Thailand TST 2017

Language: **English**

Day: **1**

Thursday, December 22, 2016

Time: 3 hours
Each problem is worth 7 points

Problem 1. In a convex pentagon, if four of the altitudes from a vertex to the opposite side pass through a point, show that the fifth altitude also pass through that point.

Problem 2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $x, y, z \in \mathbb{R}$,

$$f(f(x) + yz) = x + f(y)f(z).$$

Problem 3. (C1*) The leader chooses a binary word of length n and a $k \leq n$ and send these numbers secretly to the deputy leader. The deputy leader then writes all binary words that differ from the leader's word in exactly k places on a blackboard. The IMO representative sees the blackboard, and try to guess the leader's word. How many times does the representative need to guess to guarantee a correct answer?

Problem 4. (N1) For each positive integer n, let $S(n)$ denote the sum of digits of n in base 10. Find all polynomials $P \in \mathbb{Z}[x]$ such that

$$P(S(n)) = S(P(n)) \text{ for all } n \geq 2016.$$

Thailand TST 2017

Language: **English**

Day: **2**

Tuesday, January 24, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. (A1) Let a, b, c be positive real numbers such that $\min\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Problem 2. Two circles Γ_1, Γ_2 intersect at M, N. A line l cut Γ_1 at A, C and Γ_2 at B, D such that A, B, C, D are all distinct and lie in this order. X is a point on line MN such that M lies between X, N.The lines AX and BM intersect at P, and the lines DX and CM intersect at Q. If K is the midpoint of AD and L is the midpoint of BC then prove that KX and LM intersect at a point on line PQ.

Problem 3. There are n boys and n girls on a circle. A pair of boy and girl is said to be a balancing pair if there are an equal amount of boys and girls in both arcs between the pair. Suppose that there is a girl who is in exactly 10 balancing pairs. Prove that there is also a boy who is in exactly 10 balancing pairs.

Problem 4. (N4) Let m, n, k, l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that either

- $m = 1$ and $l = 2k$, or
- $l \mid k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Thailand TST 2017

Language: **English**

Day: **3**

Wednesday, January 25, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. (C2) Determine all positive integers n such that it is possible to arrange all positive divisors of n in a rectangular table such that - The number in each cell are all distinct. - The sum of numbers in each row is equal. - The sum of numbers in each column is equal.

Problem 2. (G2) Let ABC be a triangle with circumcircle Γ and incenter I. The perpendicular from I to BC cut BC at D, and M is the midpoint of BC. The line through I perpendicular to AI intersect AB, AC at E, F. The circumcircle of $\triangle AEF$ intersect Γ again at $X \neq A$. Prove that XD and AM intersect on Γ.

Problem 3. (A3) Determine all positive integers $n \geq 3$ such that, for any reals $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for each $k = 1, 2, \ldots, n$, there exists $x_1, x_2, \ldots, x_n \in \{-1, 1\}$ such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leq 1.$$

Thailand TST 2017

Language: **English**

Day: **4**

Tuesday, March 7, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. (N2) Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Problem 2. (A4) Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)(f(f(x^2)) + f(f(y^2)))$$

for all positive real numbers x and y.

Problem 3. (G8) Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1, CC_1 are the internal angle bisectors of triangle ABC. Let I be the incenter of triangle ABC, and H be the orthocenter of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Thailand TST 2017

Language: **English**

Day: **5**

Friday, March 10, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Let $n \geq 2$ be an integer. Killer is a game played by a dealer and n players. The game begins with the dealer designating one of the n players a killer and keeping this information a secret. Every player knows that the killer exists and is among the n players. The dealer can make as many public announcements as he wishes. Then, he secretly gives each of the n a (possibly different) name of one of the n players. The game has the property that:

(i) Alone, each player (killer included) does not know who is the killer. Each player also cannot tell with certainty who is not the killer.

(ii) If any 2 of the n players exchange information, they can determine the killer.

For example, if there are a dealer and 2 players, the dealer can announce that he will give the same name to both players if the first player is the killer, and give different names to each player if the second player is the killer.

(a) Prove that Killer can be played with a dealer and 5 players.

(b) Determine whether Killer can be played with a dealer and 4 players.

