Functional Programming Principles in Scala

Martin Odersky

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Programming Paradigms

Paradigm: In science, a *paradigm* describes distinct concepts or thought patterns in some scientific discipline.

Main programming paradigms:

- imperative programming
- functional programming
- logic programming

Orthogonal to it:

- object-oriented programming

Review: Imperative programming

Imperative programming is about

- modifying mutable variables,
- using assignments
- and control structures such as if-then-else, loops, break, continue, return.

The most common informal way to understand imperative programs is as instruction sequences for a Von Neumann computer.

Imperative Programs and Computers

There's a strong correspondence between

Mutable variables ≈ memory cells
Variable dereferences ≈ load instructions
Variable assignments ≈ store instructions
Control structures ≈ jumps

Problem: Scaling up. How can we avoid conceptualizing programs word by word?

Reference: John Backus, Can Programming Be Liberated from the von. Neumann Style?, Turing Award Lecture 1978.

Scaling Up

In the end, pure imperative programming is limited by the “Von Neumann” bottleneck:

> *One tends to conceptualize data structures word-by-word.*

We need other techniques for defining high-level abstractions such as collections, polynomials, geometric shapes, strings, documents.

Ideally: Develop *theories* of collections, shapes, strings, …

What is a Theory?

A theory consists of

- one or more data types
- operations on these types
- laws that describe the relationships between values and operations

Normally, a theory does not describe mutations!

Theories without Mutation

For instance the theory of polynomials defines the sum of two polynomials by laws such as:

```
(a*x + b) + (c*x + d) = (x+c)*x + (b+d)
```

But it does not define an operator to change a coefficient while keeping the polynomial the same!

Theories without Mutation

For instance the theory of polynomials defines the sum of two polynomials by laws such as:

```
(a*x + b) + (c*x + d) = (x+c)*x + (b+d)
```

But it does not define an operator to change a coefficient while keeping the polynomial the same!

Other example:

The theory of strings defines a concatenation operator $++$ which is associative:

```
(a ++ b) ++ c  =  a ++ (b ++ c)
```

But it does not define an operator to change a sequence element while keeping the sequence the same!

Consequences for Programming

Let's

- concentrate on defining theories for operators,
- minimize state changes,
- treat operators as functions, often composed of simpler functions.

Functional Programming

- In a *restricted* sense, functional programming (FP) means programming without mutable variables, assignments, loops, and other imperative control structures.
- In a *wider* sense, functional programming means focusing on the functions.
- In particular, functions can be values that are produced, consumed, and composed.
- All this becomes easier in a functional language.

Functional Programming Languages

- In a *restricted* sense, a functional programming language is one which does not have mutable variables, assignments, or imperative control structures.
- In a *wider* sense, a functional programming language enables the construction of elegant programs that focus on functions.
- In particular, functions in a FP language are first-class citizens. This means
  - they can be defined anywhere, including inside other functions
  - like any other value, they can be passed as parameters to functions and returned as results
  - as for other values, there exists a set operators to compose functions

Some functional programming languages

In the restricted sense:

- Pure Lisp, XSLT, XPath, XQuery, FP
- Haskell (without I/O Monad or UnsafePerformIO)

In the wider sense:

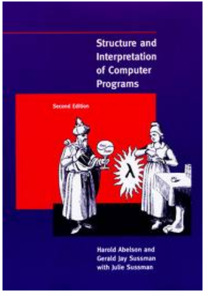
- Lisp, Scheme, Racket, Clojure
- SML, Ocaml, F#
- Haskell (full language)
- Scala
- Smalltalk, Ruby (!)

History of FP languages

1959	Lisp
1975-77	ML, FP, Scheme
1978	Smalltalk
1986	Standard ML
1990	Haskell, Erlang
1999	XSLT
2000	OCaml
2003	Scala, XQuery
2005	F#
2007	Clojure

Recommended Book (1)

Structure and Interpretation of Computer Programs. Harold Abelson and Gerald J. Sussman. 2nd edition. MIT Press 1996.

