Thailand
Team Selection Test
2015

Day 1

Thursday, December 25, 2014 | *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 | *Each problem is worth 7 points.*

1. Let $ABCD$ be a quadrilateral inscribed in the circle ω. Let the symmedians of the angle B of triangles ABD and CBD intersect ω again at point P and Q respectively. If lines CP and AB intersect at X, lines AQ and BC intersect at Y, then prove that points X, D, Y are collinear.

Note. The symmedian of the angle B of triangle ABC is the line symmetric to the median line from the point B (the line drawn from B to the midpoint of the side AC) with respect to the angle bisector of the angle B.

2. A group of seven people has the property that any six of them can be seated in a round table so that any two neighbors are friends. Prove that all seven people can also be seated in a round so that any neighbors are friends. (Assume that if A is a friend of B, then B is also friend of A.)

3. Find all prime numbers $p \le q \le r$ such that

$$p^a q^b r^c = 7^{2x} - 2^{2y} 7^{2x} - 10^{2x} + 2^{2x+2y} 25^x$$

for some positive integers a, b, c, x, y.

4. Let a, b, c be positive real numbers such that $a+b+c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Prove that

$$\frac{2a^2}{\sqrt{a^2+b^2}} + \frac{2b^2}{\sqrt{b^2+c^2}} + \frac{2c^2}{\sqrt{c^2+a^2}} \ge \sqrt{\frac{a^2+3}{2}} + \sqrt{\frac{b^2+3}{2}} + \sqrt{\frac{c^2+3}{2}}.$$

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Day 2

Friday, December 26, 2014 *Time allowed: 4 hours and 30 minute*

09.00 – 13.00 *Each problem is worth 7 points.*

1. Find the least positive integer k for which there exist polynomials $f, g \in \mathbb{Z}[X]$ (polynomials with integer coefficients) such that

$$f(X)(X+1)^{2014} + g(X)(X^{2014}+1) = k .$$

2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(x+2557) + 2557) = f(x+2557) \text{ for all } x \in \mathbb{R} .$$

3. Susan has three balls, initially with the number 56, 2014 and 2557 written on them, respectively. Each minute, Susan chooses one of the balls. If the chosen ball has the number c, and the other two balls have the numbers a and b, then Susan replaces the number c with the number $\dfrac{1}{c(a+b)}$.

Is it possible for Susan to reach a point where three balls have the number 56, 2015 and 2558?

4. Let P be a point in the interior of a triangle ABC. The three cevians AA', BB', CC' of P divide the triangle into six triangles. Prove that the circumcenters of the six triangles are concyclic if and only if P is the centroid of ΔABC.

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Day 3

Tuesday, March 10, 2015 *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

1. Let $P(x)$ be a polynomial with integer coefficients and $x_1, x_2, x_3, \dots, x_{2015}$ be distinct integers such that $1 \le P(x_1), P(x_2), P(x_3), \dots, P(x_{2015}) \le 2014$.
Prove that $P(x_1) = P(x_2) = P(x_3) = \dots = P(x_{2015})$

2. Find the maximum number of colored squares in 8x8 board which satisfy the condition: for every square in the board, there are at most one adjacent square which is colored. (the adjacent squares are squares that share a same side)

3. Let ABC be a triangle such that $\angle ACB = 60°$. Let E be a point on side AC such that $CE < BC$ and D be a point on side BC such that $\dfrac{AE}{BD} = \dfrac{BC}{CE} - 1$. Let P be the intersection of AD and BE. And Q be the intersection of circumcircle of AEP and BDP, distinct from P. Prove that QE and BC are parallel.

4. Let S be non-empty finite set of integer. For non-empty subset $T = \{t_1, t_2, t_3, \dots, t_k\}$ of S, define **score** of T is $(-2)^k \cdot \gcd(t_1, t_2, t_3, \dots, t_k)$. Prove that sum of score of all non-empty subset of S is negative.

5. Let $n \ge 2$ be a positive integer. Let $A = \{a_1, a_2, a_3, \dots, a_n\}$ and $B = \{b_1, b_2, b_3, \dots, b_n\}$ be sets of distinct positive integer.
Let $S_A = \{a_i + a_j \mid 1 \le i < j \le n\}$ and $S_B = \{b_i + b_j \mid 1 \le i < j \le n\}$. Suppose that $A \ne B$ but $S_A = S_B$, find all possible value of n.

