Talis Biomedical Statistics Course - Homework 2

Due: 5 December 2019 11:59 PM

Name: [your first and last name]
Collaborators: [list all the people you worked with]
Date: [date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

Linear algebra

The following problems are out of a textbook by Friedberg et al and is available online at https://ulissesgtz.files.wordpress.com/2019/02/stephen-h.-friedberg-arnold-j.-insel-lawrence-e.-spence-linear-algebra-pearson-2014.pdf.

Problem 1

From Friedberg et al (n.b. the notation is 'Chapter'.'Section'.'Exercise#'):

(a) Exercise 1.1.1 (p.5)

- (a) No, since $\frac{3}{6} \neq \frac{1}{4} \neq \frac{2}{2}$.
- (b) Yes, since $-3(-3, 1, 7) = (9, -2, -21)$.
- (c) Yes, since $-1(5, -6, 7) = (-5, 6, -7)$.
- (d) No, since $\frac{2}{5} \neq \frac{0}{0} \neq \frac{-5}{2}$.

(b) Exercise 1.1.3 (p.6)

Let s and t represent fixed points on a plane.

- (a) (2, -5, -1) + s(-2, 9, 7) + t(-5, 12, 2)
- (b) (3, -6, 7) + s(-5, 6, -11) + t(2, -3, -9)
- (c) (-8, 2, 0) + s(9, 1, 0) + t(14, 3, 0)
- (d) (1, 1, 1) + s(4, 4, 4) + t(-7, 3, 1)

(c) Exercise 1.1.6 (p.6)

If we take a vector going from (a, b) to (c, d), but terminating midway, we'd get a distance of $\frac{1}{2}(c - a, d - b)$. So, the midpoint is $(a, b) + \frac{1}{2}(c - a, d - b) = ((a + c)/2, (b + d)/2)$.

Problem 2

From Friedberg et al (n.b. the notation is ‘Chapter’.‘Section’.‘Exercise#’):

(a) Exercise 1.2.1 (p.12)

- (a) True. It’s condition $(VS\ 3)$.
- (b) False. If both x and y are zero vectors, then by condition $(VS\ 3)$ we’d have $x = x + y = y$.
- (c) False. Letting c be the zero vector, we have $1c = 2c$.
- (d) False. It will be false when $a = 0$.
- (e) True.
- (f) False. It’s the opposite (i.e. m rows and n columns).
- (g) False.
- (h) False. For example, we have that $x + (-x) = 0$.
- (i) True.
- (j) True.
- (k) True.

(b) Exercise 1.2.3 (p.13)

$\mathbf{M}_{13} = 3$, $\mathbf{M}_{21} = 4$, $\mathbf{M}_{22} = 5$.

(c) Exercise 1.2.6 (p.14)

We have that $\mathbf{M} = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 5 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \end{bmatrix}$. Since the inventory is doubled for everything, we can simply represent it as $2\mathbf{M}$. This means that $2\mathbf{M} - \mathbf{A}$ is the list of sold items. The total number of items sold is just the sum of this matrix, resulting in 34.

(d) Exercise 1.2.12 (p.15)

For any two even functions, $f(\cdot)$ and $g(\cdot)$, we have that $f(-t)+g(-t) = f(t)+g(t)$. Furthermore, we have that $cf(-t) = cf(t)$. Thus, it is a vector space.

Problem 3

From Friedberg et al (n.b. the notation is ‘Chapter’.‘Section’.‘Exercise#’):

(a) Exercise 1.3.2 (p.20)

(a) $\begin{bmatrix} -4 & 5 \\ 2 & -1 \end{bmatrix}$. $tr = -5$.

(b) $\begin{bmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 0 & 6 \\ 9 & -2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{bmatrix}$. $tr = 12$.

(e) $\begin{bmatrix} 1 \\ -1 \\ 3 \\ 5 \end{bmatrix}$

(f) $\begin{bmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & 6 & 7 \end{bmatrix}$

(h) $\begin{bmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{bmatrix}$. $tr = 2$.

(b) Exercise 1.3.3 (p.20)

Let $M = aA + bB$ and $N = aA^t + bB^t$. It is easy to see that $M_{ij} = aA_{ij} + bB_{ij} = N_{ji}$, and so $M^t = N$.

Problem 4

From Friedberg et al (n.b. the notation is 'Chapter'.'Section'.'Exercise#'):

(a) Exercise 1.4.1 (p.32)

(a) True. Just set the coefficients to be 0.

(b) False. By definition, it shoudl be $\{0\}$.

(c) True. Every subspace where S is a subset contains $span(S)$, and $span(S)$ is a subspace.

(d) False. This will change the system of linear equations (since the linear equations all need to act together).

(e) True.

(f) False.

