a + Aa) = 0$

$\langle\gamma_0, \gamma|_1 * (b + Ba + Ab + Bc) = 0$

$(\langle\gamma_0, \gamma|_1 * (C + Ca + BB + Ac + Cc) = 0$

$\langle\gamma_0, \gamma|_1 (Cbb + Bc) = 0$

$\langle\gamma_0, \gamma|_1 (Cc) = 0$

if we assume $|\gamma_0\rangle$ is a constant parameter,then every bipartition have 5 homogenous linear equations with 2 unknowns .and we could write these linear homogenous equation in matrix form.

$$E = \langle\gamma_1| \begin{bmatrix} b + Ba + Ab + Bc \\ C + Ca + Bb + Ac + Cc \\ Cb + Bc \\ Cc \end{bmatrix} \quad (3)$$

and this kind of matrix does not exit for satisfying $\langle\zeta, \gamma| |\Psi\rangle = 0 \ \forall\alpha$ where $|\Psi\rangle \in B$ and let $\bar{B}$ be a GES,it is requires that the system only has the trivial soultion.and it will happen when the matirx E of the sytem is full rank. r(E)=5,for all $\langle\gamma_1|$ is not same time equal to zero. this scenerio is same to in all bipartitaion cases. from above ,it could give an following theoroem

**Theorem:**Assume $\bar{B}$ is the subspace of $H_{2^3} \subset (C^2)^n$ orthogonal to spane of the vectors in B.then $\bar{B}$ is a GES,and dimention is 3 ,satisfying this condition the matrix E for any bipartitation have full rank with respect any $\gamma$.

If we extend this situation to in general cases,then we have the following theorem.

**Theorem:**Assume $\bar{B}$ is the subspace of $H_{2^n} \subset (C^2)^n$ orthogonal to spane of the vectors in B.then $\bar{B}$ is a GES,and dimention is $2^{n-1}-1$ ,satisfying this condition the matrix E for any bipartitation have full rank with respect any $\gamma$.

**EXAMPLE1.**Let the vectors spanning the subspace orthogonal to-GES are given by following:

$A|BC$

AB: $(1, a+b\alpha+c\alpha^2)_A \otimes (1, A+B\alpha+C\alpha^2)_B \ = (1, a+b\alpha+c\alpha^2, A+B\alpha+C^2, Aa+(Ba+Ab+Bc)\alpha+(Ca+Bb+Ac+Cc)\alpha^2+(Cb+Bc)\alpha^3+Cc\alpha^4)$ ,we could assign values for cofficients and rewrite it in the form :

$$(1,\alpha)_A \otimes (1,\alpha)_B(1,\alpha)_C \quad \forall \alpha \in C$$

$$(1,\alpha)_A \otimes (1,\alpha,\alpha,\alpha^2)_{BC} \quad \forall \alpha \in C$$

BC spane 3 dimentional vector spave,

$$(1,\alpha,\alpha,\alpha^2,\alpha,\alpha^2,\alpha^2,\alpha^3)_{ABC}$$

set of 4 linealrly independent polynomilays.which implies:

$$u = dimspanB$$

obiviusly,it is possible to chose u values of $\alpha$ so that the set

$$\bar{B} = \{|\Psi(\alpha_i)\rangle\}_{i=1}^4 \quad \forall \alpha \in C$$

fix vectors of $\alpha$

$$(1,\alpha_1)_A \otimes (1,\alpha_1)_B \otimes (1,\alpha_1)_C$$

$$(1,\alpha_2)_A \otimes (1,\alpha_2)_B \otimes (1,\alpha_2)_C$$

$$(1,\alpha_3)_A \otimes (1,\alpha_3)_B \otimes (1,\alpha_3)_C$$

$$(1,\alpha_4)_A \otimes (1,\alpha_4)_B \otimes (1,\alpha_4)_C$$

we need to check is there a vector in any bipartation orthogonal to all the vectors above.

$B|AC$ bipartition

AC:$(1,\alpha,\alpha,\alpha^2)$ spane 3 dimenstion vector space

$(1,\alpha,\alpha,\alpha^2,\alpha,\alpha^2,\alpha^2,\alpha^3)_{ABC}$ set of 4 linearly independent polynomilas.

$$(1,0)_A \otimes (1,0)_B \otimes (1,0)_C = (1,\alpha)_A \otimes (1,0,0,1)_{BC}$$

$$(1,1)_A \otimes (1,1)_A \otimes (1,1)_C = (1,1)_B \otimes (1,1,1,1)_{BC}$$

$$(1,-1)_A \otimes (1,-1)_A \otimes (1,-1)_C = (1,-1)_B \otimes (1,-1,-1,1)_{BC}$$

$$(1,2)_A \otimes (1,2)_A \otimes (1,2)_C = (1,2)_B \otimes (1,2,2,4)_{BC}$$

It is not hurd to see that the vector $(2,-1)_B \otimes (0,1,-1,0)_{AC}$ orthogonal to all above vectors.

when we give some specific value for $\alpha$,and the dimention of otrhocomplement of span$\bar{B}$ are diffrent value.and the diffrent possiblit of value for cofficients in cordinates is $3^9$