Talis Biomedical Statistics Course - Homework 5

Due: 15 January 2020 9:00 AM

Name: [your first and last name]
Collaborators: [list all the people you worked with]
Date: [date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

The exercises can be found in the DeGroot & Schervish's textbook, available here.

Problem 1

(a) Section 3.9 Exercise 1.

The joint pdf of X_1 and X_2 is

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1, x_2 < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $Z = X_2$. Then $x_1 = y - z$ and $x_2 = z$. The matrix $\mathbf{A}^{-1}$ of coefficients of this transformation is

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{2}$$

Therefore, $\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} = 1$. It follows from Equation (3.9.20) that the joint pdf g_0 of Y and Z at every given point (y, z) will be

$$g_0(y, z) = f(y - z, z) \tag{3}$$

Therefore, the marginal pdf g of Y can be obtained from integrating out z, i.e.

$$g(y) = \int f(y - z, z)dz \tag{4}$$

This integrand is positive only for $0 < y - z < 1$ and $0 < z < 1$. We also know that the support for y is $(0, 2)$. Therefore, for $0 < y \leq 1$, it is positive only for $0 < z < y$ and we have

$$g(y) = \int_0^y 1 \cdot dz = y \tag{5}$$

For $1 < y < 2$, it is positive only for $y - 1 < z < 1$ and we have

$$g(y) = \int_{y-1}^1 1 \cdot dz = 2 - y \tag{6}$$

Our resulting pdf is therefore

$$g(y) = \begin{cases} y & \text{for } 0 < y \leq 1 \\ 2 - y & \text{for } 1 < y < 2 \end{cases} \tag{7}$$

(b) In Python, set up a simulation to see if the results match what you derived in Part (a). Specifically, do $1,000$ draws where you draw $X_1, X_2 \overset{iid}{\sim} U(0,1)$. Then, add them together to get $Y = X_1 + X_2$. Create a histogram on the resulting 1000 Y's. Comment on how it compares to the pdf you derived.

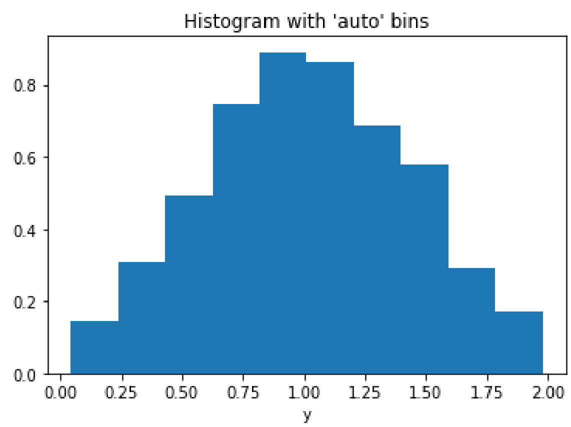

Figure 1: Histogram for 1,000 simulations, normalized to represent probabilities.

Figure 1 shows the resulting histogram from running the simulation. As expected from the pdf derived, we see that the density increases as y grows to 1 and then decreases as y continues to 2.

Problem 2

Section 3.9 Exercise 4.

Let $Z = X_1$. Then the transformation from X_1 and X_2 to Y and Z is a one-to-one transformation. The inverse transformation is

$$x_1 = z \tag{8}$$

$$x_2 = \frac{y}{z} \tag{9}$$

Therefore,

$$J = \det \begin{bmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ \frac{1}{z} & -\frac{y}{z^2} = -\frac{1}{z} \end{bmatrix} \tag{10}$$

For $0 < y < z < 1$, the joint pdf of Y and Z is

$$g(y,z) = f(z, \frac{y}{z})|J| = \left(z + \frac{y}{z}\right)\left(\frac{1}{z}\right) \tag{11}$$

It then follows that for $0 < y < 1$, the marginal pdf of Y is

$$g_Y(y) = \int_y^1 g(y,z)dz = 2(1-y) \tag{12}$$

Problem 3

(a) Section 4.1 Exercise 6.

$$\mathbb{E}\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x} \cdot 2x\, dx = 2 \tag{13}$$

(b) In Python, draw 1,000 samples from the pdf given in Part (a). Transform the values by taking $1/X$ and calculate the mean. How does it compare to the expectation you derived?

There is a method called "*Inverse transformation sampling*" that we could rely upon to sample from our pdf. However, we haven't actually learned about that in class yet. I will therefore rely on another approach that we can derive from simple intuition.

