Talis Biomedical Statistics Course - Homework 3

Due: 11 December 2019 11:59 PM

Name:	[your first and last name]
Collaborators:	[list all the people you worked with]
Date:	[date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

The exercises can be found in the DeGroot & Schervish's textbook, available here.

Problem 1

From DeGroot & Schervish:

(a) Section 2.1: Exercise 2

If $A \subset B$, then $A \cap B = A$ and $P(A \cap B) = P(A)$. So $P(A|B) = P(A)/P(B)$.

(b) Section 2.1: Exercise 4

Denote $A_i : i = 1, 2, ...$ as the event that the shopper purchases brand A on his i^{th} purchase. Similarly, let $B_i : i = 1, 2, ...$ denote the event that the shopper purchases brand B on his i^{th} purchase. Then

$$\begin{aligned} P(A_1) &= \frac{1}{2} \\ P(A_2|A_1) &= \frac{1}{3} \\ P(B_3|A_1, A_2) &= \frac{2}{3} \\ P(B_4|A_1, A_2, B_3) &= \frac{1}{3} \end{aligned}$$

The overall probability is just the product of the four probabilities, which is 1/27.

Problem 2

From DeGroot & Schervish:

(a) Section 2.2: Exercise 4

Teh probability of the sum being 7 on any given roll is 1/6. Since the rolls are independent, the probability that it will occur on three successive rolls is $(1/6)^3$.

(b) Section 2.2: Exercise 10

The probability p_j that exactly j children will have blue eyes is

$$p_j = \binom{5}{j}\left(\frac{1}{4}\right)^j\left(\frac{3}{4}\right)^{5-j} : j = 0, 1, ..., 5.$$

Our desired probability is

$$\frac{p_3 + p_4 + p_5}{p_1 + p_2 + p_3 + p_4 + p_5}$$

Problem 3

From DeGroot & Schervish:

(a) Section 2.3: Exercise 2

We partition the space of possibilities into three events B_1, B_2, B_3 as follows. Let B_1 be the event that the machine is in good working order. Let B_2 be the event that the machine is wearing down. Let B_3 be the event that it needs maintenance. We are told that $P(B_1) = 0.8$ and $P(B_2) = P(B_3) = 0.1$. Let A be the event that a part is defective. We are asked to find $P(A)$. We are told that $P(A|B_1) = 0.02, P(A|B_2) = 0.1$, and $P(A|B_3) = 0.3$. The law of total probability allows us to compute $P(A)$ as follows

$$P(A) = \sum_{j=1}^{3} P(B_j)P(A|B_j) = 0.8 * 0.02 + 0.1 * 0.1 + 0.1 * 0.3 = 0.056$$

(b) Section 2.3: Exercise 8

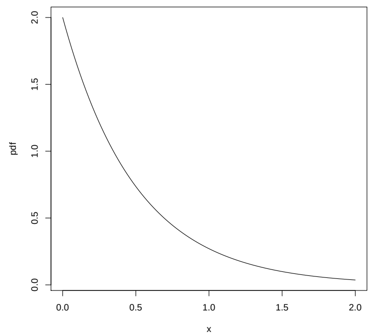
(a) If coin i is selected, the probability that the first head will be obtained on the fourth toss is $(1 - p_i)^3 p_i$. Therefore, the posterior probability that coin i was selected is

$$\pi_i = \frac{\frac{1}{5}(1 - p_i)^3 p_i}{\sum_{j=1}^{5} \frac{1}{5}(1 - p_j)^3 p_j} : i = 1, ..., 5$$

The five values are $\pi_1 = 0, \pi_2 = 0.5870, \pi_3 = 0.3478, \pi_4 = 0.0652, \pi_5 = 0$.

(b) If coin i is used, the probability that exactly three additional tosses will be required to obtain another head is $(1 - p_i)^2 p_i$. Therefore, the desired probability is

$$\sum_{i=1}^{5} \pi_i (1 - p_i)^2 p_i = 0.1291$$



Problem 4

From DeGroot & Schervish:

(a) Section 3.1: Exercise 6

The desired probability is the sum of the entries for $k = 0, 1, ..., 5$ for a binomial distribution with $n = 15$ and $p = 0.5$. The sum is 0.1509.

