Thailand
Team Selection Test
2011

Day 1

(Saturday, December 25, 2010; 09.00 - 13.00)

1. Find all pairs of positive integers (m, n) for which we can paint each unit square of an $m \times n$ in black or white, such that for any unit square, the number of squares with at least one vertex in common that are painted with the same color is odd.

2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$|f(x + y)| = |f(x) + f(y)| \text{ for all } x, y \in \mathbb{R}.$$

Then show that f satisfies the functional equation

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

3. Let ABC be an acute-angled triangle with $|AB| > |AC|$ and altitude AD. Let the circle O with diameter BC intersect AC at E, and the tangents to the circle O at B and E meet at X. If the lines AX and BC meet at Y, prove that

$$\frac{1}{|DC|} - \frac{1}{|BD|} = \frac{2}{|DY|}.$$

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Day 2

(Sunday, December 26, 2010; 09.00 - 13.00)

1. Assume that $f(x)$ and $g(x)$ are polynomials in $\mathbb{Q}[x]$ such that the product $f(x)g(x) \in \mathbb{Z}[x]$. Prove that the product of any coefficient of $f(x)$ with any coefficient of $g(x)$ is an integer.

2. Let p, q be odd primes and $m \in \mathbb{Z}^+$ such that $q^m \parallel p - 1$. Prove that for any integer $n \geq 0$, the order of p modulo q^{m+n} is q^n and $q^{m+n} \parallel p^{q^n} - 1$.

(For a prime number r, $r^n \parallel a$ means $r^n \mid a$ but $r^{n+1} \nmid a$.)

3. Let $\triangle ABC$ be a triangle with $AB \leq AC$ and let P be an interior point on the angle bisector of $\angle BAC$. Let D, E be points on the segments PC, PB respectively such that $\angle PBD = \angle PCE$. The line BD meets AC at X, and CE meets AB at Y.

Prove that $BX \leq CY$.

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Day 3
(Saturday, January 22, 2011; 09.00 - 13.30)

1. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt[3]{\frac{abc}{(a+3)(b+3)(c+3)}} \leq \frac{9}{32}\left(\left(\frac{3-a}{1+2bc}\right)^3+\left(\frac{3-b}{1+2ca}\right)^3+\left(\frac{3-c}{1+2ab}\right)^3\right).$$

2. Let ABC be an acute-angled triangle with orthocenter H and nine-point center N. Consider the points Y and Z on the sides CA and CB respectively such that the directed angles $(AC, HY) = -\frac{\pi}{3}$ and $(AB, HZ) = \frac{\pi}{3}$. Let U be the circumcenter of $\triangle HYZ$. Prove that A, N, U are collinear.

3. Consider a graph with $2n$ vertices and $2n(n-1)$ edges, $n > 1$. Prove that some vertices and edges of the graph can be colored in red in such way that each red edge connects red vertices and each red vertex belongs to exactly n red edges.

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Day 4

(Sunday, January 23, 2011; 09.00 - 13.30)

1. Consider a 16×16 matrix consisting of elements $+1$ and -1. If for any two distinct columns, the sum of products of two elements in the same row is less than or equal to 0, show that the number of $+1$ in this matrix is no more than 160.

2. Determine all prime numbers p for which there are integers a and b such that

$$p^2 = a^2 + b^2 \text{ and } p \mid a^3 + b^3 - 4.$$

3. Given a triangle ABC with circumcircle Γ and incircle of unit radius, the circle Γ_A touches AB and AC, and touches Γ internally. Define Γ_B, Γ_C in a corresponding way. Let R_A, R_B, R_C be the radii of $\Gamma_A, \Gamma_B, \Gamma_C$ respectively.

Prove that $R_A + R_B + R_C \geq 4$.

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Day 5 (APMO)
(Tuesday, March 8, 2011; 09.00 - 13.00)

1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2+b+c, b^2+c+a, c^2+a+b$ to be perfect squares.