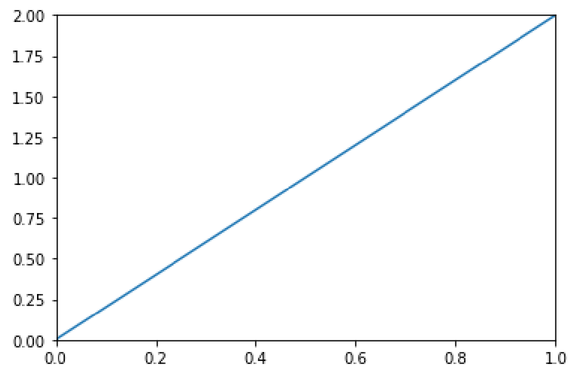

Figure 2: pdf of x.

Figure 2 shows the pdf of x. Note that it is simply a triangle, falling within the rectangle within $(0,1)$ on the x-axis and $(0,2)$ on the y axis. Since the area underneath the line represents the probability mass that is contained within each value of $x \in (0,1)$, we could just draw points within the rectangle and keep the

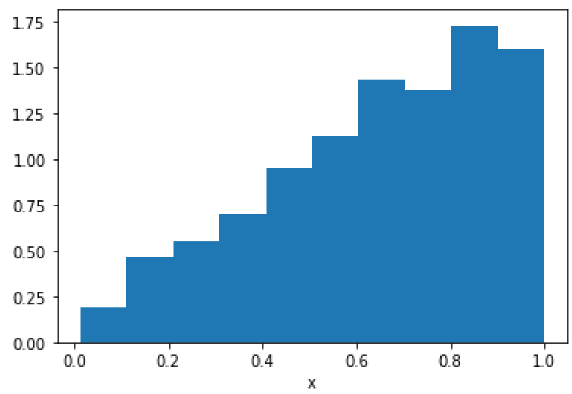

Figure 3: Histogram of x for 1,000 simulations.

point as an observation in x if the point falls within the triangle (disregarding the point and redrawing if it is above the triangle). This results in a density that approaches $2x$ as $n \to \infty$.

Figure 3 shows the histogram of 1,000 draws of x using this method. Taking the average of $1/x$ for each of the 1,000 draws results in a value of ≈ 2.28, which is 14% higher than our derived value (2).

Problem 4

(a) Section 4.1 Exercise 4.

There are a total of 8 words in the sentence and each are equally probable. Thus, the possible values of X and their respective probabilities are

x	f(x)
2	1/8
3	5/8
4	1/8
9	1/8

It then follows that

$$\mathbb{E}[X] = 2(1/8) + 3(5/8) + 4(1/8) + 9(1/8) = 15/4. \tag{14}$$

(b) Section 4.3 Exercise 2.

Using the same table as Part (a), we see that

$$\mathbb{E}[X^2] = 2^2(1/8) + 3^2(5/8) + 4^2(1/8) + 9^2(1/8) = 73/4. \tag{15}$$

Therefore,

$$var(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = \frac{73}{4} - \left(\frac{15}{4}\right)^2 = \frac{67}{16} \tag{16}$$

(c) In Python, draw $1,000$ words (with replacement) from the sentence given in Part (a) and record the number of letters in the word. Compute the variance of the result. How does it compare to the variance you derived?

The observed variance is 4.18, as opposed to the expected variance of 4.19, so they're pretty much the same.

Problem 5

Section 4.6 Exercise 5

We have that $\mathbb{E}[aX + b] = a\mu_X + b$ and $\mathbb{E}[cY + d] = c\mu_Y + d$. Therefore,

$$\begin{aligned} Cov(aX + b, cY + d) &= \mathbb{E}[[(aX + b) - (a\mu_X + b)][(cY + d) - (c\mu_Y + d)]] && (17) \\ &= \mathbb{E}[ac(X - \mu_X)(Y - \mu_Y)] && (18) \\ &= acCov(X, Y) && (19) \end{aligned}$$

Problem 6

Section 4.6 Exercise 10

$$\begin{aligned} var(X + Y) &= var(X) + var(Y) + 2Cov(X, Y) && (20) \\ var(X - Y) &= var(X) + var(Y) - 2Cov(X, Y) && (21) \end{aligned}$$

Since $Cov(X, Y) < 0$ it follows that $var(X + Y) < var(X - Y)$.

Problem 7

Section 4.7 Exercise 2

Let X denote the score of the selected student. Then

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|school]] = 0.2(80) + 0.3(76) + 0.5(84) = 80.8 \tag{22}$$

Problem 8

Section 4.7 Exercise 3

Since $\mathbb{E}[X|Y] = c$, then $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = c$ and $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|Y]] = \mathbb{E}[Y\mathbb{E}[X|Y]] = \mathbb{E}[cY] = c\mathbb{E}[Y]$. Therefore,

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = c\mathbb{E}[Y] - c\mathbb{E}[Y] = 0 \tag{23}$$