Exercise 1

$P(C=T|R=T,S=T,W=T) \propto P(R=T,S=T,W=T|C=T) \cdot P(C=T) =$
$= P(S=T|C=T) \cdot P(R=T|C=T) \cdot P(W=T|S=T,R=T) \cdot P(C=T) =$
$= 0.1 \cdot 0.8 \cdot 0.99 \cdot 0.5 = 0.0396$

Normalisation factor: $P(C=F|R=T,S=T,W=T) \propto P(S=T|C=F) \cdot P(R=T|C=F) \cdot P(W=T|S=T,R=T) \cdot P(C=F) = 0.5 \cdot 0.2 \cdot 0.99 \cdot 0.5 = 0.0495$

Renormalisation: $P(C=T|R=T,S=T,W=T) = 0.0396/(0.0396+0.0495) \approx 0.444$

$P(C=T|R=F,S=T,W=T) \propto P(R=F,S=T,W=T|C=T) \cdot P(C=T) =$
$= P(S=T|C=T) \cdot P(R=F|C=T) \cdot P(W=T|S=T,R=F) \cdot P(C=T) =$
$= 0.1 \cdot 0.2 \cdot 0.9 \cdot 0.5 = 0.009$

Normalisation factor:
$P(C=F|R=F,S=T,W=T) \propto P(R=F,S=T,W=T|C=F) \cdot P(C=F) =$
$= P(S=T|C=F) \cdot P(R=F|C=F) \cdot P(W=T|S=T,R=F) \cdot P(C=F) =$
$= 0.5 \cdot 0.8 \cdot 0.9 \cdot 0.5 = 0.18$

Renormalisation: $P(C=T|R=F,S=T,W=T) = 0.009/(0.009+0.18) \approx 0.048$

$P(R=T|C=T,S=T,W=T) \propto P(C=T,S=T,W=T|R=T) \cdot P(R=T) =$
$= P(C=T|R=T) \cdot P(S=T|C=T) \cdot P(W=T|R=T,S=T) \cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)] \propto P(R=T|C=T) \cdot P(C=T)$
$\cdot P(S=T|C=T) \cdot P(W=T|R=T,S=T) \cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)] = 0.8 \cdot 0.5 \cdot 0.1 \cdot 0.99 \cdot 0.5 = 0.0198$

Normalisation factor:
$P(R=F|C=T,S=T,W=T) \propto P(C=T,S=T,W=T|R=F) \cdot P(R=F) =$
$= P(C=T|R=F) \cdot P(S=T|C=T) \cdot P(W=T|R=F,S=T) \cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)] \propto P(R=F|C=T) \cdot P(C=T)$
$\cdot P(S=T|C=T) \cdot P(W=T|R=F,S=T) \cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)] = 0.2 \cdot 0.5 \cdot 0.1 \cdot 0.9 \cdot 0.5 = 0.0045$

Renormalisation: $P(C=T|R=F,S=T,W=T) = 0.0198/(0.0198+0.0045) \approx 0.815$

$P(R=T|C=F,S=T,W=T) \propto P(C=F,S=T,W=T|R=T) \cdot P(R=T) =$
$= P(C=F|R=T) \cdot P(S=T|C=F) \cdot P(W=T|R=T,S=T) \cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)] \propto P(R=T|C=F) \cdot P(C=F)$
$\cdot P(S=T|C=F) \cdot P(W=T|R=T,S=T) \cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)] = 0.2 \cdot 0.5 \cdot 0.5 \cdot 0.99 \cdot 0.5 = 0.02475$

Normalisation factor:
$P(R=F|C=F,S=T,W=T) \propto P(C=F,S=T,W=T|R=F) \cdot P(R=F) =$
$= P(C=F|R=F) \cdot P(S=T|C=F) \cdot P(W=T|R=F,S=T) \cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)] \propto P(R=F|C=F) \cdot P(C=F)$
$\cdot P(S=T|C=F) \cdot P(W=T|R=F,S=T) \cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)] = 0.8 \cdot 0.5 \cdot 0.5 \cdot 0.9 \cdot 0.5 = 0.09$

