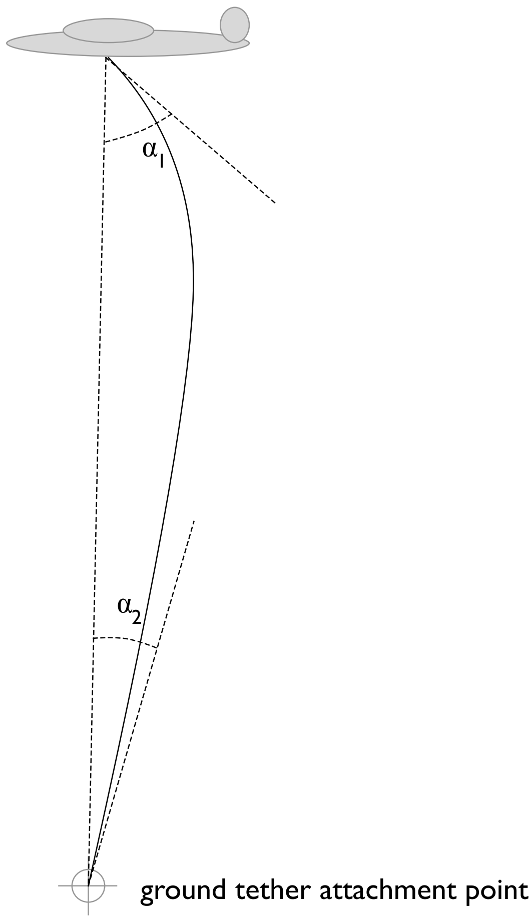
A simplified drag estimate for a tether with a belly

We must differentiate between rotary power transfer devices and other AWE style devices when considering tether drag. We will explain by looking at the following sketch



For the simplified model of the kite, we would just use the straight line tether, then add drag to the kite and ground attachment point [GAP]. How do we allocate the drag force between the kite and GAP?

Well for any power production not using the tether as a moment carrying shaft, we only worry about the angle α_1 because any drag causing α_2 is handled by the GAP as it is usually stationary, and there is no power loss associated with α_2. A simple way to assign drag is a linear allocation based on the drag's position on the tether. Any tether drag close to the kite is allocated mostly to the kite, and vice versa for the GAP.

For a soft shaft moment transfer [TRPT] the story is a bit different. The transferred shaft energy is received at the GAP which is moving. The tether drag forces associated with α_1 and α_2 both count as energy losses. Thus, the entire drag of the tether may be, as an approximation, regarded tether drag to be subtracted from the system overall performance.

The equations for both types, when assuming that the radius of rotation of the tether is $r(s)$ is

For Yoyo

$$D_t = \int_0^l \frac{1}{2} \rho \omega^2 \left[r(s) \right]^2 C_{D,t} \frac{s}{l} d \, \mathrm{d}s \qquad (1)$$

And for TRPT

$$D_t = \int_0^l \frac{1}{2}\rho\omega^2 \left[r(s)\right]^2 C_{D,t} d\,\mathrm{d}s \tag{2}$$

Where ω is the rotary speed of the rig, l is the tether length, $C_{D,t}$ is the tether drag coefficient and ρ is the air density. The free variable s is zero at the ground station side and $s = l$ at the kite.

Calculating these for a straight line tether yields (where r_0 is the looping radius at the ground and r_1 at the kite)

$$r(s) = r_0 + \frac{\left(r_1 - r_0\right)s}{l} \tag{3}$$

For Yoyo

$$D_t = \frac{1}{8}\rho\omega^2 r_1^2 C_{D,t} d l \tag{4}$$

And for TRPT

$$D_t = \frac{1}{6}\rho\omega^2 \left(r_0^2 + r_0 r_1 + r_1^2\right) C_{D,t} d l \tag{5}$$

We find this using the Maple code

```
simple_drag_trpt := simplify(int(1/2 * rho * (omega * subs(r(s) = r__0 + s * (r__1 -
r__0)/ l, r(s)))^2 * C__Dt * d, s = 0..l), assume = positive);
simple_drag_yoyo := simplify(int(1/2*rho*(omega * r__1 * x / l)^2 * C__Dt * d * x/l, x
= 0..l), assume = positive);
```

