Talis Biomedical Statistics Course - Homework 0

Due: 20 November 2019 11:59 PM

Name: [your first and last name]
Collaborators: [list all the people you worked with]
Date: [date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

Algebra

Problem 1

Simplify/factor the following expressions.

(a) Simplify $5z^2(7z^2 + 3z - 1)$

$$\begin{aligned} 5z^2(7z^2 + 3z - 1) &= 5z^2 * 7z^2 + 5z^2 * 3z - 5z^2 * 1 && (1) \\ &= 35z^4 + 15z^3 - 5z^2 && (2) \end{aligned}$$

(b) Simplify $(1 + x)(x^2 - 5x - 6)$

$$\begin{aligned} (1 + x)(x^2 - 5x - 6) &= (1 * x^2 - 1 * 5x - 1 * 6) + (x * x^2 - x * 5x - x * 6) && (3) \\ &= x^2 - 5x - 6 + x^3 - 5x^2 - 6x && (4) \\ &= x^3 - 4x^2 - 11x - 6 && (5) \end{aligned}$$

(c) Simplify $(x + 2x^2)^2$

$$\begin{aligned} (x + 2x^2)^2 &= (x + 2x^2) * (x + 2x^2) && (6) \\ &= x^2 + 2x^3 + 2x^3 + 4x^4 && (7) \\ &= 4x^4 + 4x^3 + x^2 && (8) \end{aligned}$$

(d) Simplify $\frac{2x^3 - x^2 - 12}{x+3}$

$$\begin{aligned} \frac{2x^3 - x^2 - 12}{x + 3} &= \frac{(x + 3)(2x^2 - 7x + 21) - 75}{x + 3} && (9) \\ &= 2x^2 - 7x + 21 - \frac{75}{x + 3} && (10) \end{aligned}$$

(e) Factor $(x-5)^2+2y^3(x-5)+y^6$

Let $U=(x-5)$ and $V=y^3$. Then

$$\begin{aligned}
(x-5)^2+2y^3(x-5)+y^6 &= ((x-5)+y^3)^2 && (11)\\
&= (U+V)^2 && (12)\\
&= (x-5+y^3)^2 && (13)
\end{aligned}$$

(f) Factor $16x^3+24x^2+9x$

$$\begin{aligned}
16x^3+24x^2+9x &= x(16x^2+24x+9) && (14)\\
&= x((4x)^2+2\cdot 4\cdot 3x+3^2) && (15)\\
&= x(4x+3)^2 && (16)
\end{aligned}$$

(g) Factor x^2-49y^2

$$\begin{aligned}
x^2-49y^2=(x+7y)(x-7y) && (17)\\
&& (18)
\end{aligned}$$

(h) The polynomial $3x^3-20x^2+37x-20$ has a known factor of $(x-4)$. Factor it.

$$3x^3-20x^2+37x-20=(x-4)(3x^2-8x+5) \qquad (19)$$

Problem 2

Logarithms and exponents.

(a) $3^a=\sqrt[5]{3^2}$. Solve for a.

$$\begin{aligned}
(3^a)^5 &= (\sqrt[5]{3^2})^5 && (20)\\
3^{5a} &= 3^2 && (21)\\
&\implies 5a=2 && (22)\\
& a=2/5 && (23)
\end{aligned}$$

(b) $26^{9x+5}=1$. Solve for x.

$26^0 = 1$. Therefore, we want $9x + 5 = 0$. Solving for x gives us $x = -5/9$.

(c) $2^{3x+5} = 64^{x-7}$. Solve for x.

$$\begin{aligned} 2^{3x+5} &= 64^{x-7} && (24)\\ &= (2^6)^{x-7} && (25)\\ &= 2^{6x-42} && (26) \end{aligned}$$

Therefore, we want to have $3x + 5 = 6x - 42$. Solving for x gives us $x = 47/3$.

(d) Rewrite $\log_3 27x$ as a sum of a constant and (a function of) a variable.

$$\begin{aligned} \log_3 27x &= \log_3 27 + \log_3 x && (27)\\ &= 3 + \log_3 x && (28) \end{aligned}$$

(e) Rewrite $\log_5 \frac{25^x}{y}$ as a sum of functions of two variables.

$$\begin{aligned} \log_5 \frac{25^x}{y} &= \log_5 25^x - \log_5 y && (29)\\ &= 2x - \log_5 y && (30) \end{aligned}$$

(f) Solve $\log_c 16 \cdot \log_2 c$ where c is an unknown constant.

$$\begin{aligned} \log_c 16 \cdot \log_2 c &= \frac{\log_{10} 16}{\log_{10} c} \cdot \frac{\log_{10} c}{\log_{10} 2} && (31)\\ &= \frac{\log 16}{\log 2} && (32)\\ &= \log_2 16 = 4 && (33) \end{aligned}$$

(g) Solve the equation for t and express the answer in terms of base 10 logarithm.

$$10^{2t-3} = 7 \qquad (34)$$

$$\begin{aligned} \log_{10} 7 &= 2t - 3 && (35)\\ \log_{10} 7 + 3 &= 2t && (36)\\ \frac{\log_{10} 7 + 3}{2} &= t && (37) \end{aligned}$$

Multivariable calculus

Problem 3

Calculate the following gradients.

