Lecture 1: Introduction

The goal of this class is to teach you to **solve** computation problems, and to **communicate** that your solutions are **correct** and **efficient**.

Problem

- Binary relation from **problem inputs** to **correct outputs**
- Usually don't specify every correct output for all inputs (too many!)
- Provide a verifiable **predicate** (a property) that correct outputs must satisfy
- 6.006 studies problems on large general input spaces
- Not general: small input instance
  - **Example:** In this room, is there a pair of students with same birthday?
- General: arbitrarily large inputs
  - **Example:** Given any set of n students, is there a pair of students with same birthday?
  - If birthday is just one of 365, for $n > 365$, answer always true by pigeon-hole
  - Assume resolution of possible birthdays exceeds n (include year, time, etc.)

Algorithm

- Procedure mapping each input to a **single** output (deterministic)
- Algorithm **solves** a problem if it returns a correct output for every problem input
- **Example:** An algorithm to solve birthday matching
  - Maintain a **record** of names and birthdays (initially empty)
  - Interview each student in some order
    - If birthday exists in record, return found pair!
    - Else add name and birthday to record
  - Return None if last student interviewed without success

Correctness

- Programs/algorithms have fixed size, so how to prove correct?
- For small inputs, can use case analysis
- For arbitrarily large inputs, algorithm must be **recursive** or loop in some way
- Must use **induction** (why recursion is such a key concept in computer science)
- **Example:** Proof of correctness of birthday matching algorithm
  - Induct on k: the number of students in record
  - **Hypothesis:** if first k contain match, returns match before interviewing student $k+1$
  - **Base case**: $k = 0$, first k contains no match
  - Assume for induction hypothesis holds for $k = k'$, and consider $k = k' + 1$
  - If first k' contains a match, already returned a match by induction
  - Else first k' do not have match, so if first $k' + 1$ has match, match contains $k' + 1$
  - Then algorithm checks directly whether birthday of student $k' + 1$ exists in first k' $\square$

Efficiency

- How fast does an algorithm produce a correct output?
  - Could measure time, but want performance to be machine independent
  - **Idea!** Count number of fixed-time operations algorithm takes to return
  - Expect to depend on size of input: larger input suggests longer time
  - Size of input is often called 'n', but not always!
  - Efficient if returns in **polynomial time** with respect to input
  - Sometimes no efficient algorithm exists for a problem! (See L20)
- Asymptotic Notation: ignore constant factors and low order terms
  - Upper bounds (O), lower bounds (Ω), tight bounds (Θ) $\in, =$, is, order
  - Time estimate below based on one operation per cycle on a 1 GHz single-core machine
  - Particles in universe estimated $< 10^{100}$

input	constant	logarithmic	linear	log-linear	quadratic	polynomial	exponential
n	$\Theta(1)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(n^c)$	$2^{\Theta(n^c)}$
1000	1	≈ 10	1000	$\approx$ 10,000	1,000,000	1000^c	$2^{1000} \approx 10^{301}$
Time	$1\,ns$	$10\,ns$	$1\,\mu s$	$10\,\mu s$	$1\,ms$	$10^{3c-9}\,s$	10^{281} millenia

Model of Computation

- Specification for what operations on the machine can be performed in $O(1)$ time
- Model in this class is called the **Word-RAM**
- **Machine word**: block of w bits (w is word size of a w-bit Word-RAM)
- **Memory**: Addressable sequence of machine words
- **Processor** supports many **constant time** operations on a $O(1)$ number of words (**integers**):
  - **integer** arithmetic: `(+, -, *, //, %)`
  - **logical** operators: `(&&, ||, !, ==, <, >, <=, =>)`
  - (**bitwise** arithmetic: `(&, |, <<, >>, ...)`)
  - Given word a, can **read** word at address a, **write** word to address a
- Memory address must be able to access every place in memory
  - Requirement: $w \geq$ # bits to represent largest memory address, i.e., $\log_2 n$
  - 32-bit words $\rightarrow$ max ~ 4 GB memory, 64-bit words $\rightarrow$ max ~ 16 exabytes of memory
- **Python** is a more complicated model of computation, implemented on a Word-RAM

Data Structure

- A **data structure** is a way to store non-constant data, that supports a set of operations
- A collection of operations is called an **interface**
  - **Sequence:** Extrinsic order to items (first, last, nth)
  - **Set:** Intrinsic order to items (queries based on item keys)
- Data structures may implement the same interface with different performance
- **Example: Static Array** - fixed width slots, fixed length, static sequence interface
  - `StaticArray(n)`: allocate static array of size n initialized to 0 in $\Theta(n)$ time
  - `StaticArray.get_at(i)`: return word stored at array index i in $\Theta(1)$ time
  - `StaticArray.set_at(i, x)`: write word x to array index i in $\Theta(1)$ time
- Stored word can hold the address of a larger object
- Like Python `tuple` plus `set_at(i, x)`, Python `list` is a **dynamic array** (see L02)

```
def birthday_match(students):
    '''
    Find a pair of students with the same birthday
    Input:  tuple of student (name, bday) tuples
    Output: tuple of student names or None
    '''
    n = len(students)                           # O(1)
    record = StaticArray(n)                     # O(n)
    for k in range(n):                          # n
        (name1, bday1) = students[k]            # O(1)
        # Return pair if bday1 in record
        for i in range(k):                      # k
            (name2, bday2) = record.get_at(i)   # O(1)
            if bday1 == bday2:                  # O(1)
                return (name1, name2)           # O(1)
        record.set_at(k, (name1, bday1))        # O(1)
    return None                                 # O(1)
```

Example: Running Time Analysis

- Two loops: outer $k \in \{0, \ldots, n-1\}$, inner is $i \in \{0, \ldots, k\}$
- Running time is $O(n) + \sum_{k=0}^{n-1}(O(1) + k \cdot O(1)) = O(n^2)$
- Quadratic in n is **polynomial**. Efficient? Use different data structure for record!

How to Solve an Algorithms Problem

1. Reduce to a problem you already know (use data structure or algorithm)

Search Problem (Data Structures)	Sort Algorithms	Shortest Path Algorithms
Static Array (L01)	Insertion Sort (L03)	Breadth First Search (L09)
Linked List (L02)	Selection Sort (L03)	DAG Relaxation (L11)
Dynamic Array (L02)	Merge Sort (L03)	Depth First Search (L10)
Sorted Array (L03)	Counting Sort (L05)	Topological Sort (L10)
Direct-Access Array (L04)	Radix Sort (L05)	Bellman-Ford (L12)
Hash Table (L04)	AVL Sort (L07)	Dijkstra (L13)
Balanced Binary Tree (L06-L07)	Heap Sort (L08)	Johnson (L14)
Binary Heap (L08)		Floyd-Warshall (L18)

2. Design your own (recursive) algorithm
   - Brute Force
   - Decrease and Conquer
   - Divide and Conquer
   - **Dynamic Programming** (L15-L19)
   - Greedy / Incremental

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Lecture 2: Data Structures

Data Structure Interfaces

- A **data structure** is a way to store data, with algorithms that support **operations** on the data
- Collection of supported operations is called an **interface** (also API or ADT)
- Interface is a **specification**: what operations are supported (the problem!)
- Data structure is a **representation**: how operations are supported (the solution!)
- In this class, two main interfaces: **Sequence** and **Set**

Sequence Interface (L02, L07)

- Maintain a sequence of items (order is **extrinsic**)
- Ex: $(x_0, x_1, x_2, \ldots, x_{n-1})$ (zero indexing)
- (use n to denote the number of items stored in the data structure)
- Supports sequence operations:

Container	`build(X)`	given an iterable `X`, build sequence from items in `X`
	`len()`	return the number of stored items
Static	`iter_seq()`	return the stored items one-by-one in sequence order
	`get_at(i)`	return the i^{th} item
	`set_at(i, x)`	replace the i^{th} item with x
Dynamic	`insert_at(i, x)`	add x as the i^{th} item
	`delete_at(i)`	remove and return the i^{th} item
	`insert_first(x)`	add x as the first item
	`delete_first()`	remove and return the first item
	`insert_last(x)`	add x as the last item
	`delete_last()`	remove and return the last item

- Special case interfaces:

stack	`insert_last(x)` and `delete_last()`
queue	`insert_last(x)` and `delete_first()`

Set Interface (L03-L08)

- Sequence about **extrinsic** order, set is about **intrinsic** order
- Maintain a set of items having **unique keys** (e.g., item `x` has key `x.key`)
- (Set or multi-set? We restrict to unique keys for now.)
- Often we let key of an item be the item itself, but may want to store more info than just key
- Supports set operations:

Container	`build(X)`	given an iterable `X`, build sequence from items in `X`
	`len()`	return the number of stored items
Static	`find(k)`	return the stored item with key `k`
Dynamic	`insert(x)`	add `x` to set (replace item with key `x.key` if one already exists)
	`delete(k)`	remove and return the stored item with key `k`
Order	`iter_ord()`	return the stored items one-by-one in key order
	`find_min()`	return the stored item with smallest key
	`find_max()`	return the stored item with largest key
	`find_next(k)`	return the stored item with smallest key larger than `k`
	`find_prev(k)`	return the stored item with largest key smaller than `k`

- Special case interfaces:

  dictionary | set without the Order operations

- In recitation, you will be asked to implement a Set, given a Sequence data structure.

Array Sequence

- Array is great for static operations! `get_at(i)` and `set_at(i, x)` in $\Theta(1)$ time!
- But not so great at dynamic operations...
- (For consistency, we maintain the invariant that array is full)
- Then inserting and removing items requires:
  - reallocating the array
  - shifting all items after the modified item

Data Structure	Operation, Worst Case $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Array	n	1	n	n	n

Linked List Sequence

- Pointer data structure (this is **not** related to a Python "list")
- Each item stored in a **node** which contains a pointer to the next node in sequence
- Each `node` has two fields: `node.item` and `node.next`
- Can manipulate nodes simply by relinking pointers!
- Maintain pointers to the first node in sequence (called the head)
- Can now insert and delete from the front in $\Theta(1)$ time! Yay!
- (Inserting/deleting efficiently from back is also possible; you will do this in PS1)
- But now `get_at(i)` and `set_at(i, x)` each take $O(n)$ time... :(
- Can we get the best of both worlds? Yes! (Kind of...)

Data Structure	Operation, Worst Case $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Linked List	n	n	1	n	n

Dynamic Array Sequence

- Make an array efficient for **last** dynamic operations
- Python "list" is a dynamic array
- **Idea!** Allocate extra space so reallocation does not occur with every dynamic operation
- **Fill ratio:** $0 \leq r \leq 1$ the ratio of items to space
- Whenever array is full ($r = 1$), allocate $\Theta(n)$ extra space at end to fill ratio r_i (e.g., $1/2$)
- Will have to insert $\Theta(n)$ items before the next reallocation
- A single operation can take $\Theta(n)$ time for reallocation
- However, any sequence of $\Theta(n)$ operations takes $\Theta(n)$ time
- So each operation takes $\Theta(1)$ time "on average"

Amortized Analysis

- Data structure analysis technique to distribute cost over many operations
- Operation has **amortized cost** $T(n)$ if k operations cost at most $\leq kT(n)$
- "$T(n)$ amortized" roughly means $T(n)$ "on average" over many operations
- Inserting into a dynamic array takes $\boldsymbol{\Theta(1)}$ **amortized** time
- More amortization analysis techniques in 6.046!

Dynamic Array Deletion

- Delete from back? $\Theta(1)$ time without effort, yay!
- However, can be very wasteful in space. Want size of data structure to stay $\Theta(n)$
- **Attempt:** if very empty, resize to $r = 1$. Alternating insertion and deletion could be bad...
- **Idea!** When $r < r_d$, resize array to ratio r_i where $r_d < r_i$ (e.g., $r_d = 1/4$, $r_i = 1/2$)
- Then $\Theta(n)$ cheap operations must be made before next expensive resize
- Can limit extra space usage to $(1 + \varepsilon)n$ for any $\varepsilon > 0$ (set $r_d = \frac{1}{1+\varepsilon}, r_i = \frac{r_d+1}{2}$)
- Dynamic arrays only support dynamic **last** operations in $\Theta(1)$ time
- Python List `append` and `pop` are amortized $O(1)$ time, other operations can be $O(n)$!
- (Inserting/deleting efficiently from front is also possible; you will do this in PS1)

Data Structure	Operation, Worst Case $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Array	n	1	n	n	n
Linked List	n	n	1	n	n
Dynamic Array	n	1	n	$1_{(a)}$	n

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Lecture 3: Sorting

Set Interface (L03-L08)

Container	`build(X)`	given an iterable `X`, build set from items in `X`
	`len()`	return the number of stored items
Static	`find(k)`	return the stored item with key `k`
Dynamic	`insert(x)`	add `x` to set (replace item with key `x.key` if one already exists)
	`delete(k)`	remove and return the stored item with key `k`
Order	`iter_ord()`	return the stored items one-by-one in key order
	`find_min()`	return the stored item with smallest key
	`find_max()`	return the stored item with largest key
	`find_next(k)`	return the stored item with smallest key larger than `k`
	`find_prev(k)`	return the stored item with largest key smaller than `k`

- Storing items in an array in arbitrary order can implement a (not so efficient) set
- Stored items sorted increasing by key allows:
  - faster find min/max (at first and last index of array)
  - faster finds via binary search: $O(\log n)$

Set Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$n \log n$	$\log n$	n	1	$\log n$

- But how to construct a sorted array efficiently?