(b) Section 3.1: Exercise 8

The number of red balls obtained will have a binomial distribution with parameters $n = 20$ and $p = 0.1$. The required probability can be found from adding all entries for $k = 4, 5, ..., 20$. The sum is 0.1330.

Problem 5

From DeGroot & Schervish:

(a) Section 3.2: Exercise 8

(a) We must have

$$\int f(x)dx = \int_0^\infty c\exp(-2x)dx = \frac{1}{2}c = 1$$

Therefore, $c = 2$. The pdf looks like below

(b)

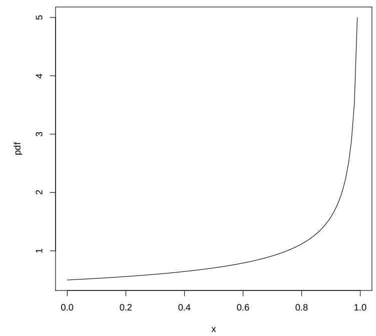
$$\int_1^2 f(x)dx = \exp(-2) - \exp(-4)$$

(b) Section 3.2: Exercise 10

We must have

$$\int f(x)dx = \int_0^1 \frac{c}{(1-x)^{1/2}}dx = 2c = 1$$

Therefore, $c = 1/2$.The pdf looks like below



Problem 6

From DeGroot & Schervish:

(a) Section 3.8: Exercise 6

From Exercise 5, we see that

$$g(y) = f\left[\frac{1}{a}(y-b)\right]\left|\frac{dx}{dy}\right| = \frac{1}{|a|}f\left(\frac{y-b}{a}\right)$$

X lies between 0 and 2 if and only if Y lies between 2 and 8. Therefore, it follows that for $2 < y < 8$

$$g(y) = \frac{1}{3}f\left(\frac{y-2}{3}\right) = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{y-2}{3} = \frac{1}{18}(y-2)$$

(b) Section 3.8: Exercise 10

For $0 < x < 2$, the cdf of X is

$$F(x) = \int_0^x f(t)dt = \int_0^x \frac{1}{2}tdt = \frac{1}{4}x^2$$

Therefore, by the probability integral transformation, we know that $U = F(X) = X^2/4$ will have a uniform distribution on the interval $[0, 1]$. Since U has this uniform distribution, we know from Exercise 9 that $Y = 2U^{1/3}$ will have the required pdf g. Therefore, teh required transformation is

$$Y = 2U^{1/3} = 2(X^2/4)^{1/3} = (2X^2)^{1/3}$$

Problem 7

From DeGroot & Schervish:

(a) Section 4.1: Exercise 8

$$\mathbb{E}[XY] = \int_0^1 \int_0^x xy \cdot 12y^2 dydx = \frac{1}{2}$$

(b) Section 4.1: Exercise 11

From Section 3.9, we know that the pdf's of Y_1 and Y_n are

$$\begin{aligned} g_1(y) &= \begin{cases} n(1-y)^{n-1} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ g_n(y) &= \begin{cases} ny^{n-1} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[Y_1] &= \int_0^1 yn(1-y)^{n-1} dy = \frac{1}{n+1} \\ \mathbb{E}[Y_n] &= \int_0^1 yny^{n-1} dy = \frac{n}{n+1} \end{aligned}$$

Problem 8

From DeGroot & Schervish:

(a) Section 4.2: Exercise 8

We see from Example 4.2.4 that

$$\mathbb{E}[X] = 8\left(\frac{10}{25}\right) = \frac{16}{5}$$

Since $Y = 8 - X$, we have that $\mathbb{E}[Y] = 8 - \mathbb{E}[X] = \frac{24}{5}$. Lastly, $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] = \frac{-8}{5}$.

(b) Section 4.2: Exercise 10

(a) Since the probability of success on any trial is $p = 1/2$, it follows from the material presented at the end of this section that the expected number of tosses is $1/p = 2$.

(b) The number of tails that will be obtained is equal to the total number of tosses minus one. Therefore, the expected number of tails is $2 - 1 = 1$.