Problem 2. Let ABC be a triangle with $\angle A = 60^\circ$ and $AB > AC$. Let O be its circumcenter, F the foot of altitude from C, and D a point on the side AB such that $BD = AC$. Suppose that the points O, F, and D are distinct. Prove that the circumcircle of $\triangle OFD$ intersects the circle centered at O with radius OF on the altitude of $\triangle ABC$ from B.

Problem 3. Genji the ninja is to jump along the real axis, starting at the point 0. In doing so, he alternates between the following two types of jumps.

Type 1: Genji jumps from the current position x to a point y in the set

$$\{x, x+1, x+2, x+3, x+4, x+5, x+6\}.$$

Type 2: Genji jumps from the current position y to $(\sqrt{2}-3)y$.

Let a and b be integers. Prove that there exists a finite sequence of jumps that allows Genji to land on the point $a + b\sqrt{2}$.

Thailand TST 2017

Language: **English**

Day: **APMO**

Tuesday, March 14, 2017

Time: 4 hours
Each problem is worth 7 points

Problem 1. We call a collection of 5 numbers *arrangeable* if they can be arranged as a, b, c, d, e so that $a - b + c - d + e = 29$. Determine all integers $n_1, n_2, \ldots, n_{2017}$ such that when we place them on a circle in clockwise order then any 5 consecutive numbers are arrangeable.

Problem 2. Let ABC be a triangle with $AB < AC$. The internal angle bisector of $\angle BAC$ intersects (ABC) again at D. The external angle bisector of $\angle BAC$ intersects the perpendicular bisector of AC at Z. Prove that the midpoint of AB lies on (ADZ).

Problem 3. Let n be a positive integer. Let A be the set of sequences $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers which sum to n and $a_j + 1$ is a power of 2 for each $j = 1, 2, \ldots, k$. Let B be the set of sequences $b_1 \geq b_2 \geq \cdots \geq b_k$ which sum to n and $b_j \geq 2b_{j+1}$ for each $j = 1, 2, \ldots, k-1$. Prove that $|A| = |B|$.

Problem 4. We call a rational number r *powerful* if it can be expressed in the form $r = \frac{p^k}{q}$ where p, q are integers such that $\gcd(p, q) = 1$. Suppose that a, b, c are rationals such that $abc = 1$ and $a^x + b^y + c^z$ is an integer for some x, y, z. Prove that a, b, c are all powerful.

Problem 5. Let n be a positive integer. We call a pair of sequences of integers $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ *exquisite* if

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq 1.$$

Find the size of the biggest collection of sequences $(a_1, a_2, \ldots, a_n)$ such that each pair of sequences in it is exquisite.

Thailand TST 2017

Language: **English**

Day: **6**

Tuesday, March 21, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Let $(2+\sqrt{3})^k = m+n\sqrt{3}$, where m, n, k are positive integers and k is odd. Prove that $\sqrt{m-1}$ is an integer.

Problem 2. (A8*) Prove that $a = \frac{4}{9}$ is the best constant such that

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

holds for all positive integer n and real numbers $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$.

Problem 3. (C6) A ferry operator operates some ferry lines between islands, so that it is impossible to split the islands into two groups such that no lines are run between them. Each year, the ferry operator will close a line between two islands A, B. To compensate, the ferry operator will open lines that run between x and B if there is a line between x and A and vice versa.

It is given that at any time, for any partition of the islands into two groups, the ferry operator will close a line between those groups some year later. Prove that some year, there will be an island that is connected to every other island by a ferry line.

Thailand TST 2017

Language: **English**

Day: **7**

Saturday, March 25, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. (A7) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$(f(x+y))^2 = 2f(x)f(y) + \max\{f(x^2) + f(y^2), f(x^2 + y^2)\}$$

for all real numbers x and y.

Problem 2. (N8) Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, ..., x_n$ such that

$$\frac{P(x_i)}{P(x_j)}$$

belongs to the interval $\left(\frac{1}{2}, 2\right)$ and is the d^{th} power of a rational number of every pair of indices i and j where $1 \leq i, j \leq n$.

Thailand TST 2017

Language: **English**

Day: **8**

Sunday, March 26, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Let $n \geq 3$ be an integer. Each edge of a complete graph with n vertices is colored in one of the three colors. There is at least one edge for each color. Determine the smallest k such that it is possible to recolor k edges in the graph so that all vertices of the graph are connected by edges of the same color.