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Day 4

Tuesday, March 10, 2015 — *Time allowed: 4 hours*

09.00 – 13.00 — *Each problem is worth 7 points.*

1. Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point $Z \neq B$. The lines ZD and ZY intersect ω again at V and W, respectively. Prove that $AB = VW$.

2. Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b \ ?$$

3. A sequence of real numbers $a_0, a_1, \ldots$ is said to be **good** if the following three conditions hold.

(i) The value of a_0 is a positive integer.

(ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \dfrac{a_i}{a_i + 2}$.

(iii) There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

4. Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let B be the set of all points on the plane that lie on at least one blue line, and R the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects B in exactly $2n-1$ points, and also intersects R in exactly $2n-1$ points.

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5. Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

(i) a_{n+2} is divisible by a_n ;

(ii) $\left|s_{n+1} - (n+1)a_n\right| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \ldots + (-1)^{n+1}a_0$.

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Day 5

Wednesday, March 18, 2015 — *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

1. Let $a, b_1, c_1, b_2, c_2, \ldots, b_n, c_n$ be real number such that for all real number x

$$x^{2n} + ax^{2n-1} + ax^{2n-2} + \ldots + ax + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)\ldots(x^2 + b_nx + c_n)$$

Prove that $c_1 = c_2 = \ldots = c_n = 1$.

2. Let ABC be an acute triangle such that $AB > AC$. The angle bisector of $\angle ABC$ intersects the circumcircle of triangle ABC at $M \neq B$. Let ω be a circle which has BM as a diameter. Let E and F be the intersections of the perpendicular bisector of AB and ω such that F lies on the same side of C with respect to AB. Let P and N be the intersections of the perpendicular bisector of BC and ω such that N lies on the same side of A with respect to BC. If EP and FN intersect at X. Prove that XB and AC are parallel.

3. Let n be a positive integer. Color every element in the set $S = \{1, 2, 3, \ldots, n\}$ with one from three colors such that there are no more than $\frac{n}{2}$ elements with the same color.
Let A be a set of all the quadruples (a, b, c, d) such that $a, b, c, d \in S$ and $a + b + c + d$ is divisible by n and a, b, c, d are all the same color.
Let B be a set of all the quadruples (a, b, c, d) such that $a, b, c, d \in S$ and $a + b + c + d$ is divisible by n, a and b have the same color, c and d also have the same color but not as same as the colors of a and b.
Prove that $|B| \geq |A|$

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Day 6

Thursday, March 19, 2015 *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

1. Determine all positive integers m, n such that $\sqrt[3]{7m^2 - 13mn + 7n^2} = |m - n| + 1$.

2. Let $a, b, c \in [0,1]$. Prove that

$$\frac{a+b}{\sqrt{2c^2+3}} + \frac{b+c}{\sqrt{2a^2+3}} + \frac{c+a}{\sqrt{2b^2+3}} \le \frac{6}{\sqrt{5}}.$$

3. Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be a point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK.

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Day 7

Monday, March 29, 2015 — *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

1. Let the triangle ABC with $AB > BC$ be inscribed into a circle ω. Let points M and N be on the sides AB and BC respectively, such that $AM = CN$. Let the lines MN and AC intersect at K and let S be the midpoint of the arc AC passing through B. Let the circle with diameter KS intersect the line MN and the bisector of the angle $\angle MKA$ at T and D respectively.

1. Prove that $MT = TN$.
2. If P is the incenter of ΔCKN and Q is the excenter of ΔAKM opposite to K, then prove that $DP = DQ$.

2. In a group of students, they are separated into 3 classrooms. It is known that for each pair of students from the different classroom, among the students of the third classroom there are exactly 10 students who are familiar with both and exactly 10 students who are unfamiliar with both. Find the total number of students in all three classrooms.