(a) Let $f(x, y) = x^2 - xy$. What is $\nabla f(x, y)$?

$$\begin{bmatrix} 2x - y \\ -x \end{bmatrix} \tag{38}$$

(b) What is the gradient of $f(x, y) = -x^4 + 4(x^2 - y^2) - 3$?

$$\begin{bmatrix} -4x^3 + 8x \\ -8y \end{bmatrix} \tag{39}$$

(c) What is the gradient of $f(x, y, z) = x - xy + z^2$?

$$\begin{bmatrix} 1 - y \\ -x \\ 2z \end{bmatrix} \tag{40}$$

(d) Find $\frac{\partial}{\partial}(\cos(t))^2 \sin(t)$.

We begin by defining $x(t) = \cos(t)$ and $y(t) = sin(t)$. Then $f(x, y) = x^2 y$. We can then apply the multivariable chain rule to get the answer, i.e.

$$\begin{aligned} \frac{\partial}{\partial t} f(x, y) &= \frac{\partial}{\partial y} f(x, y) \cdot \frac{\partial y}{\partial t} + \frac{\partial}{\partial x} f(x, y) \cdot \frac{\partial x}{\partial t} && (41) \\ &= \cos^2(t)\cos(t) + 2\cos(t)\sin(t)(-\sin(t)) && (42) \end{aligned}$$

(e) Find the Jacobian of $\begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} x + \sin(y) \\ y + \sin(x) \end{bmatrix}$

$$\mathbf{J}_F = \ = \begin{bmatrix} 1 & \cos(y) \\ \cos(x) & 1 \end{bmatrix} \tag{43}$$

(f) Let $f(a, b, c) = \cos(ab)\sin(b) + c$. Evaluate $\frac{\partial f}{\partial a}(\frac{1}{2}, \frac{\pi}{3}, 7)$

$$\frac{\partial}{\partial a} f(a, b, c) = -b\sin(b)\sin(ab) \tag{44}$$

and $\frac{\partial f}{\partial a}(\frac{1}{2}, \frac{\pi}{3}, 7) = \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{1}{2}\cdot\frac{\pi}{3}\right) = \frac{\sqrt{3}}{12}\pi$

Problem 4

Calculate the following integrals.

(a) $\int x \log(x) dx$

We apply integration by parts. Let $u = \log(x)$ and $dv = xdx$. Then

$$\begin{aligned}
\int u dv &= uv - \int v du + c && (45)\\
&= \frac{1}{2}x^2 \log(x) - \frac{1}{2}\int x dx + c && (46)\\
&= \frac{1}{4}x^2(2\log(x) - 1) + c && (47)
\end{aligned}$$

(b) $\int x^2 \sin(x) dx$

We apply integration by parts. Let $u = x^2$ and $dv = \sin(x)dx$. Then

$$\begin{aligned}
\int u dv &= uv - \int v du + c && (48)\\
&= -x^2\cos(x) + 2\int x\cos(x) dx && (49)\\
&= 2x\sin(x) - (x^2 - 2)\cos(x) + c && (50)
\end{aligned}$$

(c) $\int_{0.5}^{2.5}\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5} dx dy$

The inner integral is

$$\int_{0.5}^{3.5} \sin(xy) + \frac{6}{5} dx = \frac{-\cos(xy)}{y} + \frac{6x}{5}\Big|_{0.5}^{3.5} = \frac{-\cos(3.5y)}{y} + \frac{\cos(0.5y)}{y} + \frac{18}{5} \quad (51)$$

The integral of this function does not have a closed form. Consequently, we'd have to approximate it using e.g. the Riemann integral.

(d) $\int_0^1 \int_0^{x^2} x + 2y^2 dy dx$

$$\begin{aligned}
\int_0^1\int_0^{x^2} x + 2y^2 dy dx &= \int_0^1 xy + \frac{2}{3}y^3\Big|_0^{x^2} dx && (52)\\
&= \int_0^1 x^3 + \frac{2}{3}x^6 dx && (53)\\
&= \frac{x^4}{4} + \frac{2}{21}x^7\Big|_0^1 && (54)\\
&= \frac{1}{4} + \frac{2}{21} = \frac{29}{84} && (55)
\end{aligned}$$