—

From the AWEC2019 poster «The second law of tether scaling» we have a more accurate description of tether shape defined by the curvature of the kite both due to centrifugal forces on the tether causing a «belly» shape and also spiraling due to tether drag. This is given by a pair of differential equations

$$\ddot{r} = \frac{1}{T}\left(C(s) - \mu\omega^2 r(s)\right) \tag{6}$$

$$\ddot{\alpha} = \frac{1}{Tr} D(s)\sqrt{1 - r^2\dot{\alpha}^2} - \frac{\dot{r}\dot{\alpha}}{r} \tag{7}$$

We will not dwell too much on this result here, other than see that the first equation is independent of the second, and may be solved in an isolated manner to find the radius $r(s)$ of tether looping at any point s on the tether. The force described by $C(s)$ is assumed to be zero in this context. The other values in the first differential equation are T being the tension force on the tether, μ being the weight of the tether per unit length of tether and ω being the rotary speed of the shaft,

That equation (6) may be solved for a TRPT based soft shaft using

$$r(s) = \frac{\left(r_1 - r_0\cos\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)\right)\sin\left(\frac{\sqrt{\mu}\omega s}{\sqrt{T}}\right)}{\sin\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)} + r_0\cos\left(\frac{\sqrt{\mu}\omega s}{\sqrt{T}}\right) \tag{8}$$

For the non-TRPT case we use $r_0 = 0$ and get

$$r(s) = r_1 \frac{\sin\left(\frac{\sqrt{\mu}\omega s}{\sqrt{T}}\right)}{\sin\left(\frac{\sqrt{\mu}\omega l}{\sqrt{T}}\right)} \tag{9}$$

The Maple code to arrive at this is

```
eq := diff(r(s), s, s) = - 1 / T * mu * omega^2 * r(s);
sol1 := dsolve({eq, r(0) = r__0, r(l) = r__1});
sol2 := dsolve({eq, r(0) = 0, r(l) = r__1});
```

So far all is well, but the second differential equation (7) is hard to solve explicitly. So instead we make a simplified estimate of tether drag assuming that the belly of the tether exists, but the spiral shape does not. This simplification should be correct except it does not consider that the area of tether facing the apparent wind is reduced a little due to the spiral curvature.

We do this by combining, for the TRPT, equation (2) and (8) to get equation (11)

$$D_t = -\frac{\rho C_{D,t} d\omega}{4\sqrt{\mu}\sin^2\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)}\left(\sqrt{T}\left[\left(r_0^2 + r_1^2\right)\cos\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right) - 2r_0r_1\right]\sin\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right) + \omega\sqrt{\mu}l\left[2r_0r_1\cos\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right) - r_0^2 - r_1^2\right]\right)$$

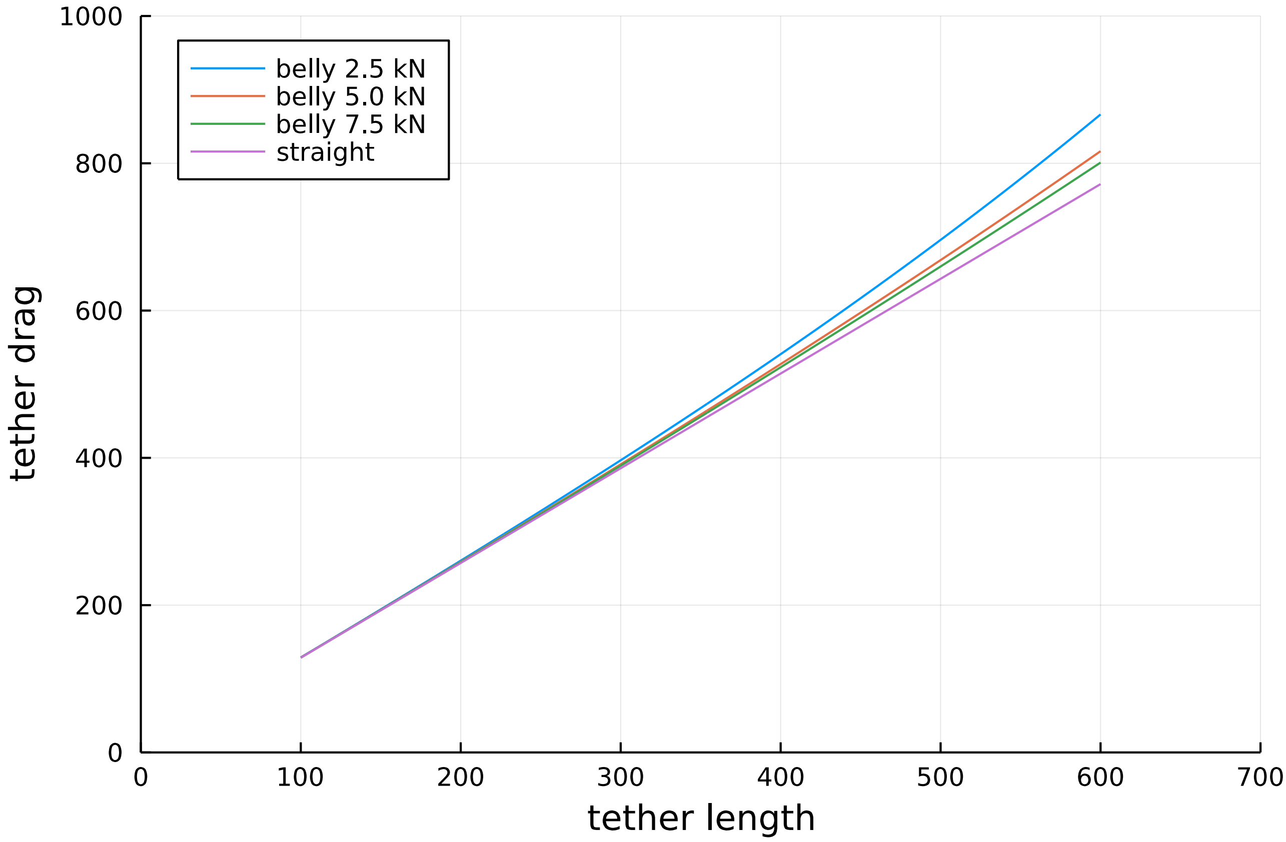
For the non-TRPT case, we combine equation (1) with equation (9) to get equation (12)

