Thailand TST 2016

Day 1

Friday, December 25, 2015

Problem 1. Santa has k piles of gifts, containing $h_1, h_2, \ldots, h_k$ gifts each. If

i) $1 \leq h_1 < h_2 < \cdots < h_k$, and

ii) no matter how Santa split a pile into two piles, there will always be two piles in the resulting $k+1$ piles that contain equal number of gifts.

Prove that for all $1 \leq s \leq k$, $h_s \leq 2s$.

Problem 2. Let ABC be an acute-angled triangle and altitudes AA_1 and BB_1 intersect at H. Consider circles ω_1 and ω_2 with centers H and B and with radii HB_1 and BB_1 respectively. Let CN and CK be the tangent lines from C to circles ω_1 and ω_2 respectively($N \neq B_1, K \neq B_1$). Prove that A_1, N and K are collinear.

Problem 3. **(N2)** Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geq 2b + 2$.

Problem 4. **(A2)** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy the following condition:

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

for all $x, y \in \mathbb{Z}$.

Language : English *Time : 3 hours*

Each problem is worth 7 points

Thailand TST 2016

Day 2

Thursday, January 14, 2016

Problem 1. **(A3)** Let n be a positive integer. Find the maximum value of the sum

$$\sum_{1 \leq r < s \leq 2n} (r - s - n) x_r x_s$$

where $-1 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, 2n$

Problem 2. Each square of $n \times n$ is colored by n different colors such that there are equal unit squares of each color. Prove that there is a row or column containing at least $\sqrt{n}$ distinct colors.

Problem 3. **(G3)** Let ABC be a triangle with $\angle C = 90°$, and H be the foot of altitude from C. Let D be a point in $\triangle HBC$ such that CH bisects AD, and let BD intersect CH at P. Let ω be the semicircle with diameter BD that intersects BC internally, and let PQ be the tangent from P to ω. Prove that lines AD and CQ intersect on ω.

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

Day 3

Friday, January 15, 2016

Problem 1. Let ω be the circumcircle of triangle ABC, and let M be a midpoint of $\overset{\frown}{BC}$, not containing A. Let D and E be tangent points of the incircle and A-excircle of the triangle ABC with line BC, respectively. Draw MD and ME intersect ω again at points $T \neq D$ and $R \neq E$, respectively. Let RI_A intersects ω at $S \neq R$, where I_A is the A-excenter of the triangle ABC. Let I be the incenter of the triangle ABC. Show that T, I, and S are collinear.

Problem 2. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{\sqrt{2}a^2b}{2a+b} + \frac{\sqrt{2}b^2c}{2b+c} + \frac{\sqrt{2}c^2a}{2c+a} \leq \frac{\sqrt{a^2+b^2}}{2ab+1} + \frac{\sqrt{b^2+c^2}}{2bc+1} + \frac{\sqrt{c^2+a^2}}{2ca+1}$$

Problem 3. (C3) Let A be a set of positive integers. We call a partition of A into two disjoint sets A_1, A_2 good if the least common multiple of all members of A_1 is equal to the greatest common denominator of all members of A_2. Find the least positive integer n such that there exists a set of n positive integers with exactly 2015 good partitions.

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

APMO

Tuesday, March 8, 2016

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(z+1)f(x+y) = f(xf(z)+y) + f(yf(z)+x),$$

for all positive real numbers x, y, z.

Language : English *Time : 4 hours*

Each problem is worth 7 points

Thailand TST 2016

Day 4

Monday, March 14, 2016

Problem 1. (G7) Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS, BQOP, CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ and RS are either parallel or concurrent to each other.

Problem 2. (A6) Let n be a fixed integer with $n \geq 2$. We say that two polynomials P and Q with real coefficients are twin if for each $i = 1, 2, \ldots, n$ the sequences

$$P(2015i), P(2015i - 1), \ldots, P(2015i - 2014) \text{ and}$$

$$Q(2015i), Q(2015i - 1), \ldots, Q(2015i - 2014)$$

are permutations of each other.

(a) Prove that there exist distinct twin polynomials of degree .

(b) Prove that there does not exist distinct twin polynomials of degree .

