Thailand
Team Selection Test
2012

Day 1

Wednesday, January 25, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let n be a positive integer and $n \geq 2$. Suppose that $\sigma \in \mathsf{Per}(n)$ and write $\sigma = C_1C_2\cdots C_k$ as a product of k disjoint cycles of length $c_1, c_2, \ldots, c_k$ respectively, then σ is called an *even permutation of degree* n if $\sum_{i=1}^{k}(c_i - 1)$ is even, otherwise, it is called an *odd permutation of degree* n. Prove that:

(1) There are exactly $\frac{n!}{2}$ even permutation of degree n.

(2) If $n \geq 3$, then σ is an even permutation of degree n if and only if there exists l cycles $C_1C_2\cdots C_l$ in $\mathsf{Per}(n)$ each of length 3 such that $\sigma = C_1C_2\cdots C_l$.

$\boxed{2}$ Find all real numbers x which satisfy the equation $\lfloor x^2 \rfloor - \lfloor -x^2 \rfloor - 8\lfloor x \rfloor + 2 = 0$.

$\boxed{3}$ Given $\triangle ABC$. Let M be the midpoint of BC. Let S and T be points on BM and CM, respectively, such that $SM = MT$. Let P and Q be points on AT and AS, respectively, such that $\angle PST = \angle BAS$ and $\angle QTS = \angle CAT$. If lines BQ and PC intersect at point R, show that $\overline{RM} \perp \overline{BC}$.

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Day 2

Thursday, January 26, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n+7) = f\big(f(n)\big) + f(n+5)$$

for all $n \in \mathbb{N}$?

$\boxed{2}$ Assume ABC is a non-equilateral triangle. Let I and O be respectively the incentre and circumcentre of the triangle ABC. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + AC$ with equality holding only simultaneously.

$\boxed{3}$ Given a regular polygon of n sides with all its diagonals drawn, and light bulbs on each vertex. There are k bulbs on initially. In each step of the game, the player erases a diagonal and changes the state of one of the two bulbs in the end of the diagonal. Determine all possible values of k for which it's possible to turn on all the light bulbs after having erased all the diagonals.

Note: the sides of the polygon are also considered to be diagonals.

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Day 3

Saturday, February 4, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let $G(V,E)$ be a simple graph of order n and size m, and $x \in V$. The *degree of vertex* x, denoted by $d(x)$, is the number of edges in E having x as one of their end vertices. If the graph $G(V,E)$ does not contain C_4 as its subgraph, prove that

(a) $\displaystyle\sum_{x\in V}\binom{d(x)}{2} \le \binom{n}{2}$

(b) $m \le \dfrac{n\left(1+\sqrt{4n-3}\right)}{4}$.

$\boxed{2}$ Let $P(x), Q(x)$ be polynomials with integer coefficients where $P(x)$ is monic. Prove that there is a polynomial $R(x)$ with integer coefficients such that $P(x) \mid R\big(Q(x)\big)$.

$\boxed{3}$ Find, with proof, the least odd integer $a > 5$ which satisfies the condition:

"there exists positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$ and $m_1 - n_1 = m_2 - n_2$ is divisible by 3".

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Day 4

Sunday, February 5, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let a, b, c be non-negative real numbers such that $a + b + c \leq 3$. Prove that

$$\frac{1}{4}a^2b\big(b\sqrt{c}+3\big)^2 + \frac{1}{4}b^2c\big(c\sqrt{a}+3\big)^2 + \frac{1}{4}c^2a\big(a\sqrt{b}+3\big)^2$$
$$\leq abc\big(ab^2+bc^2+ca^2\big) + 3\big(a^2b+b^2c+c^2a\big) \leq 12.$$

$\boxed{2}$ a) Find a continuous, strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = 2012x + 2555$ for all $x \in \mathbb{R}$.

b) Find all continuous, strictly decreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = x + 2555$ for all $x \in \mathbb{R}$.

$\boxed{3}$ Let ABC be an acute angled triangle with circumcircle (O) centered at O. Let P be a point inside $\triangle ABC$ and let AP, BP, CP intersect (O) again at A', B', C' respectively. Denoted by $B_c, C_b, A_b, B_a, C_a, A_c$ the circumcenters of triangles PBC', PCB', PAB', PBA', PCA', PAC' respectively.

Prove that the lines B_cC_b, A_bB_a, C_aA_c are concurrent at the midpoint of the segment OP.

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Day 5 (APMO)

Tuesday, March 13, 2012 — *Time Allowed: 4 hours*

09.00 – 13.00 — *Each problem is worth 7 points.*

$\boxed{1}$ Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of the triangle ABC must be 6 if the area of each of the triangles PFA, PDB and PEC is 1.