Sorting

- Given a sorted array, we can leverage binary search to make an efficient set data structure.
- **Input**: (static) array A of n numbers
- **Output**: (static) array B which is a sorted permutation of A
  - **Permutation**: array with same elements in a different order
  - **Sorted**: $B[i-1] \leq B[i]$ for all $i \in \{1, \ldots, n\}$
- Example: $[8, 2, 4, 9, 3] \rightarrow [2, 3, 4, 8, 9]$
- A sort is **destructive** if it overwrites A (instead of making a new array B that is a sorted version of A)
- A sort is **in place** if it uses $O(1)$ extra space (implies destructive: in place $\subseteq$ destructive)

Permutation Sort

- There are $n!$ permutations of A, at least one of which is sorted
- For each permutation, check whether sorted in $\Theta(n)$
- Example: $[2, 3, 1] \rightarrow \{[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]\}$

```
def permutation_sort(A):
    '''Sort A'''
    for B in permutations(A):                # O(n!)
        if is_sorted(B):                     # O(n)
            return B                         # O(1)
```

- `permutation_sort` analysis:
  - Correct by case analysis: try all possibilities (Brute Force)
  - Running time: $\Omega(n! \cdot n)$ which is **exponential** :(

Solving Recurrences

- **Substitution**: Guess a solution, replace with representative function, recurrence holds true
- **Recurrence Tree**: Draw a tree representing the recursive calls and sum computation at nodes
- **Master Theorem**: A formula to solve many recurrences (R03)

Selection Sort

- Find a largest number in prefix `A[:i + 1]` and swap it to `A[i]`
- Recursively sort prefix `A[:i]`
- Example: $[8, 2, 4, 9, 3], [8, 2, 4, 3, 9], [3, 2, 4, 8, 9], [3, 2, 4, 8, 9], [2, 3, 4, 8, 9]$

```
1  def selection_sort(A, i = None):              # T(i)
2      '''Sort A[:i + 1]'''
3      if i is None: i = len(A) - 1              # O(1)
4      if i > 0:                                 # O(1)
5          j = prefix_max(A, i)                  # S(i)
6          A[i], A[j] = A[j], A[i]               # O(1)
7          selection_sort(A, i - 1)              # T(i - 1)
8
9  def prefix_max(A, i):                         # S(i)
10     '''Return index of maximum in A[:i + 1]'''
11     if i > 0:                                 # O(1)
12         j = prefix_max(A, i - 1)              # S(i - 1)
13         if A[i] < A[j]:                       # O(1)
14             return j                          # O(1)
15     return i                                  # O(1)
```

- `prefix_max` analysis:
  - Base case: for $i = 0$, array has one element, so index of max is i
  - Induction: assume correct for i, maximum is either the maximum of `A[:i]` or `A[i]`, returns correct index in either case. □
  - $S(1) = \Theta(1), S(n) = S(n-1) + \Theta(1)$
    - Substitution: $S(n) = \Theta(n), \quad cn = \Theta(1) + c(n-1) \implies 1 = \Theta(1)$
    - Recurrence tree: chain of n nodes with $\Theta(1)$ work per node, $\sum_{i=0}^{n-1} 1 = \Theta(n)$
- `selection_sort` analysis:
  - Base case: for $i = 0$, array has one element so is sorted
  - Induction: assume correct for i, last number of a sorted output is a largest number of the array, and the algorithm puts one there; then `A[:i]` is sorted by induction □
  - $T(1) = \Theta(1), T(n) = T(n-1) + \Theta(n)$
    - Substitution: $T(n) = \Theta(n^2), \quad cn^2 = \Theta(n) + c(n-1)^2 \implies c(2n-1) = \Theta(n)$
    - Recurrence tree: chain of n nodes with $\Theta(i)$ work per node, $\sum_{i=0}^{n-1} i = \Theta(n^2)$

Insertion Sort

- Recursively sort prefix `A[:i]`
- Sort prefix `A[:i + 1]` assuming that prefix `A[:i]` is sorted by repeated swaps
- Example: $[8, 2, 4, 9, 3], [2, 8, 4, 9, 3], [2, 4, 8, 9, 3], [2, 4, 8, 9, 3], [2, 3, 4, 8, 9]$

```
1  def insertion_sort(A, i = None):            # T(i)
2      '''Sort A[:i + 1]'''
3      if i is None: i = len(A) - 1            # O(1)
4      if i > 0:                               # O(1)
5          insertion_sort(A, i - 1)            # T(i - 1)
6          insert_last(A, i)                   # S(i)
7
8  def insert_last(A, i):                      # S(i)
9      '''Sort A[:i + 1] assuming sorted A[:i]'''
10     if i > 0 and A[i] < A[i - 1]:           # O(1)
11         A[i], A[i - 1] = A[i - 1], A[i]     # O(1)
12         insert_last(A, i - 1)               # S(i - 1)
```

- `insert_last` analysis:
  - Base case: for $i = 0$, array has one element so is sorted
  - Induction: assume correct for i, if `A[i] >= A[i - 1]`, array is sorted; otherwise, swapping last two elements allows us to sort `A[:i]` by induction □
  - $S(1) = \Theta(1), S(n) = S(n-1) + \Theta(1) \implies S(n) = \Theta(n)$
- `insertion_sort` analysis:
  - Base case: for $i = 0$, array has one element so is sorted
  - Induction: assume correct for i, algorithm sorts `A[:i]` by induction, and then `insert_last` correctly sorts the rest as proved above □
  - $T(1) = \Theta(1), T(n) = T(n-1) + \Theta(n) \implies T(n) = \Theta(n^2)$

Merge Sort

- Recursively sort first half and second half (may assume power of two)
- Merge sorted halves into one sorted list (two finger algorithm)
- Example: $[7,1,5,6,2,4,9,3], [1,7,5,6,2,4,3,9], [1,5,6,7,2,3,4,9], [1,2,3,4,5,6,7,9]$

```
def merge_sort(A, a = 0, b = None):                          # T(b - a = n)
    '''Sort A[a:b]'''
    if b is None: b = len(A)                                 # O(1)
    if 1 < b - a:                                            # O(1)
        c = (a + b + 1) // 2                                 # O(1)
        merge_sort(A, a, c)                                  # T(n / 2)
        merge_sort(A, c, b)                                  # T(n / 2)
        L, R = A[a:c], A[c:b]                                # O(n)
        merge(L, R, A, len(L), len(R), a, b)                 # S(n)

def merge(L, R, A, i, j, a, b):                              # S(b - a = n)
    '''Merge sorted L[:i] and R[:j] into A[a:b]'''
    if a < b:                                                # O(1)
        if (j <= 0) or (i > 0 and L[i - 1] > R[j - 1]):      # O(1)
            A[b - 1] = L[i - 1]                              # O(1)
            i = i - 1                                        # O(1)
        else:                                                # O(1)
            A[b - 1] = R[j - 1]                              # O(1)
            j = j - 1                                        # O(1)
        merge(L, R, A, i, j, a, b - 1)                       # S(n - 1)
```

- `merge` analysis:
  - Base case: for $n = 0$, arrays are empty, so vacuously correct
  - Induction: assume correct for n, item in `A[r]` must be a largest number from remaining prefixes of `L` and `R`, and since they are sorted, taking largest of last items suffices; remainder is merged by induction □
  - $S(0) = \Theta(1), S(n) = S(n-1) + \Theta(1) \implies S(n) = \Theta(n)$
- `merge_sort` analysis:
  - Base case: for $n = 1$, array has one element so is sorted
  - Induction: assume correct for $k < n$, algorithm sorts smaller halves by induction, and then `merge` merges into a sorted array as proved above. □
  - $T(1) = \Theta(1), T(n) = 2T(n/2) + \Theta(n)$
    - Substitution: Guess $T(n) = \Theta(n \log n)$
      $cn \log n = \Theta(n) + 2c(n/2)\log(n/2) \implies cn \log(2) = \Theta(n)$
    - Recurrence Tree: complete binary tree with depth $\log_2 n$ and n leaves, level i has 2^i nodes with $O(n/2^i)$ work each, total: $\sum_{i=0}^{\log_2 n}(2^i)(n/2^i) = \sum_{i=0}^{\log_2 n} n = \Theta(n \log n)$

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Lecture 4: Hashing

Review

Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$n \log n$	$\log n$	n	1	$\log n$

- **Idea!** Want faster search and dynamic operations. Can we `find(k)` faster than $\Theta(\log n)$?
- Answer is no (lower bound)! (But actually, yes...!?)

Comparison Model

- In this model, assume algorithm can only differentiate items via comparisons
- **Comparable items**: black boxes only supporting comparisons between pairs
- Comparisons are $<, \leq, >, \geq, =, \neq$, outputs are binary: True or False
- **Goal:** Store a set of n comparable items, support `find(k)` operation
- Running time is **lower bounded** by # comparisons performed, so count comparisons!

Decision Tree

- Any algorithm can be viewed as a **decision tree** of operations performed
- An internal node represents a **binary comparison**, branching either True or False
- For a comparison algorithm, the decision tree is binary (draw example)
- A leaf represents algorithm termination, resulting in an algorithm **output**
- A **root-to-leaf path** represents an **execution of the algorithm** on some input
- Need at least one leaf for each **algorithm output**, so search requires $\geq n + 1$ leaves

Comparison Search Lower Bound

- What is worst-case running time of a comparison search algorithm?
- running time $\geq$ # comparisons $\geq$ max length of any root-to-leaf path $\geq$ height of tree
- What is minimum height of any binary tree on $\geq n$ nodes?
- Minimum height when binary tree is complete (all rows full except last)
- Height $\geq \lceil \lg(n+1) \rceil - 1 = \Omega(\log n)$, so running time of any comparison sort is $\Omega(\log n)$
- Sorted arrays achieve this bound! Yay!
- More generally, height of tree with $\Theta(n)$ leaves and max branching factor b is $\Omega(\log_b n)$
- To get faster, need an operation that allows super-constant $\omega(1)$ branching factor. How??

Direct Access Array

- Exploit Word-RAM $O(1)$ time random access indexing! Linear branching factor!
- **Idea!** Give item **unique** integer key k in $\{0, \ldots, u-1\}$, store item in an array at index k
- Associate a meaning with each index of array
- If keys fit in a machine word, i.e. $u \leq 2^w$, worst-case $O(1)$ find/dynamic operations! Yay!
- 6.006: assume input numbers/strings fit in a word, unless length explicitly parameterized
- Anything in computer memory is a binary integer, or use (static) 64-bit address in memory
- But space $O(u)$, so really bad if $n \ll u$... :(
- **Example:** if keys are ten-letter names, for one bit per name, requires $26^{10} \approx 17.6$ TB space
- How can we use less space?

Hashing

- **Idea!** If $n \ll u$, map keys to a smaller range $m = \Theta(n)$ and use smaller direct access array
- **Hash function**: $h(k) : \{0, \ldots, u-1\} \to \{0, \ldots, m-1\}$ (also hash map)
- Direct access array called **hash table**, $h(k)$ called the **hash** of key k
- If $m \ll u$, no hash function is injective by pigeonhole principle

- Always exists keys a, b such that $h(a) = h(b) \rightarrow$ **Collision**! :(
- Can't store both items at same index, so where to store? Either:
  - store somewhere else in the array (**open addressing**)
    - complicated analysis, but common and practical
  - store in another data structure supporting dynamic set interface (**chaining**)

Chaining

- **Idea!** Store collisions in another data structure (a chain)
- If keys roughly evenly distributed over indices, chain size is $n/m = n/\Omega(n) = O(1)$!
- If chain has $O(1)$ size, all operations take $O(1)$ time! Yay!
- If not, many items may map to same location, e.g. $h(k) =$ constant, chain size is $\Theta(n)$:(
- Need good hash function! So what's a good hash function?

Hash Functions

Division (bad): $h(k) = (k \bmod m)$

- Heuristic, good when keys are uniformly distributed!
- m should avoid symmetries of the stored keys
- Large primes far from powers of 2 and 10 can be reasonable
- Python uses a version of this with some additional mixing
- If $u \gg n$, every hash function will have some input set that will a create $O(n)$ size chain
- **Idea!** Don't use a fixed hash function! Choose one randomly (but carefully)!

Universal (good, theoretically): $h_{ab}(k) = (((ak + b) \bmod p) \bmod m)$

- Hash Family $\mathcal{H}(p, m) = \{h_{ab} \mid a, b \in \{0, \ldots, p-1\} \text{ and } a \neq 0\}$
- Parameterized by a fixed prime $p > u$, with a and b chosen from range $\{0, \ldots, p-1\}$
- $\mathcal{H}$ is a **Universal** family: $\Pr_{h \in \mathcal{H}}\{h(k_i) = h(k_j)\} \leq 1/m \quad \forall k_i \neq k_j \in \{0, \ldots, u-1\}$
- Why is universality useful? Implies short chain lengths! (in expectation)
- X_{ij} indicator random variable over $h \in \mathcal{H}$: $X_{ij} = 1$ if $h(k_i) = h(k_j)$, $X_{ij} = 0$ otherwise
- Size of chain at index $h(k_i)$ is random variable $X_i = \sum_j X_{ij}$
- Expected size of chain at index $h(k_i)$

$$\begin{aligned}
\mathop{\mathbb{E}}_{h \in \mathcal{H}}\{X_i\} = \mathop{\mathbb{E}}_{h \in \mathcal{H}}\left\{\sum_j X_{ij}\right\} = \sum_j \mathop{\mathbb{E}}_{h \in \mathcal{H}}\{X_{ij}\} &= 1 + \sum_{j \neq i} \mathop{\mathbb{E}}_{h \in \mathcal{H}}\{X_{ij}\} \\
&= 1 + \sum_{j \neq i} (1) \Pr_{h \in \mathcal{H}}\{h(k_i) = h(k_j)\} + (0) \Pr_{h \in \mathcal{H}}\{h(k_i) \neq h(k_j)\} \\
&\leq 1 + \sum_{j \neq i} 1/m = 1 + (n-1)/m
\end{aligned}$$

- Since $m = \Omega(n)$, load factor $\alpha = n/m = O(1)$, so $O(1)$ **in expectation**!