Renormalisation: $P(C=T|R=F,S=T,W=T) = 0.02475/(0.02475+0.09) \approx 0.216$

Exercise 2 & 3

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0.53
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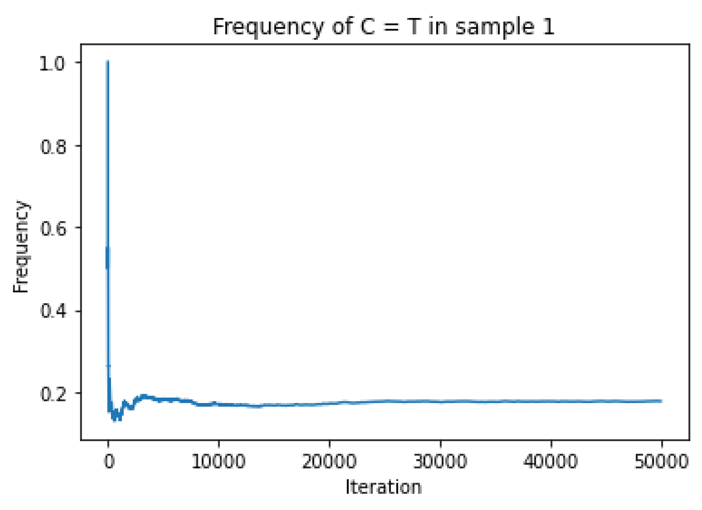
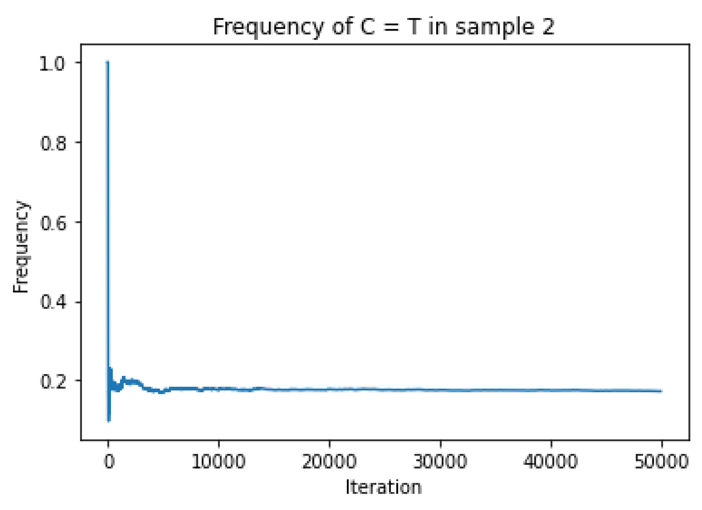
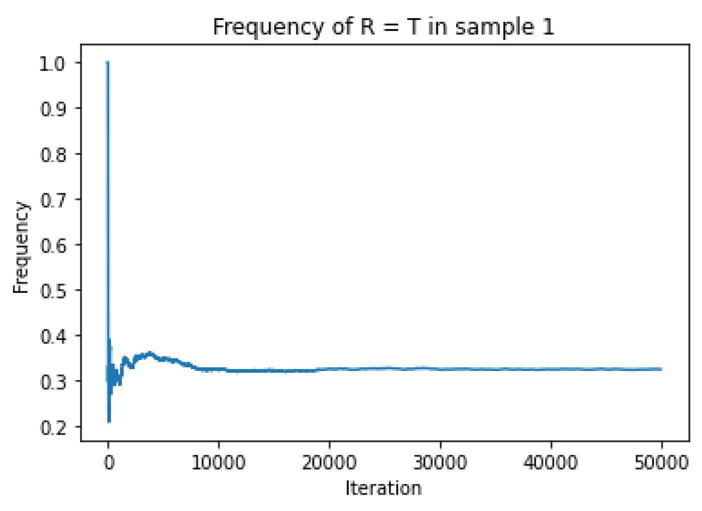
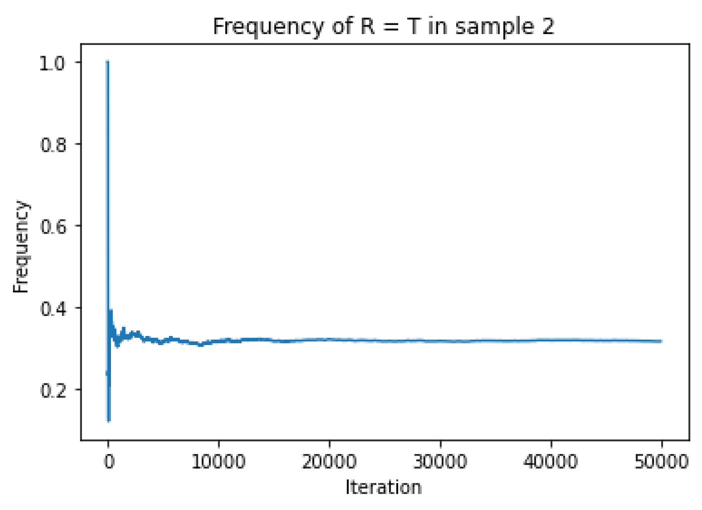
The value of P(R = T | S = T, W = T) estimated based on 100 Gibbs samples was equal to 0.53. To check the quality of the estimation, we need to investigate convergence of our chain.