2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

3. Let ABC be an acute triangle with $\angle BAC = 30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2 respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2 respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

4. Let n be a fixed positive odd integer. Take $m+2$ distinct points $P_0, P_1, \ldots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:

(1) $P_0 = (0,1), P_{m+1} = (n+1, n)$, and for each integer i, $1 \le i \le m$ both x- and y-coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).

(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.

(3) For each pair i, j with $0 \le i \le j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 points.

Determine the maximum possible value that m can take.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following 2 conditions:

(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

(2) For every pair of real numbers x and y,

$$f\big(xf(y)\big) + yf(x) = xf(y) + f(xy)$$

is satisfied.

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Day 6

(Monday, March 14, 2011; 09.00 - 13.30)

1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and its circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

2. Is it possible to color all rational numbers with one of two colors such that if $x, y \in \mathbb{Q}$, $x \neq y$ and $xy = 1$ or $x + y \in \{0, 1\}$, then x and y must be different colors?

3. A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \cdots + x_n \geq 0$ for all $n \geq 1$.

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Day 7

(Tuesday, March 15, 2011; 09.00 - 13.30)

1. An 8×8 array is divided into 64 unit squares. In some of the unit squares a diagonal is drawn such that no two diagonals share a common point (not even an endpoint). Find the maximum number of diagonals that can be drawn under these circumstances.

2. A circle O touches a triangle ABC at A and D on the segment BC. From D drop a feet to the segment AC at E. Let F be a point on the circle O in the side of EC that is different from D such that $\angle EFC = 2\angle ECD$. Suppose CF meets the circle O again at G. Prove that AG is parallel to BC.

3. Find all the smallest number n such that there exists polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2.$$

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Day 8

(Wednesday, March 23, 2011; 09.00 - 13.30)

1. Let a, b, c be real number with $ab + bc + ca = 1$. Prove that

$$\frac{(a+b)^2+1}{c^2+2} + \frac{(b+c)^2+1}{a^2+2} + \frac{(c+a)^2+1}{b^2+2} \geq 3.$$

2. Determine the smallest positive integer n for which there exists a set $\{a_1, a_2, \ldots, a_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right)\cdots\left(1 - \frac{1}{a_n}\right) = \frac{51}{2010}.$$

3. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Day 9

(Thursday, March 24, 2011; 09.00 - 13.30)

1. Let D and E be interior points of sides AB and AC of a triangle ABC, respectively, so that the line DE intersects the extension of BC at F. Prove that $\triangle ADE$ is similar to the triangle whose vertices are the circumcenters of $\triangle ABC, \triangle DFB$ and $\triangle EFC$.

2. A tourist preparing to visit Dreamland finds out that

a) there are 1024 cities in Dreamland and each city is labeled with a distinct integer from the set $\{0, 1, 2, \ldots, 1023\}$;

b) two cities with numbers m and n are connected by a single road if and only if the binary expansions of m and n are different in exactly one digit;

c) during his visit, 8 roads will be closed for repairing.

Prove that the tourist can organize a closed path by traveling through the roads that remain opened and passing through every city exactly once.

3. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

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Day 10

(Wednesday, March 30, 2011; 09.00 - 13.30)

1. Let $a, b, c > 0$. Prove that

$$\frac{(a+b)^2}{c(a+b+2c)} + \frac{(b+c)^2}{a(b+c+2a)} + \frac{(c+a)^2}{b(c+a+2b)} \geq \sqrt{\frac{3a(b+c)}{b^2+4bc+c^2}} + \sqrt{\frac{3b(c+a)}{c^2+4ca+a^2}} + \sqrt{\frac{3c(a+b)}{a^2+4ab+b^2}}.$$

2. For any integer $n \geq 2$ denote by A_n the set of solutions of the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \cdots + \left\lfloor \frac{x}{n} \right\rfloor.$$

Prove that $A := \bigcup_{n\geq 2} A_n$ is finite and find $\max A$.