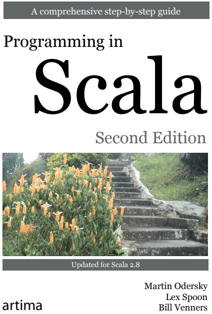


A classic. Many parts of the course and quizzes are based on it, but we change the language from Scheme to Scala.

The full text can be downloaded here.

Recommended Book (2)

Programming in Scala. Martin Odersky, Lex Spoon, and Bill Venners. 2nd edition. Artima 2010.



A comprehensive step-by-step guide

Programming in
Scala
Second Edition

Updated for Scala 2.8

Martin Odersky
Lex Spoon
Bill Venners

artima

The standard language introduction and reference.

Recommended Book (3)

Scala for the Impatient

A faster paced introduction to Scala for people with a Java background.

The first part of the book is available for free downlaod

Why Functional Programming?

Functional Programming is becoming increasingly popular because it offers an attractive method for exploiting parallelism for multicore and cloud computing.

To find out more, see the video of my 2011 Oscon Java keynote

Working Hard to Keep it Simple

(16.30 minutes).

The slides for the video are available separately.

Elements of Programming

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Elements of Programming

Every non-trivial programming language provides:

- primitive expressions representing the simplest elements
- ways to *combine* expressions
- ways to *abstract* expressions, which introduce a name for an expression by which it can then be referred to.

The Read-Eval-Print Loop

Functional programming is a bit like using a calculator

An interactive shell (or REPL, for Read-Eval-Print-Loop) lets one write expressions and responds with their value.

The Scala REPL can be started by simply typing

```
> scala
```

Expressions

Here are some simple interactions with the REPL

```
scala> 87 + 145
232
```

Functional programming languages are more than simple calcululators because they let one define values and functions:

```
scala> def size = 2
size: => Int

scala> 5 * size
10
```

Evaluation

A non-primitive expression is evaluated as follows.

1. Take the leftmost operator
2. Evaluate its operands (left before right)
3. Apply the operator to the operands

A name is evaluated by replacing it with the right hand side of its definition

The evaluation process stops once it results in a value

A value is a number (for the moment)

Later on we will consider also other kinds of values

Example

Here is the evaluation of an arithmetic expression:

```
(2 * pi) * radius
```

Example

Here is the evaluation of an arithmetic expression:

```
(2 * pi) * radius
```

```
(2 * 3.14159) * radius
```

Example

Here is the evaluation of an arithmetic expression:

```
(2 * pi) * radius
```

```
(2 * 3.14159) * radius
```

```
6.28318 * radius
```

Example

Here is the evaluation of an arithmetic expression:

```
(2 * pi) * radius

(2 * 3.14159) * radius

6.28318 * radius

6.28318 * 10
```

Example

Here is the evaluation of an arithmetic expression:

```
(2 * pi) * radius

(2 * 3.14159) * radius

6.28318 * radius

6.28318 * 10

62.8318
```

Parameters

Definitions can have parameters. For instance:

```
scala> def square(x: Double) = x * x
square: (Double)Double

scala> square(2)
4.0

scala> square(5 + 4)
81.0

scala> square(square(4))
256.0

def sumOfSquares(x: Double, y: Double) = square(x) + square(y)
sumOfSquares: (Double,Double)Double
```

Parameter and Return Types

Function parameters come with their type, which is given after a colon

```
def power(x: Double, y: Int): Double = ...
```

If a return type is given, it follows the parameter list.

Primitive types are as in Java, but are written capitalized, e.g:

`Int`	32-bit integers
`Double`	64-bit floating point numbers
`Boolean`	boolean values `true` and `false`

Evaluation of Function Applications

Applications of parameterized functions are evaluated in a similar way as operators:

1. Evaluate all function arguments, from left to right
2. Replace the function application by the function's right-hand side, and, at the same time
3. Replace the formal parameters of the function by the actual arguments.

