Thailand October Camp 2014 Quizzes
IPST

Number Theory Quiz (1.5 hours)

9:00-10:30 am, 20 October 2014

1. Find all primes $1 < p < 100$ such that the equation $x^2 - 6y^2 = p$ has an integer solution (x, y).

2. Determine the least integer $n > 1$ such that the quadratic mean of the first n positive integers is an integer.

   Note: the quadratic mean of $a_1, a_2, \ldots, a_n$ is defined to be $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$.

Algebra Quiz (1.5 hours)

4:00-5:30 pm, 27 October 2014

1. Prove that the Fibonacci sequence (F_n) defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 1$ is a divisibility sequence, that is, if $m \mid n$ then $F_m \mid F_n$ for all positive integers m and n.

2. Let (F_n) be the Fibonacci sequence (as defined in problem 1) and let f be a polynomial of degree 1006 such that $f(k) = F_k$ for all $k \in \{1008, \ldots, 2014\}$. Prove that
$$233 \mid f(2015) + 1.$$

Inequalities Quiz (1.5 hours)

2:30-4:00 pm, 28 October 2014

1. Let x, y, z be positive real numbers satisfying $x + y + z = \frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x}$. Prove that
$$\frac{3}{2} \leqslant \frac{3}{\sqrt[3]{xyz}(1+\sqrt[3]{xyz})} \leqslant \frac{1}{x(y+1)} + \frac{1}{y(z+1)} + \frac{1}{z(x+1)}.$$

2. Let $a, b, c \geqslant 1$. Prove that
$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} \geqslant \frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}.$$

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Functional Equations Quiz (1.5 hours)

2:30-4:00 pm, 29 October 2014

1. Let A and B be nonempty sets and let $f : A \to B$. Prove that the following statements are equivalent:
   (i) f is onto B
   (ii) For every set C and and every functions $g, h : B \to C$, if $g \circ f = h \circ f$ then $g = h$.
2. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $f : \mathbb{N} \to \mathbb{R}$. Prove that there is an infinite subset A of $\mathbb{N}$ such that f is increasing on A or f is decreasing on A.

Combinatorics Quiz (1.5 hours)

2:30-4:00 pm, 29 October 2014

1. A sequence $a_0, a_1, \dots, a_n, \dots$ of positive integers is constructed as follows:
   - If the last digit of a_n is less than or equal to 5, then this digit is deleted and a_{n+1} is the number consisting of the remaining digits. (If a_{n+1} contains no digits, the process stops.)
   - Otherwise, $a_{n+1} = 9a_n$.

   Can one choose a_0 so that this sequence is infinite?
2. Let C be the set of all 100-digit numbers consisting of only the digits 1 and 2. For any number in C, we may transform the number by considering any 10 consecutive digits $x_0x_1x_2 \dots x_9$ and transform it into $x_5x_6 \dots x_9x_0x_1 \dots x_4$. We say that two numbers in C are similar if one of them can be reached from the other by performing finitely many such transformations. Let D be a subset of C such that any two numbers in D are not similar. Determine the maximum possible size of D.

Geometry Quiz (1.5 hours)

2:30-4:00 pm, 31 October 2014

1. Let D be a point inside an acute triangle ABC such that $\angle ADC = \angle A + \angle B, \angle BDA = \angle B + \angle C$ and $\angle CDB = \angle C + \angle A$. Prove that
$$\frac{AB \cdot CD}{AD} = \frac{AC \cdot CB}{AB}.$$
2. In any $\triangle ABC$, ℓ is any line through C and points P, Q. If BP, AQ are perpendicular to the line ℓ and M is the midpoint of the line segment AB, then prove that $MP = MQ$.

Thailand October Camp 2014 - Day 1
IPST
28 October 2014

Number Theory (2 hours)

1. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of two squares of integers.

2. Find all integer solutions to the equation $y^2 = 2x^4 + 17$.

3. Find the maximum number of colors used in coloring integers n from 49 to 94 such that if a, b (not necessarily different) have the same color but c has a different color, then c does not divide $a + b$.