3. Let $ABCDE$ be a convex pentagon such that $\overline{BC} \parallel \overline{AE}, AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

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Day 11

(Thursday, March 31, 2011; 09.00 - 13.30)

1. Consider the recurrence relation

$$\begin{aligned} &K_0 = 1 \\ &K_n = 1 + \min\left\{2K_{\lfloor \frac{n}{2} \rfloor}, 3K_{\lfloor \frac{n}{3} \rfloor}\right\}, n \geq 0. \end{aligned}$$

Prove or disprove the statement that "$K_n \geq n$ for all integers $n \geq 0$".

2. Given $n \geq 4$ points in the plane with the property that the distance between any two points is an integer. Prove that at least $\frac{1}{6}$ of these distances are divisible by 3.

3. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i'th player. Prove that

$$\sum_{i=1}^{n}(w_i - l_i)^3 \geq 0.$$

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Day 12

(Sunday, April 17, 2011; 09.00 - 13.30)

1. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$36 \leq 4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) \leq 48.$$

2. Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, P_2, \dots, P_n$ onto lines $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1X_2\dots X_n$ on sides $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \frac{X_2X_3}{P_2P_3}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

3. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n \mid P(m) - P(k)$ implies $n \mid m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

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Day 13

(Monday, April 18, 2011; 09.00 - 13.30)

1. Let I be the incenter of $\triangle ABC$ and the incircle touches the sides AB, BC, CA at the points X, Y, Z respectively. Let H' be the orthocenter of $\triangle XYZ$. If the line IH' and BC intersect at D and the line AD and XH' intersect at P, prove that $PY = YX$.

2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exists N flags forming a diverse set.

3. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f\big(g(n)\big) = f(n) + 1$ and $g\big(f(n)\big) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

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Day 14

(Friday, April 29, 2011; 09.00 - 13.30)

1. Let a, b, c be positive real numbers. Prove that

$$\frac{4a^4}{(a+b)(a^2+b^2)} + \frac{4b^4}{(b+c)(b^2+c^2)} + \frac{4c^4}{(c+a)(c^2+a^2)} \geq \frac{5b^3-a^3}{ab+3b^2} + \frac{5c^3-b^3}{bc+3c^2} + \frac{5a^3-c^3}{ca+3a^2}.$$

2. Let I be the incenter of $\triangle ABC$ and let l be a tangent to the incircle, not a side of $\triangle ABC$, that meets the sides AB, BC, and the extension of the side CA at the points X, Y, Z respectively. Let AY intersects CX at P, and the lines IP and BZ meet at the point Q. Prove that if A, C, Y, X are concyclic, then $ZI^2 = ZQ \cdot ZB$.

3. Let $n \geq 2$ be an integer. A group of people is called *n-compact* if for every person in that group there are n people different from him who are acquainted with one another. Find the maximum possible number N such that every n-compact group of N people contains a subgroup of $n+1$ people who are acquainted with one another.

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Day 15

(Saturday, April 30, 2011; 09.00 - 13.30)

1. Let $f : (0, \infty) \to \mathbb{R}$ satisfy the functional equation

$$f(x+y) = f\left(\frac{x+y}{xy}\right) f(xy) \text{ for all } x, y \in (0, \infty).$$

Show that f satisfies the functional equation

$$f(xy) = f(x) + f(y) \text{ for all } x, y \in (0, \infty).$$

2. Let a and n be two positive integers such that the prime factors of a are all greater than n. Prove that $n!$ divides $(a-1)(a^2-1)\cdots(a^n-1)$.

3. Let A_1, A_2, A_3, A_4 be four points in the plane where A_4 is the centroid of the triangle $A_1A_2A_3$. Find a point A_5 in the plane that maximizes the ratio

$$\frac{\min\limits_{1 \le i < j < k \le 5} A_iA_jA_k}{\max\limits_{1 \le i < j < k \le 5} A_iA_jA_k}.$$