Dynamic

- If n/m far from 1, rebuild with new randomly chosen hash function for new size m
- Same analysis as dynamic arrays, cost can be **amortized** over many dynamic operations
- So a hash table can implement dynamic set operations in expected amortized $O(1)$ time! :)

Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$n \log n$	$\log n$	n	1	$\log n$
Direct Access Array	u	1	1	u	u
Hash Table	$n_{(e)}$	$1_{(e)}$	$1_{(a)(e)}$	n	n

6.006 Introduction to Algorithms

Spring 2020

Lecture 5: Linear Sorting

Review

- Comparison search lower bound: any decision tree with n nodes has height $\geq \lceil \lg(n+1) \rceil - 1$
- Can do faster using random access indexing: an operation with linear branching factor!
- **Direct access array** is fast, but may use a lot of space $(\Theta(u))$
- Solve space problem by mapping (**hashing**) key space u down to $m = \Theta(n)$
- **Hash tables** give **expected** $O(1)$ time operations, **amortized** if dynamic
- Expectation input-independent: choose hash function randomly from **universal** hash family
- Data structure overview!
- Last time we achieved faster find. Can we also achieve faster sort?

Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$n \log n$	$\log n$	n	1	$\log n$
Direct Access Array	u	1	1	u	u
Hash Table	$n_{(e)}$	$1_{(e)}$	$1_{(a)(e)}$	n	n

Comparison Sort Lower Bound

- Comparison model implies that algorithm decision tree is binary (constant branching factor)
- Requires # leaves $L \geq$ # possible outputs
- Tree height lower bounded by $\Omega(\log L)$, so worst-case running time is $\Omega(\log L)$
- To sort array of n elements, # outputs is $n!$ permutations
- Thus height lower bounded by $\log(n!) \geq \log((n/2)^{n/2}) = \Omega(n \log n)$
- So merge sort is optimal in comparison model
- Can we exploit a direct access array to sort faster?

Direct Access Array Sort

- **Example:** [5, 2, 7, 0, 4]
- Suppose all keys are **unique** non-negative integers in range $\{0, \ldots, u-1\}$, so $n \leq u$
- Insert each item into a direct access array with size u in $\Theta(n)$
- Return items in order they appear in direct access array in $\Theta(u)$
- Running time is $\Theta(u)$, which is $\Theta(n)$ if $u = \Theta(n)$. Yay!

```
def direct_access_sort(A):
    "Sort A assuming items have distinct non-negative keys"
    u = 1 + max([x.key for x in A])    # O(n) find maximum key
    D = [None] * u                     # O(u) direct access array
    for x in A:                        # O(n) insert items
        D[x.key] = x
    i = 0
    for key in range(u):               # O(u) read out items in order
        if D[key] is not None:
            A[i] = D[key]
            i += 1
```

- What if keys are in larger range, like $u = \Omega(n^2) < n^2$?
- **Idea!** Represent each key k by tuple (a, b) where $k = an + b$ and $0 \leq b < n$
- Specifically $a = \lfloor k/n \rfloor < n$ and $b = (k \bmod n)$ (just a 2-digit base-n number!)
- This is a built-in Python operation `(a, b) = divmod(k, n)`
- **Example:** [17, 3, 24, 22, 12] $\Rightarrow$ [(3,2), (0,3), (4,4), (4,2), (2,2)] $\Rightarrow$ [32, 03, 44, 42, 22]$_{(n=5)}$
- How can we sort tuples?

Tuple Sort

- Item keys are tuples of equal length, i.e. item $x.key = (x.k_1, x.k_2, x.k_2, \ldots)$.
- Want to sort on all entries **lexicographically**, so first key k_1 is most significant
- How to sort? **Idea!** Use other **auxiliary sorting algorithms** to separately sort each key
- (Like sorting rows in a spreadsheet by multiple columns)
- What order to sort them in? Least significant to most significant!
- **Exercise:** [3**2**, 0**3**, 4**4**, 4**2**, 2**2**] $\implies$ [**4**2, **2**2, **3**2, **0**3, **4**4] $\implies$ [03, 22, 32, 42, 44]$_{(n=5)}$

- **Idea!** Use tuple sort with **auxiliary direct access array sort** to sort tuples (a, b).
- **Problem!** Many integers could have the same a or b value, even if input keys distinct
- Need sort allowing **repeated keys** which preserves input order
- Want sort to be **stable**: repeated keys appear in output in same order as input
- Direct access array sort cannot even sort arrays having repeated keys!
- Can we modify direct access array sort to admit multiple keys in a way that is stable?

Counting Sort

- Instead of storing a single item at each array index, store a chain, just like hashing!
- For stability, chain data structure should remember the order in which items were added
- Use a **sequence** data structure which maintains insertion order
- To insert item x, `insert_last` to end of the chain at index $x.key$
- Then to sort, read through all chains in sequence order, returning items one by one

```
1  def counting_sort(A):
2      "Sort A assuming items have non-negative keys"
3      u = 1 + max([x.key for x in A])    # O(n) find maximum key
4      D = [[] for i in range(u)]         # O(u) direct access array of chains
5      for x in A:                        # O(n) insert into chain at x.key
6          D[x.key].append(x)
7      i = 0
8      for chain in D:                    # O(u) read out items in order
9          for x in chain:
10             A[i] = x
11             i += 1
```

Radix Sort

- **Idea!** If $u < n^2$, use tuple sort with **auxiliary counting sort** to sort tuples (a, b)
- Sort least significant key b, then most significant key a
- Stability ensures previous sorts stay sorted
- Running time for this algorithm is $O(2n) = O(n)$. Yay!
- If every key $< n^c$ for some positive $c = \log_n(u)$, every key has at most c digits base n
- A c-digit number can be written as a c-element tuple in $O(c)$ time
- We sort each of the c base-n digits in $O(n)$ time
- So tuple sort with **auxiliary counting sort** runs in $O(cn)$ time in total
- If c is constant, so each key is $\leq n^c$, this sort is linear $O(n)$!

```
def radix_sort(A):
    "Sort A assuming items have non-negative keys"
    n = len(A)
    u = 1 + max([x.key for x in A])             # O(n) find maximum key
    c = 1 + (u.bit_length() // n.bit_length())
    class Obj: pass
    D = [Obj() for a in A]
    for i in range(n):                          # O(nc) make digit tuples
        D[i].digits = []
        D[i].item = A[i]
        high = A[i].key
        for j in range(c):                      # O(c) make digit tuple
            high, low = divmod(high, n)
            D[i].digits.append(low)
    for i in range(c):                          # O(nc) sort each digit
        for j in range(n):                      # O(n) assign key i to tuples
            D[j].key = D[j].digits[i]
        counting_sort(D)                        # O(n) sort on digit i
    for i in range(n):                          # O(n) output to A
        A[i] = D[i].item
```

Algorithm	Time $O(\cdot)$	In-place?	Stable?	Comments
Insertion Sort	n^2	Y	Y	$O(nk)$ for k-proximate
Selection Sort	n^2	Y	N	$O(n)$ swaps
Merge Sort	$n \log n$	N	Y	stable, optimal comparison
Counting Sort	$n + u$	N	Y	$O(n)$ when $u = O(n)$
Radix Sort	$n + n \log_n(u)$	N	Y	$O(n)$ when $u = O(n^c)$

6.006 Introduction to Algorithms

Spring 2020

Lecture 6: Binary Trees I

Previously and New Goal

Sequence Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Array	n	1	n	n	n
Linked List	n	n	1	n	n
Dynamic Array	n	1	n	$1_{(a)}$	n
Goal	n	$\log n$	$\log n$	$\log n$	$\log n$

Set Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$n \log n$	$\log n$	n	1	$\log n$
Direct Access Array	u	1	1	u	u
Hash Table	$n_{(e)}$	$1_{(e)}$	$1_{(a)(e)}$	n	n
Goal	$n \log n$	$\log n$	$\log n$	$\log n$	$\log n$

How? Binary Trees!

- Pointer-based data structures (like Linked List) can achieve **worst-case** performance
- Binary tree is pointer-based data structure with three pointers per node
- Node representation: `node.{item, parent, left, right}`
- **Example:**

```
1            _________<A>_____        node   | <A> | <B> | <C> | <D> | <E> | <F> |
2         __<B>_____       <C>        item   |  A  |  B  |  C  |  D  |  E  |  F  |
3      __<D>      <E>                 parent |  -  | <A> | <A> | <B> | <B> | <D> |
4     <F>                             left   | <B> | <C> |  -  | <F> |  -  |  -  |
5                                     right  | <C> | <D> |  -  |  -  |  -  |  -  |
```

Terminology

- The **root** of a tree has no parent (**Ex:** `<A>`)
- A **leaf** of a tree has no children (**Ex:** `<C>`, `<E>`, and `<F>`)
- Define **depth**(`<X>`) of node `<X>` in a tree rooted at `<R>` to be length of path from `<X>` to `<R>`
- Define **height**(`<X>`) of node `<X>` to be max depth of any node in the **subtree** rooted at `<X>`
- **Idea:** Design operations to run in $O(h)$ time for root height h, and maintain $h = O(\log n)$
- A binary tree has an inherent order: its **traversal order**
  - every node in node `<X>`'s left subtree is **before** `<X>`
  - every node in node `<X>`'s right subtree is **after** `<X>`
- List nodes in traversal order via a recursive algorithm starting at root:
  - Recursively list left subtree, list self, then recursively list right subtree
  - Runs in $O(n)$ time, since $O(1)$ work is done to list each node
  - **Example:** Traversal order is `(<F>, <D>, <B>, <E>, <A>, <C>)`
- Right now, traversal order has no meaning relative to the stored items
- Later, assign semantic meaning to traversal order to implement Sequence/Set interfaces

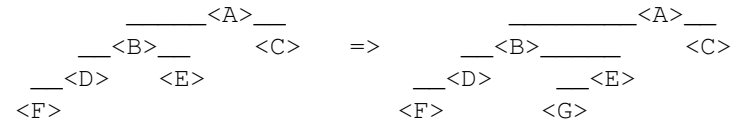
---

Tree Navigation

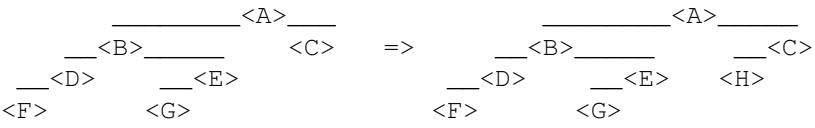
- **Find first** node in the traversal order of node `<X>`'s subtree (last is symmetric)
  - If `<X>` has left child, recursively return the first node in the left subtree
  - Otherwise, `<X>` is the first node, so return it
  - Running time is $O(h)$ where h is the height of the tree
  - **Example:** first node in `<A>`'s subtree is `<F>`
- **Find successor** of node `<X>` in the traversal order (predecessor is symmetric)
  - If `<X>` has right child, return first of right subtree
  - Otherwise, return lowest ancestor of `<X>` for which `<X>` is in its left subtree
  - Running time is $O(h)$ where h is the height of the tree
  - **Example:** Successor of: `<B>` is `<E>`, `<E>` is `<A>`, and `<C>` is `None`

Dynamic Operations

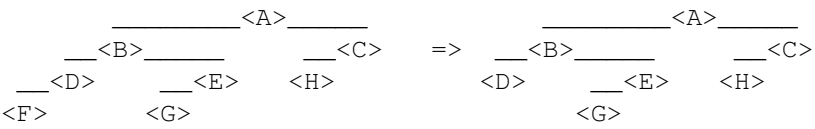
- Change the tree by a single item (only add or remove leaves):
  - add a node after another in the traversal order (before is symmetric)
  - remove an item from the tree
- **Insert** node <Y> after node <X> in the traversal order
  - If <X> has no right child, make <Y> the right child of <X>
  - Otherwise, make <Y> the left child of <X>'s successor (which cannot have a left child)
  - Running time is $O(h)$ where h is the height of the tree
- **Example:** Insert node <G> before <E> in traversal order



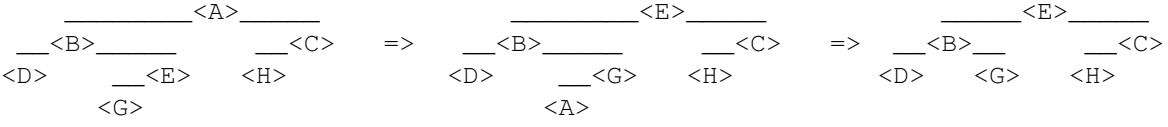
- **Example:** Insert node <H> after <A> in traversal order



- **Delete** the item in node <X> from <X>'s subtree
  - If <X> is a leaf, detach from parent and return
  - Otherwise, <X> has a child
    - If <X> has a left child, swap items with the predecessor of <X> and recurse
    - Otherwise <X> has a right child, swap items with the successor of <X> and recurse
  - Running time is $O(h)$ where h is the height of the tree
  - **Example:** Remove <F> (a leaf)



  - **Example:** Remove <A> (not a leaf, so first swap down to a leaf)



Application: Set

- **Idea! Set Binary Tree** (a.k.a. **Binary Search Tree / BST**):
  Traversal order is sorted order increasing by key
  - Equivalent to **BST Property**: for every node, every key in left subtree $\leq$ node's key $\leq$ every key in right subtree
- Then can find the node with key k in node `<X>`'s subtree in $O(h)$ time like binary search:
  - If k is smaller than the key at `<X>`, recurse in left subtree (or return `None`)
  - If k is larger than the key at `<X>`, recurse in right subtree (or return `None`)
  - Otherwise, return the item stored at `<X>`
- Other Set operations follow a similar pattern; see recitation

