Talis Biomedical Statistics Course - Homework 6

Due: 22 January 2020 9:00 AM

Name: [your first and last name]
Collaborators: [list all the people you worked with]
Date: [date of submission]

By turning in this assignment, I agree by the **Stanford honor code** and declare that all of this is my own work.

The exercises can be found in the DeGroot & Schervish's textbook, available here.

Problem 1

Section 7.5 Exercise 2

Example 7.5.4 shows us that the MLE is $\bar{x}_n$. Applying it to this exercise gives us $58/70 = 29/35$.

Problem 2

Section 7.5 Exercise 3

The likelihood function for this given sample is

$$L(\theta) = p^{58}(1-p)^{12} \tag{1}$$

Restricting ourselves to the interval $1/2 \leq p \leq 2/3$, we see it is maximized at $p = 2/3$.

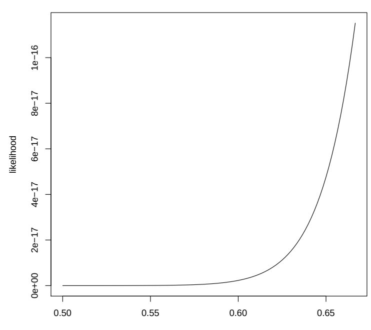

Figure 1: Likelihood for different values of θ.

Problem 3

Section 7.5 Exercise 6

Let $\theta = \sigma^2$. We can write out the log-likelihood as

$$\begin{aligned} \ell(\theta) &= \log f_n(x_1, \ldots, x_n|\theta) && (2) \\ &= \log \left(\frac{1}{(2\pi\theta)^{n/2}} \exp \left(-\frac{1}{2\theta} \sum_{i=1}^{n} (x_i - \mu)^2 \right) \right) && (3) \end{aligned}$$

Taking the derivative gives us

$$\frac{\partial}{\partial\theta} \ell(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} (x_i - \mu)^2 \tag{4}$$

Setting the derivative equal to 0 and solving for θ gives us

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 \tag{5}$$

Note that this is essentially the same estimator as the one provided in Equation 7.5.5, but with $\bar{x}_n$ switched out with the known mean μ.

Problem 4

(a) Section 7.5 Exercise 9

For $0 < x_i < 1 : i = 1, \ldots, n$, the log-likelihood is

$$\begin{aligned} \ell(\theta) &= \log f_n(x_1, \ldots, x_n|\theta) && (6) \\ &= \log \left(\theta^n \left(\prod_{i=1}^{n} x_i \right)^{\theta - 1} \right) && (7) \end{aligned}$$

Taking the derivative gives us

$$\frac{\partial}{\partial} \ell(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i \tag{8}$$

Setting the derivative equal to 0 and solving for θ gives us

$$\hat{\theta}_n = -\frac{n}{\sum_{i=1}^{n} \log x_i}. \tag{9}$$

It should be noted that $\hat{\theta} > 0$.

(b) Suppose that $X_1, \dots, X_n$ form a random sample from a distribution with the pdf given in Part (a). Find the MLE of $e^{-1/\theta}$. *n.b. you'll want to read Section 7.6 before answering this part.*

The MLE of $\exp(-1/\theta)$ is $\exp(1/\hat{\theta})$, where $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$ is the MLE of θ. Specifically, we have

$$\exp\left(\sum_{i=1}^{n} \log x_i/n\right) = \exp\left(\log\left[\prod_{i=1}^{n} x_i\right]^{1/n}\right) = \left(\prod_{i=1}^{n} x_i\right)^{1/n} \tag{10}$$

Problem 5

Section 7.6 Exercise 3

Recall that the median of any distribution is m where $\int_{-\infty}^{m} f(x)dx = 1/2$. Substituting in the pdf for the exponential distribution gives us

$$\int_0^m \beta \exp(-\beta x)dx = \frac{1}{2} \tag{11}$$

We solve for m by solving the integral and subsequently solving for m. Solving the integral gives us

$$\int_0^m \beta \exp(-\beta x)dx = -\exp(-\beta x)\Big|_0^m = -\exp(-\beta m) + 1 \tag{12}$$

Setting that equal to $1/2$ and solving for m gives us $m = (\log 2)/\beta$. It follows that $\hat{m} = (\log 2)/\hat{\beta}$. To find $\hat{\beta}$, we need to derive the log-likelihood, set it equal to 0, and solve for β. The log-likelihood is

$$\begin{aligned} \ell(\beta) &= \log f_n(x_1, \dots, x_n|\beta) & (13) \\ &= \log\left(\beta^n \exp(-\beta \sum_{i=1}^{n} x_i)\right) & (14) \end{aligned}$$

Taking the derivative gives us

$$\frac{\partial}{\partial}\ell(\beta) = \frac{n}{\beta} - \sum_{i=1}^{n} x_i \tag{15}$$

Setting the derivative equal to 0 and solving for β gives us

$$\hat{\beta}_n = n/\sum_{i=1}^{n} x_i = 1/\bar{x}_n \tag{16}$$

Thus, our MLE is $\hat{m} = (\log 2)\bar{x}_n$.

Problem 6

Section 7.6 Exercise 4

The probability that a given lamp will fail in a period of T hours is $p = 1 - \exp(\beta T)$. The probability that exactly x lamps will fail is

$$\binom{n}{n} p^x (1-p)^{n-x} \tag{17}$$

Example 7.5.4 shows that the MLE for p is $\hat{p} = x/n$. Since $\beta = -\log(1-p)/T$, it follows that the MLE for β is $\hat{\beta} = -\log(1 - x/n)/T$.