Example

```
sumOfSquares(3, 2+2)
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
3 * 3 + square(4)
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
3 * 3 + square(4)
9 + square(4)
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
3 * 3 + square(4)
9 + square(4)
9 + 4 * 4
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
3 * 3 + square(4)
9 + square(4)
9 + 4 * 4
9 + 16
```

Example

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4)
square(3) + square(4)
3 * 3 + square(4)
9 + square(4)
9 + 4 * 4
9 + 16
25
```

The substitution model

This scheme of expression evaluation is called the *substitution model*.

The idea underlying this model is that all evaluation does is *reduce an expression to a value*.

It can be applied to all expressions, as long as they have no side effects.

The substitution model is formalized in the *λ-calculus*, which gives a foundation for functional programming.

Termination

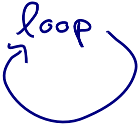
- *Does every expression reduce to a value (in a finite number of steps)?*

Termination

- *Does every expression reduce to a value (in a finite number of steps)?*
- *No. Here is a counter-example*

```
def loop: Int = loop
```

`loop` → loop → . . . .



Changing the evaluation strategy

The interpreter reduces function arguments to values before rewriting the function application.

One could alternatively apply the function to unreduced arguments.

For instance:

```
sumOfSquares(3, 2+2)
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For instance:

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sumOfSquares(3, 2+2)
square(3) + square(2+2)
3 * 3 + square(2+2)
9 + square(2+2)
```

Changing the evaluation strategy

The interpreter reduces function arguments to values before rewriting the function application.

One could alternatively apply the function to unreduced arguments.

For instance:

```
sumOfSquares(3, 2+2)
square(3) + square(2+2)
3 * 3 + square(2+2)
9 + square(2+2)
9 + (2+2) * (2+2)
```

Changing the evaluation strategy

The interpreter reduces function arguments to values before rewriting the function application.

One could alternatively apply the function to unreduced arguments.

For instance:

```
sumOfSquares(3, 2+2)
square(3) + square(2+2)
3 * 3 + square(2+2)
9 + square(2+2)
9 + (2+2) * (2+2)
9 + 4 * (2+2)
```

Changing the evaluation strategy

The interpreter reduces function arguments to values before rewriting the function application.

One could alternatively apply the function to unreduced arguments.

For instance:

```
sumOfSquares(3, 2+2)
square(3) + square(2+2)
3 * 3 + square(2+2)
9 + square(2+2)
9 + (2+2) * (2+2)
9 + 4 * (2+2)
9 + 4 * 4
```

Changing the evaluation strategy

The interpreter reduces function arguments to values before rewriting the function application.

One could alternatively apply the function to unreduced arguments.

For instance:

```
sumOfSquares(3, 2+2)
square(3) + square(2+2)
3 * 3 + square(2+2)
9 + square(2+2)
9 + (2+2) * (2+2)
9 + 4 * (2+2)
9 + 4 * 4
25
```

Call-by-name and call-by-value

The first evaluation strategy is known as *call-by-value*, the second is is known as *call-by-name*.

Both strategies reduce to the same final values as long as

- the reduced expression consists of pure functions, and
- both evaluations terminate.

Call-by-value has the advantage that it evaluates every function argument only once.

Call-by-name has the advantage that a function argument is not evaluated if the corresponding parameter is unused in the evaluation of the function body.

Call-by-name vs call-by-value

Question: Say you are given the following function definition:

```
def test(x: Int, y: Int) = x * x
```

For each of the following function applications, indicate which evaluation strategy is fastest (has the fewest reduction steps)

CBV fastest	CBN fastest	same #steps	
○	○	○	`test(2, 3)`
○	○	○	`test(3+4, 8)`
○	○	○	`test(7, 2*4)`
○	○	○	`test(3+4, 2*4)`

Call-by-name vs call-by-value

```
def test(x: Int, y: Int) = x * x

test(2, 3)
test(3+4, 8)
test(7, 2*4)
test(3+4, 2*4)
```

Evaluation Strategies and Termination

August 31, 2012

Call-by-name, Call-by-value and termination

You know from the last module that the call-by-name and call-by-value evaluation strategies reduce an expression to the same value, as long as both evaluations terminate.

But what if termination is not guaranteed?

We have:

- If CBV evaluation of an expression e terminates, then CBN evaluation of e terminates, too.
- The other direction is not true

Non-termination example

Question: Find an expression that terminates under CBN but not under CBV.

Non-termination example

Let's define

