15th Thailand Mathematical Olympiad - Day 1
Nakhon Ratchasima
6 May 2018

Time: 4.5 hours

1. Let the incircle of triangle ABC tangent to BC, CA, AB at D, E, F respectively. Let P and Q be the midpoint of DF and DE respectively. Let PC intersect DE at R and BQ intersect DF at S. Prove that

   a) Points B, C, P, Q lie on a circle.

   b) Points P, Q, R, S lie on a circle.

2. Show that there are no functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + f(y)) = f(x) + y^2$$
   for all real number x and y.

3. Karakade has three flash drives of each of the six capacities $1, 2, 4, 8, 16, 32$ GB. She gives each of her 6 servants three flash drives of different capacities.

   Prove that either there are two capacities where each servant has at most one of the two capacities, or all servants have flash drives with different sum of capacities.

4. Let a, b, c be nonzero real numbers such that $a + b + c = 0$. Determine the maximum possible value of
$$\frac{a^2b^2c^2}{(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2)}$$

5. Let a, b be positive integers such that $5 \nmid a, b$ and $5^5 \mid a^5 + b^5$. What is the minimum possible value of $a + b$?

15th Thailand Mathematical Olympiad - Day 2
Nakhon Ratchasima
7 May 2018

Time: 4.5 hours

6. Let A be the set of all triples (x, y, z) of positive integers satisfying $2x^2 + 3y^3 = 4z^4$.

    a) Show that if $(x, y, z) \in A$ then 6 divides all of x, y, z.

    b) Show that A is an infinite set.

7. We color each number in the set $S = \{1, 2, ..., 61\}$ with one of 25 colors, where it is not necessary that every color gets used. Let m be the number of non-empty subsets of S such that every number in the subset has the same color. What is the minimum value of m?

8. There are $2n + 1$ tickets, each with a unique positive integer as the ticket number. It is known that the sum of all ticket numbers is more than 2330, but the sum of any n ticket numbers is at most 1165. What is the maximum value of n?

9. Let the incircle of $\triangle ABC$ be tangent to AB at D. Let P be a point on BC different from B and C. Let K and L be incenters of $\triangle ABP$ and $\triangle ACP$ respectively. Suppose the circumcircle of $\triangle KPL$ cuts AP again at Q. Prove that $AD = AQ$

10. Let a, b, c be nonzero real numbers. Suppose that functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$af(x + y) + bf(x - y) = cf(x) + g(y)$$

for all real numbers x and y such that $y > 2018$. Show that there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x - y) = 2f(x) + h(y)$$

for all real numbers x and y.

15th Thailand Mathematical Olympiad - Unofficial Solutions

1. From simple angle chasing, we get $\angle BPQ + \angle BCQ = (90 + \frac{A+B}{2}) + \frac{C}{2} = 180^\circ$ Therefore, B, C, P, Q are on the same circle. From $\angle QBP = \angle QCP$ and $\angle BPD = \angle DQC$, we get $\angle PSQ = \angle PRQ$. Therefore, P, Q, R, S are on the same circle.

2. Suppose that there is a function f. Let $P(x, y)$ denote the statement $f(x + f(y)) = f(x) + y^2$. $P(x, 0)$ gives $f(x + f(0)) = f(x)$ so $f(f(0)) = f(0)$. Comparing $P(x, 0)$ and $P(x, f(0))$ gives $f(0)^2 = 0$, therefore $f(0) = 0$.

   Now look at $P(0, x)$: $f(f(x)) = x^2$, $P(0, f(x))$: $f(x^2) = f(x)^2$, and $P(f(x), x)$: $f(2f(x)) = 2x^2$

   Substituting x with $f(x)$ in the last equation gives $f(2x^2) = 2f(x)^2$, while $P(f(x), x)$ gives $f(f(2f(x))) = f(2x^2)$, so
   $$4f(x)^2 = 2f(x)^2$$
   for all real x, so $f(x) \equiv 0$. However this is clearly not a solution, which is the desired contradiction.

   REMARK: There is also a solution by looking at $f(\mathbb{R})$.