Exercise 4 & 5



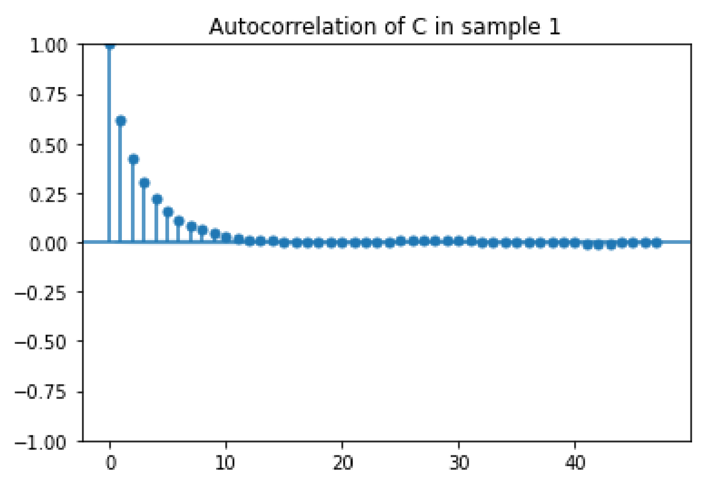


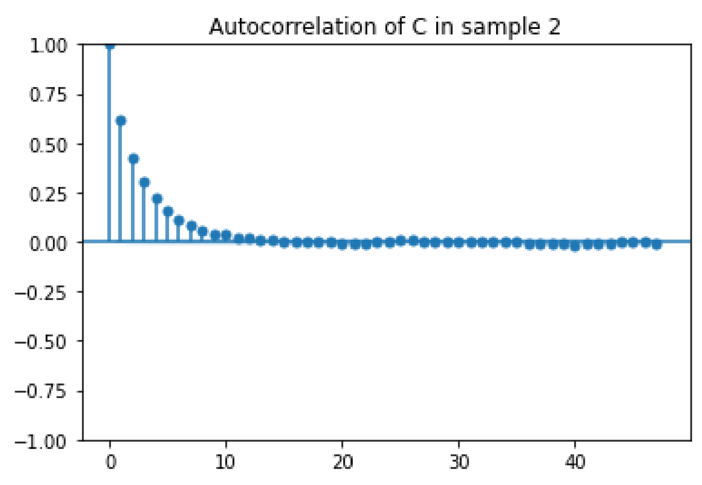


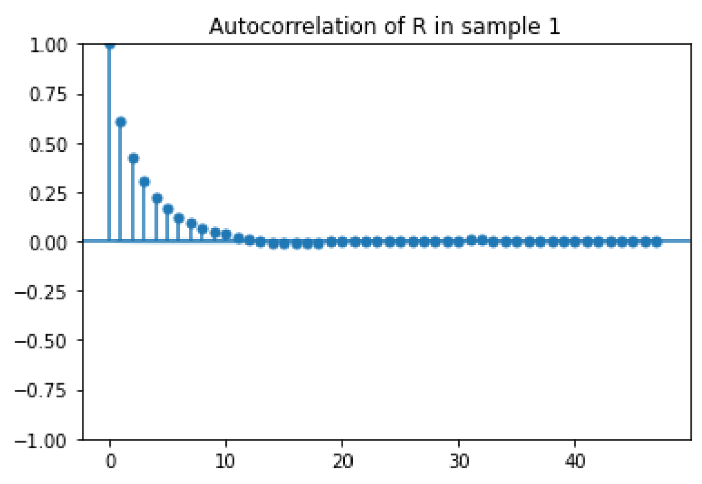


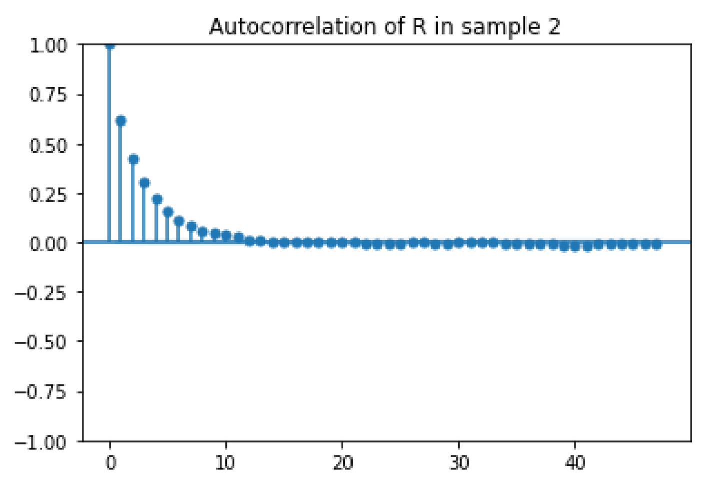
Based on the plots above we can conclude that relative frequencies of both C and R stabilise after around 10000 iterations - a number that will be used later as a burn-in time. Notably, the frequencies in first iterations are very divergent, which means implementing burn-in can significantly improve our estimations.

