Talis Biomedical Statistics Course - Homework 7

Due: 29 January 2020 9:00 AM

Name:	[your first and last name]
Collaborators:	[list all the people you worked with]
Date:	[date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

The exercises can be found in the DeGroot & Schervish's textbook, available here.

Problem 1

Section 8.1 Exercise 1

Let $U = \max\{X_1, ..., X_n\}$. Then its cdf is

$$\begin{aligned} F(u) &= P(U \leq u) = P(\max\{X_1, ..., X_n\} \leq u) && (1) \\ &= P(X_1 \leq u)P(X_2 \leq u) \cdots P(X_n \leq u) && (2) \\ &= \left(\frac{u}{\theta}\right)^n && (3) \end{aligned}$$

where $0 \leq u \leq \theta$. Since $P(U \leq \theta) = 1$, the event $|U - \theta| \leq 0.1\theta$ is the same as the event $U > 0.9\theta$. Thus, we have that $P(U > 0.9\theta) = 1 - F(0.9\theta) = 1 - 0.9^n$. Setting it equal to 0.95 and solving for n give us $n = \log(0.05)/\log(0.9) \approx 28.43$. Rounding up, we have that we need at least 29 observations.

Problem 2

Section 8.2 Exercise 4

Let r denote the radius of the circle. The point (X, Y) will lie inside the circle if and only if $X^2 + Y^2 < r^2$. Recall that X^2 and Y^2 follow a $\chi^2(1)$ distribution. Their sum follows a $\chi^2(2)$ distribution. We can look at the end of the book (or use statistical programming software) to compute the value u such that $P(X^2 + Y^2 \leq u) = 0.99$, giving us $u = 9.210$.

Problem 3

Section 8.3 Exercise 2

We can verify that the matrices for (a), (b) and (e) are orthogonal, since in each case the sum of the squares of the elements in each row is 1 and the sum of the products of the corresponding terms in any two different rows is 0. Matrix (c) is not orthogonal because the sum of squares for the bottom row is not 1. Matrix (d) is not orthogonal since the sum of the products for rows 1 and 2 is not 0.

Problem 4

Section 8.4 Exercise 2

Recall from Equation (8.4.4) that $U = \frac{n^{1/2}(\bar{X}_n-\mu)}{\left(\frac{S_n^2}{n-1}\right)^{1/2}}$. Since $\hat{\mu} = \bar{X}_n$ and $\hat{\sigma}^2 = S_n^2/n$, applying it to this problem gives us

$$\begin{aligned} P(\hat{\mu} > \mu + k\hat{\sigma}) &= P\left(\frac{\bar{X}_n - \mu}{\hat{\sigma}} > k\right) = P\left(\frac{U}{(n-1)^{1/2}} > k\right) && (4) \\ &= P\left(U > k(n-1)^{1/2}\right) && (5) \end{aligned}$$

We know from the chapter that U has a t-distribution with $n-1$ degrees of freedom and are given that $n = 17$. We must therefore choose k such that

$$P\left(U > k(n-1)^{1/2}\right) = P\left(U > 4k\right) = 0.95 \tag{6}$$

Looking at the end of the book (or using statistical software), we see that $P\left(U > -1.746\right) = 0.95$, which means that $4k = -1.746$ or that $k = -0.4365$.

Problem 5

Section 8.5 Exercise 4

Recall that $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ follows a standard normal distribution. Thus, we have that

$$P\left(-1.96 < \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < 1.96\right) = 0.95 \tag{7}$$

which can be rewritten as

$$P\left(\bar{X}_n - \frac{1.96\sigma}{\sqrt{n}} < \mu < \bar{X}_n + \frac{1.96\sigma}{\sqrt{n}}\right) = 0.95 \tag{8}$$

This confidence interval has a length of $2\frac{1.96\sigma}{\sqrt{n}}$. It follows that $2\frac{1.96\sigma}{\sqrt{n}} < 0.01\sigma$ if and only if $\sqrt{n} > 392$. Equivalently, we need $n > 392^2$.

Problem 6

Section 8.8 Exercise 3

For a Poisson distribution, we have that $f(x|\theta) = exp(-\theta)\theta^x/x!$. Thus,

$$\ell(\theta|x) = \log\ f(x|\theta) = -\theta + x \log\, \theta - \log\, x! \tag{9}$$

Taking the second derivative of this gives us

$$\frac{\partial^2}{\partial\theta^2}\ell(\theta|x) = -\frac{x}{\theta^2} \tag{10}$$

and by Equation (8.8.3) we have that the Fisher information is

$$I(\theta) = -\mathbb{E}_\theta\left[\frac{\partial^2}{\partial\theta^2}\ell(\theta|x)\right] = \frac{\mathbb{E}[x]}{\theta^2} = \frac{1}{\theta} \tag{11}$$