Thailand
Team Selection Test
2014

Day 1

Thursday, December 26, 2013 — *Time Allowed: 4 hours*

09.00 – 13.00 — *Each problem is worth 7 points.*

$\boxed{1}$ Let a, b, c be positive real numbers such that $a+b+c=abc$. Prove that

$$\frac{a(a+b)+b(b+c)+c(c+a)-3}{5} \geq \frac{ab^2+bc^2+ca^2}{a+b+c} \geq \sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}.$$

$\boxed{2}$ Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function satisfying these conditions:

(i) $f(x+y)\leq f(x)+f(y)$ for all $x, y \in (0,\infty)$.

(ii) $f(2556x)=2556f(x)$ for all $x\in(0,\infty)$.

Prove that $f(x)=f(1)\cdot x$ for all $x\in(0,\infty)$.

$\boxed{3}$ Let $\triangle ABC$ be a triangle and γ be an incircle of $\triangle ABC$ which touches side BC, CA, AB at points A_1, B_1, C_1 respectively. From the tangency point of γ with the unique circle ω_A passing through points B, C, drawing line through A_1 and meets ω_A at M_A. Define M_B, M_C analogously. Prove that

(i) A_1M_A, B_1M_B, C_1M_C are concurrent at a point called K,

(ii) K, I and O are collinear and satisfy the equation

$$\frac{KO}{KI}=\frac{R(M_AM_BM_C)}{r(ABC)}$$

where $R(XYZ)$ is the radius of circumcircle of triangle XYZ and $r(XYZ)$ is the radius of incircle of triangle XYZ.

$\boxed{4}$ Let n be a positive integer. $2^{2n-1}+1$ odd numbers are chosen from the integer between 2^{2n} and 2^{3n}. Prove that there exists the integers x, y which x doesn't divide y^2 and y doesn't divide x^2.

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Day 2

Friday, December 27, 2013 — *Time Allowed: 3 hours*

09.00 – 12.00 — *Each problem is worth 7 points.*

$\boxed{1}$ Let ABC be a scalene triangle with incenter I and excenters I_b, I_c opposite to the vertices B, C respectively. Let D be the intersection point of the perpendiculars from I_b to AC and from I_c to AB. If the bisectors of the angles BI_bD, CI_cD intersect at G, and the line through G parallel to AI intersect I_bI_c at H, prove that the circle centered at G with radius GH is tangent to the circumcircle of $\triangle ABC$.

$\boxed{2}$ Consider the sequence of integers (a_n) and (b_n) such that $|a_{n+2} - a_n| \leq 2$ for all $n \in \mathbb{N}$ and $a_m + a_n = b_{m^2+n^2}$ for all $m, n \in \mathbb{N}$. Prove that there are at most six distinct numbers in the sequence (a_n).

$\boxed{3}$ Find all odd primes p such that both of the numbers

$$1 + p + p^2 + p^3 + \cdots + p^{p-2} + p^{p-1} \quad \text{and} \quad 1 - p + p^2 - p^3 + \cdots - p^{p-2} + p^{p-1}$$

are primes.

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Day 3

Saturday, January 25, 2014 — *Time Allowed: 4 hours*

09.00 – 13.00 — *Each problem is worth 7 points.*

$\boxed{1}$ Let M be an arbitrary point on the circumcircle of $\triangle ABC$ and let the tangents from this point to the incircle of the triangle meet the sideline BC at X_1, X_2. Prove that the second intersection of the circumcircle of $\triangle MX_1X_2$ with the circumcircle of $\triangle ABC$ (distinct from M) coincides with the tangency point of the circumcircle with mixtilinear incircle in angle A. (As usual, the A-mixtilinear incircle names the circle tangent to AB, AC and to the circumcircle of $\triangle ABC$ internally.)

$\boxed{2}$ Let a, b, c be positive real numbers. Prove that

$$\begin{aligned}\sqrt[3]{a^2b^2}+\sqrt[3]{b^2c^2}+\sqrt[3]{c^2a^2} &\le \frac{a(b+c)}{\sqrt[3]{a^4}+\sqrt[3]{b^2c^2}}+\frac{b(c+a)}{\sqrt[3]{b^4}+\sqrt[3]{c^2a^2}}+\frac{c(a+b)}{\sqrt[3]{c^4}+\sqrt[3]{a^2b^2}}\\ &\le \sqrt[3]{a^4}+\sqrt[3]{b^4}+\sqrt[3]{c^4}.\end{aligned}$$

$\boxed{3}$ In a 17×17 matrix, each entry is written an integer from 1 to 17. Each number from 1 to 17 is written in exactly 17 entries. Prove that it is possible to find a row or a column of the matrix consisting of at least 5 different numbers.