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

Day 5

Wednesday, March 16, 2016

Problem 1. Hillary and Donald play a game on a set of $n \geq 5$ points, no three of which are collinear, with players alternating turns. In each turn, a player choose two points that have not been joined and join them with a line segment. A player wins when after his/her turn, all points are an endpoint of at least one line segment. If Hillary goes first, then for which values of n will Donald win regardless of how Hillary plays?

Problem 2. (G4) Let ABC be a triangle and M the midpoint of BC. A circle that pass through A and M intersect AB, AC at P, Q respectively. T is the point such that $BPTQ$ is a parallelogram. If T lies on the circumcircle of $\triangle ABC$, find all possible values of $\dfrac{BT}{BM}$.

Problem 3. (N4) Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive integers with $a_0, b_0 \geq 2$ such that

$$a_{n+1} = \gcd(a_n, b_n) + 1, \text{ and } b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1$$

for all $n \geq 0$. Prove that the sequence $\{a_n\}$ is eventually periodic, i.e. there exists integers N, t such that for all $n \geq N$, $a_{n+t} = a_n$.

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

Day 6

Friday, March 25, 2016

Problem 1. Find a monic polynomial $P(x)$ with rational coefficients such that $P(\sqrt[3]{3}) = \sqrt[3]{2}$ with smallest degree, or prove that no such polynomial exists.

Problem 2. Dumbledore has n bags of candies, containing a total of n^2 candies. Dumbledore can choose two bags containing a total of even candies, and transfer candies between them so that there are equal candies in each. Find all n such that no matter how the candies are distributed in the starting position, Dumbledore can always end up having all bags containing equal candies.

Problem 3. Let ω_1, ω_2 be circles intersecting at point A, B. Line l_1 through A intersect ω_1, ω_2 again at C, E, and line l_2 through B intersect ω_1, ω_2 again at D, F. Point G is on l_1 between A, E, and point H is on l_2 between B, F. Lines CH, DG intersect ω_1 again at I, J, and lines FG, EH intersect ω_2 again at M, N. Lines CH, FG intersect at K, and lines DG, EH intersect at L. Assume that points $A, B, \ldots, N$ are all distinct. Prove that I, J, K, L, M, N are concyclic.

Language : English

Time : 4 hours and 30 minutes

Each problem is worth 7 points

Thailand TST 2016

Day 7

Saturday, March 26, 2016

Problem 1. Let ABC be a triangle with its circumcircle ω. Let D be a point on AB. Let Γ be the circle that touches lines DB and DC and externally tangents to ω and lines DB and DC, respectively. Let X be the intersection point of MN and the external angle bisector of $\angle ABC$. Show that AX is the angle bisector of $\angle BAC$.

Problem 2. Let $a_1 = 11^{11}, a_2 = 12^{12}, a_3 = 13^{13}$, and $a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|$ for all $n \geq 4$. Determine $a_{14^{14}}$.

Problem 3. **(A5)** Let $2\mathbb{Z}+1$ be the set of odd integers. Find all functions $f : \mathbb{Z} \to 2\mathbb{Z}+1$ such that

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for all $x, y \in \mathbb{Z}$.

Language : English

Time : 4 hours and 30 minutes

Each problem is worth 7 points

Thailand TST 2016

Day 8

Wednesday, April 20, 2016

Problem 1. Find all complex numbers $z = a + bi$ such that $a, b \in \mathbb{Z}$ and satisfy the following equation

$$3|z|^2 + |z + 4 + 2i|^2 + 6(z + i - z)^2 = 14$$

Problem 2. Find all continuous periodic functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following condition:

$$(1 + f(x)f(y))f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

Problem 3. Let n be an integer such that $n \geq 2$. Color each square of an $n \times n$ table either red or blue. A domino is called *chic* if it covers squares of different colors ,and *cool* if it covers two squares of the same color.

Find the largest integer k such that no matter how the table is colored, we can always put k non-overlapping dominoes on the table such that they are either all chic or all cool.

Problem 4. For a 2015-tuple $(a_1, a_2, \ldots, a_{2015})$, we can choose $1 \leq m, n \leq 2015$ such that a_m is even, and replace a_m, a_n with $\frac{a_m}{2}, a_n + \frac{a_m}{2}$ respectively.

