Searching for Additive-Multiplicative Magic Squares

Krit Boonsiriseth, CJ Quines, Steven Raphael

Abstract

It is unknown whether there are 6×6 magic squares that are simultaneously additive and multiplicative. With less than 10,000 hours of single-core compute, we find over 750,000 6×6 semi-magic squares, which are magic except for their diagonals. The best semi-magic square is almost magic, except for one diagonal with a different sum. Code is public at `https://github.com/talkon/magic-squares`.

Types of magic squares

- All squares will be $n \times n$ grids of distinct positive integers.
- Let $\boldsymbol{S}$ be the **magic sum** and $\boldsymbol{P}$ the **magic product**.
- An *axis* is a row, column, or diagonal. An $n \times n$ square has $2n + 2$ axes.
- An axis can have two *magic properties*:
  If its sum is S, the axis *has magic sum*.
  If its product is P, the axis *has magic product*.
  An axis is *magic* if it has both magic sum and product.
- An (additive-multiplicative) **semi-magic square** has magic rows and columns.
- An (additive-multiplicative) **magic square** is a semi-magic square that also has magic diagonals.

Finding magic squares is hard, so progress is specified in terms of semi-magic squares, whose diagonals have additional magic properties:

- A **0-type** square has no diagonals with magic properties.
- An $\boldsymbol{S}$**-type** square has one diagonal with magic sum.
- A $\boldsymbol{P}$**-type** square has one diagonal with magic product.
- An $\boldsymbol{SP}$**-type** square has one magic diagonal.
- An $\boldsymbol{S+P}$**-type** square is one where one diagonal has magic sum, and the other diagonal has magic product, etc.

Best known progress

Year	Author	Size	Type	S	P
2007	Morgenstern	6×6	0	327	$2^{10}3^4 5^3 7^2 = 508{,}032{,}000$
2007	Morgenstern	6×6	$S+P$	289	$2^{11}3^4 5^3 7^1 11^1 = 1{,}596{,}672{,}000$
2008	Morgenstern	6×6	$S+S$	360	$2^{10}3^6 5^3 7^1 13^1 = 8{,}491{,}392{,}000$
2010	Shirakawa	7×7	$S+S$	310	$2^{10}3^4 5^2 7^2 11^1 13^1 = 14{,}529{,}715{,}200$
2010	Shirakawa	7×7	$SP+P$	380	$2^{11}3^5 5^2 7^2 11^1 13^1 = 87{,}178{,}291{,}200$
2016	Miquel	7×7	$SP+SP$	465	$2^{10}3^7 5^3 7^2 11^1 = 150{,}885{,}504{,}000$

Table 1. Best known progress for finding additive-multiplicative magic squares. [1]

126	66	50	90	48	1	84
20	70	16	54	189	110	6
100	2	22	98	36	72	135
96	60	81	4	10	49	165
3	63	30	176	120	45	28
99	180	14	25	7	108	32
21	24	252	18	55	80	15

Table 2. Miquel's 7×7 additive-multiplicative magic square.

Algorithm overview

1. **Enumeration**: Fix S and P, then enumerate all sets of 6 integers with this sum and product. We call such sets *vectors*.
2. **Arrangement**: Through backtracking, find all ways to arrange 12 vectors to form 6 rows and 6 columns to make a semi-magic square.
3. **Postprocessing**: For each semi-magic square, count how many potential diagonals have sum S and product P.

Backtracking pseudocode

```
Search(g, t)
  if t is a solution, return
  for axis in rows, cols
    if axis = cols and t.rows.vecs = ∅, break
    if length(t.axis.vecs) = 6, continue
    for vec in t.axis.valid
      if max(g.vecs[vec]) < max(t.unmatched), break
      if max(t.unmatched) ∉ vec and (axis = cols or t.rows.vecs ≠ ∅), continue
      u = copy(t)
      u.axis = t.axis ∪ {vec}
      u.unmatched = t.unmatched ⊕ g.bitarrays[vec]
      minVec = 0 if length(t′.rows) > 0 or length(t′.cols) > 0 else vec
      FillValids(g, t.rows, u.rows, [axis = cols], vec, minVec)
      FillValids(g, t.cols, u.cols, [axis = rows], vec, minVec)
      Search(g, u)

FillValids(g, oldAxis, newAxis, numInters, newVec, minVec)
  newAxis.valid = ∅
  for vec in oldAxis.valid
    if vec < minVec, continue
    if g.inters[newVec][vec] ≠ numInters, continue
    newAxis.valid = newAxis.valid ∪ {vec}
```

Implementation notes

- Enumeration and postprocessing time is negligible. We wrote it in Python.
- Assignment is task-parallelizable. Run several copies; overhead is negligble.
- Key heuristic: relabel elements by frequency (less frequent → larger label), then choose vectors by maximum unmatched label.
- FillValids is critical. We used AVX-512 to vectorize it.

