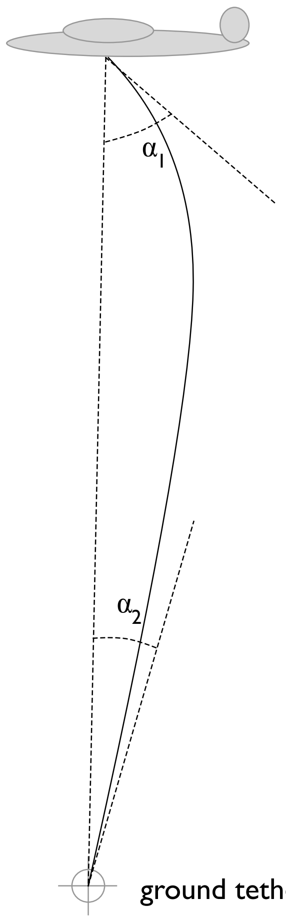
A simplified drag estimate for a tether with a belly

We must differentiate between rotary power transfer devices and other AWE style devices when considering tether drag. We will explain by looking at the following sketch



For the simplified model of the kite, we would just use the straight line tether, then add drag to the kite and ground attachment point [GAP]. How do we allocate the drag force between the kite and GAP?

Well for any power production not using the tether as a moment carrying shaft, we only worry about the angle α_1 because any drag causing α_2 is handled by the GAP as it is usually stationary, and there is no power loss associated with α_2. A simple way to assign drag is a linear allocation based on the drag's position on the tether. Any tether drag close to the kite is allocated mostly to the kite, and vice versa for the GAP.

For a soft shaft moment transfer [TRPT] the story is a bit different. The transferred shaft energy is received at the GAP which is moving. The tether drag forces associated with α_1 and α_2 both count as energy losses. Thus, the entire drag of the tether may be, as an approximation, regarded tether drag to be subtracted from the system overall performance.

The equations for both types, when assuming that the radius of rotation of the tether is $r(s)$ is

For Yoyo

$$D_t = \int_0^l \frac{1}{2} \rho \omega^2 \left[r(s) \right]^2 C_{D,t} \frac{s}{l} d\, \mathrm{d}s \tag{1}$$

And for TRPT

$$D_t = \int_0^l \frac{1}{2}\rho\omega^2 \left[r(s)\right]^2 C_{D,t} d\,\mathrm{d}s \tag{2}$$

Where ω is the rotary speed of the rig, l is the tether length, $C_{D,t}$ is the tether drag coefficient and ρ is the air density. The free variable s is zero at the ground station side and $s = l$ at the kite. Note that the calculation is done for zero twist angle and therefore is a worst caser estimate. The tether drag will approach that of non-TRPT plants when shaft twist is approximately 90 degrees, then become even lower between 90 and 180 degrees.

Calculating these for a straight line tether yields (where r_0 is the looping radius at the ground and r_1 at the kite)

$$r(s) = r_0 + \frac{\left(r_1 - r_0\right) s}{l} \tag{3}$$

For Yoyo

$$D_t = \frac{1}{8}\rho\omega^2 r_1^2 C_{D,t} d l \tag{4}$$

And for TRPT

$$D_t = \frac{1}{6}\rho\omega^2 \left(r_0^2 + r_0 r_1 + r_1^2\right) C_{D,t} d l \tag{5}$$

