Thailand TST 2013

Day 1

Date : 24 December 2012 *Time Allowed : 4.5 hours*

Time : 09.00-13.30 *Each problem is worth 7 points*

Problem 1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) - f(x) = (x + f(y))^{2013} - x^{2013}$$

for every real numbers x, y.

Problem 2. Let n be a positive integer and let $G = \{z \in \mathbb{C} \mid z^n = 1\}$. Determine all functions $f : G \to \mathbb{Z}$ which satisfies the following conditions.

(i) $f(z) = 1$ if and only if $z = 1$.

(ii) $f(z^k) = \dfrac{f(z)}{\gcd(f(z), k)}$ for every $z \in G$ and positive integer k.

Problem 3. Let I be the incenter of triangle ABC. Let the incircle of ΔABC touches sides BC, CA, AB at D, E, F respectively. A circle k cut segments EF, FD, DE at $\{X_1, X_2\}$, $\{X_3, X_4\}$, $\{X_5, X_6\}$ respectively. Suppose that lines X_1X_4, X_2X_5 and X_3X_6 pass through center G of k. Prove that.

(i) Points A, D, G are colinear.

(ii) If the line through G parallel to DE cuts BC at P and the line through G parallel to DF cuts BC at Q. Then $IP = IQ$.

Thailand TST 2013

Day 2

Date : 17 January 2013 *Time Allowed : 4.5 hours*

Time : 09.00-13.30 *Each problem is worth 7 points*

Problem 1. Determine all increasing functions $f : \mathbb{Z}^+ \to \mathbb{R}$ such that

$$f(mn) = f(m)f(n)$$

for every positive integers m, n.

Problem 2. Determine all ordered pairs (x, y) of positive integers such that

$$x^3 + y^3 = 4(x^2y + y^2x - 5)$$

Problem 3. A stone is placed on Cartesian coordinate plane. If the stone is at position (x, y), it can be moved as the following.

(i) For any positive integer z, it can be moved to position $(x - z, y - z)$.

(ii) It can be moved to position $(3x, y), (3y, x)$.

Find all positive m, n which if a stone is placed on (m, n), then it can be moved to $(0, 0)$ in a finite number of moves.

Thailand TST 2013

Day 3

Date : 21 January 2013 *Time Allowed : 4.5 hours*

Time : 09.00-13.30 *Each problem is worth 7 points*

Problem 1. Let S be a set of students with $|S| \geqslant 4$. Suppose that there exists a positive integer m which $3 \leqslant m \leqslant |S| - 1$ such that for each $A \subseteq S$ which $|A| = m$, there exists unique student who is friend of every students in A. (Friendship is always mutual.) Prove that

(i) there exists subset $B \subseteq S$ which $|B| = m + 1$ and any two students are friends.

(ii) $|S| = m + 1$.

Problem 2. Let ABC be a triangle. M is the midpoint of arc BC of circumcircle of triangle ABC, not containing A. Let I be the incenter of triangle ABC and points E, F are projections from I to lines MB, MC respectively. Prove that $IE + IF \leqslant AM$.

Problem 3. Find all ordered pairs (a, b) of positive integers which satisfies

$$n \mid a^n + b^{n+1} \qquad \text{for all positive integers } n$$

Source : China Western Mathematical Olympiad 2011

Thailand TST 2013

Day 4

Date : 23 January 2013 *Time Allowed : 4.5 hours*

Time : 9.00-13.00 *Each problem is worth 7 points*

Problem 1. Let x, y, z be positive reals. Prove that

$$\frac{x^2}{y(x+y)+z(x+z)}+\frac{y^2}{z(y+z)+x(y+x)}+\frac{z^2}{x(z+x)+y(z+y)}$$
$$\geqslant \frac{x}{(x+y)+(x+z)}+\frac{y}{(y+z)+(y+x)}+\frac{z}{(z+x)+(z+y)}.$$

Problem 2. Let O, I be the circumcenter and incenter of scalene triangle ABC respectively. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F respecitvely. Let AP, BQ, CR be the angle bisectors of triangle ABC where P, Q, R lie on BC, CA, AB respectively.

If the reflection of line OI across DE, DF intersect at X. Prove that points P, Q, R, X are concyclic.

Problem 3. There is $k \geqslant 2$ piles of coins, each pile having $n_1, n_2, n_3, ..., n_k$ coins respectively. The only permitted moves are selecting two piles, having a, b coins where $a \geqslant b$ and move b coins from the first pile (which originally has a coins) to the second pile.

Determine the necessary and sufficient conditions for $n_1, n_2, n_3, ..., n_k$ which it is possible to move all coins to the same pile, using finite number of permitted moves.

Source : Romania National Olympiad 2012

Thailand TST 2013

Day 5

Date : 24 January 2013 *Time Allowed : 4.5 hours*

Time : 9.00-13.00 *Each problem is worth 7 points*

Problem 1. Let $P(x)$ be an irreducible polynomial (over $\mathbb{Q}$) with rational coefficients. Suppose that there exists irrational number α which $P(\alpha) = P(-\alpha) = 0$. Prove that there exists irreducible polynomial $Q(x)$ with rational coefficients such that $P(x) = Q(x^2)$.