(b) Exercise 1.4.2 parts a-c (p.33)

(a) From the original system, we have

$$\begin{array}{ccccccccc} x_1 & - & x_2 & - & 2x_3 & - & x_4 & = & -3 \\ & & & & x_3 & + & 2x_4 & = & 4 \\ & & & & 4x_3 & + & 8x_4 & = & 16 \end{array}$$

. Solving this gives us $\{(5 + x - 3t, x, 4 - 2t, t) : s, t \in \mathbb{F}\}$

(b) $\{-2, -4, -3\}$.

(c) No solution.

(c) Exercise 1.4.4 parts a-c (p.33)

(a) Yes. Solving $x_1(1, 3, 0)+x_2(2, 4, -1) = (-2, 0, 3)$ gives us the solution $(4, -3)$.

(b) Yes.

(c) No.

(d) Exercise 1.4.5 parts a-c (p.34)

(a) Yes. Let the span be denoted $r(1, 0, 2) + s(-1, 1, 1) : r, s \in \mathbb{F}$. Then $r = 1, s = -1$ gives us the point $(2, -1, 1)$.

(b) No.

(c) No.

(e) Exercise 1.4.16 (p.35)

If we have that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n$, then we have that $(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$. This means that $a_i = b_i \, \forall i$.

Problem 5

From Friedberg et al (n.b. the notation is 'Chapter'.'Section'.'Exercise#'):

(a) Exercise 1.5.1 (p.40)

(a) False. Example: $S = \{(1, 0), (2, 0), (0, 1)\}$. This is a linearly dependent set but $(0, 1)$ is not a combination of the other two.

(b) True, since $c\vec{0} = \vec{0}$.

(c) False.

(d) False. Example: $S = \{(1,0),(2,0),(0,1)\}$ is linearly dependent, but $S = \{(1,0),(0,1)\}$ is linearly independent.

(e) True.

(f) True.

(b) Exercise 1.5.2 parts a-c (p.40)

(a) Linearly dependent, since $-2\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix}$

(b) Linearly independent.

(c) Linearly independent.

(c) Exercise 1.5.17 (p.42)

Let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ be the columns of $\mathbf{M}$. Let $a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_n\mathbf{C}_n = \vec{0}$. Then, by looking at the last entry of $\mathbf{C}_n$, we see that a_n has to be 0 since the last entry is non-zero. Similarly, we see that $a_{n-1} = 0$ by looking at the penultimate entry of $\mathbf{C}_{n-1}$. This continues all the way back to $a_1 = 0$.

Problem 6

From Friedberg et al (n.b. the notation is 'Chapter'.'Section'.'Exercise#'):

(a) Exercise 2.1.1 (p.74)

(a) True.

(b) False.

(c) False.

(d) True.

(e) False.

(f) False.

(g) True.

(h) False.

(b) Exercise 2.1.2 (p.74)

We have that

$$\begin{aligned} T((a_1,a_2,a_3)+(b_1,b_2,b_3)) &= T(a_1+b_1,a_2+b_2,a_3+b_3) \\ &= (a_1+b_1-a_2-b_2,2a_3+2b_3) \\ &= (a_1-a_2,2a_3)+(b_1-b_2,2b_3) \\ &= T(a_1,a_2,a_3)+T(b_1,b_2,b_3) \end{aligned}$$

. Furthermore, we have $T(ca_1,ca_2,ca_3) = (c(a_1-a_2),2ca_3) = cT(a_1,a_2,a_3)$. $N(T) = \{(a_1,a_a,0)\}$ with basis $\{(1,1,0)\}$ and $R(T)=\mathbb{R}^2$ with basis $\{(1,0),(0,1)\}$.

(c) Exercise 2.1.4 (p.74)

$N(T) = \left\{\begin{bmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right\}$ with basis

$$\left\{\begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

and $R(T) = \left\{\begin{bmatrix} x & t \\ 0 & 0 \end{bmatrix}\right\}$ with basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

(d) Exercise 2.1.37 (p.78)

We must show that $T(cx) = cT(x)$. Let's define $c = \frac{a}{b} \in \mathbb{Q}$. Then

$$T(x) = T(\underbrace{\frac{1}{b}x + \frac{1}{b}x + \cdots + \frac{1}{b}x}_{b \text{ times}}) = bT(\frac{1}{b}x)$$

implying that $T(\frac{1}{b}x) = \frac{1}{b}T(x)$. Thus,

$$T(cx) = T(\frac{a}{b}x) = T(\underbrace{\frac{1}{b}x + \frac{1}{b}x + \cdots + \frac{1}{b}x}_{a \text{ times}}) = aT(\frac{1}{b}x) = \frac{a}{b}T(x) = cT(x)$$