Lecture 2: Linear algebra

STATS 101: Foundations of Statistics

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Announcements

- 30 students submitted homework
- Next assignment will be posted tonight (due 12/4 @ 11:59pm)
- Re: prizes for top 3 students
  - Amazon gift cards
  - Extra points are awarded for:
    1. Class participation (e.g. asking/answering questions, etc)
    2. Catching & correcting errata/typos
    3. Answering questions / participating in discussions on Piazza
  - Blinded top 3 scores will be posted to course website
- No class next week (Happy Thanksgiving!)

Outline

All things linear algebra

- Basic concepts
- Matrix multiplication
- Operations and Properties
- Matrix Calculus

Basic concepts

Consider the following equations:

$$\begin{aligned} 4x_1 - 5x_2 &= -13 \qquad (1) \\ -2x_1 + 3x_2 &= 9 \qquad (2) \end{aligned}$$

Let's solve for x_1 and x_2.

Basic concepts

Consider the following equations:

$$
\begin{align}
4x_1 - 5x_2 &= -13 \tag{1} \\
-2x_1 + 3x_2 &= 9 \tag{2}
\end{align}
$$

Let's solve for x_1 and x_2.

We can write this system of equations more compactly in matrix notation, e.g.

$$\mathbf{Ax} = \mathbf{b} \tag{3}$$

where $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$

Basic concepts

Some basic notation:

- We denote a matrix with m rows and n columns as $\mathbf{A} \in \mathbb{R}^{m \times n}$, where each entry in the matrix is a real number.
- We denote a vector with n entries as $\mathbf{x} \in \mathbb{R}^n$.
  - By convention, we typically think of a vector as a 1 column matrix.
- We denote the i^{th} element of a vector $\mathbf{x}$ as x_i, e.g.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{4}$$

Basic concepts

Some basic notation:

- We denote each entry in a matrix $\mathbf{A}$ by a_{ij}, corresponding to the i^{th} row and j^{th} column, e.g.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{5}$$

- We denote the *transpose* of a matrix as $\mathbf{A}^\top$, e.g.

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{6}$$

Basic concepts

Some basic notation:

- We denote the j^{th} column of $\mathbf{A}$ by $\mathbf{a}_j$ or $\mathbf{A}_{.j}$, e.g.

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \tag{7}$$

- We denote the i^{th} row of $\mathbf{A}$ by $\mathbf{a}_i^\top$ or $\mathbf{A}_{i.}$.

$$\mathbf{A} = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix} \tag{8}$$

n.b. This isn't universal, though should be clear from its presentation and use.

Matrix multiplication

Given two matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$, we can multiply them by

$$\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p} : \mathbf{C}_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik} \mathbf{B}_{kj} \tag{9}$$

n.b. The dimensions have to be compatible for matrix multiplication to be valid (e.g. the number of columns in **A** must be equal to the number of rows in **B**).

Matrix multiplication

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the quantity $\mathbf{x}^\top \mathbf{y} \in \mathbb{R}$ (aka *dot product* or *inner product*) is a scalar given by

$$\mathbf{x}^\top \mathbf{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \tag{10}$$

Note: For vectors, we always have that $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$. This is not generally true for matrices.

Matrix multiplication

Given $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$, the quantity $\mathbf{x}^\top \mathbf{y} \in \mathbb{R}^{m \times n}$ (aka *outer product*) is a matrix given by

$$\mathbf{x}\mathbf{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix} \quad (11)$$

Matrix multiplication

Example: Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix such that all columns are equal to some vector $\mathbf{x} \in \mathbb{R}^m$. Using outer products, we can represent $\mathbf{A}$ compactly as

$$
\begin{aligned}
\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{x} \\ | & | & & | \end{bmatrix} &= \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m & \cdots & x_m \end{bmatrix} && (12) \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} && (13) \\
&= \mathbf{x}\mathbf{1}^\top && (14)
\end{aligned}
$$

Matrix-vector products

Given $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^n$, their product is a vector $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$.

Matrix-vector products

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There are two ways of interpreting this:

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{x} \\ \mathbf{a}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{a}_m^\top \mathbf{x} \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \quad (16)$$

$$= \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n \quad (17)$$

Matrix-vector products

Example:

Define $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$.