3. Consider all polynomials $P(x)$ with real cofficients that have the following property: for any two real number x and y one has

$$|\, y^2 - P(x) \,| \le 2\,|\, x \,| \text{ if and only if } |\, x^2 - P(y) \,| \le 2\,|\, y \,|.$$

Determine all possible values of $P(0)$.

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Day 8

Tuesday, March 31, 2015 — *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

1. Find all non-decreasing functions $f:\mathbb{N}\to\mathbb{N}$ such that $f(1)=1$ and

$$f(n+f(n)+1)=f(n)+1\,,\; f(m)f(n)\le f(2mn+m+n) \text{ for all } m,n\in\mathbb{N}\,.$$

2. Find the necessary and sufficient conditions for a prime $p\in\mathbb{N}$ to be of the from $p=x^2+xy+y^2$ for some $x,y\in\mathbb{Z}$.

3. Let $ABCD$ be a convex quadrilateral with $\angle B=\angle D=90^\circ$. Let H be the foot of the perpendicular from A to BD. Choose points S and T on the sides AB and AD, respectively, so that H lies inside triangle SCT and $\angle SHC-\angle BSC=90^\circ$, $\angle THC-\angle DTC=90^\circ$. Prove that the line BD is tangent to the circumcircle of triangle SHT.

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Day 9

Thursday, April 16, 2015 | *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 | *Each problem is worth 7 points.*

1. Let $S=\{1,2,3,\ldots,2015\}$. Each number in S is colored either red or blue. In an operation, we are allowed to choose a number in S, and change the color of that number along with all numbers that are not relatively prime to it. Suppose that initially all of the numbers are red. Determine whether we can perform finitely many operations so that all of the numbers become blue.

2. Let n be an integer. Prove that there exists exactly one sequence of integers $x_0,x_1,x_2,\ldots$ such that

$$n=x_0(-2015)^0+x_1(-2015)^1+x_2(-2015)^2+\ldots \text{ and } 0\le x_i<2015 \text{ for all } i.$$

3. Let $ABCD$ be a convex quadrilateral, and let M be a point inside the quadrilateral. Suppose that the projections of M onto the sides AB,BC,CD,DA lie on a circle with center O. Let N be the reflection of M with respect to O. Prove that the projections of B onto the lines AM,AN,CM,CN also lie on a circle.

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Day 10

Friday, April 17, 2015 *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

1. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = 180^\circ - 2\angle BCD$. The incircle of triangle ABF is tangent to the sides AB and AD at points P and Q, respectively. Prove that the circumcircle of triangle APQ is tangent to the circumcircle of triangle BCD.

2. Determine all positive real numbers x, y, z, w that satisfy the following equation:

$$\frac{x^2 - yw}{y + 2z + w} + \frac{y^2 - zx}{z + 2w + x} + \frac{z^2 - wy}{w + 2x + y} + \frac{w^2 - xz}{x + 2y + z} = 0\,.$$

3. Determine all integers a, b such that $a^3 - a + 9 = 5b^2$.

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Day 11

Wednesday, April 29, 2015 | *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 | *Each problem is worth 7 points.*

1. Let ABC be a triangle. The point K, L and M lie on the segments BC,CA and AB, respectively, such that the lines AK, BL and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK and CKL whose inradii sum up to at least the inradius of the triangle ABC.

2. For a sequence $x_1, x_2, ..., x_n$ of real numbers, we define its **price** as

$$\max_{1 \le i \le n} | x_1 + x_2 + ... + x_n |.$$

Given n real numbers, David and George want to arrange them into a sequence with a low price. Diligent David checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $| x_1 |$ is as small as possible; among the remaining numbers, he chooses x_2 such that $| x_1 + x_2 |$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimize the value of $| x_1 + x_2 + ... + x_i |$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real number, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

3. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(m)+n) + f(m) = f(n) + f(3m) + 2014 \text{ for all integer } m \text{ and } n.$$

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Day 12

Thursday, April 30, 2015 *Time allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

2. Let $n>1$ be a given integer. Prove that infinitely many term if the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor$$

are odd.
(For a real number x , $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

3. Let ABC be a fixed acute0angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E,F) **interesting**, if the quadrilateral $KSAT$ is cyclic.
Suppose that the pairs (E_1,F_1) and (E_2,F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$