```
def first(x: Int, y: Int) = x
```

and consider the expression `first(1, loop)`.

Under CBN:

```
first(1, loop)
```

Under CBV:

```
first(1, loop)
```

Scala's evaluation strategy

Scala normally uses call-by-value.

But if the type of a function parameter starts with => it uses call-by-name.

Example:

```scala
def constOne(x: Int, y: => Int) = 1
```

Let's trace the evaluations of

```scala
constOne(1+2, loop)
```

and

```scala
constOne(loop, 1+2)
```

Trace of `constOne(1 + 2, loop)`

```
constOne(1 + 2, loop)
```

Trace of `constOne(loop, 1 + 2)`

```
constOne(loop, 1 + 2)
```

Conditionals and Value Definitions

August 31, 2012

Conditional Expressions

To express choosing between two alternatives, Scala has a conditional expression `if-else`.

It looks like a `if-else` in Java, but is used for expressions, not statements.

Example:

```
def abs(x: Int) = if (x >= 0) x else -x
```

`x >= 0` is a *predicate*, of type `Boolean`.

Boolean Expressions

Boolean expressions `b` can be composed of

```
true  false       // Constants
!b                // Negation
b && b           // Conjunction
b || b           // Disjunction
```

and of the usual comparison operations:

```
e <= e, e >= e, e < e, e > e, e == e, e != e
```

Rewrite rules for Booleans

Here are reduction rules for Boolean expressions (`e` is an arbitrary expression):

```
!true        -->   false
!false       -->   true
true && e    -->   e
false && e   -->   false
true || e    -->   true
false || e   -->   e
```

Note that `&&` and `||` do not always need their right operand to be evaluated.

We say, these expressions use “short-circuit evaluation”.

Exercise: Formulate rewrite rules for if-else

$\text{if } (b)\ e_1 \text{ then } e_2$

$\text{if } (\text{true})\ e_1 \text{ else } e_2 \longrightarrow e_1$

$\text{if } (\text{false})\ e_1 \text{ else } e_2 \longrightarrow e_2$

Value Definitions

We have seen that function parameters can be passed by value or be passed by name.

The same distinction applies to definitions.

The `def` form is “by-name”, its right hand side is evaluated on each use.

There is also a `val` for, which is “by-value”. Example:

```
val x = 2
val y = square(x)
```

The right-hand side of a `val` definition is evaluated at the point of the definition itself.

Afterwards, the name refers to the value.

For instance, `y` above refers to `4`, not `square(2)`.

Value Definitions and Termination

The difference between `val` and `def` becomes apparent when the right hand side does not terminate. Given

```
def loop: Boolean = loop
```

A definition

```
def x = loop
```

is OK, but a definition

```
val x = loop
```

will lead to an infinite loop.

Exercise

Write functions `and` and `or` such that for all argument expressions `x` and `y`:

```
and(x, y)    ==    x && y
or(x, y)     ==    x || y
```

(do not use `||` and `&&` in your implementation)

What are good operands to test that the equalities hold?

Example: Square roots with Newton's method

August 31, 2012

Task

We will define in this session a function

```
/** Calculates the square root of parameter x */
def sqrt(x: Double): Double = ...
```

The classical way to achieve this is by successive approximations using Newton's method.

Method

To compute `sqrt(x)`:

- Start with an initial *estimate* `y` (let's pick `y = 1`).
- Repeatedly improve the estimate by taking the mean of `y` and `x/y`.

Example: $x = 2$

Estimation	Quotient	Mean
1	2 / 1 = 2	1.5
1.5	2 / 1.5 = 1.333	1.4167
1.4167	2 / 1.4167 = 1.4118	1.4142
1.4142	...	...

Implementation in Scala (1)

First, define a function which computes one iteration step