3. Suppose there are two servants A and B such that the sum of capacities of A's flash drives is equal to of B's. (Else the problem is already done.) It's easy to see that A and B must have the exact same set of flash drives, say, with capacities $\{x, y, z\}$. Now pick a capacity $w \notin \{x, y, z\}$, and consider the servant that is not A or B, and does not have the w-flash drive. This servant must have at least one of the flash drives with capacities $\{x, y, z\}$, say the x-flash drive. We can now choose w and x as the two capacities.

4. ANSWER: 4/27.

   Plugging $c = -a - b$ in the problem, we want to find the maximum value of
   $$\frac{a^2b^2(a+b)^2}{(a^2+ab+b^2)^3}.$$
   From
   $$\begin{aligned}\frac{4}{27} - \frac{a^2b^2(a+b)^2}{(a^2+ab+b^2)^3} &= \frac{4(a^2+ab+b^2)^3 - 27a^2b^2(a+b)^2}{27(a^2+ab+b^2)^3} \\ &= \frac{(a-b)^2(2a+b)^2(2b+a)^2}{27(a^2+ab+b^2)^3} \\ &\geqslant 0,\end{aligned}$$
   it follows that the maximum value of $\frac{a^2b^2c^2}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$ is $\frac{4}{27}$ and this holds when (a, b, c) is $(x, x, -2x)$ and its permutations.

5. ANSWER: 625.

   From $x^5 \equiv x \pmod 5$, it follows that $5 \mid a + b$, where it's easy to show that $5 \mid a^4 - a^3b + a^2b^2 - ab^3 + b^4$ but $25 \nmid a^4 - a^3b + a^2b^2 - ab^3 + b^4$. Therefore, $5^4 \mid a + b$. Hence, the minimum value of $a + b$ is 625 which is attained at, for example, $(a, b) = (1, 624)$.

   REMARK: This problem is basically the LIFTING THE EXPONENT lemma.

6. a) It suffices to show that 2 and 3 divides each of x, y, z separately.

   To show that 2 divides each of x, y, z, first $3y^3 = 2(2z^4 - x^2)$ implies $2 \mid y$, so $y = 2y_1$ for some $y_1 \in \mathbb{Z}_{\geqslant 0}$. Then, $x^2 = 2(6y_1^3 + z^4)$ implies $2 \mid x$, hence $x = 2x_1$ for some $x' \in \mathbb{Z}_{\geqslant 0}$. Finally, this implies $z^4 = 2n^2 - 6m^3$ so $2 \mid z$.

   To show that 3 divides each of x, y, z, note that we have $4z^4 \equiv 2x^2 \pmod 3$, but $4z^4 \mod 3 \in \{0, 1\}$ and $2x^2 \mod 3 \in \{0, 2\}$, so $2x^2 \equiv 4z^4 \equiv 0 \mod 3$, hence $3 \mid x, z$. Now let $x = 3x_2, z = 3z_2$ where $x_2, z_3 \in \mathbb{Z}_{\geqslant 0}$ to get $y^3 = 3(36z_2^4 - x_2^2)$, so $3 \mid y$.

   b) $(144t^6, 24t^3, 12t^3) \in A$ for all positive integers t.

7. ANSWER: 119.

   Let x_i be the number of $a \in S$ that is colored with the i^{th} color. It's easy to see that $m = \sum\limits_{i=1}^{25} (2^{x_i} - 1)$.

   If there are $i, j \in \{1, 2, ..., 25\}$ such that $x_i - x_j \geqslant 2$, then the adjustment $(x_i, x_j) \rightarrow (x_i - 1, x_j + 1)$ decreases the value of m, hence when m is minimal, $|x_i - x_j| \leqslant 1$ for all i, j, which forces the multiset $\{x_1, \ldots, x_{25}\}$ to contain 11 threes and 14 twos.

   Hence $m \geqslant 11 \cdot (2^3 - 1) + 14 \cdot (2^2 - 1) = 119$, and this lower bound is obviously attainable.

