Lecture 2: Probability

STATS 101: Foundations of Statistics

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Announcements

- 30 students submitted homework
- New course textbook: *DeGroot & Schervish*
- Next assignment will be posted tonight (due 12/4 @ 11:59pm)
- Re: prizes for top 3 students
  - Amazon gift cards
  - Extra points are awarded for:
    1. Class participation (e.g. asking/answering questions, etc)
    2. Catching & correcting errata/typos
    3. Answering questions / participating in discussions on Piazza
  - Blinded top 3 scores will be posted to course website
- No class next week (Happy Thanksgiving!)

Outline

All things linear algebra

- Sample space
- Probability function
- Probability space
- Random variables

Pre-requisites

Warning: I am assuming

- Fluency with algebra, calculus
- Familiarity with linear algebra
- Comfort with mathematical notation

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Warning: This lecture pace is fast.

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More notation:

- $\emptyset$ is the *empty set*. Can be denoted as $\emptyset = \{\}$.
- $\cup_{i=1}^{\infty} B_i$ is the union of sets B_i. Formally,
  - $\cup_{i=1}^{\infty} B_i = \{s \in S : s \in B_i \forall\, i$
- $B \subseteq S$ means B is a *subset* of the sample space.
- *Heads*, without curly braces, is an *element* of set B.
- $B^C = S \setminus B$ is the complement of set B

Probability function

A *probability function* is a function $P : \mathcal{B} \to [0, 1]$, where

- $P(S) = 1$
- $P\left(\cup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$ when $B_1, B_2, \ldots$ are disjoint

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Example: For flipping a coin, we have

$$\mathcal{B} = 2^S = \{\emptyset, \{Heads\}, \{Tails\}, \{Heads, Tails\}\} \tag{2}$$

This implies that

$$P(B) = \begin{cases} 1 & B = \{Heads, Tails\} \\ \frac{1}{2} & B = \{Heads\} \\ \frac{1}{2} & B = \{Tails\} \\ 0 & B = \emptyset \end{cases} \tag{3}$$

n.b. The power set is a *'set of sets'*

Probability function domains

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- $B \in \mathcal{B} \Rightarrow B^C \in \mathcal{B}$
- $B_1, B_2, \ldots \in \mathcal{B} \Rightarrow \cup_{i=1}^{\infty} B_i \in \mathcal{B}$

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Example:

- The *discrete* σ-algebra:
  $\mathcal{B} = 2^S = \{\emptyset, \{Heads\}, \{Tails\}, \{Heads, Tails\}\}$
- The *trivial* σ-algebra: $\mathcal{B} = \emptyset \cup S = \{\emptyset, \{Heads, Tails\}\}$

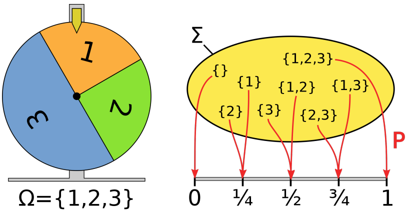
n.b. For uncountable sets, we use the *Borel* σ-algebra.

Probability space

Def:
A *probability space* is a triple $(S, \mathcal{B}, P)$.

- S is the set of possible singleton events
- $\mathcal{B}$ is the set of questions to ask P
- P maps sets into probabilities

n.b. They represent the ingredients needed to talk about probabilities



Probability functions

Some properties of $P(\cdot)$

- $P(B) = 1 - P(B^C)$
- $P(\emptyset) = 0$, since $P(\emptyset) = 1 - P(S)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, implying that
  - $P(A \cup B) \leq P(A) + P(B)$
  - $P(A \cap B) \geq P(A) + P(B) - 1$

Conditional probability

For events A and B where $P(B) > 0$, the *conditional probability* of A given B (denoted $P(A|B)$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{4}$$

Example: In an agricultural region with 1000 farms, we want to know if the farm has vineyards or cork trees.