```
def sqrtIter(guess: Double, x: Double): Double =
  if (isGoodEnough(guess, x)) guess
  else sqrtIter(improve(guess, x), x)
```

Note that `sqrtIter` is *recursive*, its right-hand side calls itself.

Recursive functions need an explicit return type in Scala.

For non-recursive functions, the return type is optional

Implementation in Scala (2)

Second, define a function `improve` to improve an estimate and a test to check for terminatation:

```
def improve(guess: Double, x: Double) =
  (guess + x / guess) / 2

def isGoodEnough(guess: Double, x: Double) =
  abs(guess * guess - x) < 0.001
```

Implementation in Scala (3)

Third, define the `sqrt` function:

```
def sqrt(x: Double) = srqtIter(1.0, x)
```

Exercise

1. The `isGoodEnough` test is not very precise for small numbers and can lead to non-termination for very large numbers. Explain why.
2. Design a different version of `isGoodEnough` that does not have these problems.
3. Test your version with some very very small and large numbers, e.g.

```
0.001
0.1e-20
1.0e20
1.0e50
```

Blocks and Lexical Scope

August 31, 2012

Nested functions

It's good functional programming style to split up a task into many small functions.

But the names of functions like `sqrtIter`, `improve`, and `isGoodEnough` matter only for the *implementation* of `sqrt`, not for its *usage*.

Normally we would not like users to access these functions directly.

We can achieve this and at the same time avoid "name-space pollution" by putting the auxciliary functions inside `sqrt`.

The sqrt Function, Take 2

```
def sqrt(x: Double) = {
  def sqrtIter(guess: Double, x: Double): Double =
    if (isGoodEnough(guess, x)) guess
    else sqrtIter(improve(guess, x), x)

  def improve(guess: Double, x: Double) =
    (guess + x / guess) / 2

  def isGoodEnough(guess: Double, x: Double) =
    abs(square(guess) - x) < 0.001

  sqrtIter(1.0, x)
}
```

Blocks in Scala

- A block is delimited by braces `{ ... }`.

```
{ val x = f(3)
  x * x
}
```

- It contains a sequence of definitions or expressions.
- The last element of a block is an expression that defines its value.
- This return expression can be preceded by auxiliary definitions.
- Blocks are themselves expressions; a block may appear everywhere an expression can.

Blocks and Visibility

```
val x = 0
def f(y: Int) = y + 1
val result = {
  val x = f(3)
  x * x
}
```

- The definitions inside a block are only visible from within the block.
- The definitions inside a block *shadow* definitions of the same names outside the block.

Exercise: Scope Rules

Question: What is the value of `result` in the following program?

```
val x = 0
def f(y: Int) = y + 1
val result = {
  val x = f(3)
  x * x
} + x
```

x = 4

16

Possible answers:

- ○ `0`
- ● `16`
- ○ `32`
- ○ `reduction does not terminate`

Lexical Scoping

Definitions of outer blocks are visible inside a block unless they are shadowed.

Therefore, we can simplify `sqrt` by eliminating redundant occurrences of the `x` parameter, which means everywhere the same thing:

The sqrt Function, Take 3

```
def sqrt(x: Double) = {
  def sqrtIter(guess: Double): Double =
    if (isGoodEnough(guess)) guess
    else sqrtIter(improve(guess))

  def improve(guess: Double) =
    (guess + x / guess) / 2

  def isGoodEnough(guess: Double) =
    abs(square(guess) - x) < 0.001

  sqrtIter(1.0)
}
```

Semicolons

In Scala, semicolons at the end of lines are in most cases optional

You could write

```
val x = 1;
```

but most people would omit the semicolon.

On the other hand, if there are more than one statements on a line, they need to be separated by semicolons:

```
val y = x + 1; y * y
```

Semicolons and infix operators

One issue with Scala's semicolon convention is how to write expressions that span several lines. For instance

```
someLongExpression
+ someOtherExpression
```

would be interpreted as *two* expressions:

```
someLongExpression;
+ someOtherExpression
```

Semicolons and infix operators

There are two ways to overcome this problem.

You could write the multi-line expression in parentheses, because semicolons are never inserted inside `(...)`:

```
(someLongExpression
 + someOtherExpression)
```

Or you could write the operator on the first line, because this tells the Scala compiler that the expression is not yet finished:

```
someLongExpression +
someOtherExpression
```

Summary

You have seen simple elements of functional programing in Scala.

- arithmetic and boolean expressions
- conditional expressions if-else
- functions with recursion
- nesting and lexical scope

You have learned the difference between the call-by-name and call-by-value evaluation strategies.

You have learned a way to reason about program execution: reduce expressions using the substitution model.

This model will be an important tool for the coming sessions.

Tail Recursion

Review: Evaluating a Function Application

One simple rule : One evaluates a function application $f(e_1, ..., e_n)$

- by evaluating the expressions $e_1, \ldots, e_n$ resulting in the values $v_1, ..., v_n$, then
- by replacing the application with the body of the function `f`, in which
- the actual parameters $v_1, ..., v_n$ replace the formal parameters of `f`.

Application Rewriting Rule

This can be formalized as a *rewriting of the program itself*:

$$\begin{array}{ll} & \mathtt{def}\ \mathtt{f}(\mathtt{x}_1, ..., \mathtt{x}_n) = \mathtt{B};\ \ ...\ \mathtt{f}(\mathtt{v}_1, ..., \mathtt{v}_n) \\ \rightarrow & \\ & \mathtt{def}\ \mathtt{f}(\mathtt{x}_1, ..., \mathtt{x}_n) = \mathtt{B};\ \ ...\ [\mathtt{v}_1/\mathtt{x}_1, ..., \mathtt{v}_n/\mathtt{x}_n]\,\mathtt{B} \end{array}$$

Here, $[\mathtt{v}_1/\mathtt{x}_1, ..., \mathtt{v}_n/\mathtt{x}_n]\,\mathtt{B}$ means:

The expression `B` in which all occurrences of $\mathtt{x}_\mathtt{i}$ have been replaced by $\mathtt{v}_\mathtt{i}$.

$[\mathtt{v}_1/\mathtt{x}_1, ..., \mathtt{v}_n/\mathtt{x}_n]$ is called a *substitution*.

Rewriting example:

Consider `gcd`, the function that computes the greatest common divisor of two numbers.

Here's an implementation of `gcd` using Euclid's algorithm.

```
def gcd(a: Int, b: Int): Int =
  if (b == 0) a else gcd(b, a % b)
```

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

```
gcd(14, 21)
```

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

```
gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
```

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

→ `if (21 == 0) 14 else gcd(21, 14 % 21)`

→ `if (false) 14 else gcd(21, 14 % 21)`

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

```
gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
→ if (false) 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
```

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

```
gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
→ if (false) 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
→ gcd(21, 14)
```

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

→ `if (21 == 0) 14 else gcd(21, 14 % 21)`

→ `if (false) 14 else gcd(21, 14 % 21)`

→ `gcd(21, 14 % 21)`

→ `gcd(21, 14)`

→ `if (14 == 0) 21 else gcd(14, 21 % 14)`

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

$\rightarrow$ `if (21 == 0) 14 else gcd(21, 14 % 21)`

$\rightarrow$ `if (false) 14 else gcd(21, 14 % 21)`

$\rightarrow$ `gcd(21, 14 % 21)`

$\rightarrow$ `gcd(21, 14)`

$\rightarrow$ `if (14 == 0) 21 else gcd(14, 21 % 14)`

$\twoheadrightarrow$ `gcd(14, 7)`

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

$\rightarrow$ `if (21 == 0) 14 else gcd(21, 14 % 21)`

$\rightarrow$ `if (false) 14 else gcd(21, 14 % 21)`

$\rightarrow$ `gcd(21, 14 % 21)`

$\rightarrow$ `gcd(21, 14)`

$\rightarrow$ `if (14 == 0) 21 else gcd(14, 21 % 14)`

$\twoheadrightarrow$ `gcd(14, 7)`

$\twoheadrightarrow$ `gcd(7, 0)`

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

→ `if (21 == 0) 14 else gcd(21, 14 % 21)`

→ `if (false) 14 else gcd(21, 14 % 21)`

→ `gcd(21, 14 % 21)`

→ `gcd(21, 14)`

→ `if (14 == 0) 21 else gcd(14, 21 % 14)`

↠ `gcd(14, 7)`

↠ `gcd(7, 0)`

→ `if (0 == 0) 7 else gcd(0, 7 % 0)`

Rewriting example:

`gcd(14, 21)` is evaluated as follows:

`gcd(14, 21)`

→ `if (21 == 0) 14 else gcd(21, 14 % 21)`

→ `if (false) 14 else gcd(21, 14 % 21)`

→ `gcd(21, 14 % 21)`

→ `gcd(21, 14)`

→ `if (14 == 0) 21 else gcd(14, 21 % 14)`

↠ `gcd(14, 7)`

↠ `gcd(7, 0)`

→ `if (0 == 0) 7 else gcd(0, 7 % 0)`

→ `7`

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