8. ANSWER: 10.

   First, note that the set of ticket numbers $\{101, 102, ..., 121\}$ shows that 10 is attainable. Now we'll show that $n \leqslant 10$. Let the ticket numbers be $x_1 < x_2 < ... < x_{2n+1}$

   From the problem, we get

$$x_1 + ... + x_{2n+1} > 2330 \tag{1}$$
$$x_{n+2} + ... + x_{2n+1} \leqslant 1165 \tag{2}$$

   So, $x_1 + ... + x_{n+1} > x_{n+2} + ... + x_{2n+1}$, which implies

$$x_1 > \sum_{i=1}^{n} (x_{n+1+i} - x_{i+1}) \geqslant \sum_{i=1}^{n} n.$$

   Therefore $x_1 \geqslant n^2 + 1$, and it's easy to show that $x_i \geqslant n^2 + i$ for all 1. From (2), we get $1165 \geqslant \sum\limits_{i=1}^{n} n^2 + n + 1 + i$, which is $1165 \geqslant \frac{2n^3+3n^2+3n}{2}$, so $n \leqslant 10$.

9. Let the incircles of $\triangle ABP$ and $\triangle ACP$ be tangent to BC at M and N respectively. It's easy to see that $\angle LPK = 90°$. Let $\angle APK = x, \angle LPA = 90° - x$. Let O be the midpoint of KL, so O is the circumcenter of $KPLQ$. By Power of Point, we get $AQ \cdot AP = AO^2 - OK^2$.

   Applying law of cosine to $\triangle AOK$ and $\triangle BOK$ gives

$$AK^2 + AL^2 = 2(AO^2 + OK^2).$$

   Applying law of cosine to $\triangle APK$ and $\triangle APL$, we get

$$AK^2 = AP^2 + PK^2 - 2AP \cdot PK \cdot \cos x$$
$$AL^2 = AP^2 + PL^2 - 2AP \cdot PL \cdot \sin x$$

Adding the two equations gives

$$AK^2 + AL^2 = 2AP^2 + PK^2 + PL^2 - 2AP(PK \cdot \cos x + PL \cdot \sin x).$$

From $\angle LPK = 90^\circ$, we get $PK^2 + PL^2 = 4OK^2$. Therefore,

$$\begin{array}{rcl} 2(AO^2 + OK^2) & = & 2AP^2 + 4OK^2 - 2AP(PK \cdot \cos x + PL \cdot \sin x) \\ AO^2 - OK^2 & = & AP(AP - PK \cos x + PL \cdot \sin x) \\ AP \cdot AQ & = & AP(AP - PK \cos x + PL \cdot \sin x) \end{array}$$

Finally, from $PK \cdot \cos x = PM = \frac{AP+PB-AB}{2}$ and $PL \cdot \sin x = PN = \frac{AP+PC-AC}{2}$,

$$AQ = \frac{AB + AC - BP - CP}{2} = \frac{AB + AC - BC}{2} = AD.$$

10. We show that the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(k) = \frac{1}{c}\big(g(y_k + 2k) - 2g(y_k + k) + g(y_k)\big),$$

where $y_k = 2019 + 2|k|$ works. Fix a $k \in \mathbb{R}$ and note that $y_k, y_k + k, y_k + 2k > 2018$. Comparing $(x, y) = (x, y_k)$ and $(x + k, y_k + k)$ in the original equation gives

$$a\big(f(x + y_k + 2k) - f(x + y_k)\big) = c\big(f(x + k) - f(x)\big) + g(y_k + k) - g(y_k). \qquad (3)$$

Substituting $(x, y_k) \to (x - k, y_k + k)$ in (3) gives

$$a\big(f(x + y_k + 2k) - f(x + y_k)\big) = c\big(f(x) - f(x - k)\big) + g(y_k + 2k) - g(y_k + k).$$

Therefore, for all $x \in \mathbb{R}$,

$$f(x + k) + f(x - k) - 2f(x) = \frac{1}{c}\big(g(y_k + 2k) - 2g(y_k + k) + g(y_k)\big) = h(k),$$

hence h satisfy the required condition.