Thailand October Camp 2013 - Day 1
IPST
31 October 2013

Inequalities (3 hours)

1. Let x, y, z be positive real numbers. Prove that [sic]

$$4(x^2 + y^2 + z^2) \geqslant 3(xy + yz + zx).$$

2. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}} \geqslant \sqrt{\frac{2ab}{3a+b+2c}} + \sqrt{\frac{2bc}{3b+c+2a}} + \sqrt{\frac{2ca}{3c+a+2b}}.$$

3. For positive real numbers a, b, c such that $abc = 1$, prove that

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1}{(a+b+c+1)^2} + \frac{3}{8}\sqrt[3]{\frac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}} \geqslant 1.$$

Number Theory (3 hours)

1. Find all 3-tuples of positive integers (a, b, c) such that

$$(2^a - 1)(3^b - 1) = c!$$

2. Prove that there are infinitely many positive integer solutions to $x^8 = n! + 1$. [sic]

3. Let $s(n)$ denote the sum of digits of a positive integer n. Prove that $s(9^n) > 9$ for all $n \geqslant 3$.

Thailand October Camp 2013 - Day 2
IPST
1 November 2013

Combinatorics (3 hours)

1. Find the number of ways to put a number in every unit square of a 3×3 square such that any number is divisible by the number directly to the top and the number directly to the left of it, and the top-left number is 1 and the bottom-right number is 2013.

2. Find the number of permutations $(a_1, a_2, \ldots, a_{2013})$ of $(1, 2, \ldots, 2013)$ such that there are exactly two indices $i \in \{1, 2, \ldots, 2012\}$ where $a_i < a_{i+1}$.

3. Let S be the set of all 3-tuples (a, b, c) of positive integers such that $a + b + c = 2013$. Find
$$\sum_{(a,b,c)\in S} abc.$$

Algebra and Functional Equations (3 hours)

1. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(1 + xf(y)) = yf(x + y)$$
for all $x, y \in \mathbb{R}^+$.

2. In a triangle ABC, let $x = \cos \frac{A-B}{2}, y = \cos \frac{B-C}{2}, z = \cos \frac{C-A}{2}$. Prove that
$$x^4 + y^4 + z^4 \leqslant 1 + 2x^2y^2z^2.$$

3. Define $a_k = 2^{2^{k-2013}} + k$ for all integers k. Simplify
$$(a_0 + a_1)(a_1 - a_0)(a_2 - a_1) \cdots (a_{2013} - a_{2012}).$$

Thailand October Camp 2013 - Day 3
IPST
2 November 2013

Geometry (3 hours)

1. In a triangle ABC, $AC = BC$ and D is the midpoint of AB. Let E be an arbitrary point on line AB which is not B or D. Let O be the circumcenter of $\triangle ACE$ and F the intersection of the perpendicular from E to BC and the perpendicular to DO at D.

   Prove that the acute angle between BC and BF does not depend on the choice of point E.

2. In a triangle ABC, let the incircle with incenter I be tangent to BC at A_1, CA at B_1, and AB at C_1. Denote the intersection of AA_1 and BB_1 by G, the intersection of AC and A_1C_1 by X, and the intersection of BC and B_1C_1 by Y. Prove that $IG \perp XY$.

3. Let O be the incenter of a tangential quadrilateral $ABCD$. Prove that the orthocenters of $\triangle AOB, \triangle BOC, \triangle COD, \triangle DOA$ are collinear.