We find this using the Maple code

```
simple_drag_trpt := simplify(int(1/2 * rho * (omega * subs(r(s) = r__0 + s * (r__1 -
r__0)/ l, r(s)))^2 * C__Dt * d, s = 0..l), assume = positive);
simple_drag_yoyo := simplify(int(1/2*rho*(omega * r__1 * x / l)^2 * C__Dt * d * x/l, x
= 0..l), assume = positive);
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From the AWEC2019 poster «The second law of tether scaling» we have a more accurate description of tether shape defined by the curvature of the kite both due to centrifugal forces on the tether causing a «belly» shape and also spiraling due to tether drag.

As time has passed and the understanding of the problem has since become better, we will redo these calculations with greater accuracy.

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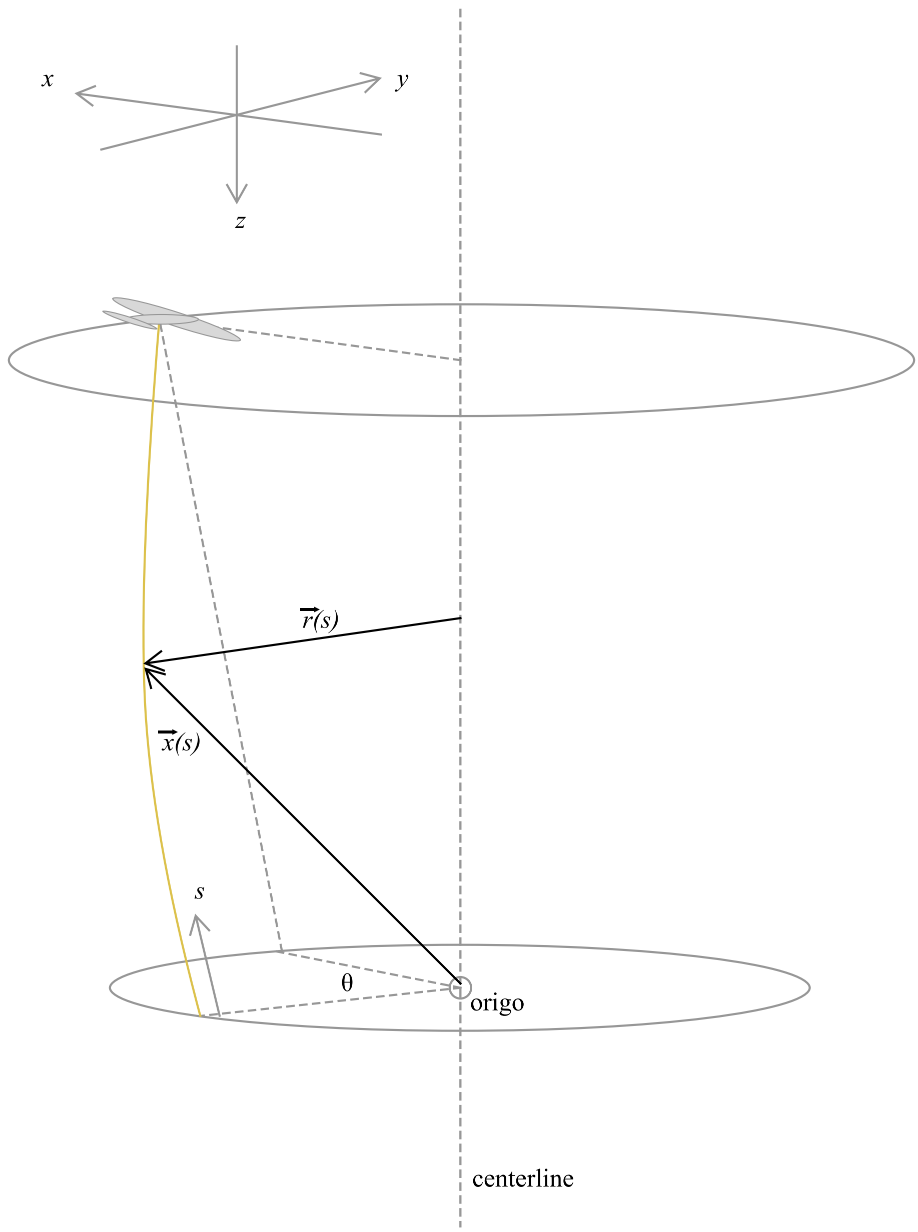
The tether will be curved twice. One due to the centrifugal forces working on the tether itself. Next because of tether drag. To calculate the exact shape of the tether requires differential equations of the tether shape to be solved. We can try to solve this as a quasi static movement. All the tether is rotating at a constant speed ω but the shape of the tether is constant and rotating around the centerline of looping.

In a first stab at calculating the drag of a tether for TRPT with a differential equation, we will assume that there is no bending of the tether due to drag, only due to mass and centrifugal forces.

The big step forward compared to the previous attempts will be that the calculation will take into account the shaft twist θ.



Our path function or the tether is $\vec{x}(s)$ where s is the point on the tether, starting from zero and ending at the tether length. This means that

$$\left| \frac{\partial \vec{x}(s)}{\partial s} \right| = 1$$

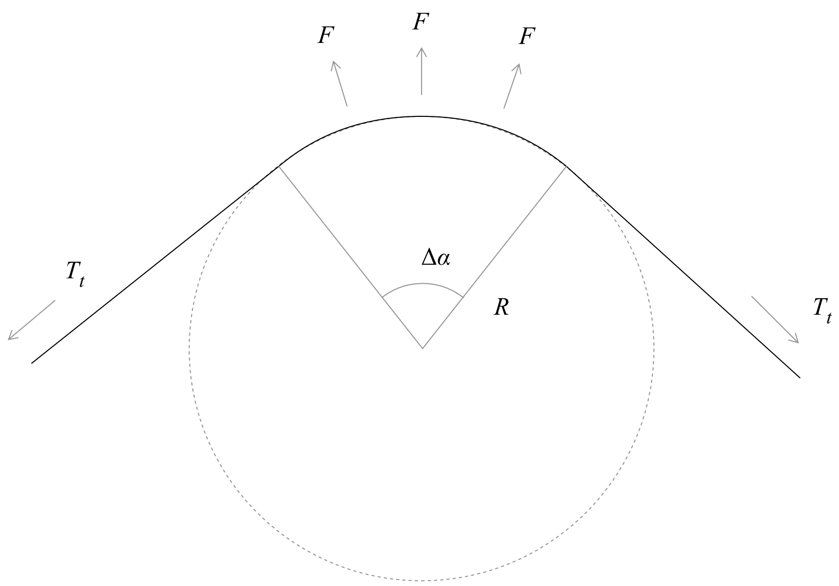
The curvature of $\vec{x}(s)$ must be everywhere just right to compensate for centrifugal forces. All of the tether is moving in a quasi static way at rotational speed ω. Our shaft tension T is constant, but measured in the direction of the centerline. If we say $T_t(s)$ is the tension in the tether at s, we get

