A simplified drag estimate for a tether with a belly

We know that the drag of a straight tether [stiff tether model] there one end of the tether is fixed and the other moving at the speed of the kite is given by the integral summing the drag at each point of the tether, assuming a straight radius, with one end rotating at radius r_0 and the other r_1

$$D_t = \int_0^l \frac{1}{2}\rho\omega^2\left(r_0 + \frac{(r_1 - r_0)\,x}{l}\right)^2 C_{D,t} d\,\mathrm{d}x$$

Solved to

$$D_t = \frac{1}{6}\rho\omega^2\left(r_0^2 + r_0 r_1 + r_1^2\right) C_{D,t} d l$$

The same equation may be used for a single tether kite with $r_0 = 0$.

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From the AWEC2019 poster «The second law of tether scaling» we have a more accurate description of tether shape defined by the curvature of the kite both due to centrifugal forces on the tether causing a «belly» shape and also spiraling due to tether drag. This is given by a pair of differential equations

$$\ddot{r} = \frac{1}{T}\left(C(s) - \mu\omega^2 r(s)\right)$$

$$\ddot{\alpha} = \frac{1}{Tr} D(s)\sqrt{1 - r^2\dot{\alpha}^2} - \frac{\dot{r}\dot{\alpha}}{r}$$

We will not dwell too much on this result here, other than see that the first equation is independent of the second, and may be solved in an isolated manner to find the radius $r(s)$ of tether looping at any point s on the tether. The force described by $C(s)$ is assumed to be zero in this context. The other values in the first differential equation are T being the tension force on the tether, μ being the weight of the tether per unit length of tether and ω being the rotary speed of the shaft,

That first equation may be solved for a tether rotating with radius r_0 and r_1 at either end, leading to

$$r\,(s) = -\left(r_0\,\cos\left(\omega\,l\sqrt{\mu}\frac{1}{\sqrt{T}}\right) - r_1\right)\sin\left(\omega\,s\sqrt{\mu}\frac{1}{\sqrt{T}}\right)\left(\sin\left(\omega\,l\sqrt{\mu}\frac{1}{\sqrt{T}}\right)\right)^{-1} + r_0\,\cos\left(\omega\,s\sqrt{\mu}\frac{1}{\sqrt{T}}\right)$$

$$r(s) = \frac{\left(r_1 - r_0\cos\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)\right)\sin\left(\frac{\sqrt{\mu}\omega s}{\sqrt{T}}\right)}{\sin\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)} + r_0\cos\left(\frac{\sqrt{\mu}\,\omega\,s}{\sqrt{T}}\right)$$

If the tether is attached at the ground in a single point $r_0 = 0$ we rather get

$$r(s) = r_1 \frac{\sin\left(\frac{\sqrt{\mu}\omega s}{\sqrt{T}}\right)}{\sin\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)}$$