---

Application: Sequence

- **Idea! Sequence Binary Tree**: Traversal order is sequence order
- How do we find i^{th} node in traversal order of a subtree? Call this operation `subtree_at(`i`)`
- Could just iterate through entire traversal order, but that's bad, $O(n)$
- However, if we could compute a subtree's **size** in $O(1)$, then can solve in $O(h)$ time
  - How? Check the size n_L of the left subtree and compare to i
  - If $i < n_L$, recurse on the left subtree
  - If $i > n_L$, recurse on the right subtree with $i' = i - n_L - 1$
  - Otherwise, $i = n_L$, and you've reached the desired node!
- Maintain the size of each node's subtree at the node via **augmentation**
  - Add `node.size` field to each `node`
  - When adding new leaf, add $+1$ to `a.size` for all ancestors a in $O(h)$ time
  - When deleting a leaf, add -1 to `a.size` for all ancestors a in $O(h)$ time
- Sequence operations follow directly from a fast `subtree_at(i)` operation
- Naively, `build(X)` takes $O(nh)$ time, but can be done in $O(n)$ time; see recitation

So Far

Set Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Binary Tree	$n \log n$	h	h	h	h
Goal	$n \log n$	$\log n$	$\log n$	$\log n$	$\log n$

Sequence Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Binary Tree	n	h	h	h	h
Goal	n	$\log n$	$\log n$	$\log n$	$\log n$

Next Time

- Keep a binary tree **balanced** after insertion or deletion
- Reduce $O(h)$ running times to $O(\log n)$ by keeping $h = O(\log n)$

6.006 Introduction to Algorithms

Spring 2020

Lecture 7: Binary Trees II: AVL

Last Time and Today's Goal

Sequence Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic		
	`build(X)`	`get_at(i)` `set_at(i,x)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Binary Tree	n	h	h	h	h
AVL Tree	n	$\log n$	$\log n$	$\log n$	$\log n$

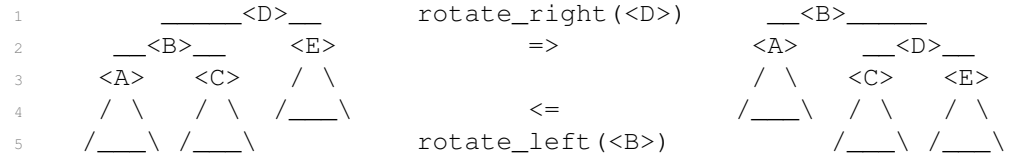
Set Data Structure	Operations $O(\cdot)$				
	Container	Static	Dynamic	Order	
	`build(X)`	`find(k)`	`insert(x)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Binary Tree	$n \log n$	h	h	h	h
AVL Tree	$n \log n$	$\log n$	$\log n$	$\log n$	$\log n$

Height Balance

- How to maintain height $h = O(\log n)$ where n is number of nodes in tree?
- A binary tree that maintains $O(\log n)$ height under dynamic operations is called **balanced**
  - There are many balancing schemes (Red-Black Trees, Splay Trees, 2-3 Trees, . . .)
  - First proposed balancing scheme was the **AVL Tree** (Adelson-Velsky and Landis, 1962)

Rotations

- Need to reduce height of tree without changing its traversal order, so that we represent the same sequence of items
- How to change the structure of a tree, while preserving traversal order? **Rotations!**



- A rotation relinks $O(1)$ pointers to modify tree structure and maintains traversal order

Rotations Suffice

- **Claim:** $O(n)$ rotations can transform a binary tree to any other with same traversal order.
- **Proof:** Repeatedly perform last possible right rotation in traversal order; resulting tree is a canonical chain. Each rotation increases depth of the last node by 1. Depth of last node in final chain is $n-1$, so at most $n-1$ rotations are performed. Reverse canonical rotations to reach target tree. □
- Can maintain height-balance by using $O(n)$ rotations to fully balance the tree, but slow :(
- We will keep the tree balanced in $O(\log n)$ time per operation!

AVL Trees: Height Balance

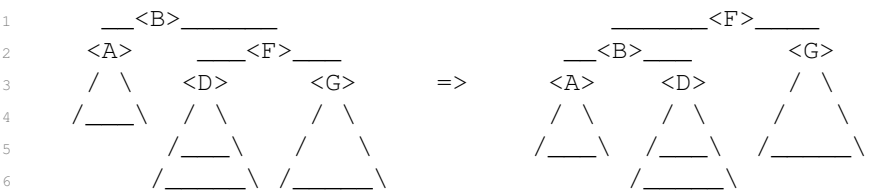
- AVL trees maintain **height-balance** (also called the **AVL Property**)
  - A node is **height-balanced** if heights of its left and right subtrees differ by at most 1
  - Let **skew** of a node be the height of its right subtree minus that of its left subtree
  - Then a node is height-balanced if its skew is $-1, 0,$ or 1

- **Claim:** A binary tree with height-balanced nodes has height $h = O(\log n)$ (i.e., $n = 2^{\Omega(h)}$)
- **Proof:** Suffices to show fewest nodes $F(h)$ in any height h tree is $F(h) = 2^{\Omega(h)}$

  $F(0) = 1,\ F(1) = 2,\ F(h) = 1 + F(h-1) + F(h-2) \geq 2F(h-2) \implies F(h) \geq 2^{h/2}$ □

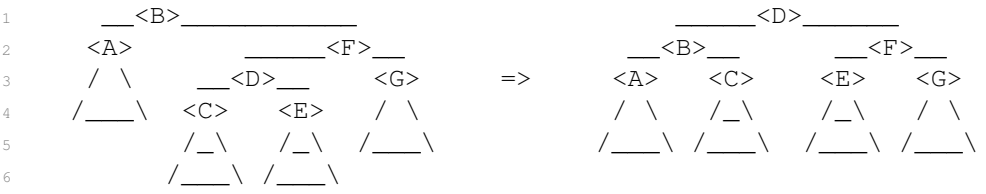
- Suppose adding or removing leaf from a height-balanced tree results in imbalance
  - Only subtrees of the leaf's ancestors have changed in height or skew
  - Heights changed by only ± 1, so skews still have magnitude ≤ 2
  - **Idea:** Fix height-balance of ancestors starting from leaf up to the root
  - Repeatedly rebalance lowest ancestor that is not height-balanced, wlog assume skew 2

- **Local Rebalance:** Given binary tree node <B>:
  - whose skew 2 and
  - every other node in <B>'s subtree is height-balanced,
  - then <B>'s subtree can be made height-balanced via one or two rotations
  - (after which <B>'s height is the same or one less than before)
- **Proof:**
  - Since skew of <B> is 2, <B>'s right child <F> exists
  - **Case 1:** skew of <F> is 0 or **Case 2:** skew of <F> is 1
    - Perform a left rotation on <B>



    - Let $h = \text{height}(\texttt{<A>})$. Then $\text{height}(\texttt{<G>}) = h + 1$ and $\text{height}(\texttt{<D>})$ is $h + 1$ in Case 1, h in Case 2
    - After rotation:
      - the skew of <B> is either 1 in Case 1 or 0 in Case 2, so <B> is height balanced
      - the skew of <F> is -1, so <F> is height balanced
      - the height of <B> before is $h + 3$, then after is $h + 3$ in Case 1, $h + 2$ in Case 2

  - **Case 3:** skew of <F> is -1, so the left child <D> of <F> exists
    - Perform a right rotation on <F>, then a left rotation on <B>



    - Let $h = \text{height}(\texttt{<A>})$. Then $\text{height}(\texttt{<G>}) = h$ while $\text{height}(\texttt{<C>})$ and $\text{height}(\texttt{<E>})$ are each either h or $h - 1$
    - After rotation:
      - the skew of <B> is either 0 or -1, so <B> is height balanced
      - the skew of <F> is either 0 or 1, so <F> is height balanced
      - the skew of <D> is 0, so D is height balanced
      - the height of <B> is $h + 3$ before, then after is $h + 2$

- **Global Rebalance:** Add or remove a leaf from height-balanced tree T to produce tree T'. Then T' can be transformed into a height-balanced tree T'' using at most $O(\log n)$ rotations.
- **Proof:**
  - Only ancestors of the affected leaf have different height in T' than in T
  - Affected leaf has at most $h = O(\log n)$ ancestors whose subtrees may have changed
  - Let <x> be lowest ancestor that is not height-balanced (with skew magnitude 2)
  - If a leaf was added into T:
    - Insertion increases height of <x>, so in Case 2 or 3 of Local Rebalancing
    - Rotation decreases subtree height: balanced after one rotation
  - If a leaf was removed from T:
    - Deletion decreased height of one child of <x>, not <x>, so only imbalance
    - Could decrease height of <x> by 1; parent of <x> may now be imbalanced
    - So may have to rebalance every ancestor of <x>, but at most $h = O(\log n)$ of them
- So can maintain height-balance using only $O(\log n)$ rotations after insertion/deletion!
- But requires us to evaluate whether possibly $O(\log n)$ nodes were height-balanced

---

Computing Height

- How to tell whether node <x> is height-balanced? Compute heights of subtrees!
- How to compute the height of node <x>? Naive algorithm:
  - Recursively compute height of the left and right subtrees of <x>
  - Add 1 to the max of the two heights
  - Runs in $\Omega(n)$ time, since we recurse on every node :(
- **Idea:** Augment each node with the height of its subtree! (Save for later!)
- Height of <x> can be computed in $O(1)$ time from the heights of its children:
  - Look up the stored heights of left and right subtrees in $O(1)$ time
  - Add 1 to the max of the two heights
- During dynamic operations, we must **maintain** our augmentation as the tree changes shape
- Recompute subtree augmentations at every node whose subtree changes:
  - Update relinked nodes in a rotation operation in $O(1)$ time (ancestors don't change)
  - Update all ancestors of an inserted or deleted node in $O(h)$ time by walking up the tree

Steps to Augment a Binary Tree

- In general, to augment a binary tree with a **subtree property** `P`, you must:
  - State the subtree property `P(<X>)` you want to store at each node `<X>`
  - Show how to compute `P(<X>)` from the augmentations of `<X>`'s children in $O(1)$ time
- Then stored property `P(<X>)` can be maintained without changing dynamic operation costs

---

Application: Sequence

- For sequence binary tree, we needed to know subtree **sizes**
- For just inserting/deleting a leaf, this was easy, but now need to handle rotations
- Subtree size is a subtree property, so can maintain via augmentation
  - Can compute size from sizes of children by summing them and adding 1

Conclusion

- Set AVL trees achieve $O(\lg n)$ time for all set operations,
  except $O(n \log n)$ time for build and $O(n)$ time for iter
- Sequence AVL trees achieve $O(\lg n)$ time for all sequence operations,
  except $O(n)$ time for build and iter

Application: Sorting

- Any Set data structure defines a sorting algorithm: build (or repeatedly insert) then iter
- For example, Direct Access Array Sort from Lecture 5
- AVL Sort is a new $O(n \lg n)$-time sorting algorithm

6.006 Introduction to Algorithms

Spring 2020

Lecture 8: Binary Heaps

Priority Queue Interface

- Keep track of many items, quickly access/remove the most important
  - Example: router with limited bandwidth, must prioritize certain kinds of messages
  - Example: process scheduling in operating system kernels
  - Example: discrete-event simulation (when is next occurring event?)
  - Example: graph algorithms (later in the course)
- Order items by key = priority so **Set interface** (not Sequence interface)
- Optimized for a particular subset of Set operations:

`build(X)`	build priority queue from iterable `X`
`insert(x)`	add item x to data structure
`delete_max()`	remove and return stored item with largest key
`find_max()`	return stored item with largest key

- (Usually optimized for max or min, not both)
- Focus on `insert` and `delete_max` operations: `build` can repeatedly `insert`; `find_max()` can `insert(delete_min())`

Priority Queue Sort

- Any priority queue data structure translates into a sorting algorithm:
  - `build(A)`, e.g., `insert` items one by one in input order
  - Repeatedly `delete_min()` (or `delete_max()`) to determine (reverse) sorted order
- All the hard work happens inside the data structure
- Running time is $T_{\texttt{build}} + n \cdot T_{\texttt{delete\_max}} \leq n \cdot T_{\texttt{insert}} + n \cdot T_{\texttt{delete\_max}}$
- Many sorting algorithms we've seen can be viewed as priority queue sort:

Priority Queue Data Structure	Operations $O(\cdot)$			Priority Queue Sort		
	`build(A)`	`insert(x)`	`delete_max()`	Time	In-place?	
Dynamic Array	n	$1_{(a)}$	n	n^2	Y	Selection Sort
Sorted Dynamic Array	$n \log n$	n	$1_{(a)}$	n^2	Y	Insertion Sort
Set AVL Tree	$n \log n$	$\log n$	$\log n$	$n \log n$	N	AVL Sort
Goal	n	$\log n_{(a)}$	$\log n_{(a)}$	$n \log n$	Y	Heap Sort

Priority Queue: Set AVL Tree

- Set AVL trees support `insert(x)`, `find_min()`, `find_max()`, `delete_min()`, and `delete_max()` in $O(\log n)$ time per operation
- So priority queue sort runs in $O(n \log n)$ time
  - This is (essentially) AVL sort from Lecture 7
- Can speed up `find_min()` and `find_max()` to $O(1)$ time via subtree augmentation
- But this data structure is complicated and resulting sort is not in-place
- Is there a simpler data structure for just priority queue, and in-place $O(n \lg n)$ sort?
  YES, binary heap and heap sort
- Essentially implement a Set data structure on top of a Sequence data structure (array), using what we learned about binary trees

---

Priority Queue: Array

- Store elements in an **unordered** dynamic array
- `insert(x)`: append x to end in amortized $O(1)$ time
- `delete_max()`: find max in $O(n)$, swap max to the end and remove
- `insert` is quick, but `delete_max` is slow
- Priority queue sort is selection sort! (plus some copying)

Priority Queue: Sorted Array

- Store elements in a **sorted** dynamic array
- `insert(x)`: append x to end, swap down to sorted position in $O(n)$ time
- `delete_max()`: delete from end in $O(1)$ amortized
- `delete_max` is quick, but `insert` is slow
- Priority queue sort is insertion sort! (plus some copying)
- Can we find a compromise between these two array priority queue extremes?