$$T_t(s)\left|\frac{\partial \overrightarrow{x}(s)}{\partial s}\cdot\hat{k}\right| = T$$

Note that $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are the unit vectors of our coordinate system.

If the tether is not normal to the vector going from the centerline to the kite, some of the centrifugal forces will be taken up by the tether inline tension. As these additions to tether tension are expected to be much smaller than the tether tension, we will assume these are zero, and only take into account any curvature in the direction or $\vec{r}(s)$ which points from the centerline to a point on our path.

$$\vec{r}(s) = \overrightarrow{x}(s)\cdot\left(\hat{\imath}+\hat{\jmath}\right)$$



We will look at force per meter tether when a tether is curved with a radius R. We are looking for an expression for force per meter. Looking at the figure the length of the curved segment is $\Delta s = \Delta\alpha R$ and if we assume $\Delta\alpha$ is small, all F will be pointing in the same direction. In that case the sum force should be, by Newtons first law;

$$\sum F \approx 2\sin\frac{\Delta\alpha}{2}T_t$$

For small $\Delta\alpha$ we may approximate this to

$$\sum F \approx \Delta\alpha T_t$$

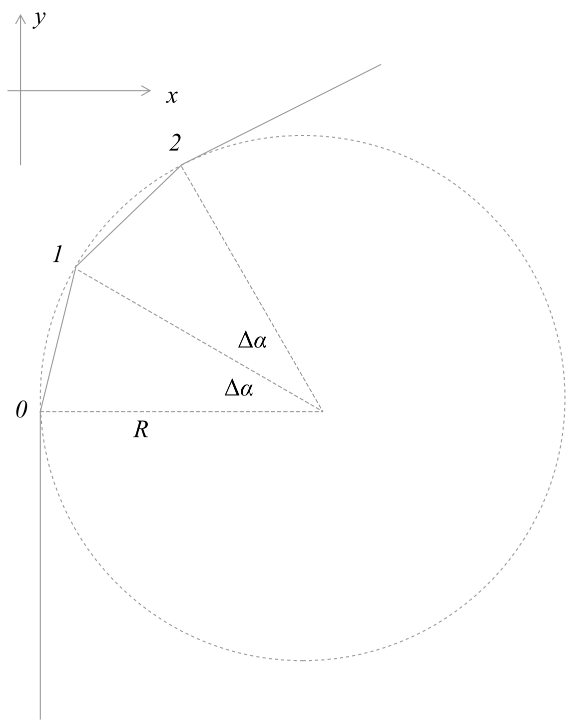
Furthermore we replace $\Delta\alpha$ and divide by Δs

$$\frac{\sum F}{\Delta s} \approx \frac{T_t}{R}$$

Which should give us

$$\frac{\partial F}{\partial s} = \frac{T_t(s)}{R}$$

Next we will look at the relationship between the double derivative of the path of the shape and the radius R. We start by a simple discretized figure of the tether curved over a cylinder with radius R. We also added a coordinate system where the tether starts aligned to the y-axis, then increases with x.



We will assume we have a function f giving the value of the x coordinate based on the tether length s. The discretized version of the derivative at point 0 is

$$\frac{\Delta f_0}{\Delta s} \approx \frac{R\ \Delta\alpha\ \sin\frac{\Delta\alpha}{2}}{\Delta s} \approx \frac{R\ \Delta\alpha\ \frac{\Delta\alpha}{2}}{\Delta s}$$