Problem 2. Determine all positive integer n such that

$$\left\lfloor \frac{1000000}{n} \right\rfloor - \left\lfloor \frac{1000000}{n+1} \right\rfloor = 1$$

Source : Modified from Japan Mathematical Olympiad Preliminary 2012

Problem 3. Let ABC be a triangle which $AB > AC$. Let the incircle of triangle ABC touches BC, CA, AB at D, E, F respectively. The angle bisector of $\angle BAC$ cuts DE, DF at K, L respectively. Let M be the midpoint of BC and let H be the feet of altitude from A to BC. Prove that $\angle MLK = \angle MHK$.

Thailand TST 2013

Day 6

Date : 16 March 2013 *Time Allowed : 4.5 hours*

Time : 9.00-13.30 *Each problem is worth 7 points*

Problem 1. Let $P_1, P_2, ..., P_n$ $(n \geqslant 3)$ be points on a unit circle. Suppose that the product of distances from arbitrary point Q to $P_1, P_2, ..., P_n$ is less than or equal to 2. Prove that $P_1, P_2, ..., P_n$ are vertices of an regular n-gon.

Problem 2. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$f\left(2x + \frac{1}{1+x+y}\right) = f(x) + f\left(x + \frac{1}{1+x+y}\right)$$

for any positive reals x, y.

Problem 3. Let S be the set of all positive integers which have exactly 11 digits. Let $A \subseteq S$. Call an element x of A *lonely* if and only if there don't exists $y, z \in A$ (not necessarily distinct) which $y + z$ divides x. Suppose that A has at most 10 lonely number. Determine the maximum possible number of elements of A.

Thailand TST 2013

Day 7

Date : 20 March 2013 *Time Allowed : 4.5 hours*

Time : 9.00-13.30 *Each problem is worth 7 points*

Problem 1. Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Source : IMO Shortlist 2012 N2

Problem 2. Let a, b, c be positive reals satisfying $abc = 1$. Prove that

$$\frac{1}{1 + a^4 + (b^2 + 1)^2} + \frac{1}{b^4 + (c^2 + 1)^2} + \frac{1}{c^4 + (a^2 + 1)^2} \leqslant \frac{a}{2b + c + 3} + \frac{b}{2c + a + 3} + \frac{c}{2a + b + 3}$$

Problem 3. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Source : IMO Shortlist 2012 G6

Thailand TST 2013

Day 8

Date : 31 March 2013 — *Time Allowed : 4.5 hours*

Time : 9.00-13.30 — *Each problem is worth 7 points*

Problem 1. In a 2556×2556 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain. *Source : Slightly Modified from IMO Shortlist 2012 C3*

Problem 2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x+y) + f(x-y))^2 = 4f(x)^2$$

for all reals x, y.

Problem 3. Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

Source : IMO Shortlist 2012 N6

Thailand TST 2013

Day 9

Date : 2 April 2013 | *Time Allowed : 4.5 hours*

Time : 9.00-13.30 | *Each problem is worth 7 points*

Problem 1. There are $n \geqslant 4$ parallel line segments lying on the same plane where for any three segments, there exists a line which cuts all three segments. Prove that there exists a line which cuts all the n segments.

Problem 2. Let $a, b, c > 0$. Prove that

$$32\left(\frac{1}{7+(a-3)^2}+\frac{1}{7+(b-3)^2}+\frac{1}{7+(c-3)^2}\right) \leqslant \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}+6$$

Problem 3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Source : IMO Shortlist 2012 G3

Thailand TST 2013

Day 10

Date : 3 April 2013 | *Time Allowed : 4.5 hours*

Time : 9.00-13.30 | *Each problem is worth 7 points*

Problem 1. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Source : IMO Shortlist 2012 G4

Problem 2. Let $(a_1, a_2, ..., a_{2n})$ be a permutation of $\{1, 2, ..., 2n\}$ which for each $i \in \{1, 2, ..., 2n-1\}$, value of $|a_{i+1} - a_i|$ are all distinct. Prove that $a_1 - a_{2n} = n$ if and only if $1 \leqslant a_{2k} \leqslant n$ for each $k = 1, 2, ..., n$

Problem 3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

Source : IMO Shortlist 2012 A6

Thailand TST 2013

Day 11

Date : 5 April 2013 *Time Allowed : 4.5 hours*

Time : 9.00-13.30 *Each problem is worth 7 points*

Problem 1. For any sets $X, Y \subseteq \mathbb{Q}$, let $X + Y$ denotes the set $\{x + y : x \in X, y \in Y\}$. Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A + B, B + C, C + A$ are disjoint?

Source : Slightly modified from IMO Shortlist 2012 A2

Problem 2. An integer a is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.

b) Decide whether $a = 2$ is friendly.

Source : IMO Shortlist 2012 N4

Problem 3. Let S be a finite subset of $\mathbb{Z}^+$ which the smallest and the largest elements are relatively prime. Let S_n be the set of all positive integers which can be expressed as the sum of at most n (not necessarily distinct) elements of S. Let a be the largest element of S. Prove that there exists a positive integer k such that for any positive integer $m > k$, $|S_{m+1}| - |S_m| = a$.