```
factorial(4)
```

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

```
factorial(4)
```

→ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

`factorial(4)`

→ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

↠ `4 * factorial(3)`

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

`factorial(4)`

→ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

↠ `4 * factorial(3)`

↠ `4 * (3 * factorial(2))`

Another rewriting example:

Consider `factorial`:

```scala
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

`factorial(4)`

$\rightarrow$ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

$\twoheadrightarrow$ `4 * factorial(3)`

$\twoheadrightarrow$ `4 * (3 * factorial(2))`

$\twoheadrightarrow$ `4 * (3 * (2 * factorial(1)))`

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

```
factorial(4)
```

$\rightarrow$ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

$\twoheadrightarrow$ `4 * factorial(3)`

$\twoheadrightarrow$ `4 * (3 * factorial(2))`

$\twoheadrightarrow$ `4 * (3 * (2 * factorial(1)))`

$\twoheadrightarrow$ `4 * (3 * (2 * (1 * factorial(0)))`

Another rewriting example:

Consider `factorial`:

```scala
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

`factorial(4)`

$\rightarrow$ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

$\twoheadrightarrow$ `4 * factorial(3)`

$\twoheadrightarrow$ `4 * (3 * factorial(2))`

$\twoheadrightarrow$ `4 * (3 * (2 * factorial(1)))`

$\twoheadrightarrow$ `4 * (3 * (2 * (1 * factorial(0)))`

$\twoheadrightarrow$ `4 * (3 * (2 * (1 * 1)))`

Another rewriting example:

Consider `factorial`:

```
def factorial(n: Int): Int =
  if (n == 0) 1 else n * factorial(n - 1)
```

`factorial(4)`

$\rightarrow$ `if (4 == 0) 1 else 4 * factorial(4 - 1)`

$\twoheadrightarrow$ `4 * factorial(3)`

$\twoheadrightarrow$ `4 * (3 * factorial(2))`

$\twoheadrightarrow$ `4 * (3 * (2 * factorial(1)))`

$\twoheadrightarrow$ `4 * (3 * (2 * (1 * factorial(0))))`

$\twoheadrightarrow$ `4 * (3 * (2 * (1 * 1)))`

$\twoheadrightarrow$ `120`

What are the differences between the two sequences?

Tail Recursion

Implementation Consideration: If a function calls itself as its last action, the function's stack frame can be reused. This is called *tail recursion*.

⇒ Tail recursive functions are iterative processes.

In general, if the last action of a function consists of calling a function (which may be the same), one stack frame would be sufficient for both functions. Such calls are called *tail-calls*.

Tail Recursion in Scala

In Scala, only directly recursive calls to the current function are optimized.

One can require that a function is tail-recursive using a `@tailrec` annotation:

```scala
@tailrec
def gcd(a: Int, b: Int): Int = ...
```

If the annotation is given, and the implementation of `gcd` were not tail recursive, an error would be issued.

Exercise: Tail recursion

Design a tail recursive version of `factorial`.