At point 1

$$\frac{\Delta f_1}{\Delta s} \approx \frac{R\ \Delta\alpha\ \sin\frac{3}{2}\Delta\alpha}{\Delta s} \approx \frac{R\ \Delta\alpha\ \frac{3\Delta\alpha}{2}}{\Delta s}$$

To find the discretized double derivative we have

$$\frac{\frac{\Delta f_1}{\Delta s} - \frac{\Delta f_0}{\Delta s}}{\Delta s} \approx \frac{\frac{R\ \Delta\alpha\ \frac{3\,\Delta\alpha}{2}}{\Delta s} - \frac{R\ \Delta\alpha\ \frac{\Delta\alpha}{2}}{\Delta s}}{\Delta s}$$

$$\frac{\frac{\Delta f_1}{\Delta s} - \frac{\Delta f_0}{\Delta s}}{\Delta s} \approx \frac{R\ \Delta\alpha^2}{\Delta s^2} = \frac{R\left(\frac{\Delta s}{R}\right)^2}{\Delta s^2} = \frac{1}{R}$$

Which leads ut to believe that

$$\frac{\partial^2 f(s)}{\partial s^2} = \frac{1}{R}$$

We can use this to get rid of references to R and rather use the double derivatives of the path function

$$\frac{\partial F}{\partial s} = \frac{\partial^2 f(s)}{\partial s^2} T_t(s)$$

We will assume the tether must be curved in response to two effects; the first being the centripetal forces required to withstand the rotation and the second being the tether drag. We will sum these together in a differential fashion giving force per unit length of tether.

The unit length vector in the direction of $\vec{r}(s)$ is

$$\hat{r}(s) = \frac{\vec{r}(s)}{\left|\vec{r}(s)\right|}$$

The «centrifugal» forces vector per tether length s are given by, with μ being tether mass per meter;

$$\vec{c}(s) = \left[\mu \left|\vec{r}(s)\right| \omega^2\right] \hat{r}(s)$$

$$\vec{c}(s) = \mu \omega^2 \, \vec{r}(s)$$

The drag forces are dependent on the front facing area per length of the tether relative to the apparent wind. First we find a normal vector facing in the direction of the drag vector

$$\hat{d}(s) = \frac{A_{90} \, \vec{r}(s)}{\left|A_{90} \, \vec{r}(s)\right|}$$

$$\hat{d}(s) = \frac{A_{90} \, \vec{r}(s)}{\left|\vec{r}(s)\right|}$$

The matrix A is the rotation matrix -90 degrees around the z-axis