Array as a Complete Binary Tree

- **Idea:** interpret an array as a complete binary tree, with maximum 2^i nodes at depth i except at the largest depth, where all nodes are **left-aligned**

```
d0          _______0_____
d1      ____0_____    __0__
d2    __0__    __0    0    0
d3    0    0    0
```

- Equivalently, complete tree is filled densely in reading order: root to leaves, left to right
- Perspective: **bijection** between arrays and complete binary trees

```
Q = [0,1,2,3,4,5,6,7,8,9]
d0   0                       ->          _______0_____
d1     1 2                   ->      ____1____    __2__
d2         3 4 5 6           ->    __3__    __4    5    6
d3                 7 8 9     ->    7    8    9
```

- Height of complete tree perspective of array of n item is $\lceil \lg n \rceil$, so **balanced** binary tree

---

Implicit Complete Tree

- Complete binary tree structure can be **implicit** instead of storing pointers
- Root is at index 0
- Compute neighbors by index arithmetic:

$$\begin{aligned} \text{left}(i) &= 2i + 1 \\ \text{right}(i) &= 2i + 2 \\ \text{parent}(i) &= \left\lfloor \frac{i-1}{2} \right\rfloor \end{aligned}$$

Binary Heaps

- **Idea:** keep larger elements higher in tree, but only locally
- **Max-Heap Property** at node i: $Q[i] \geq Q[j]$ for $j \in \{\text{left}(i), \text{right}(i)\}$
- **Max-heap** is an array satisfying max-heap property at all nodes
- **Claim:** In a max-heap, every node i satisfies $Q[i] \geq Q[j]$ for **all nodes** j in $\text{subtree}(i)$
- Proof:
  - Induction on $d = \text{depth}(j) - \text{depth}(i)$
  - Base case: $d = 0$ implies $i = j$ implies $Q[i] \geq Q[j]$ (in fact, equal)
  - $\text{depth}(\text{parent}(j)) - \text{depth}(i) = d - 1 < d$, so $Q[i] \geq Q[\text{parent}(j)]$ by induction
  - $Q[\text{parent}(j)] \geq Q[j]$ by Max-Heap Property at $\text{parent}(j)$ □
- In particular, max item is at root of max-heap

Heap Insert

- Append new item x to end of array in $O(1)$ amortized, making it next leaf i in reading order
- `max_heapify_up`(i): swap with parent until Max-Heap Property
  - Check whether $Q[\text{parent}(i)] \geq Q[i]$ (part of Max-Heap Property at $\text{parent}(i)$)
  - If not, swap items $Q[i]$ and $Q[\text{parent}(i)]$, and recursively `max_heapify_up`$(\text{parent}(i))$
- Correctness:
  - Max-Heap Property guarantees all nodes $\geq$ descendants, except $Q[i]$ might be $>$ some of its ancestors (unless i is the root, so we're done)
  - If swap necessary, same guarantee is true with $Q[\text{parent}(i)]$ instead of $Q[i]$
- Running time: height of tree, so $\Theta(\log n)$!

Heap Delete Max

- Can only easily remove last element from dynamic array, but max key is in root of tree
- So swap item at root node $i = 0$ with last item at node $n - 1$ in heap array
- `max_heapify_down`(i): swap root with larger child until Max-Heap Property
  - Check whether $Q[i] \geq Q[j]$ for $j \in \{\text{left}(i), \text{right}(i)\}$ (Max-Heap Property at i)
  - If not, swap $Q[i]$ with $Q[j]$ for child $j \in \{\text{left}(i), \text{right}(i)\}$ with maximum key, and recursively `max_heapify_down`(j)
- Correctness:
  - Max-Heap Property guarantees all nodes $\geq$ descendants, except $Q[i]$ might be $<$ some descendants (unless i is a leaf, so we're done)
  - If swap is necessary, same guarantee is true with $Q[j]$ instead of $Q[i]$
- Running time: height of tree, so $\Theta(\log n)$!

Heap Sort

- Plugging max-heap into priority queue sort gives us a new sorting algorithm
- Running time is $O(n \log n)$ because each `insert` and `delete_max` takes $O(\log n)$
- But often include two improvements to this sorting algorithm:

In-place Priority Queue Sort

- Max-heap Q is a prefix of a larger array A, remember how many items $|Q|$ belong to heap
- $|Q|$ is initially zero, eventually $|A|$ (after inserts), then zero again (after deletes)
- `insert()` absorbs next item in array at index $|Q|$ into heap
- `delete_max()` moves max item to end, then abandons it by decrementing $|Q|$
- In-place priority queue sort with Array is exactly Selection Sort
- In-place priority queue sort with Sorted Array is exactly Insertion Sort
- In-place priority queue sort with binary Max Heap is **Heap Sort**

Linear Build Heap

- Inserting n items into heap calls `max_heapify_up(i)` for `i` from 0 to $n-1$ (root down):

$$\text{worst-case swaps} \approx \sum_{i=0}^{n-1} \text{depth}(i) = \sum_{i=0}^{n-1} \lg i = \lg(n!) \geq (n/2)\lg(n/2) = \Omega(n \lg n)$$

- **Idea!** Treat full array as a complete binary tree from start, then `max_heapify_down(i)` for `i` from $n-1$ to 0 (leaves up):

$$\text{worst-case swaps} \approx \sum_{i=0}^{n-1} \text{height}(i) = \sum_{i=0}^{n-1} (\lg n - \lg i) = \lg \frac{n^n}{n!} = \Theta\left(\lg \frac{n^n}{\sqrt{n}(n/e)^n}\right) = O(n)$$

- So can `build` heap in $O(n)$ time
- (Doesn't speed up $O(n \lg n)$ performance of heap sort)

Sequence AVL Tree Priority Queue

- Where else have we seen linear build time for an otherwise logarithmic data structure? Sequence AVL Tree!
- Store items of priority queue in Sequence AVL Tree in **arbitrary order** (insertion order)
- Maintain max (and/or min) augmentation:
  `node.max` = pointer to node in subtree of `node` with maximum key
  - This is a subtree property, so constant factor overhead to maintain
- `find_min()` and `find_max()` in $O(1)$ time
- `delete_min()` and `delete_max()` in $O(\log n)$ time
- `build(A)` in $O(n)$ time
- Same bounds as binary heaps (and more)

Set vs. Multiset

- While our Set interface assumes no duplicate keys, we can use these Sets to implement Multisets that allow items with duplicate keys:
  - Each item in the Set is a Sequence (e.g., linked list) storing the Multiset items with the same key, which is the key of the Sequence
- In fact, without this reduction, binary heaps and AVL trees work directly for duplicate-key items (where e.g. `delete_max` deletes *some* item of maximum key), taking care to use $\leq$ constraints (instead of $<$ in Set AVL Trees)

6.006 Introduction to Algorithms

Spring 2020

Lecture 9: Breadth-First Search

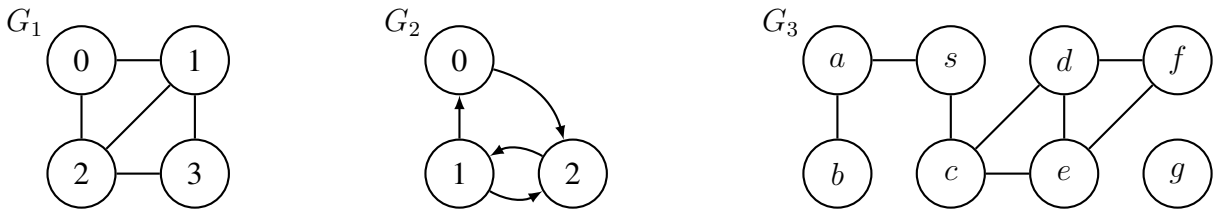
New Unit: Graphs!

- Quiz 1 next week covers lectures L01 - L08 on Data Structures and Sorting
- Today, start new unit, lectures L09 - L14 on Graph Algorithms

Graph Applications

- Why? Graphs are everywhere!
- any network system has direct connection to graphs
- e.g., road networks, computer networks, social networks
- the state space of any discrete system can be represented by a transition graph
- e.g., puzzle & games like Chess, Tetris, Rubik's cube
- e.g., application workflows, specifications

Graph Definitions



- Graph $G = (V, E)$ is a set of vertices V and a set of pairs of vertices $E \subseteq V \times V$.
- **Directed** edges are ordered pairs, e.g., (u, v) for $u, v \in V$
- **Undirected** edges are unordered pairs, e.g., $\{u, v\}$ for $u, v \in V$ i.e., (u, v) and (v, u)
- In this class, we assume all graphs are **simple**:
  - **edges are distinct**, e.g., (u, v) only occurs once in E (though (v, u) may appear), and
  - edges are **pairs of distinct vertices**, e.g., $u \neq v$ for all $(u, v) \in E$
  - Simple implies $|E| = O(|V|^2)$, since $|E| \leq \binom{|V|}{2}$ for undirected, $\leq 2\binom{|V|}{2}$ for directed

Neighbor Sets/Adjacencies

- The **outgoing neighbor set** of $u \in V$ is $\text{Adj}^+(u) = \{v \in V \mid (u, v) \in E\}$
- The **incoming neighbor set** of $u \in V$ is $\text{Adj}^-(u) = \{v \in V \mid (v, u) \in E\}$
- The **out-degree** of a vertex $u \in V$ is $\deg^+(u) = |\text{Adj}^+(u)|$
- The **in-degree** of a vertex $u \in V$ is $\deg^-(u) = |\text{Adj}^-(u)|$
- For undirected graphs, $\text{Adj}^-(u) = \text{Adj}^+(u)$ and $\deg^-(u) = \deg^+(u)$
- Dropping superscript defaults to outgoing, i.e., $\text{Adj}(u) = \text{Adj}^+(u)$ and $\deg(u) = \deg^+(u)$

Graph Representations

- To store a graph $G = (V, E)$, we need to store the outgoing edges $\text{Adj}(u)$ for all $u \in V$
- First, need a Set data structure Adj to map u to $\text{Adj}(u)$
- Then for each u, need to store $\text{Adj}(u)$ in another data structure called an **adjacency list**
- Common to use **direct access array** or **hash table** for Adj, since want lookup fast by vertex
- Common to use **array** or **linked list** for each $\text{Adj}(u)$ since usually only iteration is needed[1]
- For the common representations, Adj has size $\Theta(|V|)$, while each $\text{Adj}(u)$ has size $\Theta(\deg(u))$
- Since $\sum_{u \in V} \deg(u) \leq 2|E|$ by handshaking lemma, graph storable in $\Theta(|V| + |E|)$ space
- Thus, for algorithms on graphs, **linear time** will mean $\Theta(|V| + |E|)$ (linear in size of graph)

Examples

- Examples 1 and 2 assume vertices are labeled $\{0, 1, \ldots, |V| - 1\}$, so can use a direct access array for Adj, and store $\text{Adj}(u)$ in an array. Example 3 uses a hash table for Adj.

```
Ex 1 (Undirected)  | Ex 2 (Directed)  | Ex 3 (Undirected)
G1 = [             | G2 = [           | G3 = {
    [2, 1],    # 0 |     [2],     # 0 |     a: [s, b],    b: [a],
    [2, 0, 3], # 1 |     [2, 0],  # 1 |     s: [a, c],    c: [s, d, e],
    [1, 3, 0], # 2 |     [1],     # 2 |     d: [c, e, f], e: [c, d, f],
    [1, 2],    # 3 | ]                |     f: [d, e],    g: [],
]                  |                  | }
```

- Note that in an undirected graph, connections are symmetric as every edge is outgoing twice

[1] A hash table for each $\text{Adj}(u)$ can allow checking for an edge $(u, v) \in E$ in $O(1)_{(e)}$ time

Paths

- A **path** is a sequence of vertices $p = (v_1, v_2, \ldots, v_k)$ where $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.
- A path is **simple** if it does not repeat vertices[2]
- The **length** $\ell(p)$ of a path p is the number of edges in the path
- The **distance** $\delta(u, v)$ from $u \in V$ to $v \in V$ is the minimum length of any path from u to v, i.e., the length of a **shortest path** from u to v
  (by convention, $\delta(u, v) = \infty$ if u is not connected to v)

Graph Path Problems

- There are many problems you might want to solve concerning paths in a graph:

- SINGLE_PAIR_REACHABILITY(G, s, t): is there a path in G from $s \in V$ to $t \in V$?
- SINGLE_PAIR_SHORTEST_PATH(G, s, t): return distance $\delta(s, t)$, and a shortest path in $G = (V, E)$ from $s \in V$ to $t \in V$
- SINGLE_SOURCE_SHORTEST_PATHS(G, s): return $\delta(s, v)$ for all $v \in V$, and a **shortest-path tree** containing a shortest path from s to every $v \in V$ (defined below)

- Each problem above is **at least as hard** as every problem above it
  (i.e., you can use a black-box that solves a lower problem to solve any higher problem)
- We won't show algorithms to solve all of these problems
- Instead, show one algorithm that solves the **hardest** in $O(|V| + |E|)$ time!

Shortest Paths Tree

- How to return a shortest path from source vertex s for every vertex in graph?
- Many paths could have length $\Omega(|V|)$, so returning every path could require $\Omega(|V|^2)$ time
- Instead, for all $v \in V$, store its **parent** $P(v)$: second to last vertex on a shortest path from s
- Let $P(s)$ be null (no second to last vertex on shortest path from s to s)
- Set of parents comprise a **shortest paths tree** with $O(|V|)$ size!
  (i.e., reversed shortest paths back to s from every vertex reachable from s)

[2] A path in 6.006 is a "walk" in 6.042. A "path" in 6.042 is a simple path in 6.006.