$\boxed{4}$ Let a, b, c be positive integers for which $ac = b^2 + b + 1$. Prove that the equation $ax^2 - (2b+1)xy + cy^2 = 1$ has an integer solution.

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Day 4

Sunday, January 26, 2014 *Time Allowed: 4 hours*

09.00 – 13.00 *Each problem is worth 7 points.*

$\boxed{1}$ Let ABC be an acute-angled triangle with circumcenter O, orthocenter H, and nine-point center N. Let P be the second intersection of the line AO and the circumcircle of $\triangle OBC$, and Q be the reflection of A in BC. Show that the midpoint of the segment PQ lies on the line AN.

$\boxed{2}$ Every two of n towns in a country are connected by one way or two way road. It is known that for every k towns, there exists a round trip passing through each of these k towns exactly once. Find the maximal possible number of one way roads in this country.

$\boxed{3}$ Find all positive integers n for which

$$\left(1^4 + \frac{1}{4}\right)\left(2^4 + \frac{1}{4}\right)\cdots\left(n^4 + \frac{1}{4}\right)$$

is the square of a rational number.

$\boxed{4}$ Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$

$$f(xy) + f(x + y) = 1 + f(x)f(y).$$

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Day 5 (APMO)

Tuesday, March 11, 2014 — *Time Allowed: 4 hours*

09.00 – 13.00 — *Each problem is worth 7 points.*

$\boxed{1}$ For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \quad \text{and} \quad S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

$\boxed{2}$ Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its *representative*. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of at least one of A, B, C.

$\boxed{3}$ Find all positive integers n such that for any integer k there exists an integer a for which $a^2 + a - k$ is divisible by n.

$\boxed{4}$ Let n and b be positive integers. We say n is b-*discerning* if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.

(a) Prove that 8 is a 100-discerning.

(b) Prove that 9 is not a 100-discerning.

$\boxed{5}$ Circles ω and Ω meet at points A and B. Let M be the midpoint of arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let l_P be the tangent line to ω at P, and let l_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines l_P, l_Q and AB is tangent to Ω.

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Day 6

Monday, March 17, 2014 | *Time Allowed: 4 hours and 30 minutes*
09.00 – 13.30 | *Each problem is worth 7 points.*

$\boxed{1}$ Let ω be the circle of an acute-angled triangle ABC. Denote by M and N the midpoints of the sides AB and AC respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

$\boxed{2}$ Let a, b, c and d be positive real numbers such that $a^3 + b^3 + c^3 + d^3 \leq 4$. Prove that

$$\frac{1}{\sqrt{abc}} + \frac{1}{\sqrt{abd}} + \frac{1}{\sqrt{acd}} + \frac{1}{\sqrt{bcd}} \geq a + b + c + d.$$

$\boxed{3}$ Let $R(x) = \dfrac{F(x)}{G(x)}$ be a rational function with $F(x), G(x) \in \mathbb{Z}[x]$ and $F(x), G(x)$ have no common root modulo p for all primes p. Consider the rational function

$$Q(x) = \underbrace{R(R(\ldots(R(x))))}_{n \text{ times}}$$

where $n \in \mathbb{N}$. Prove that if there is an integer k such that $Q(k) = k$, then $R(R(k)) = k$. (2 points for proving the case $G(x) = 1$ (constant polynomial))

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Day 7

Tuesday, March 18, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let a function $f : \mathbb{Q} \to \mathbb{Z}$ be given. Prove that there exist two distinct rational numbers p and q such that

$$\frac{f(p) + f(q)}{2} \leq f\left(\frac{p+q}{2}\right).$$

$\boxed{2}$ Let $ABCD$ be a quadrilateral with AC bisects A and BD bisects B. A rhombus $KLMN$ is inscribed in the quadrilateral $ABCD$ where all vertices of the rhombus lie on different sides of $ABCD$. If ϕ denotes the non-obtuse angle of the rhombus, prove that $\phi \leq \max\{\angle BAD, \angle ABC\}$.

$\boxed{3}$ There are n piles of book, each pile with at least one book. Peter comes along and rearranges the books into $n+1$ piles, each with at least one book. Call a book *lucky* if it ends up in a pile with fewer books than it was before. Prove that there are at least two lucky books.