Calculate $\mathbf{y} = \mathbf{A}\mathbf{x}$.

Matrix-matrix products

Given $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}$, their product is a matrix $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.

Matrix-matrix products

Given $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}$, their product is a matrix $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.

Similar to before, we can think of this in two ways:

Interpretation # 1

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \cdots \mathbf{a}_1^\top \mathbf{b}_p \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \cdots \mathbf{a}_2^\top \mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_m^\top \mathbf{b}_1 & \mathbf{a}_m^\top \mathbf{b}_2 & \cdots \mathbf{a}_m^\top \mathbf{b}_p \end{bmatrix} \tag{19}$$

Matrix-matrix products

Interpretation # 2

$$\mathbf{C} = \mathbf{AB} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \cdots & \mathbf{A}\mathbf{b}_p \\ | & | & & | \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix}\mathbf{B} = \begin{bmatrix} — & \mathbf{a}_1^\top\mathbf{B} & — \\ — & \mathbf{a}_2^\top\mathbf{B} & — \\ & \vdots & \\ — & \mathbf{a}_m^\top\mathbf{B} & — \end{bmatrix} \tag{22}$$

Matrix multiplication properties

- Associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- Distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{BC}$
- Not commutative: $\mathbf{AB} \neq \mathbf{BA}$

Matrix multiplication properties

Demonstrating *associativity*:

We just need to show that $((\mathbf{AB})\mathbf{C})_{ij} = (\mathbf{A}(\mathbf{BC}))_{ij}$:

$$
\begin{aligned}
((\mathbf{AB})\mathbf{C})_{ij} &= \sum_{k=1}^{p} (\mathbf{AB})_{ik}\mathbf{C}_{kj} = \sum_{k=1}^{p} \left(\sum_{l=1}^{n} \mathbf{A}_{il}\mathbf{B}_{lk} \right) \mathbf{C}_{kj} && (23) \\
&= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} \mathbf{A}_{il}\mathbf{B}_{lk}\mathbf{C}_{kj} \right) = \sum_{l=1}^{n} \left(\sum_{k=1}^{p} \mathbf{A}_{il}\mathbf{B}_{lk}\mathbf{C}_{kj} \right) && (24) \\
&= \sum_{l=1}^{n} \mathbf{A}_{il} \left(\sum_{k=1}^{p} \mathbf{B}_{lk}\mathbf{C}_{kj} \right) = \sum_{l=1}^{n} \mathbf{A}_{il}(\mathbf{BC})_{lj} && (25) \\
&= (\mathbf{A}(\mathbf{BC}))_{ij} && (26)
\end{aligned}
$$

The identity matrix:

The *identity matrix*, denoted $\mathbf{I} \in \mathbb{R}^{n \times n}$ is a square matrix with 1's in the diagnoal and 0's everywhere else, i.e.

$$\mathbf{I}_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{27}$$

Operations & properties

The identity matrix:

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$$\mathbf{I}_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{27}$$

It has the property

$$\mathbf{AI} = \mathbf{A} = \mathbf{IA} \ \forall \mathbf{A} \in \mathbb{R}^{m \times n} \tag{28}$$

n.b. The dimensionality of $\mathbf{I}$ is typically inferred (e.g. $n \times n$ vs $m \times m$)

Operations & properties

The diagonal matrix: The *diagonal matrix*, denoted $\mathbf{D} = diag(d_1, d_2, ldots, d_n)$ is a matrix where all non-diagonal elements are 0, i.e.

$$\mathbf{D}_{ij} = \begin{cases} d_i & i = j \\ 0 & i \neq j \end{cases} \tag{29}$$

Clearly, $\mathbf{I} = diag(1, 1, ..., 1)$.

The transpose

The *transpose* of a matrix results from "*flipping*" the rows and columns, i.e.

$$(\mathbf{A}^\top)_{ij} = \mathbf{A}_{ji} \tag{30}$$

Consequently, for $\mathbf{A} \in \mathbb{R}^{m \times n}$ we have that $\mathbf{A}^\top \in \mathbb{R}^{n \times m}$.

Some properties:

- $(\mathbf{A}^\top)^\top = \mathbf{A}$
- $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$
- $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

Symmetry

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^\top$.