Breadth-First Search (BFS)

- How to compute $\delta(s, v)$ and $P(v)$ for all $v \in V$?
- Store $\delta(s, v)$ and $P(v)$ in Set data structures mapping vertices v to distance and parent
- (If no path from s to v, do not store v in P and set $\delta(s, v)$ to ∞)
- **Idea!** Explore graph nodes in increasing order of distance
- **Goal:** Compute **level sets** $L_i = \{v \mid v \in V \text{ and } d(s, v) = i\}$ (i.e., all vertices at distance i)
- Claim: Every vertex $v \in L_i$ must be adjacent to a vertex $u \in L_{i-1}$ (i.e., $v \in \text{Adj}(u)$)
- Claim: No vertex that is in L_j for some $j < i$, appears in L_i
- **Invariant:** $\delta(s, v)$ and $P(v)$ have been computed correctly for all v in any L_j for $j < i$

- Base case ($i = 1$): $L_0 = \{s\}$, $\delta(s, s) = 0$, $P(s) =$ `None`
- Inductive Step: To compute L_i:
  - for every vertex u in L_{i-1}:
    - for every vertex $v \in \text{Adj}(u)$ that does not appear in any L_j for $j < i$:
      - add v to L_i, set $\delta(s, v) = i$, and set $P(v) = u$
- Repeatedly compute L_i from L_j for $j < i$ for increasing i until L_i is the empty set
- Set $\delta(s, v) = \infty$ for any $v \in V$ for which $\delta(s, v)$ was not set

---

- Breadth-first search correctly computes all $\delta(s, v)$ and $P(v)$ by induction
- Running time analysis:
  - Store each L_i in data structure with $\Theta(|L_i|)$-time iteration and $O(1)$-time insertion (i.e., in a dynamic array or linked list)
  - Checking for a vertex v in any L_j for $j < i$ can be done by checking for v in P
  - Maintain δ and P in Set data structures supporting dictionary ops in $O(1)$ time (i.e., direct access array or hash table)
  - Algorithm adds each vertex u to ≤ 1 level and spends $O(1)$ time for each $v \in \text{Adj}(u)$
  - Work upper bounded by $O(1) \times \sum_{u \in V} \deg(u) = O(|E|)$ by handshake lemma
  - Spend $\Theta(|V|)$ at end to assign $\delta(s, v)$ for vertices $v \in V$ not reachable from s
  - So breadth-first search runs in linear time! $O(|V| + |E|)$
- Run breadth-first search from s in the graph in Example 3

6.006 Introduction to Algorithms

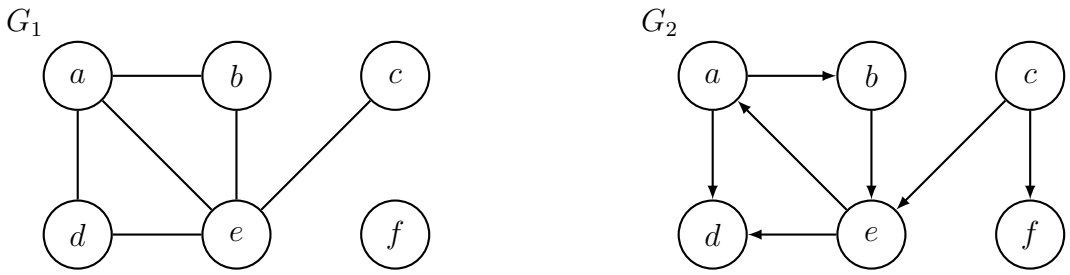
Spring 2020

Lecture 10: Depth-First Search

Previously

- Graph definitions (directed/undirected, simple, neighbors, degree)
- Graph representations (Set mapping vertices to adjacency lists)
- Paths and simple paths, path length, distance, shortest path
- Graph Path Problems
  - `Single_Pair_Reachability(`G`,`s`,`t`)`
  - `Single_Source_Reachability(`G`,`s`)`
  - `Single_Pair_Shortest_Path(`G`,`s`,`t`)`
  - `Single_Source_Shortest_Paths(`G`,`s`)` (SSSP)
- Breadth-First Search (BFS)
  - algorithm that solves Single Source Shortest Paths
  - with appropriate data structures, runs in $O(|V| + |E|)$ time (linear in input size)

Examples



Depth-First Search (DFS)

- Searches a graph from a vertex s, similar to BFS
- Solves Single Source Reachability, **not** SSSP. Useful for solving other problems (later!)
- Return (not necessarily shortest) parent tree of parent pointers back to s

---

- **Idea!** Visit outgoing adjacencies recursively, but never revisit a vertex
- i.e., follow any path until you get stuck, backtrack until finding an unexplored path to explore
- $P(s) =$ `None`, then run $\text{visit}(s)$, where
- $\text{visit}(u)$:
  - for every $v \in \text{Adj}(u)$ that does not appear in P:
    - set $P(v) = u$ and recursively call $\text{visit}(v)$
  - (DFS finishes visiting vertex u, for use later!)

- **Example:** Run DFS on G_1 and/or G_2 from a

Correctness

- **Claim:** DFS visits v and correctly sets $P(v)$ for every vertex v reachable from s
- **Proof:** induct on k, for claim on only vertices within distance k from s
  - Base case ($k = 0$): $P(s)$ is set correctly for s and s is visited
  - Inductive step: Consider vertex v with $\delta(s, v) = k' + 1$
  - Consider vertex u, the second to last vertex on some shortest path from s to v
  - By induction, since $\delta(s, u) = k'$, DFS visits u and sets $P(u)$ correctly
  - While visiting u, DFS considers $v \in \text{Adj}(u)$
  - Either v is in P, so has already been visited, or v will be visited while visiting u
  - In either case, v will be visited by DFS and will be added correctly to P □

Running Time

- Algorithm visits each vertex u at most once and spends $O(1)$ time for each $v \in \text{Adj}(u)$
- Work upper bounded by $O(1) \times \sum_{u \in V} \deg(u) = O(|E|)$
- Unlike BFS, not returning a distance for each vertex, so DFS runs in $O(|E|)$ time

Full-BFS and Full-DFS

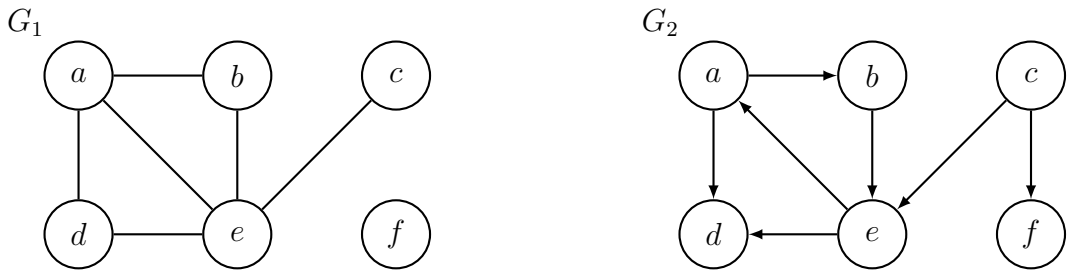
- Suppose want to explore entire graph, not just vertices reachable from one vertex
- **Idea!** Repeat a graph search algorithm A on any unvisited vertex

---

- Repeat the following until all vertices have been visited:
  - Choose an arbitrary unvisited vertex s, use A to explore all vertices reachable from s

---

- We call this algorithm **Full-A**, specifically Full-BFS or Full-DFS if A is BFS or DFS
- Visits every vertex once, so both Full-BFS and Full-DFS run in $O(|V| + |E|)$ time
- **Example:** Run Full-DFS/Full-BFS on G_1 and/or G_2



Graph Connectivity

- An **undirected** graph is ***connected*** if there is a path connecting every pair of vertices
- In a directed graph, vertex u may be reachable from v, but v may not be reachable from u
- Connectivity is more complicated for directed graphs (we won't discuss in this class)
- `Connectivity(`G`)`: is undirected graph G connected?
- `Connected_Components(`G`)`: given undirected graph $G = (V, E)$, return partition of V into subsets $V_i \subseteq V$ (***connected components***) where each V_i is connected in G and there are no edges between vertices from different connected components
- Consider a graph algorithm A that solves Single Source Reachability
- **Claim:** A can be used to solve Connected Components
- **Proof:** Run Full-A. For each run of A, put visited vertices in a connected component □

Topological Sort

- A ***Directed Acyclic Graph (DAG)*** is a directed graph that contains no directed cycle.
- A ***Topological Order*** of a graph $G = (V, E)$ is an ordering f on the vertices such that: every edge $(u, v) \in E$ satisfies $f(u) < f(v)$.
- **Exercise:** Prove that a directed graph admits a topological ordering if and only if it is a DAG.
- How to find a topological order?
- A ***Finishing Order*** is the order in which a Full-DFS **finishes visiting** each vertex in G
- **Claim:** If $G = (V, E)$ is a DAG, the reverse of a finishing order is a topological order
- **Proof:** Need to prove, for every edge $(u, v) \in E$ that u is ordered before v, i.e., the visit to v finishes before visiting u. Two cases:
  - If u visited before v:
    - Before visit to u finishes, will visit v (via (u, v) or otherwise)
    - Thus the visit to v finishes before visiting u
  - If v visited before u:
    - u can't be reached from v since graph is acyclic
    - Thus the visit to v finishes before visiting u □

Cycle Detection

- Full-DFS will find a topological order if a graph $G = (V, E)$ is acyclic
- If reverse finishing order for Full-DFS is not a topological order, then G must contain a cycle
- Check if G is acyclic: for each edge (u, v), check if v is before u in reverse finishing order
- Can be done in $O(|E|)$ time via a hash table or direct access array
- To return such a cycle, maintain the set of **ancestors** along the path back to s in Full-DFS
- **Claim:** If G contains a cycle, Full-DFS will traverse an edge from v to an ancestor of v.
- **Proof:** Consider a cycle $(v_0, v_1, \ldots, v_k, v_0)$ in G
  - Without loss of generality, let v_0 be the first vertex visited by Full-DFS on the cycle
  - For each v_i, before visit to v_i finishes, will visit v_{i+1} and finish
  - Will consider edge (v_i, v_{i+1}), and if v_{i+1} has not been visited, it will be visited now
  - Thus, before visit to v_0 finishes, will visit v_k (for the first time, by v_0 assumption)
  - So, before visit to v_k finishes, will consider (v_k, v_0), where v_0 is an ancestor of v_k □

6.006 Introduction to Algorithms

Spring 2020

Lecture 11: Weighted Shortest Paths

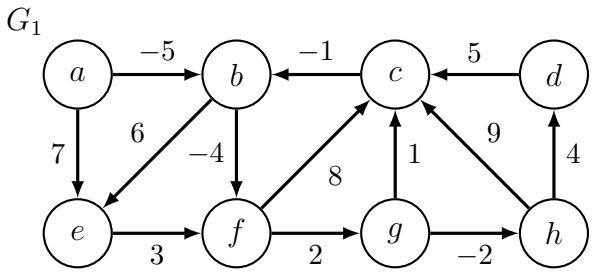
Review

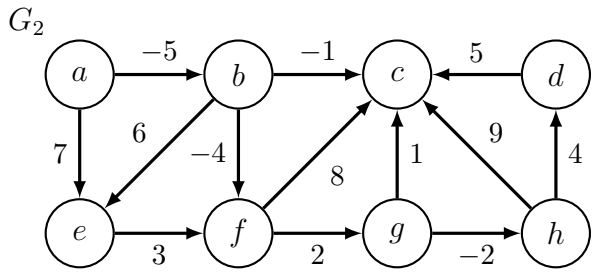
- Single-Source Shortest Paths with BFS in $O(|V| + |E|)$ time (return distance per vertex)
- Single-Source Reachability with BFS or DFS in $O(|E|)$ time (return only reachable vertices)
- Connected components with Full-BFS or Full-DFS in $O(|V| + |E|)$ time
- Topological Sort of a DAG with Full-DFS in $O(|V| + |E|)$ time
- **Previously:** distance = number of edges in path **Today:** generalize meaning of distance

Weighted Graphs

- A **weighted graph** is a graph $G = (V, E)$ together with a weight function $w : E \to \mathbb{Z}$
- i.e., assigns each edge $e = (u, v) \in E$ an integer weight: $w(e) = w(u, v)$
- Many applications for edge weights in a graph:
  - distances in road network
  - latency in network connections
  - strength of a relationship in a social network
- Two common ways to represent weights computationally:
  - Inside graph representation: store edge weight with each vertex in adjacency lists
  - Store separate Set data structure mapping each edge to its weight
- We assume a representation that allows querying the weight of an edge in $O(1)$ time

Examples





Weighted Paths

- The **weight** $w(\pi)$ of a path π in a weighted graph is the sum of weights of edges in the path
- The **(weighted) shortest path** from $s \in V$ to $t \in V$ is path of minimum weight from s to t
- $\delta(s,t) = \inf\{w(\pi) \mid \text{path } \pi \text{ from } s \text{ to } t\}$ is the **shortest-path weight** from s to t
- (Often use "distance" for shortest-path weight in weighted graphs, not number of edges)
- As with unweighted graphs:
  - $\delta(s,t) = \infty$ if no path from s to t
  - Subpaths of shortest paths are shortest paths (or else could splice in a shorter path)
- Why infimum not minimum? Possible that no finite-length minimum-weight path exists
- When? Can occur if there is a negative-weight cycle in the graph, Ex: (b, f, g, c, b) in G_1
- A **negative-weight cycle** is a path π starting and ending at same vertex with $w(\pi) < 0$
- $\delta(s,t) = -\infty$ if there is a path from s to t through a vertex on a negative-weight cycle
- If this occurs, don't want a shortest path, but may want the negative-weight cycle

Weighted Shortest Paths Algorithms

- Next four lectures: algorithms to find shortest-path weights in weighted graphs
- (No parent pointers: can reconstruct shortest paths tree in linear time after. Next page!)
- Already know one algorithm: Breadth-First Search! Runs in $O(|V| + |E|)$ time when, e.g.:
  - graph has positive weights, and all weights are the same
  - graph has positive weights, and sum of all weights at most $O(|V| + |E|)$
- For general weighted graphs, we don't know how to solve SSSP in $O(|V| + |E|)$ time
- But if your graph is a **Directed Acyclic Graph** you can!