		Cork Trees	
		Yes	No
Vineyard	Yes	200	50
	No	150	600

Table: Frequency counts

Conditional probability

Example: In an agricultural region with 1000 farms, we want to know if the farm has vineyards or cork trees.

		Cork Trees	
		Yes	No
Vineyard	Yes	20%	5%
	No	15%	60%

Table: Joint probabilities

Questions:

- What is the probability of seeing cork trees in a farm with vineyards?
- Among farms with cork trees or vineyards, what is the probability of having both?

Conditional probability

Let's assume the following joint probabilties

		Cork Trees	
		Yes	No
Vineyard	Yes	25%	25%
	No	25%	25%

We have that $P(A \cap B) = P(A) \cdot P(B)$, meaning that they are *independent*

Law of total probability

Let $B_1, B_2, \ldots B_k \in \mathcal{B}$ and $P(B_i) > 0 : i = 1, \ldots, k$. The *law of total probability* states that

$$P(A) - \sum_{i=1}^{k} P(B_i)P(A|B_i) \tag{5}$$

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The *conditional law of total probability* states that

$$P(A|C) - \sum_{i=1}^{k} P(B_i|C) P(A|B_i, C) \tag{6}$$

Bayes' Theorem

Let $B_1, B_2, \dots B_k \in \mathcal{B}$, $P(B_i) > 0 : i = 1, \dots, k$, and $P(A) > 0$. Then Bayes' Theorem states that for $i = 1, \dots, k$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)} \tag{7}$$

n.b. Can be proven using the def of conditional probability

Bayes' Theorem

Example: You take a test for disease X, which has 90% sensitivity and a FPR of 10%. Past genetic screening has indicated that you have a 1 in $10,000$ chance of having the disease. What is the probability of having disease X?

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$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \quad (8) \\ &= \frac{(0.9)(0.0001)}{(0.9)(0.0001) + (0.1)(0.9999)} = 0.0009 \quad (9) \end{aligned}$$

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Notes:

- $P(B_1)$ is often referred to as the *prior* probability
- $P(B_1|A)$ is often referred to as the *posterior* probability

Random variables

A *random variable* is a (Borel measureable) function
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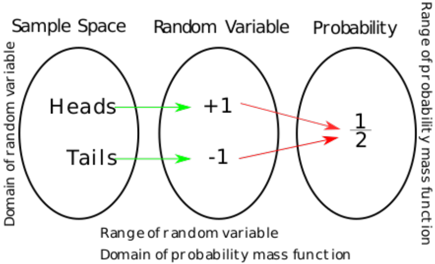
$$X(s) = \begin{cases} 1 & \text{if } s = Heads \\ 0 & \text{if } s = Tails \end{cases} \tag{10}$$



Cumulative distribution function

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we have

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \tag{12}$$

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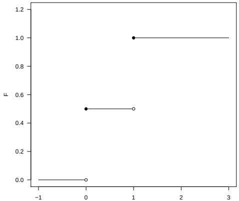

Cumulative distribution function

n.b. We have two ways of thinking about probabilities:

1. Probability functions
2. Cumulative distribution functions

Question: Which one should we use?

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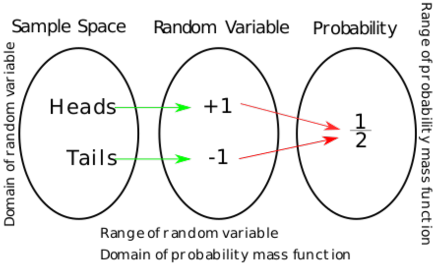
The Correspondence Theorem: Let $P_X(\cdot)$ and $P_Y(\cdot)$ be probability functions and $F_X(\cdot)$ and $F_Y(\cdot)$ be their associated cdfs. Then

$$P_X(\cdot) = P_Y(\cdot) \iff F_X(\cdot) = F_Y(\cdot) \tag{13}$$

Cumulative distribution function

Some properties for cdfs:

- $\lim_{x \Rightarrow -\infty} F(x) = 0$
- $\lim_{x \Rightarrow \infty} F(x) = 1$
- $F(\cdot)$ is non-decreasing
- $F(\cdot)$ is right-continuous