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Day 8

Tuesday, March 25, 2014 | *Time Allowed: 4 hours and 30 minutes*
09.00 – 13.30 | *Each problem is worth 7 points.*

$\boxed{1}$ Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that for all $k \in \mathbb{Z}$

$$f(k) \leq 2557 \quad \text{and} \quad f(k) \leq \frac{f(k-1) + f(k+1)}{2}.$$

$\boxed{2}$ Is there an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for each integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square? Justify your answer.

$\boxed{3}$ Let ABC be a scalene triangle with incircle $I(r)$ (centered at I with radius r) tangent to the sides BC, CA, AB at X, Y, Z respectively. Let X_1, Y_1, Z_1 be the images of X, Y, Z under the homothety $h(I, 2r)$ respectively (X_1, Y_1, Z_1 lie on the rays $\overrightarrow{IX}, \overrightarrow{IY}, \overrightarrow{IZ}$ respectively, such that $IX_1 = IY_1 = IZ_1 = 2r$). Prove that

a) AX_1, BY_1, CZ_1 are concurrent, say, at Q;

b) If P is the intersection point of the reflection of the line AQ respect to AI and the line OI where O is the circumcenter of $\triangle ABC$, then $\angle PCI = \angle ICQ$.

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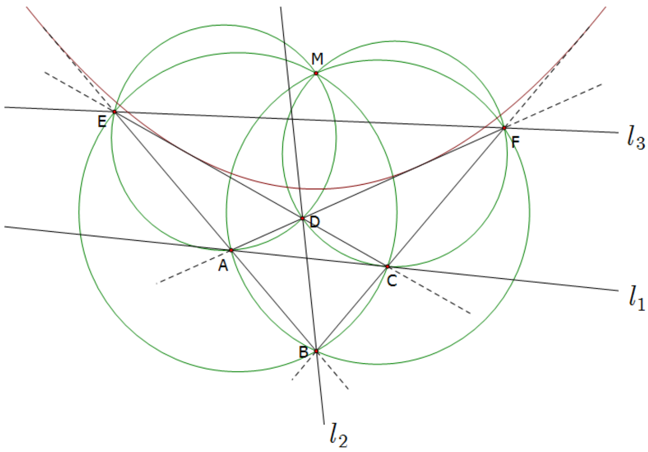
Day 9

Wednesday, March 26, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let a, b and c be positive integers such that $0 < a^2 + b^2 - abc \leq c$. Prove that $a^2 + b^2 - abc$ is a perfect square.

$\boxed{2}$ Prove that the nine-point circle of the triangle formed by the diagonals of a complete quadrilateral passes through the Miquel point of that quadrilateral. (Hint: Consider the focus of a parabola tangent to the quadrilateral.)



(M is the Miquel point and l_1, l_2, l_3 are the diagonals of a complete quadrilateral $ABCDEF$.)

$\boxed{3}$ **3.1** (4 points) Prove that every convex polyhedron which has no quadrilateral or pentagonal faces must have at least 4 triangular faces.

3.2 (3 points) Let P be a set consisting of 2557 distinct prime numbers. Let A be the set of all possible products of 1278 elements of P, and B be the set of all possible products of 1279 elements of P. Prove the existence of a one-to-one function f from A to B with the property that a divides $f(a)$ for all $a \in A$.

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Day 10

Sunday, March 30, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $x^2 + f(y) \mid xf(x) + y$ for all positive integers x and y.

$\boxed{2}$ Let G be a finite undirected graph. We can perform the following two operations on G.

(i) If a vertex V has an odd number of edges connected to it, we can delete V (and all the edges connected to it).

(ii) We can create a copy V' of every vertex V. In this operation, the two copies V' and W' are connected by an edge if and only if the original vertices V and W are connected by an edge, and each copy V' has an edge connecting it to the original vertex V. No other edges appear or disappear.

Prove that it is possible to apply a finite sequence of operations on G so that the resulting graph contains no edges.

$\boxed{3}$ Definition *A tangential quadrilateral is a convex quadrilateral with an incircle, i.e., a circle inside the quadrilateral that is tangent to all four sides.*

In a tangential quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides AB and CD intersect at E, the extensions of opposite sides BC and AD intersect at F, and assume that exactly one of the triangles CEF and AEF is outside of the quadrilateral $ABCD$. Let the incircle in triangle AEF be tangent to AE and AF at K and L respectively, and the incircle in triangle CEF be tangent to BF and DE at M and N respectively. Prove that

(a) K, L, M, N are concyclic;

(b) $ABCD$ is cyclic if and only if KN and LM are perpendicular.