Restrictions		SSSP Algorithm		
Graph	Weights	Name	Running Time $O(\cdot)$	Lecture
General	Unweighted	BFS	$\|V\| + \|E\|$	L09
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$	L11 (Today!)
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$	L12
General	Non-negative	Dijkstra	$\|V\| \log \|V\| + \|E\|$	L13

Shortest-Paths Tree

- For BFS, we kept track of parent pointers during search. Alternatively, compute them after!
- If know $\delta(s, v)$ for all vertices $v \in V$, can construct shortest-path tree in $O(|V| + |E|)$ time
- For weighted shortest paths from s, only need parent pointers for vertices v with finite $\delta(s, v)$

- Initialize empty P and set $P(s) =$ `None`
- For each vertex $u \in V$ where $\delta(s, v)$ is finite:
  - For each outgoing neighbor $v \in \text{Adj}^+(u)$:
    - If $P(v)$ not assigned and $\delta(s, v) = \delta(s, u) + w(u, v)$:
      - There exists a shortest path through edge (u, v), so set $P(v) = u$
- Parent pointers may traverse cycles of zero weight. Mark each vertex in such a cycle.
- For each unmarked vertex $u \in V$ (including vertices later unmarked):
  - For each $v \in \text{Adj}^+(u)$ where v is marked and $\delta(s, v) = \delta(s, u) + w(u, v)$:
    - Unmark vertices in cycle containing v by traversing parent pointers from v
    - Set $P(v) = u$, breaking the cycle

- **Exercise:** Prove this algorithm correctly computes parent pointers in linear time
- Because we can compute parent pointers afterward, we focus on computing distances

DAG Relaxation

- **Idea!** Maintain a distance estimate $d(s, v)$ (initially ∞) for each vertex $v \in V$, that always upper bounds true distance $\delta(s, v)$, then gradually lowers until $d(s, v) = \delta(s, v)$
- When do we lower? When an edge violates the triangle inequality!
- **Triangle Inequality:** the shortest-path weight from u to v cannot be greater than the shortest path from u to v through another vertex x, i.e., $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$ for all $u, v, x \in V$
- If $d(s, v) > d(s, u) + w(u, v)$ for some edge (u, v), then triangle inequality is violated :(
- Fix by lowering $d(s, v)$ to $d(s, u) + w(u, v)$, i.e., **relax** (u, v) to satisfy violated constraint
- **Claim:** Relaxation is **safe**: maintains that each $d(s, v)$ is weight of a path to v (or ∞) $\forall v \in V$
- **Proof:** Assume $d(s, v')$ is weight of a path (or ∞) for all $v' \in V$. Relaxing some edge (u, v) sets $d(s, v)$ to $d(s, u) + w(u, v)$, which is the weight of a path from s to v through u. □

- Set $d(s, v) = \infty$ for all $v \in V$, then set $d(s, s) = 0$
- Process each vertex u in a topological sort order of G:
  - For each outgoing neighbor $v \in \text{Adj}^+(u)$:
    - If $d(s, v) > d(s, u) + w(u, v)$:
      - relax edge (u, v), i.e., set $d(s, v) = d(s, u) + w(u, v)$

- **Example:** Run DAG Relaxation from vertex a in G_2

Correctness

- **Claim:** At end of DAG Relaxation: $d(s, v) = \delta(s, v)$ for all $v \in V$
- **Proof:** Induct on k: $d(s, v) = \delta(s, v)$ for all v in first k vertices in topological order
  - Base case: Vertex s and every vertex before s in topological order satisfies claim at start
  - Inductive step: Assume claim holds for first k' vertices, let v be the $(k' + 1)^{\text{th}}$
  - Consider a shortest path from s to v, and let u be the vertex preceding v on path
  - u occurs before v in topological order, so $d(s, u) = \delta(s, u)$ by induction
  - When processing u, $d(s, v)$ is set to be no larger ($\leq$) than $\delta(s, u) + w(u, v) = \delta(s, v)$
  - But $d(s, v) \geq \delta(s, v)$, since relaxation is safe, so $d(s, v) = \delta(s, v)$ □
- Alternatively:
  - For any vertex v, DAG relaxation sets $d(s, v) = \min\{d(s, u) + w(u, v) \mid u \in \text{Adj}^-(v)\}$
  - Shortest path to v must pass through some incoming neighbor u of v
  - So if $d(s, u) = \delta(s, u)$ for all $u \in \text{Adj}^-(v)$ by induction, then $d(s, v) = \delta(s, v)$ □

Running Time

- Initialization takes $O(|V|)$ time, and Topological Sort takes $O(|V| + |E|)$ time
- Additional work upper bounded by $O(1) \times \sum_{u \in V} \deg^+(u) = O(|E|)$
- Total running time is linear, $O(|V| + |E|)$

6.006 Introduction to Algorithms

Spring 2020

Lecture 12: Bellman-Ford

Previously

- Weighted graphs, shortest-path weight, negative-weight cycles
- Finding shortest-path tree from shortest-path weights in $O(|V| + |E|)$ time
- DAG Relaxation: algorithm to solve SSSP on a weighted DAG in $O(|V| + |E|)$ time
- SSSP for graph with negative weights
  - Compute $\delta(s, v)$ for all $v \in V$ ($-\infty$ if v reachable via negative-weight cycle)
  - If a negative-weight cycle reachable from s, return one

Warmups

- **Exercise 1:** Given undirected graph G, return whether G contains a negative-weight cycle
- **Solution:** Return **Yes** if there is an edge with negative weight in G in $O(|E|)$ time :o
- So for this lecture, we restrict our discussion to **directed graphs**

- **Exercise 2:** Given SSSP algorithm A that runs in $O(|V|(|V| + |E|)$ time, show how to use it to solve SSSP in $O(|V||E|)$ time
- **Solution:** Run BFS or DFS to find the vertices reachable from s in $O(|E|)$ time
  - Mark each vertex v not reachable from s with $\delta(s, v) = \infty$ in $O(|V|)$ time
  - Make graph $G' = (V', E')$ with only vertices reachable from s in $O(|V| + |E|)$ time
  - Run A from s in G'.
  - G' is connected, so $|V'| = O(|E'|) = O(|E|)$ so A runs in $O(|V||E|)$ time
- Today, we will find a SSSP algorithm with this running time that works for general graphs!

Restrictions		SSSP Algorithm		
Graph	Weights	Name	Running Time $O(\cdot)$	Lecture
General	Unweighted	BFS	$\|V\| + \|E\|$	L09
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$	L11
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$	L12 (Today!)
General	Non-negative	Dijkstra	$\|V\| \log \|V\| + \|E\|$	L13

Simple Shortest Paths

- If graph contains cycles and negative weights, might contain negative-weight cycles :(
- If graph does not contain negative-weight cycles, shortest paths are simple!
- **Claim 1:** If $\delta(s, v)$ is finite, there exists a shortest path to v that is **simple**
- **Proof:** By contradiction:
  - Suppose no simple shortest path; let π be a shortest path with fewest vertices
  - π not simple, so exists cycle C in π; C has non-negative weight (or else $\delta(s, v) = -\infty$)
  - Removing C from π forms path π' with fewer vertices and weight $w(\pi') \leq w(\pi)$ □
- Since simple paths cannot repeat vertices, finite shortest paths contain at most $|V| - 1$ edges

Negative Cycle Witness

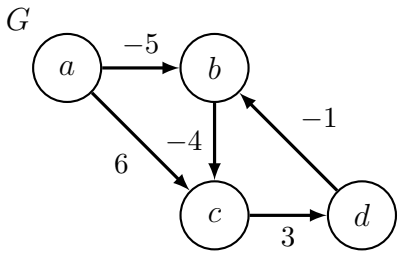
- ***k*-Edge Distance** $\delta_k(s, v)$: the minimum weight of any path from s to v using $\leq k$ edges
- **Idea!** Compute $\delta_{|V|-1}(s, v)$ and $\delta_{|V|}(s, v)$ for all $v \in V$
  - If $\delta(s, v) \neq -\infty$, $\delta(s, v) = \delta_{|V|-1}(s, v)$, since a shortest path is simple (or nonexistent)
  - If $\delta_{|V|}(s, v) < \delta_{|V|-1}(s, v)$
    - there exists a shorter non-simple path to v, so $\delta_{|V|}(s, v) = -\infty$
    - call v a (negative cycle) **witness**
  - However, there may be vertices with $-\infty$ shortest-path weight that **are not witnesses**
- **Claim 2:** If $\delta(s, v) = -\infty$, then v is reachable from a witness
- **Proof:** Suffices to prove: every negative-weight cycle reachable from s contains a witness
  - Consider a negative-weight cycle C reachable from s
  - For $v \in C$, let $v' \in C$ denote v's predecessor in C, where $\sum_{v \in C} w(v', v) < 0$
  - Then $\delta_{|V|}(s, v) \leq \delta_{|V|-1}(s, v') + w(v', v)$ (RHS weight of some path on $\leq |V|$ vertices)
  - So $\sum_{v \in C} \delta_{|V|}(s, v) \leq \sum_{v \in C} \delta_{|V|-1}(s, v') + \sum_{v \in C} w(v', v) < \sum_{v \in C} \delta_{|V|-1}(s, v)$
  - If C contains no witness, $\delta_{|V|}(s, v) \geq \delta_{|V|-1}(s, v)$ for all $v \in C$, a contradiction □

Bellman-Ford

- **Idea!** Use **graph duplication**: make multiple copies (or levels) of the graph
- $|V| + 1$ levels: vertex v_k in level k represents reaching vertex v from s using $\leq k$ edges
- If edges only increase in level, resulting graph is a DAG!

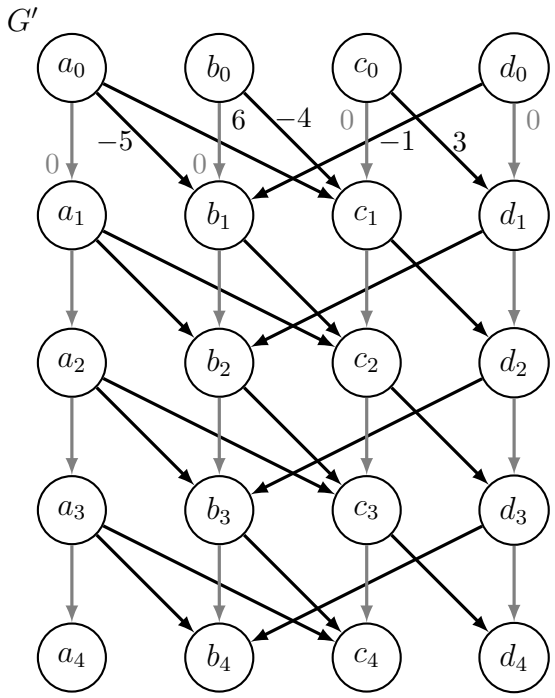
- Construct new DAG $G' = (V', E')$ from $G = (V, E)$:
  - G' has $|V|(|V| + 1)$ vertices v_k for all $v \in V$ and $k \in \{0, \ldots, |V|\}$
  - G' has $|V|(|V| + |E|)$ edges:
    - $|V|$ edges (v_{k-1}, v_k) for $k \in \{1, \ldots, |V|\}$ of weight zero for each $v \in V$
    - $|V|$ edges (u_{k-1}, v_k) for $k \in \{1, \ldots, |V|\}$ of weight $w(u, v)$ for each $(u, v) \in E$
- Run DAG Relaxation on G' from s_0 to compute $\delta(s_0, v_k)$ for all $v_k \in V'$
- For each vertex: set $d(s, v) = \delta(s_0, v_{|V|-1})$
- For each witness $u \in V$ where $\delta(s_0, u_{|V|}) < \delta(s_0, u_{|V|-1})$:
  - For each vertex v reachable from u in G:
    - set $d(s, v) = -\infty$

Example



$\delta(a_0, v_k)$

$k \setminus v$	a	b	c	d
0	0	∞	∞	∞
1	0	$\mathbf{-5}$	$\mathbf{6}$	∞
2	0	-5	$\mathbf{-9}$	$\mathbf{9}$
3	0	-5	-9	$\mathbf{-6}$
4	0	$\mathbf{-7}$	-9	-6
$\delta(a, v)$	0	$-\infty$	$-\infty$	$-\infty$



Correctness

- **Claim 3:** $\delta(s_0, v_k) = \delta_k(s, v)$ for all $v \in V$ and $k \in \{0, \ldots, |V|\}$
- **Proof:** By induction on k:
  - Base case: true for all $v \in V$ when $k = 0$ (only v_0 reachable from s_0 is $v = s$)
  - Inductive Step: Assume true for all $k < k'$, prove for $k = k'$

$$\begin{aligned}\delta(s_0, v_{k'}) &= \min\{\delta(s_0, u_{k'-1}) + w(u_{k'-1}, v_{k'}) \mid u_{k'-1} \in \text{Adj}^-(v_{k'})\} \\ &= \min\{\{\delta(s_0, u_{k'-1}) + w(u, v) \mid u \in \text{Adj}^-(v)\} \cup \{\delta(s_0, v_{k'-1})\}\} \\ &= \min\{\{\delta_{k'-1}(s, u) + w(u, v) \mid u \in \text{Adj}^-(v)\} \cup \{\delta_{k'-1}(s, v)\}\} \quad \text{(by induction)} \\ &= \delta_{k'}(s, v)\end{aligned}$$