Quantile function

Let X be a continuous rv and one-to-one over the the possible values of X. Then

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : p \leq F(x)\} \tag{14}$$

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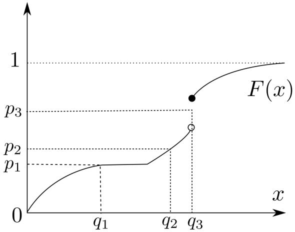
Is the quantile function of X. Let X be a *discrete* rv and one-to-one over the the possible values of X. Then $F^{-1}(p)$ states that we take the smallest value of x.

Example:



Nature of random variables

A random variable X is

- *Discrete* if $\exists f_X : \mathbb{R} \to [0,1] \ni F_X(x) = \sum_{t \leq x} f_X(t), x \in \mathbb{R}$
  - f_X is referred to as the probability mass function (pmf)
- *Continuous* if $\exists f_X : \mathbb{R} \to \mathbb{R}_+ \ni F_X(x) = \int_{-\infty}^{x} f_X(t)dt, x \in \mathbb{R}$
  - f_X is referred to as the probability density function (pdf).
  - n.b. We can have multiple pdf's consistent with the same cdf.
  - n.b. For any specific value of a continuous random variable, its probability is 0, i.e. $P(\{x\}) = 0\, \forall x \in \mathbb{R}$.

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n.b. pmf's and pdf's sum to 1, i.e.

- $f : \mathbb{R} \to [0,1]$ is the pmf of a discrete RV iff $\sum_{x \in \mathbb{R}} f(x) = 1$
- $f : \mathbb{R} \to \mathbb{R}_+$ is the pdf of a continuous RV iff $\int_{-\infty}^{\infty} f(x)dx = 1$

Nature of random variables

Example #1: Coin tossing

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \tag{15}$$

Here, F_X is a step function with pmf

$$f_X(x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Example #2: Uniform distribution on (0,1)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \tag{17}$$

Here, F_X is a continuous function. Two consistent pdfs include

$$f_X(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

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Transformations of random variables

Suppose $Y = g(X)$, where $g : \mathbb{R} \to \mathbb{R}$ and X is a *discrete* rv with cdf F_X.

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Since the function is applied to a rv, Y is also a random variable with probability function

$$f_Y(y) = P_Y(g(X) = y) = \sum_{x:g(x)=y} f_X(x) \tag{20}$$

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Example:

Let X be a uniform random variable on $\{-n, -n+1, ..., n-1, n\}$. Then $Y = |X|$ has mass function

$$f_Y(y) = \begin{cases} \frac{1}{2n+1} & \text{if } x = 0 \\ \frac{2}{2n+1} & \text{if } x \neq 0 \end{cases} \tag{21}$$

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$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int x : g(x) \leq y f_X(x) dx \tag{22}$$

We can get the probability function by taking the derivative

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Example:

Let X be a uniform rv on $[-1, 1]$. Then $Y = X^2$ has cdf

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) = P_X(X^2 \leq y) = P_X(-y^{1/2} X \leq y^{1/2}) \\ &= \int_{-y^{1/2}}^{y^{1/2}} f(x) dx = y^{1/2} \end{aligned} \tag{24}$$

and $f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{1}{2y^{1/2}}$

Affine transformations

Suppose $Y = g(X) = aX + b, a > 0, b \in \mathbb{R}$. Then

$$P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) \tag{25}$$

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If $a < 0$, then

$$P(Y \leq y) = P(aX+b \leq y) = P\left(X \geq \frac{y-b}{a}\right) = 1-F_X\left(\frac{y-b}{a}\right) \tag{26}$$

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In general, as long as the transformation $Y = g(X)$ is monotonic, then

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{\partial}{\partial y}g^{-1}(y)\right| \tag{27}$$

References

- Grinstead & Snell Chapters 1,2,4
- DeGroot & Schervish Chapters 1,2,3