Search methodology

- We deployed our algorithm on MIT SuperCloud, selecting certain values of P to search over.
- The choice of P is based on some heuristics and from data from the previous values of P searched. For example, values of P with "nicely decreasing" exponents, like $2^{13}3^8 5^4 7^2 11^1$, tend to have more semi-magic squares.
- For a given value of P, we stop the search after a target number of nodes is reached (as opposed to exhaustive search).
- We used a total of 414.32 days of single-core compute time on arrangement.

Results

Our search found a total of $758{,}949$ additive-multiplicative semi-magic squares. This includes squares of type $SP+P$, $SP+S$, SP, and $P+P$ (in order of decreasing rarity). As far as we know, each of these types of squares have not been found prior to our search.

Type	Squares	Smallest P	S	Notes
$SP+SP$	0			None known
$SP+P$	1	$2^{13}3^5 5^3 7^2 13^1 = 158{,}505{,}984{,}000$	632	**New type**
$SP+S$	9	$2^{11}3^6 5^3 7^2 13^1 = 118{,}879{,}488{,}000$	577	**New type**
SP	519	$2^{11}3^6 5^3 7^1 11^1 = 14{,}370{,}048{,}000$	492	**New type**
$P+P$	721	$2^{11}3^5 5^3 7^2 13^1 = 5{,}660{,}928{,}000$	433	**New type**
$S+P$	3,531	$2^{11}3^4 5^3 7^1 11^1 = 1{,}596{,}672{,}000$	289	Previously found
$S+S$	6,492	$2^{10}3^5 5^3 7^1 11^1 = 2{,}395{,}008{,}000$	386	
P	178,153	$2^8 3^6 5^3 7^2 = 1{,}143{,}072{,}000$	392	
S	427,869	$2^7 3^4 5^3 7^2 13^1 = 825{,}552{,}000$	346	
0	758,949	$2^{10}3^4 5^3 7^2 = 508{,}032{,}000$	327	Previously found

Table 3. Types of squares found in our search

112	200	81	16	28	195
320	39	98	54	96	25
60	30	40	42	208	252
78	147	240	120	15	32
27	24	160	260	105	56
35	192	13	140	180	72

252	42	96	125	36	26
52	15	108	128	225	49
28	64	50	39	126	270
30	210	117	84	16	120
135	90	196	12	104	40
80	156	10	189	70	72

Table 4. **Left:** smallest (and only) 6×6 $SP+P$-type square found; the diagonal from top-left to bottom-right has the wrong sum. **Right:** smallest 6×6 $SP+S$-type square found; the diagonal from bottom-left to top-right has the wrong product.

Discussion

- **Estimated remaining search time.** Around 1 in 830 and 1 in 2,600 possible diagonals have the magic sum and product, respectively. There are 5,400 ways to pick diagonals of a semi-magic square, so there is a
$$5400 \times \frac{1}{830^2} \times \frac{1}{2600^2} \approx 1 \text{ in 860 million}$$
chance that a semi-magic square is magic. At our current search rate of ≈ 45 seconds per semi-magic square, this will take **1,200 years** of CPU time.
- **Current bottlenecks.** Enumeration uses too much memory, and diagonals with correct sum and/or product become rarer as P gets larger.
- **Directions for future work.** Enumeration code could be optimized to reduce memory usage or modified to only enumerate small sums. Arrangement complexity is hard to improve, but any improvements will be very helpful. Searching for 5×5 magic squares can also be attempted using our code. Contribute at `https://github.com/talkon/magic-squares`!

Acknowledgements and References

The authors acknowledge the MIT SuperCloud and Lincoln Laboratory Supercomputing Center for providing HPC resources that have contributed to the research results reported within this poster.

[1] Boyer, C. The smallest possible additive-multiplicative magic square. Retrieved from `http://www.multimagie.com/English/SmallestAddMult.htm`