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Day 11

Monday, March 31, 2014 *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

$\boxed{1}$ Let x, y and z be positive real numbers. Prove that

$$\frac{3x}{4x^2+4y^2+z^2}+\frac{3y}{4y^2+4z^2+x^2}+\frac{3z}{4z^2+4x^2+y^2} \le \sqrt{\frac{1}{x^2+xy+y^2}+\frac{1}{y^2+yz+z^2}+\frac{1}{z^2+zx+x^2}}.$$

$\boxed{2}$ Given a cyclic quadrilateral $ABCD$, let M be the set of 16 centers of all incircles and excircles of the triangles BCD, ACD, ABD and ABC. Prove that there exists two sets K, L such that each set consists of four parallel lines and any lines in $K \cup L$ contains exactly four points of M.

$\boxed{3}$ Solve in integers the equation

$$xy - 7\sqrt{x^2+y^2} = 1.$$

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Day 12

Monday, April 21, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let ω be the circumcircle centered at O of a non-isosceles triangle ABC. Let M (different from O) be the midpoint of the side BC. If the circumcircle of the triangle AMO intersects ω for the second time at D, prove that:

a. The intersection point of tangent lines to ω at the points A and D lies on the line BC;

b. The triangles AMB, ACD and DMB are similar.

$\boxed{2}$ For each positive integer k, let $L(k)$ be the largest prime divisor of k. Prove that there exist infinitely many positive integers n such that

$$L(n^4 + n^2 + 1) = L((n + 1)^4 + (n + 1)^2 + 1).$$

$\boxed{3}$ Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the relation

$$f(f(f(n))) = f(n + 1) + 1$$

for all $n \in \mathbb{N}_0$. Here $\mathbb{N}_0$ is the set of all nonnegative integers.

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Day 13

Tuesday, April 22, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Prove that among any 2000 distinct real numbers there exist four numbers $x > y$, $z > w$ with $x \neq z$ or $y \neq w$ such that

$$\left| \frac{x-y}{z-w} - 1 \right| < \frac{1}{100000}.$$

$\boxed{2}$ In a triangle ABC with $\angle B > \angle C$, let P and Q be two different points on the line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there is an interior point D on the segment BQ such that $PD = PB$. Let the ray AD intersect the circumcircle of $\triangle ABC$ at $R \neq A$. Prove that $QB = QR$.

$\boxed{3}$ In a country, some pairs of cities are connected by direct two-way flights and it is possible to go from any city to any other by a sequence of flights. Define the *distance* between two cities to be the least possible number of flights required to go from one of them to the other. Assume that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

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Day 14

Tuesday, April 29, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ A certain graph has mn edges and it is known that its edges can be painted in m colors in such a way that for any vertex all edges adjacent to this vertex have different colors. Prove that the edges can be painted in such a way that for any vertex all edges adjacent to this vertex have different colors and that there exist exactly n edges of each color.

$\boxed{2}$ Let a, b and c be positive real numbers. Prove that

$$\frac{a}{5+a^4+b^3}+\frac{b}{5+b^4+c^3}+\frac{c}{5+c^4+a^3} \leq \frac{1}{7}\left(\sqrt{\frac{a^2+2b^2}{a^2+ab+bc}}+\sqrt{\frac{b^2+2c^2}{b^2+bc+ca}}+\sqrt{\frac{c^2+2a^2}{c^2+ca+ab}}\right).$$

$\boxed{3}$ Prove that if a convex n-gon ($n \geq 3$) can cover any triangle with sides not exceeding 1, then its area is at least $\frac{1}{2}\cos 10^\circ$.

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Day 15

Wednesday, April 30, 2014 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Prove that every positive rational numbers can be represented in the form of

$$\frac{a^3+b^3}{c^3+d^3}$$

where a, b, c, d are positive integers.

$\boxed{2}$ Let I, I_A and O be the incenter, A-excenter and circumcenter of a triangle ABC respectively. Let M be the midpoint of the arc BC not containing A and K be the midpoint of the arc AM of the circumcircle ω of $\triangle ABC$. Let P be the second intersection point of KI and ω, and Q be the second intersection point of KI_A and ω. If the lines AM and BC intersect at N, prove that P, Q, N are collinear.

$\boxed{3}$ Prove that every positive integer is the difference of two relatively prime composite positive integers.