It is *anti-symmetric* if $\mathbf{A} = -\mathbf{A}^\top$.

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It is easy to show that $\mathbf{A} + \mathbf{A}^\top$ is symmetric and $\mathbf{A} - \mathbf{A}^\top$ is anti-symmetric. Consequently, we have that

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}T\top) + \frac{1}{2}(\mathbf{A} - \mathbf{A}T\top) \tag{31}$$

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Symmetric matrices tend to be denoted as $\mathbf{A} \in \mathbb{S}^n$.

Trace

The *trace* of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, denoted $tr(\mathbf{A})$ or $tr\mathbf{A}$ is the sum of the diagonal elements, i.e.

$$tr\mathbf{A} = \sum_{i=1}^{n} \mathbf{A}_{ii} \tag{32}$$

The trace has the following properties:

- For $\mathbf{A} \in \mathbb{R}^{n \times n}$, $tr\mathbf{A} = tr\mathbf{A}^\top$
- For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, $tr(\mathbf{A} + \mathbf{B}) = tr\mathbf{A} + tr\mathbf{B}$
- For $\mathbf{A} \in \mathbb{R}^{n \times n}, c \in \mathbb{R}$, $tr(c\mathbf{A}) = c\, tr\mathbf{A}$
- For $\mathbf{A}, \mathbf{B} \ni \mathbf{AB} \in \mathbb{R}^{n \times n}$, $tr\mathbf{AB} = tr\mathbf{BA}$
- For $\mathbf{A}, \mathbf{B}, \mathbf{C} \ni \mathbf{ABC} \in \mathbb{R}^{n \times n}$, $tr\mathbf{ABC} = tr\mathbf{BCA} = tr\mathbf{CAB}$, and so on for more matrices

Trace

Example: Proving that $tr\mathbf{AB} = tr\mathbf{BA}$

$$\begin{aligned}
tr\mathbf{AB} &= \sum_{i=1}^{m} (\mathbf{AB})_{ii} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ji} \right) && (33) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{B}_{ji}\mathbf{A}_{ij} && (34) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \mathbf{B}_{ji}\mathbf{A}_{ij} \right) = \sum_{j=1}^{n} (\mathbf{BA})_{jj} && (35) \\
&= tr\mathbf{BA} && (36)
\end{aligned}$$

Norms

A *norm* of a vector $\mathbf{x}$, denoted $||\mathbf{x}||$ is a measure of the "*length*" of the vector. For example, the ℓ_2-norm (aka Euclidean norm) is

$$||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{37}$$

n.b. $||\mathbf{x}||_2^2 = \mathbf{x}^\top \mathbf{x}$, i.e. the squared norm of a vector is the dot product with itself.

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Other norms:

- ℓ_1-norm, i.e. $||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|$.
- ℓ_∞-norm, i.e. $||\mathbf{x}||_\infty = \max_i |x_i|$.
- ℓ_p-norm, i.e. $||\mathbf{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

Norms

Formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying four properties:

1. $\forall \mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) \geq 0$ (non-negativity).
2. $f(\mathbf{x}) = 0$ iff $\mathbf{x} = 0$ (definiteness).
3. $\forall \mathbf{x} \in \mathbb{R}^n, c \in \mathbb{R}, f(c\mathbf{x}) = |c| f(\mathbf{x})$ (homogeneity).
4. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ (triangle inequality).

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Norms can also be defined for matrices, e.g. The Frobenius norm,

$$||\mathbf{A}||^F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{A}_{ij}^2} = \sqrt{tr(\mathbf{A}^\top \mathbf{A})} \tag{38}$$

Linear independence

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \in \mathbb{R}^m$ is *(linearly) dependent* if one of the vectors $\mathbf{x}_i$ can be represented as a linear combination of the remaining vectors, i.e.

$$\mathbf{x}_n = \sum_{i=1}^{n-1} \alpha_i \mathbf{x}_i \tag{39}$$

for some scalar values $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathbb{R}$

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Example: Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \tag{40}$$

Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?

Rank

The *column rank* of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the largest subset of columns of $\mathbf{A}$ that are linearly independent.

- The column rank is always $\leq n$.

The *row rank* of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the largest subset of rows of $\mathbf{A}$ that are linearly independent.