$$A_{90} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we calculate the length of the derivative of the tether projected to a plane normal to the apparent wind, remembering that the norm of the derivative is one;

$$K_{\text{app}}(s) = \left| \frac{\partial \vec{x}(s)}{\partial s} \times \hat{d} \right|$$

All of this gives us a drag force vector per unit length of tether, with diameter d and drag coefficient $C_{D,t}$:

$$\vec{d}(s) = \frac{1}{2} \rho \omega^2 \left|\vec{r}(s)\right|^2 C_{D,t} d K_{\text{app}} \, \hat{d}$$

The amount of forces that must be overcome by tether curvature is

$$\vec{f}(s) = \vec{c}(s) + \vec{d}(s)$$

The force per length $\vec{f}(s)$ will have a component normal to the tether and another component inline with the tether. Only the normal part will add to the curvature of the tether. Also, the tether is only curved in that one direction, so the curvature in the direction perpendicular to that should be zero

$$\frac{\partial^2 \overrightarrow{x}(s)}{\partial s^2} \cdot \left[\frac{\partial \overrightarrow{x}(s)}{\partial s} \times \vec{f}(s)\right] = 0 \qquad \text{(I)}$$

The part of $\vec{f}(s)$ that is normal to the tether is (noting that the norm of the derivative is one)

$$\vec{f}_n(s) = \frac{\partial \overrightarrow{x}(s)}{\partial s} \times \left[\frac{\partial \overrightarrow{x}(s)}{\partial s} \times \vec{f}(s)\right]$$

$$\hat{f}_n(s) = \frac{\vec{f}_(s)}{\left|\vec{f}_n(s)\right|}$$

Newtons first equation in differential form along the tether thus ends up

$$\hat{f}_n(s) \cdot \frac{\partial^2 f(s)}{\partial s^2} T_t(s) = \hat{f}_n(s) \cdot \vec{f}_n(s)$$

$$\vec{f}_n(s) \cdot \frac{\partial^2 f(s)}{\partial s^2} T_t(s) = \vec{f}_n(s) \cdot \vec{f}_n(s)$$

$$\vec{f}_n(s) \cdot \frac{\partial^2 f(s)}{\partial s^2} = \frac{1}{T} \left| \frac{\partial \overrightarrow{x}(s)}{\partial s} \cdot \hat{k} \right| \left(\vec{f}_n(s) \cdot \vec{f}_n(s)\right) \qquad \text{(II)}$$

To find the x and y coordinates of $\overrightarrow{x}(s)$ we can use these differential equations, and to find the z coordinate function we must also use

$$\left|\frac{\partial \overrightarrow{x}(s)}{\partial s}\right| = 1$$

