Talis Biomedical Statistics Course - Homework 3

Due: 11 December 2019 11:59 PM

Name: [your first and last name]
Collaborators: [list all the people you worked with]
Date: [date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

Linear algebra

Problem 1

When is it true? Fill in each blank with '*always*', '*sometimes*', or '*never*'. Justify your choice.

(a) A nonsingular matrix is always invertible.

This is by definition.

(b) A square matrix is sometimes full-rank.

The definition of a square matrix is independent of the definition of full-rank.

(c) If $\mathbf{AB} = 0$, then $\mathbf{BA}$ is sometimes a zero matrix.

Take $\mathbf{A} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathbf{AB} = 0$, but $\mathbf{BA} \neq 0$. On the other hand, if $\mathbf{A} = 0$ and $\mathbf{B} = 0$, then both $\mathbf{AB}$ and $\mathbf{AB}$ are zero matrices.

(d) The rank of $\mathbf{A} + \mathbf{B}$ is sometimes greater than $rank(A)$.

If $\mathbf{B} = 0$, then $\mathbf{A} + \mathbf{B} = A$ and hence $rank(\mathbf{A} + \mathbf{B}) = rank(\mathbf{A})$. On the other hand, consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $\mathbf{A} + \mathbf{B} = \mathbf{I}$, $rank(\mathbf{A} + \mathbf{B}) = 2 > rank(\mathbf{A}) = 1$.

(e) If $\mathbf{A}^2$ is invertible, then $\mathbf{A}$ is always invertible.

We first note that for $\mathbf{A}^2$ to be a valid matrix product, $\mathbf{A}$ needs to be square. Assume $\mathbf{A} \in \mathbb{R}^{n \times n}$. We then have $rank(\mathbf{A}^2) \leq rank(\mathbf{A}) \leq \min(n, n) = n$. Since $rank(\mathbf{A}^2) = n$ for $\mathbf{A}^2$ to be invertible, we must have $\mathbf{A} = n$. Thus, $\mathbf{A}$ is full-rank and hence invertible.

(f) If the linear equation $\mathbf{y} = \mathbf{Ax}$ has a unique solution, then $\mathbf{A}$ is sometimes square.

If $\mathbf{A} = \mathbf{I}$, then $\mathbf{Ax} = \mathbf{y}$ has a unique solution. However, $\mathbf{A}$ need not necessarily be square for $\mathbf{Ax} = \mathbf{y}$ to have a solution. Consider $\mathbf{A} = \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix}$ for some $\mathbf{b}$. The system of equations $\mathbf{Ax} = \mathbf{y}$ has a unique solution $\mathbf{x} = \mathbf{b}$ even though $\mathbf{A}$ is not square.

Problem 2

True or False. Fill in each blank with '*True*' or '*False*'. Justify your answer.

(a) A diagonalizable matrix $\mathbf{A}$ is nonsingular. False

As a counterexample, the matrix $\mathbf{A} = 0$ is diagonalizable but singular.

(b) A nonsingular matrix $\mathbf{A}$ is diagonalizable. False

As a counterexample, the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is nonsingular but is not diagonalizable.

(c) A positive square matrix $\mathbf{A}$ is positive definite. False

As a counterexample, the matrix $\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$ is positive, but for $\mathbf{x} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, $\mathbf{x}^T\mathbf{Ax} = -2$. Thus, $\mathbf{A}$ is not positive definite.

(d) A square matrix $\mathbf{A}$ with real and positive eigenvalues is positive definite. False

As a counterexample, consider $\mathbf{A} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}$. The eigenvalue of A is 1 (with multiplicity two), but for $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}^T\mathbf{Ax} = -8$. Hence $\mathbf{A}$ is not positive definite.

Problem 3

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Given the matrices above, compute each matrix operation (if it is defined). If an expression is undefined, explain why.

(a) $-2\mathbf{A}$

$$\begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$$

(b) $\mathbf{B} - 2\mathbf{A}$

$$\begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$$

(c) $\mathbf{AC}$

Not defined, since the dimensions are incompatible.

(d) $\mathbf{CD}$

$$\begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$$

Problem 4

(a) Find the inverse of $\mathbf{A} = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

(b) Let $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$. Is $\mathbf{A}$ invertible?

Yes by the Invertible Matrix Theorem. Neither column of the matrix is a multiple of the other, so they are linearly independent. Also, the determinant is non-zero.

Problem 5

Compute the following determinants.

(a) $\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}$

4*0 - (-1*-2) = -2

(b) $\begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

3(3*-1 - 5*2) - 0(2*-1 - 2*0) + 4(2*5 - 3*0) = -39 + 40 = 1

Problem 6

Let $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of $\mathbf{A}$?

Recall that a vector $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ if $\mathbf{Ax} = \lambda\mathbf{x}$, where λ is the scalar eigenvalue. Thus,

$$\mathbf{Au} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

while

$$\mathbf{Av} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Therefore, $\mathbf{u}$ is an eigenvector, but $\mathbf{v}$ is not.

Problem 7

Let $\mathbf{A} = \mathbf{PDP}^{-1}$, where

$$\mathbf{P} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

. Compute $\mathbf{A}^4$

Recall that $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$. To use this equation, we first need to calculate $\mathbf{P}^{-1}$.

$$\mathbf{P}^{-1} = \frac{1}{|\mathbf{P}|} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

We therefore have

$$\begin{aligned} \mathbf{PD}^4\mathbf{P}^{-1} &= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix} \end{aligned}$$

Problem 8

Find the singular values of $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

We first need to compute $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. We then compute the eigenvalues and eigenvectors for $\mathbf{A}^T\mathbf{A}$, giving us $\lambda_1 = 18$ and $\lambda_2 = 0$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = 0$.