Prove that starting with $(1, 2, \ldots, 2015)$, we can arrive at any permutation of $(1, 2, \ldots, 2015)$.

Problem 5. Let ABC be a triangle such that $BC < CA < AB$. Let the circle with center B and radius BC intersects the side AC and the circumcircle of triangle ABC at points D and E, respectively. Prove that $AB \perp DE$

Problem 6. Prove that for any prime numbers p and positive integers k, there exists a positive integer n such that the decimal representation of p^n contains a string of k consecutive same digits.

Language : English *Time : 4 hours*

Each problem is worth 7 points

Thailand TST 2016

Day 9

Thursday, April 21, 2016

Problem 1. (N6) Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any $m, n \in \mathbb{Z}_{>0}$, we write $f^n(m) = f(f(\cdots f(m)\cdots))$, where there are n fs. Suppose that f has the following two properties:

(i) If $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$.

(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

Problem 2. (C6) Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exists infinitely many positive integers that are not clean.

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

Day 10

Thursday, April 28, 2016

Problem 1. **(C1)** In Lineland, there are n towns arranged on a straight line. Each town has a left-facing bulldozer placed to the left of the town and a right-facing bulldozer placed to the right of the town. The sizes of all 2n bulldozers are distinct.

When two bulldozers facing each other meet, the bigger one will push the smaller one off the road. However, bulldozers aren't designed for protection from the back, so if a bulldozer meets the back of another bulldozer, the former will push the latter off the road.

For towns A, B, where A is to the left of B, we say town A can *sweep* town B if the right-facing bulldozer of town A can move to town B without being pushed off the road, similarly, we say town B can sweep town A if the left-facing bulldozer of town B can move to town A without being pushed off the road. Prove that there is exactly one town which cannot be swept by any other town.

Problem 2. Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{abc}\Big(\sum_{cyc}\frac{3a\sqrt{a}}{2a+b}\Big) \leq \sum_{cyc}\frac{a^2(a+2b)}{\sqrt{b^2(b+c)+abc}}$$

Problem 3. Let ABC be an acute triangle with orthocenter H. Y, Z are points on AC, AB such that $\angle HYC = \angle HZB = 60^\circ$. Let U be the circumcenter of $\triangle HYZ$, and N the nine-point center of $\triangle ABC$. Prove that A, U, N are collinear.

Language : English *Time : 4 hours and 30 minutes*

Each problem is worth 7 points

Thailand TST 2016

Day 11

Friday, April 29, 2016

Problem 1. Let $ABCD$ be a convex quadrilateral such that $AC \perp BD$. Prove that there exist points P, Q, R, S on the sides AB, BC, CD, DA, respectively, such that $PR \perp QS$ and the area of the quadrilateral $PQRS$ is exactly half of that of the quadrilateral $ABCD$.

Problem 2. (N7*)

a) Prove that there does not exist a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\gcd(f(m) + n, f(n) + m) = 1$ for all $m, n \in \mathbb{Z}_{>0}$

b) Prove that there exists a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\gcd(f(m) + n, f(n) + m) \leq 2$ for all $m, n \in \mathbb{Z}_{>0}$.

Remark. The real N7 asked to find the least k such that there exists a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that satisfies $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m, n \in \mathbb{Z}_{>0}$.

Problem 3. Let n, k, and λ be positive integers. Let G be a simple graph on n vertices with the following properties:

- Every vertex v of G has degree k.
- Every pair of adjacent vertices in G has exactly λ common neighbors in G.

Suppose that the remainders of n, k, and λ when divided by 4 are 1, 2, and 2, respectively. Show that there exist four distinct vertices A, B, C, D of G such that A is adjacent to all B, C, and D, and such that the three points B, C, D are pairwise non-adjacent.

Remark. A graph G is said to be *simple* if G does not contain any loops or multiple edges. For two different vertices u and v in G, a *common neighbor* w of u and v is a vertex w in G such that w is adjacent to both u and v.

Language : English | *Time : 4 hours and 30 minutes*

Each problem is worth 7 points