Problem 2. (G7) Let I be the incenter of a non-equilateral triangle ABC, I_A be the A-excenter, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B, and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on the line OI where O is the circumcircle of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Prove that $\angle XIY = 120^\circ$.

Thailand TST 2017

Language: **English**

Day: **9**

Sunday, April 2, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Alice and Bob plays a game where

- Alice draws some n distinct circles in the plane. This will create some finite regions.
- Bob color each finite region red or blue.
- Alice may choose to change the color of a single region.

Alice wins if there is an even number of red regions in each circle, else Bob wins. Determine all n such that Alice has a winning strategy.

Problem 2. Let $\{x_n\}$ be a sequence satisfying $x_1 = 2$ and

$$x_n = 4\nu_3(n) + 2 - \frac{2}{x_{n-1}}$$

for all $n \geq 2$. Prove that each positive rational number appears exactly once in this sequence.

Problem 3. (A5)

(a) Prove that for any positive integer n, there exists integers a, b satisfying $0 < b \leq \sqrt{n} + 1$ and $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$.

(b) Show that there are infinitely many positive integers n such that there does not exist integers a, b satisfying $0 < b \leq \sqrt{n}$ and $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$.

Thailand TST 2017

Language: **English**

Day: **10**

Monday, April 3, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Show that there are infinitely many pairs of distinct prime numbers (m, n) such that

$$3^n \equiv 3 \pmod{m} \text{ and } 3^m \equiv 3 \pmod{n}.$$

Problem 2. (G3) Let $B = (-1, 0)$ and $C = (1, 0)$. A set S of points in the plane is called nice if

(i) There exists a point $T \in S$ such that for any $P \in S$, the line segment TP lies entirely inside S.

(ii) For any triangle $\triangle XYZ$, there is a unique point $A \in S$ such that $\triangle ABC$ is similar to $\triangle XYZ$ (not necessarily in order)

Prove that there exists two nice subsets S, S' of $\{(x, y) \mid x \geq 0 \wedge y \geq 0\}$ such that for any $\triangle XYZ$, if the points from (ii) are A, A' then $BA \times BA'$ is constant.

Problem 3. (C5) Given a regular n-gon, determine the maximum number of diagonals that can be chosen such that for each pair of chosen diagonals, they either (i) does not intersect in the interior of the n-gon or (ii) are perpendicular.

Thailand TST 2017

Language: **English**

Day: **11**

Tuesday, April 11, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Let P be a point inside $\triangle ABC$. Let A_1, B_1, C_1 be points in the interiors of the segments PA, PB, PC respectively. Let $BC_1 \cap CB_1 = \{A_2\}$, $CA_1 \cap AC_1 = \{B_2\}$, and $AB_1 \cap BA_1 = \{C_2\}$. Let U be the intersection of the lines A_1B_1 and A_2B_2, and V be the intersection of the lines A_1C_1 and A_2C_2. Show that the lines UC_2,VB_2, and AP are concurrent.

Problem 2. Let X be a point inside a convex 100-gon that does not lie on any side or diagonal of the 100-gon. Pete and Bazil play a game as follows: First, Pete marks two vertices of the 100-gon then they alternately mark a vertex of the 100-gon. If after a player's turn, X is inside the polygon having all vertices marked, then that player loses. Prove that Pete has a winning strategy.

Problem 3. (N6) Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, $f(m) + f(n) - mn \neq 0$ and

$$f(m) + f(n) - mn \mid mf(m) + nf(n).$$

Thailand TST 2017

Language: **English**

Day: **12**

Wednesday, April 12, 2017

Time: 4 hours 30 minutes
Each problem is worth 7 points

Problem 1. Prove that for any positive real numbers a, b, c,

$$\frac{a}{\sqrt{3ab+bc}} + \frac{b}{\sqrt{3bc+ca}} + \frac{c}{\sqrt{3ca+ab}} \geq \frac{3}{2}.$$

Problem 2. (G5) Let D be the projection of A onto the Euler line of $\triangle ABC$. A circle ω through A, D intersect lines AB, AC at X, Y. Let S be the center of ω, P the projection of A onto BC, and M the midpoint of BC. Prove that the circumcenter of $\triangle XSY$ is equidistant from P and M.

Problem 3. The elements of the set H are finite sequences whose elements are from $1, 2, 3$. Suppose that no element of H is a subsequence of any other element of H. Show that H is finite.