Functional Equations (2 hours)

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y))$$

for all $x, y \in \mathbb{R}$

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x - y)) = f(x) - f(y) + f(x)f(y) - xy$$

for all $x, y \in \mathbb{R}$

Thailand October Camp 2014 - Day 2
IPST
29 October 2014

Geometry (2 hours)

1. Let O be the circumcenter of an acute $\triangle ABC$ which has altitude AD. Let AO intersect the circumcircle of $\triangle BOC$ again at X. If E and F are points on lines AB and AC such that $\angle XEA = \angle XFA = 90°$, then prove that the line DX bisects the segment EF.

2. Let $ABCDEF$ be a hexagon inscribed in a circle (with vertices in that order) with $\angle B + \angle C > 180°$ and $\angle E + \angle F > 180°$. Let the lines AB and CD intersect at X and the lines AF and DE intersect at S. Let XY and ST be the diameters of the circumcircles of $\triangle BCX$ and $\triangle EFS$ respectively. If U is the intersection point of the lines BX and ES and V is the intersection point of the lines BY and ET, prove that the lines UV, XY and ST are all parallel.

3. Two circles S_1, S_2 intersect at M and N. Let $ABCD$ be a rectangle whose vertices A and C lie on S_1 and vertices B and D lies on S_2. Prove that the intersection of the diagonals of $\square ABCD$ lies on the line MN.

Inequalities (2 hours)

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{a+\sqrt{(a+b)(a+c)}} + \frac{b}{b+\sqrt{(b+c)(b+a)}} + \frac{c}{c+\sqrt{(c+a)(c+b)}} \leqslant \frac{2a^2+ab}{(b+\sqrt{ca}+c)^2} + \frac{2b^2+bc}{(c+\sqrt{ab}+a)^2} + \frac{2c^2+ca}{(a+\sqrt{bc}+c)^2}.$$

2. Let $a, b, c \in (0, 1)$ with $a + b + c = 1$. Prove that

$$\frac{a^5+b^5}{a^3+b^3} + \frac{b^5+c^5}{b^3+c^3} + \frac{c^5+a^5}{c^3+a^3} \geqslant \frac{a}{8+b^3+c^3} + \frac{b}{8+c^3+a^3} + \frac{c}{8+a^3+b^3}.$$

3. Let a, b, c be positive real numbers. Prove that

$$\frac{3(ab+bc+ca)}{2(a^2b^2+b^2c^2+c^2a^2)} \leqslant \frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \leqslant \frac{a+b+c}{2abc}.$$

Thailand October Camp 2014 - Day 3
IPST
31 October 2014

Algebra (2 hours)

1. Let a, b, c be positive real numbers. It is known that the system of equations

$$a^2x + b^2y + c^2z = 1 \quad \text{and} \quad xy + yz + zx = 1$$

   has only one solution (x, y, z). Prove that a, b, c are the sides of a certain triangle.

2. Fix a sequence $a_1, a_2, \ldots$ of integers satisfying the following condition: for all prime numbers p and all positive integers k, we have

$$a_{pk+1} = pa_k - 3a_p + 13.$$

   Determine all possible values of a_{2013}.

Combinatorics (2 hours)

1. Let A be a subset of $\{1, 2, \ldots, 1000000\}$ such that for any $x, y \in A$ with $x \neq y$, we have $xy \notin A$. Determine the maximum possible size of A.

2. Determine the number of sequences of points $(x_1, y_1), (x_2, y_2), \ldots, (x_{4570}, y_{4570})$ on the plane satisfying the following two properties:

   (i) $\{x_1, x_2, \ldots, x_{4570}\} = \{1, 2, \ldots, 2014\}$ and $\{y_1, y_2, \ldots, y_{4570}\} = \{1, 2, \ldots, 2557\}$.

   (ii) For each $i = 1, 2, \ldots, 4569$, exactly one of $x_i = x_{i+1}$ and $y_i = y_{i+1}$ holds.