Exercise 6









We can observe that lag-10 correlation is close to 0 for all the datasets. Therefore, we can expect to get approximately independent samples when choosing every 10-th value of (C, R) drawn by the Gibbs sampler.

Exercise 7 & 8

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0.33
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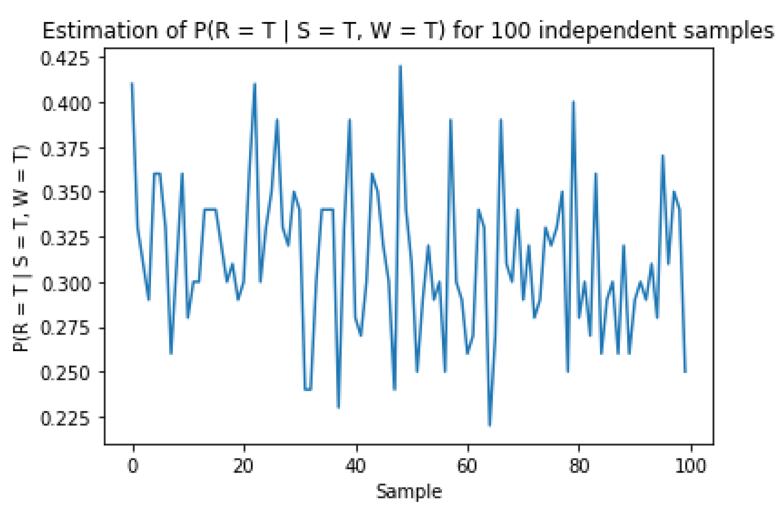
The value of P(R = T | S = T, W = T) estimated based on 100 Gibbs samples was equal to 0.33 when using random seed value equal to 43. However, the results of the estimation vary to some degree depending on the choice of the random seed. To test that variance, I decided to repeat the sampling process 100 times and compare the estimations.



Even though the mean of the estimations is close to the value obtained for random seed=43, the variance is high. It suggests that drawing 100 Gibbs samples might not be enough to approximate P(R = T | S = T, W = T).

Exercise 9

$P(R=T|S=T,W=T) \propto P(S=T,W=T|R=T) \cdot P(R=T) =$
$= [P(C=F|R=T) \cdot P(S=T|C=F) + P(C=T|R=T) \cdot P(S=T|C=T)] \cdot P(W=T|R=T,S=T) \cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)]$
$\propto [P(R=T|C=F) \cdot P(C=F) \cdot P(S=T|C=F) + P(R=T|C=T) \cdot P(C=T) \cdot P(S=T|C=T)] \cdot P(W=T|R=T,S=T)$
$\cdot [P(R=T|C=T) \cdot P(C=T) + P(R=T|C=F) \cdot P(C=F)] = [0.2 \cdot 0.5 \cdot 0.5 + 0.8 \cdot 0.5 \cdot 0.1] \cdot 0.99 \cdot 0.5 = 0.04455$

Normalisation factor:
$P(R=F|S=T,W=T) \propto P(S=T,W=T|R=F) \cdot P(R=F) =$
$= [P(C=F|R=F) \cdot P(S=T|C=F) + P(C=T|R=F) \cdot P(S=T|C=T)] \cdot P(W=T|R=F,S=T) \cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)]$
$\propto [P(R=F|C=F) \cdot P(C=F) \cdot P(S=T|C=F) + P(R=F|C=T) \cdot P(C=T) \cdot P(S=T|C=T)] \cdot P(W=T|R=F,S=T)$
$\cdot [P(R=F|C=T) \cdot P(C=T) + P(R=F|C=F) \cdot P(C=F)] = [0.8 \cdot 0.5 \cdot 0.5 + 0.2 \cdot 0.5 \cdot 0.1] \cdot 0.9 \cdot 0.5 = 0.0945$

Renormalisation: $P(R=T|S=T,W=T) = 0.04455/(0.04455+0.0945) \approx 0.32$

The value obtained analytically is close to the one obtained from Gibbs sampler with implemented burn-in and thinning out. In our case, implementing burn-in and thinning out drastically improved the estimation and made it available to obtain a satisfying approximation with as little as 100 samples.