We now separate $\overrightarrow{x}$ into three separate functions for each axis, so that $\overrightarrow{x}(s) = \left\langle x_x(s), x_y(s), x_z(s)\right\rangle$.

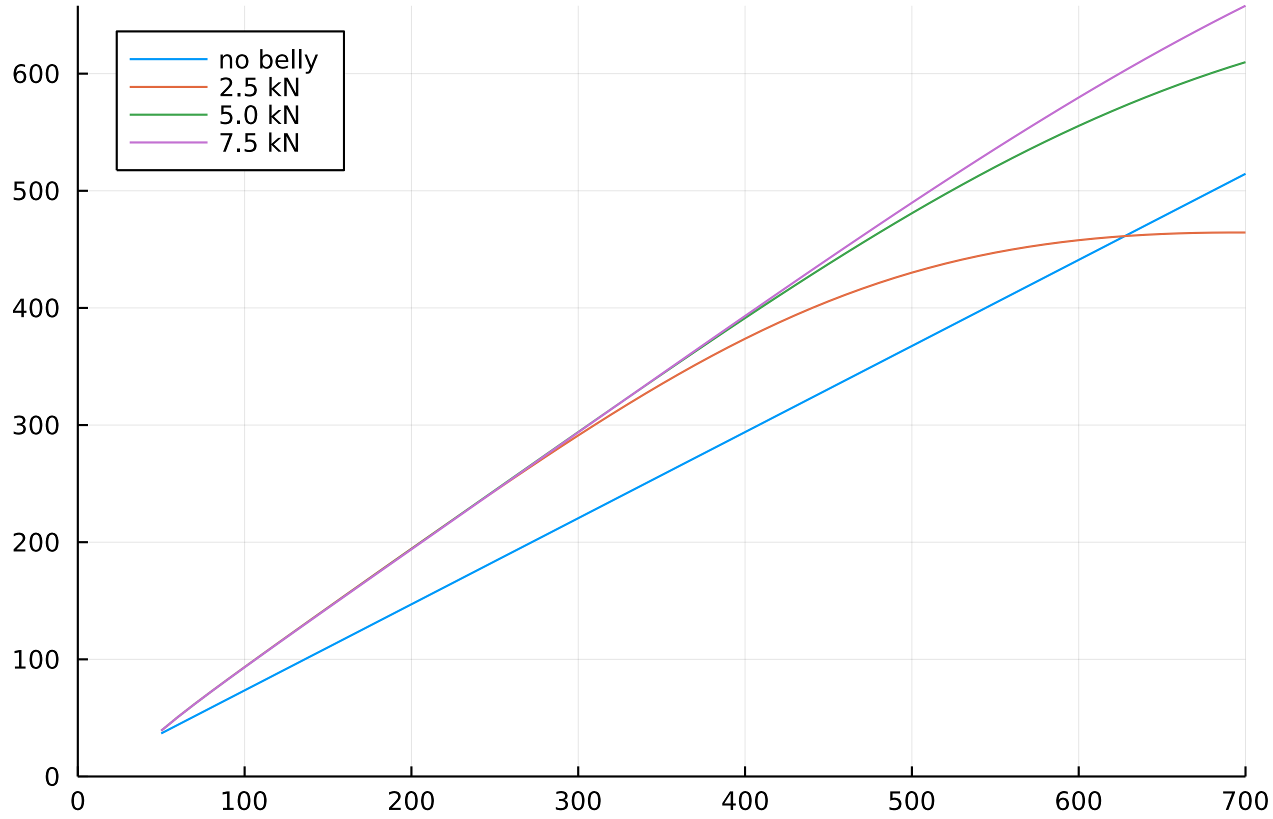
$$\left[\frac{\partial x_x(s)}{\partial s}\right]^2 + \left[\frac{\partial x_y(s)}{\partial s}\right]^2 + \left[\frac{\partial x_z(s)}{\partial s}\right]^2 = 1 \qquad \text{(III)}$$

We need three equations with the second derivative of $\overrightarrow{x}(s)$, so we derivate the last equation by s

$$\frac{\partial x_x(s)}{\partial s}\frac{\partial^2 x_x(s)}{\partial s^2} + \frac{\partial x_y(s)}{\partial s}\frac{\partial^2 x_y(s)}{\partial s^2} + \frac{\partial x_z(s)}{\partial s}\frac{\partial^2 x_z(s)}{\partial s^2} = 0 \qquad \text{(IIIb)}$$

The equations (I), (II) and (IIIb) may be fed into a numerical differential equation solver, after solving a system of three unknown double derivative functions, using matrix inversion. The differential equation solver though should use (III) to solve $x_z(s)$ to avoid drift due to numerical errors.

To see if this looks plausible, we compare some drag values for a 3 mm tether, at lengths 200 m, 400 m, 600 at tensions 2.5 kN, 5.0 kN and 7.5 kN, r_0 = 10 m and r_1 = 40 m, $C_{D,t}$ = 1.0, ω = 1.0, ρ = 1.225, μ = 0.005, twist zero degrees.



The plot confirms that the «belly» aware tether drag approximations are similar to the straight line approximation, and deviate more with longer tethers and less tension, as expected.

Julia code to produce the plot

```
using TetherDragODESolver
let d = 0.003, r0 = 10.0, r1 = 40.0, omega = 1.0, tensions = 2_500.0:2_500:7_500.0,
theta = deg2rad(0), ll = float.(50:10:700)
         plot(legend = :topleft, ylims = (0, Inf), xlims = (0, Inf))
         plot!(ll, [0.5 * 1.225 * r1^2 * omega^2 * 1.0 * l * d * 1 // 4 for l = ll],
lab = "no belly")
         reduce((acc, tension) -> plot!(acc, ll,
[solved_moment_loss(solve_tether(omega, tension, r0, r1, l, theta, d, mu = 0.005)) ./
r1 for l = ll], lab = "$(tension / 1000) kN"), tensions, init = plot!())
       end
```