The Maple code to arrive at this is

```
eq := diff(r(s), s, s) = - 1 / T * mu * omega^2 * r(s);
sol1 := dsolve({eq, r(0) = r__0, r(l) = r__1});
sol2 := dsolve({eq, r(0) = 0, r(l) = r__1});
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So far all is well, but the second differential equation is hard to solve explicitly. So instead we make a simplified estimate of tether drag assuming that the belly of the tether exists, but the spiral shape does not. This simplification should be correct except it does not consider that the area of tether facing the apparent wind is reduced a little due to the spiral curvature. In that case, the drag is computed by

$$D_t = \int_0^l \frac{1}{2} \rho C_{D,T} d \left[\omega\, r(s)\right]^2 ds$$

The solution to this for a shaft with r_0 and r_1 is

$$D_t = -\frac{\rho C_{D,t} d \omega}{4\sqrt{\mu}\sin^2\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)} \left(\sqrt{T}\left[\left(r_0^2 + r_1^2\right)\cos\left(\frac{\omega\sqrt{\mu}\,l}{\sqrt{T}}\right) - 2r_0 r_1\right]\sin\left(\frac{\omega\sqrt{\mu}\,l}{\sqrt{T}}\right) + \omega\sqrt{\mu}\,l\left[2r_0 r_1\cos\left(\frac{\omega\sqrt{\mu}\,l}{\sqrt{T}}\right) - r_0^2 - r_1^2\right]\right)$$

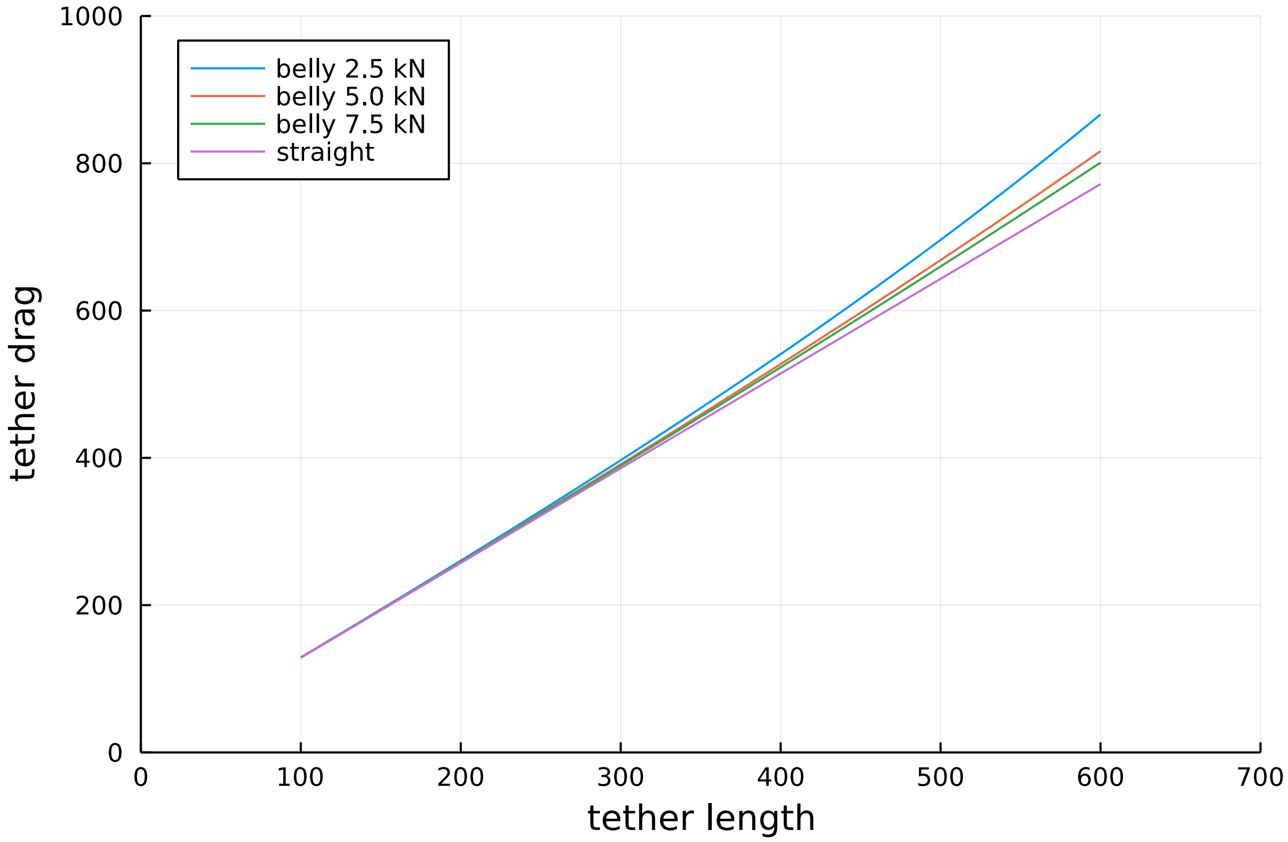
For a kite with $r_0 = 0$ the result is

$$D_t = \frac{\rho\omega C_{D,t} d\left(\omega\sqrt{\mu}\,l - \sqrt{T}\cos\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)\sin\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)\right)}{4\sqrt{\mu}\sin^2\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)}$$

The maple code to arrive at this is

```
drag_shaft := simplify(int(1/2 * rho * (omega * subs(sol1, r(s)))^2 * C__Dt * d, s =
0..l), assume = positive);
drag_single :=  simplify(int(1/2 * rho * (omega * subs(sol2, r(s)))^2 * C__Dt * d, s =
0..l), assume = positive);
```

To see if this looks plausible, we compare some drag values for a 3 mm tether, at lengths 200 m, 400 m, 600 at tensions 2.5 kN, 5.0 kN and 7.5 kN, r_0 = 10 m and r_1 = 40 m, $C_{D,t}$ = 1.0, ω = 1.0, ρ = 1.225, μ = 0.005.



The plot confirms that the «belly» aware tether drag approximations are similar to the straight line approximation, and deviate more with longer tethers and less tension, as expected.

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