$$D_t = \frac{r_1^2\rho C_{D,t}d\left(\omega^2\mu^{\frac{3}{2}}l^2 + T\sqrt{\mu} - 2\omega\sqrt{T}\mu l\cos\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)\sin\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right) - T\sqrt{\mu}\cos^2\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right) - \sqrt{T}\cos\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)\sin\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)\right)}{8\mu^{\frac{3}{2}}l\sin^2\left(\frac{\omega\sqrt{\mu}l}{\sqrt{T}}\right)}$$

The maple code to arrive at this is

```
drag_trpt := simplify(int(1/2 * rho * (omega * subs(sol1, r(s)))^2 * C__Dt * d, s = 
0..l), assume = positive);
drag_other :=  simplify(int(1/2 * rho * (omega * subs(sol2, r(s)))^2 * C__Dt * d * s / 
l, s = 0..l), assume = positive);
```

To see if this looks plausible, we compare some drag values for a 3 mm tether, at lengths 200 m, 400 m, 600 at tensions 2.5 kN, 5.0 kN and 7.5 kN, r_0 = 10 m and r_1 = 40 m, $C_{D,t}$ = 1.0, ω = 1.0, ρ = 1.225, μ = 0.005.



The plot confirms that the «belly» aware tether drag approximations are similar to the straight line approximation, and deviate more with longer tethers and less tension, as expected.

A Julia implementation of the two drag functions is given by

```
function belly_drag_shaft(rho, c_d_t, r_0, r_1, mu, d, l, omega, tension)
  -rho * c_d_t * (((r_0 ^ 2 + r_1 ^ 2) * cos(omega * sqrt(mu) * l * tension ^ (-1//2))
- 2 * r_0 * r_1) * sqrt(tension) * sin(omega * sqrt(mu) * l * tension ^ (-1//2)) + 2 *
(r_0 * r_1 * cos(omega * sqrt(mu) * l * tension ^ (-1//2)) - r_0 ^ 2 / 2 - r_1 ^ 2 / 2)
* l * sqrt(mu) * omega) * mu ^ (-1//2) * d * omega / sin(omega * sqrt(mu) * l * tension
^ (-1//2)) ^ 2 / 4
end

function belly_drag_fixed_gap(rho, c_d_t, r_1, mu, d, l, omega, tension)
  -rho * r_1 ^ 2 * c_d_t * d * (2 * omega * cos(omega * sqrt(mu) * l * tension ^
(-1//2)) * sqrt(tension) * sin(omega * sqrt(mu) * l * tension ^ (-1//2)) * l * mu -
omega ^ 2 * l ^ 2 * mu ^ (3//2) + tension * cos(omega * sqrt(mu) * l * tension ^
(-1//2)) ^ 2 * sqrt(mu) - tension * sqrt(mu)) / sin(omega * sqrt(mu) * l * tension ^
(-1//2)) ^ 2 * mu ^ (-3//2) / l / 8
end
```