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- The row rank is always $\leq m$.

n.b. Column rank is always equal to row rank. Thus, we refer to both as the *rank* of the matrix.

- For $\mathbf{A} \in \mathbb{R}^{m \times n}$, if $rank(\mathbf{A}) = \min(m, n)$, then $\mathbf{A}$ is said to be of *full rank*.
- For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $rank(\mathbf{A}) = rank(\mathbf{A}^\top$.
- For $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}$, $rank(\mathbf{AB}) \leq \min(rank(\mathbf{A}), rank(\mathbf{B}))$.
- For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, $rank(\mathbf{A} + \mathbf{B}) \leq rank(\mathbf{A}) + rank(\mathbf{B})$

Matrix inverse

The *inverse* of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is denoted $\mathbf{A}^{-1}$, and is unique such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1} \tag{41}$$

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n.b. Not all matrices have inverses (e.g. $m \times n$ matrices).

Def:
A is *invertible* or *non-singular* if $\mathbf{A}^{-1}$ exists.
Otherwise, it is *non-invertible* or *singular*.

1. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
2. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
3. $(\mathbf{A}^{-1})^{\top} = (\mathbf{A}^{\top})^{-1}$
    - This matrix is sometimes denoted $\mathbf{A}^{-\top}$

Orthogonal Matrices

Def:

- A vector $\mathbf{x} \in \mathbb{R}^n$ is *normalized* if $||\mathbf{x}||_2 = 1$
- Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{x}^\top \mathbf{y} = 0$
- A square matrix $\mathbf{U} \in \mathbb{R}^{n \times n}$ is *orthogonal* or *orthonormal* if all its columns are:
  1. Orthogonal to each other
  2. Normalized

We therfore have that

$$\mathbf{U}^\top \mathbf{U} = \mathbf{I} = \mathbf{U}\mathbf{U}^\top \tag{42}$$

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Another nice property:

$$||\mathbf{U}\mathbf{x}||_2 = ||\mathbf{x}||_2 \ \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{U} \in \mathbb{R}^{n \times n} \text{ orthogonal} \tag{43}$$

Range

Def:
The *span* of a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is

$$\text{span}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}) = \left\{ v : v = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i, \alpha_i \in \mathbb{R} \right\} \qquad (44)$$

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n.b. If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is linearly independent, then $\text{span}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}) = \mathbb{R}^n$.

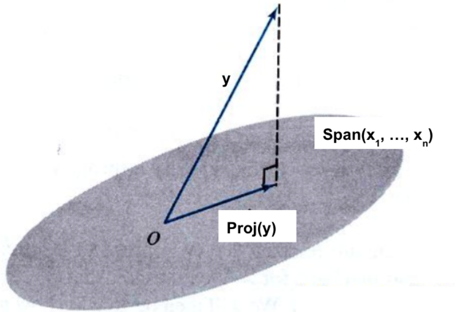
Example:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{45}$$

Projection

Def:
The *projection* of a vector $\mathbf{y} \in \mathbb{R}^m$ onto $\text{span}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}) = \mathbb{R}^n$ is

$$\text{Proj}(\mathbf{y}; \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}) = \underset{\mathbf{v} \in \text{span}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\})}{\arg\min} ||\mathbf{y} - \mathbf{v}||_2 \tag{46}$$



Range

Def:

The *range* of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, denoted $\mathcal{R}(\mathbf{A})$ is the span of the columns of $\mathbf{A}$, i.e.

$$\mathcal{R}(\mathbf{A}) = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{R}^n\} \tag{47}$$

Assuming that $\mathbf{A}$ is full rank and $n < m$, the projection of $\mathbf{y} \in \mathbb{R}^m$ onto $\mathcal{R}(\mathbf{A})$ is

$$\begin{aligned} \text{Proj}(\mathbf{y}; \mathbf{A}) &= \arg\min_{\mathbf{v} \in \mathcal{R}(\mathbf{A})} ||\mathbf{v} - \mathbf{y}||_2 & (48) \\ &= \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{y} & (49) \end{aligned}$$

Nullspace

Def:
The *nullspace* of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, denoted $\mathcal{N}(\mathbf{A})$ is the set of all vectors that equal 0 when ultiplied by $\mathbf{A}$, i.e.