$\boxed{2}$ Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes.

$\boxed{3}$ Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

$\boxed{4}$ Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

Here we denote by XY the length of the line segment XY.

$\boxed{5}$ Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \ldots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{n - a_i a_j} \leq \frac{n}{2}$$

must hold.

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Day 6

Wednesday, March 28, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Does there exists a convex n-gon such that all its sides are equal and all vertices lie on the graph of $y = x^2$, where

(a) $n = 2012$;

(b) $n = 2013$?

$\boxed{2}$ Determine all pairs of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$g\big(f(x+y)\big) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

$\boxed{3}$ Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

Day 7

Saturday, March 31, 2012 *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 *Each problem is worth 7 points.*

$\boxed{1}$ Let a, b and c be positive real numbers satisfying $a^2 + b^2 + c^2 = 3$ and $\min(a + b, b + c, c + a) > \sqrt{2}$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

$\boxed{2}$ Let $n > 2$ be a positive integer and let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial.

(a) Prove that $\Phi_n(a) > 0$ for all real numbers a.

(b) Compute (with proof) $\Phi_n(1)$ and $\Phi_n(-1)$.

$\boxed{3}$ Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

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Day 8

Wednesday, April 4, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ For any positive integer d, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example, we have $f(1) = 1, f(5) = 16$ and $f(6) = 12$). Prove that for every integer $k \geq 0$, the number $f(2^k)$ divides $f(2^{k+1})$.

$\boxed{2}$ Let G be a graph with the clique number $\omega(G) = 100$. Prove that there exists a partition V_1, V_2 of $V(G)$ such that $\omega(G[V_1]) = \omega(G[V_2]) = 50$.

$\boxed{3}$ Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meet AC at point F, and BC at point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

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Day 9

Thursday, April 5, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Prove that there exists a sequence of 1000 consecutive positive integers containing exactly 25 primes.

$\boxed{2}$ Let $\Im$ be a convex figure. Prove that there is a point C in $\Im$ such that for any chord AB of $\Im$ through C, we have

$$|AC| \leq \frac{2}{3}|AB|.$$

(A *chord of convex figure* is a line segment with endpoints on the boundary of the figure.)

$\boxed{3}$ Determine all pairs of functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1.$$

for every positive integer n. Here, $f^k(n)$ mean $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

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Day 10

Thursday, April 19, 2012 *Time Allowed: 3 hours*

13.00 – 16.00 *Each problem is worth 7 points.*

$\boxed{1}$ For each $n = 1, 2, 3, \ldots$, prove that there are infinitely many monic polynomials of degree n with integer coefficients and all zeros in the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Furthermore, show that all the zeros of such polynomials have modulus zero or one.

$\boxed{2}$ Let a and b be positive integers. Suppose that $\gcd\big(\Phi_a(t), \Phi_b(t)\big) > 1$ for some integer t. Prove that $\dfrac{a}{b}$ is an integral power of a prime number, that is, $\dfrac{a}{b} = p^k$ for a prime number p and an integer k.

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Day 11

Thursday, April 26, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Let a, b, c be positive real numbers. Prove that

$$\frac{a\sqrt{b^3+c^3}}{b^2+c^2}+\frac{b\sqrt{c^3+a^3}}{c^2+a^2}+\frac{c\sqrt{a^3+b^3}}{a^2+b^2}\geq\frac{1}{\sqrt{2\left(\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(c+a)}\right)}}.$$

$\boxed{2}$ For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a)-t(n), t(n+a+1)-t(n+1), \ldots, t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

$\boxed{3}$ Let $\Im$ be a set of subsets of $\{1, 2, 3, \ldots, n\}$ such that

(1) every element in $\Im$ is a 3-element subset; and

(2) the intersection of any two elements of $\Im$ contains at most one element.

Let f denote the maximum number of elements that $\Im$ can have. Prove that

$$n^2-4n\leq 6f\leq n^2-n.$$

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Day 12

Friday, April 27, 2012 — *Time Allowed: 4 hours and 30 minutes*

09.00 – 13.30 — *Each problem is worth 7 points.*

$\boxed{1}$ Find all functions f from the set of real numbers to itself for which there is a real-valued function g on the set of real numbers such that

$$f(x+y)f(x-y) = (x-y)g(x+y)$$

for all real numbers x and y.

$\boxed{2}$ Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

$\boxed{3}$ On a square table of 2011 by 2011 cells, we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell, we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configuration, what is the largest value of k?