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = 0\} \tag{50}$$

Some properties:

- $\{w : w = u + v, u \in \mathcal{R}(\mathbf{A}^\top), v \in \mathcal{R}(\mathbf{A})\} = \mathbb{R}^n$
- $\mathcal{R}(\mathbf{A}^\top) \bigcap \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$

This is referred to as *orthogonal complements*, denoted as $\mathcal{R}(\mathbf{A}^\top) = \mathcal{N}(\mathbf{A})^\perp$

Determinant

Def:
The *determinant* of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, denoted $|\mathbf{A}|$ or det $\mathbf{A}$ is a function det: $\mathbb{R}^{n\times n} \rightarrow \mathbb{R}$.

Let $\mathbf{A}_{\setminus i,\setminus j} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix that results from deleting the i^{th} row and j^{th} column. The general (recursive) formula for the determinant is

$$\begin{aligned} |\mathbf{A}| &= \textstyle\sum_{i=1}^{n}(-1)^{i+j} a_{ij} |\mathbf{A}_{\setminus i,\setminus j}| \quad (\forall j \in 1, ..., n) \\ &= \textstyle\sum_{j=1}^{n}(-1)^{i+j} a_{ij} |\mathbf{A}_{\setminus i,\setminus j}| \quad (\forall i \in 1, ..., n) \end{aligned} \tag{51}$$

Determinant

Given a matrix

$$\mathbf{A} = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_n^\top & — \end{bmatrix} \tag{52}$$

and a set $\mathbf{S} \subset \mathbb{R}^n$,

$$\mathbf{S} = \{\mathbf{v} \in \mathbb{R}^n : v = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i \text{ where } 0 \leq \alpha_i \leq 1, i = 1, ..., n\} \tag{53}$$

$|\mathbf{A}|$ is the volume of $\mathbf{S}$.

Determinant

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \tag{54}$$

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And $|\mathbf{A}| = -7$

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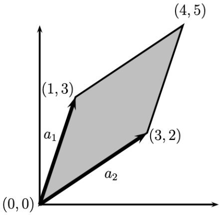

Determinant

Properties of determinants:

- For $\mathbf{A} \in \mathbb{R}^{n \times n}, |\mathbf{A}| = |\mathbf{A}^\top|$
- For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}, |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
- For $\mathbf{A} \in \mathbb{R}^{n \times n}, |\mathbf{A}| = 0$ iff $\mathbf{A}$ is singular (i.e. non-invertible).
- For $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{A}$ non-singular, $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$

Quadratic form

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, the *quadratic form* is the scalar value

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i (\mathbf{A}\mathbf{x})_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} \mathbf{A}_{ij} x_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{A}_{ij} x_i x_j \quad (56)$$

Quadratic form

Some properties involving quadratic form:

- A symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ is *positive definite* if for a non-zero $\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^\top \mathbf{A}\mathbf{x} > 0$
- A symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ is *positive semi-definite* if for a non-zero $\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^\top \mathbf{A}\mathbf{x} \geq 0$
- A symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ is *negative definite* if for a non-zero $\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^\top \mathbf{A}\mathbf{x} < 0$
- A symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ is *negative semi-definite* if for a non-zero $\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^\top \mathbf{A}\mathbf{x} \leq 0$
- A symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ is *indefinite* if it is neither positive nor negative semidefinite

n.b. Positive definite and negative definite matrices always have full rank.

Eigenvalues & eigenvectors

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an *eigenvalue* of $\mathbf{A}$ with corresponding *eigenvector* $\mathbf{x} \in \mathbb{C}^n$ if

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} : \mathbf{x} \neq 0 \tag{57}$$

n.b. The eigenvector is (usually) normalized to have length 1

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$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} : \mathbf{x} \neq 0 \tag{57}$$

n.b. The eigenvector is (usually) normalized to have length 1

We can write all of the eigenvector equations simultaneously as

$$\mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda} \tag{58}$$

where

$$\mathbf{X} \in \mathbb{R}^{n \times n} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix}, \quad \boldsymbol{\Lambda} = diag(\lambda_1, ..., \lambda_n) \tag{59}$$

Some properties:

- $tr\mathbf{A} = \sum_{i=1}^{n} \lambda_i$
- $|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$
- The rank of **A** is equal to the number of non-zero eigenvalues of **A**.
- If **A** is non-singular, then $1/\lambda_i$ is an eigenvalue of $\mathbf{A}^{-1}$ with correspondng eigenvector $\mathbf{x}_i$, i.e. $\mathbf{A}^{-1}\mathbf{x}_i = (1/\lambda_i)\mathbf{x}_i$
- The eigenvalues of a diagonal matrix $D = diag(d_1, ..., d_n)$ are just its diagonal entries $d_1, ..., d_n$

Eigenvalues & eigenvectors

Example: For $\mathbf{A} \in \mathbb{S}^n$ with ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$,

$$\max_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^\top \mathbf{A} \mathbf{x} \text{ subject to } ||\mathbf{x}||_2^2 = 1 \tag{60}$$

is solved with $\mathbf{x}_1$ corresponding to λ_1. Similarly,

$$\min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^\top \mathbf{A} \mathbf{x} \text{ subject to } ||\mathbf{x}||_2^2 = 1 \tag{61}$$

is solved with $\mathbf{x}_n$ corresponding to λ_n.

Matrix calculus

Given $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, the *gradient* of f wrt $\mathbf{A} \in \mathbb{R}^{m\times n}$ is

$$\nabla_{\mathbf{A}} f(\mathbf{A}) \in \mathbb{R}^{m\times n} = \begin{bmatrix} \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{11}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{12}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{1n}} \\ \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{21}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{22}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{m1}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{m2}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{mn}} \end{bmatrix} \tag{62}$$

Some properties

- $\nabla_{\mathbf{x}}(f(\mathbf{x}) + g(\mathbf{x})) = \nabla_{\mathbf{x}} f(\mathbf{x}) + \nabla_{\mathbf{x}} g(\mathbf{x})$
- For $c \in \mathbb{R}, \nabla_{\mathbf{x}}(c\, f(\mathbf{x})) = c\nabla_{\mathbf{x}}(f(\mathbf{x}))$

The Hessian

Given $f : \mathbb{R}^n \to \mathbb{R}$, the *Hessian* of f wrt $\mathbf{x} \in \mathbb{R}^n$ is

$$\nabla_{\mathbf{x}}^2 f(\mathbf{x}) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{bmatrix} \tag{63}$$

n.b. The Hessian is always symmetric, since $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i}$

Least squares

Given $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m \ni b \notin \mathcal{R}(A)$, we want to find $\mathbf{x} \in \mathbb{R}^n$ as close as possible to $\mathbf{b}$ (via the Euclidean norm),

$$\begin{aligned} ||\mathbf{Ax} - \mathbf{b}||_2^2 &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) && (64) \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{b}^\top \mathbf{b} && (65) \end{aligned}$$

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Taking the gradient wrt $\mathbf{x}$, we have

$$\begin{aligned} \nabla_{\mathbf{x}}(\mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{b}^\top \mathbf{b}) &= \nabla_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - \nabla_{\mathbf{x}} 2\mathbf{b}^\top \mathbf{Ax} + \nabla_{\mathbf{x}} \mathbf{b}^\top \mathbf{b} && (66) \\ &= \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} && (67) \end{aligned}$$

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$$\begin{aligned} \nabla_{\mathbf{x}}(\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{b}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{b}) &= \nabla_{\mathbf{x}}\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - \nabla_{\mathbf{x}} 2\mathbf{b}^\top \mathbf{A}\mathbf{x} + \nabla_{\mathbf{x}}\mathbf{b}^\top \mathbf{b} && (66) \\ &= \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{A}^\top \mathbf{b} && (67) \end{aligned}$$

Setting this expression equal to zero and solving for $\mathbf{x}$ gives the normal equations,

$$\mathbf{x} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} \qquad (68)$$

SVD

TODO: go into this.

References

Some textbooks on linear algebra:

- *Linear Algebra (Jim Hefferon)*
- *Introduction to Applied Linear Algebra (Boyd & Vandenberghe)*
- *Linear Algebra (Cherney, Denton et al.)*
- *Linear Algebra (Hoffman & Kunze)*
- *Fundamentals of Linear Algebra (Carrell)*
- *Linear Algebra (S. Friedberg A. Insel L. Spence)*