□

- **Claim 4:** At the end of Bellman-Ford $d(s, v) = \delta(s, v)$
- **Proof:** Correctly computes $\delta_{|V|-1}(s, v)$ and $\delta_{|V|}(s, v)$ for all $v \in V$ by Claim 3
  - If $\delta(s, v) \neq -\infty$, correctly sets $d(s, v) = \delta_{|V|-1}(s, v) = \delta(s, v)$
  - Then sets $d(s, v) = -\infty$ for any v reachable from a witness; correct by Claim 2 □

Running Time

- G' has size $O(|V|(|V| + |E|))$ and can be constructed in as much time
- Running DAG Relaxation on G' takes linear time in the size of G'
- Does $O(1)$ work for each vertex reachable from a witness
- Finding reachability of a witness takes $O(|E|)$ time, with at most $O(|V|)$ witnesses: $O(|V||E|)$
- (Alternatively, connect **super node** x to witnesses via 0-weight edges, linear search from x)
- Pruning G at start to only subgraph reachable from s yields $O(|V||E|)$-time algorithm

Extras: Return Negative-Weight Cycle or Space Optimization

- **Claim 5:** Shortest $s_0 - v_{|V|}$ path π for any witness v contains a negative-weight cycle in G
- **Proof:** Since π contains $|V| + 1$ vertices, must contain at least one cycle C in G
  - C has negative weight (otherwise, remove C to make path π' with fewer vertices and $w(\pi') \leq w(\pi)$, contradicting witness v) □
- Can use just $O(|V|)$ space by storing only $\delta(s_0, v_{k-1})$ and $\delta(s_0, v_k)$ for each k from 1 to $|V|$
- Traditionally, Bellman-Ford stores only one value per vertex, attempting to relax every edge in $|V|$ rounds; but estimates do not correspond to k-Edge Distances, so analysis trickier
- But these space optimizations don't return a negative weight cycle

6.006 Introduction to Algorithms

Spring 2020

Lecture 13: Dijkstra's Algorithm

Review

- Single-Source Shortest Paths on weighted graphs
- Previously: $O(|V| + |E|)$-time algorithms for small positive weights or DAGs
- Last time: Bellman-Ford, $O(|V||E|)$-time algorithm for **general graphs** with **negative weights**
- Today: faster for **general graphs** with **non-negative edge weights**, i.e., for $e \in E, w(e) \geq 0$

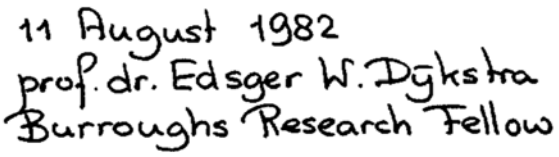
Restrictions		SSSP Algorithm		
Graph	Weights	Name	Running Time $O(\cdot)$	Lecture
General	Unweighted	BFS	$\|V\| + \|E\|$	L09
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$	L11
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$	L12
General	Non-negative	Dijkstra	$\|V\| \log \|V\| + \|E\|$	L13 (Today!)

Non-negative Edge Weights

- **Idea!** Generalize BFS approach to weighted graphs:
  - Grow a sphere centered at source s
  - Repeatedly explore closer vertices before further ones
  - But how to explore closer vertices if you don't know distances beforehand? :(
- **Observation 1:** If weights non-negative, monotonic distance increase along shortest paths
  - i.e., if vertex u appears on a shortest path from s to v, then $\delta(s, u) \leq \delta(s, v)$
  - Let $V_x \subset V$ be the subset of vertices reachable within distance $\leq x$ from s
  - If $v \in V_x$, then any shortest path from s to v only contains vertices from V_x
  - Perhaps grow V_x one vertex at a time! (but growing for every x is slow if weights large)
- **Observation 2:** Can solve SSSP fast if given order of vertices in increasing distance from s
  - Remove edges that go against this order (since cannot participate in shortest paths)
  - May still have cycles if zero-weight edges: repeatedly collapse into single vertices
  - Compute $\delta(s, v)$ for each $v \in V$ using DAG relaxation in $O(|V| + |E|)$ time

Dijkstra's Algorithm

- Named for famous Dutch computer scientist **Edsger Dijkstra** (actually Dÿkstra!)



- **Idea!** Relax edges from each vertex in increasing order of distance from source s
- **Idea!** Efficiently find next vertex in the order using a data structure
- **Changeable Priority Queue** Q on items with keys and unique IDs, supporting operations:

`Q.build(X)`	initialize Q with items in iterator `X`
`Q.delete_min()`	remove an item with minimum key
`Q.decrease_key(id, k)`	find stored item with ID `id` and change key to `k`

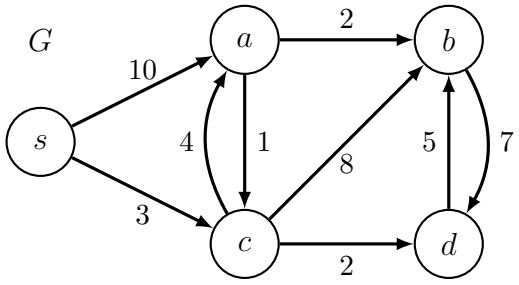
- Implement by **cross-linking** a Priority Queue Q' and a Dictionary D mapping IDs into Q'
- Assume vertex IDs are integers from 0 to $|V| - 1$ so can use a direct access array for D
- For brevity, say item `x` is the tuple `(x.id, x.key)`

- Set $d(s, v) = \infty$ for all $v \in V$, then set $d(s, s) = 0$
- Build changeable priority queue Q with an item $(v, d(s, v))$ for each vertex $v \in V$
- While Q not empty, delete an item $(u, d(s, u))$ from Q that has minimum $d(s, u)$
  - For vertex v in outgoing adjacencies $\text{Adj}^+(u)$:
    - If $d(s, v) > d(s, u) + w(u, v)$:
      - Relax edge (u, v), i.e., set $d(s, v) = d(s, u) + w(u, v)$
      - Decrease the key of v in Q to new estimate $d(s, v)$

- Run Dijkstra on example

Example

Delete v from Q	$d(s,v)$ s	a	b	c	d
s	**0**	∞	∞	∞	∞
c		10	∞	**3**	∞
d		7	11		**5**
a		**7**	10		
b			**9**		
$\delta(s,v)$	0	7	9	3	5



Correctness

- **Claim:** At end of Dijkstra's algorithm, $d(s,v) = \delta(s,v)$ for all $v \in V$
- **Proof:**
  - If relaxation sets $d(s,v)$ to $\delta(s,v)$, then $d(s,v) = \delta(s,v)$ at the end of the algorithm
    - Relaxation can only decrease estimates $d(s,v)$
    - Relaxation is safe, i.e., maintains that each $d(s,v)$ is weight of a path to v (or ∞)
  - Suffices to show $d(s,v) = \delta(s,v)$ when vertex v is removed from Q
    - Proof by induction on first k vertices removed from Q
    - Base Case ($k = 1$): s is first vertex removed from Q, and $d(s,s) = 0 = \delta(s,s)$
    - Inductive Step: Assume true for $k < k'$, consider k'th vertex v' removed from Q
    - Consider some shortest path π from s to v', with $w(\pi) = \delta(s,v')$
    - Let (x,y) be the first edge in π where y is not among first $k'-1$ (perhaps $y = v'$)
    - When x was removed from Q, $d(s,x) = \delta(s,x)$ by induction, so:

$$\begin{aligned} d(s,y) &\le \delta(s,x) + w(x,y) && \text{relaxed edge } (x,y) \text{ when removed } x \\ &= \delta(s,y) && \text{subpaths of shortest paths are shortest paths} \\ &\le \delta(s,v') && \text{non-negative edge weights} \\ &\le d(s,v') && \text{relaxation is safe} \\ &\le d(s,y) && v' \text{ is vertex with minimum } d(s,v') \text{ in } Q \end{aligned}$$

    - So $d(s,v') = \delta(s,v')$, as desired □

Running Time

- Count operations on changeable priority queue Q, assuming it contains n items:

Operation	Time	Occurrences in Dijkstra
`Q.build(X)` $(n = \|X\|)$	B_n	1
`Q.delete_min()`	M_n	$\|V\|$
`Q.decrease_key(id, k)`	D_n	$\|E\|$

- Total running time is $O(B_{|V|} + |V| \cdot M_{|V|} + |E| \cdot D_{|V|})$
- Assume pruned graph to search only vertices reachable from the source, so $|V| = O(|E|)$

Priority Queue Q' on n items	Q Operations $O(\cdot)$			Dijkstra $O(\cdot)$ $n = \|V\| = O(\|E\|)$
	`build(X)`	`delete_min()`	`decrease_key(id, k)`	
Array	n	n	1	$\|V\|^2$
Binary Heap	n	$\log n_{(a)}$	$\log n$	$\|E\| \log \|V\|$
Fibonacci Heap	n	$\log n_{(a)}$	$1_{(a)}$	$\|E\| + \|V\| \log \|V\|$

- If graph is **dense**, i.e., $|E| = \Theta(|V|^2)$, using an Array for Q' yields $O(|V|^2)$ time
- If graph is **sparse**, i.e., $|E| = \Theta(|V|)$, using a Binary Heap for Q' yields $O(|V| \log |V|)$ time
- A Fibonacci Heap is theoretically good in all cases, but is not used much in practice
- We won't discuss Fibonacci Heaps in 6.006 (see 6.854 or CLRS chapter 19 for details)
- You should assume Dijkstra runs in $O(|E| + |V| \log |V|)$ time when using in theory problems

Summary: Weighted Single-Source Shortest Paths

Restrictions		SSSP Algorithm	
Graph	Weights	Name	Running Time $O(\cdot)$
General	Unweighted	BFS	$\|V\| + \|E\|$
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$
General	Non-negative	Dijkstra	$\|V\| \log \|V\| + \|E\|$
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$

- What about All-Pairs Shortest Paths?
- Doing a SSSP algorithm $|V|$ times is actually pretty good, since output has size $O(|V|^2)$
- Can do better than $|V| \cdot O(|V| \cdot |E|)$ for general graphs with negative weights (next time!)

6.006 Introduction to Algorithms

Spring 2020

Lecture 14: Johnson's Algorithm

Previously

Restrictions		SSSP Algorithm	
Graph	Weights	Name	Running Time $O(\cdot)$
General	Unweighted	BFS	$\lvert V\rvert + \lvert E\rvert$
DAG	Any	DAG Relaxation	$\lvert V\rvert + \lvert E\rvert$
General	Non-negative	Dijkstra	$\lvert V\rvert \log \lvert V\rvert + \lvert E\rvert$
General	Any	Bellman-Ford	$\lvert V\rvert \cdot \lvert E\rvert$

All-Pairs Shortest Paths (APSP)

- **Input:** directed graph $G = (V, E)$ with weights $w : E \to \mathbb{Z}$
- **Output:** $\delta(u, v)$ for all $u, v \in V$, or abort if G contains negative-weight cycle
- Useful when understanding whole network, e.g., transportation, circuit layout, supply chains...
- Just doing a SSSP algorithm $|V|$ times is actually pretty good, since output has size $O(|V|^2)$
  - $|V| \cdot O(|V| + |E|)$ with BFS if weights positive and bounded by $O(|V| + |E|)$
  - $|V| \cdot O(|V| + |E|)$ with DAG Relaxation if acyclic
  - $|V| \cdot O(|V| \log |V| + |E|)$ with Dijkstra if weights non-negative or graph undirected
  - $|V| \cdot O(|V| \cdot |E|)$ with Bellman-Ford (general)
- **Today:** Solve APSP in any weighted graph in $|V| \cdot O(|V| \log |V| + |E|)$ time

Approach

- **Idea:** Make all edge weights non-negative while **preserving shortest paths**!
- i.e., reweight G to G' with no negative weights, where a shortest path in G is shortest in G'
- If non-negative, then just run Dijkstra $|V|$ times to solve APSP
- **Claim:** Can compute distances in G from distances in G' in $O(|V|(|V| + |E|))$ time
  - Compute shortest-path tree from distances, for each $s \in V'$ in $O(|V| + |E|)$ time (L11)
  - Also shortest-paths tree in G, so traverse tree with DFS while also computing distances
  - Takes $O(|V| \cdot (|V| + |E|))$ time (which is less time than $|V|$ times Dijkstra)
- But how to make G' with non-negative edge weights? Is this even possible??
- **Claim:** Not possible if G contains a negative-weight cycle
- **Proof:** Shortest paths are simple if no negative weights, but not if negative-weight cycle □
- Given graph G with negative weights but no negative-weight cycles,
  can we make edge weights non-negative while preserving shortest paths?

Making Weights Non-negative

- **Idea!** Add negative of smallest weight in G to every edge! All weights non-negative! :)
- **FAIL:** Does not preserve shortest paths! Biases toward paths traversing fewer edges :(
- **Idea!** Given vertex v, add h to all **outgoing edges** and subtract h from all **incoming edges**
- **Claim:** Shortest paths are preserved under the above reweighting
- **Proof:**
  - Weight of every path starting at v changes by h
  - Weight of every path ending at v changes by $-h$
  - Weight of a path passing through v **does not change** (locally) □
- This is a very general and useful trick to transform a graph while preserving shortest paths!

- Even works with multiple vertices!
- Define a **potential function** $h : V \to \mathbb{Z}$ mapping each vertex $v \in V$ to a potential $h(v)$
- Make graph G': same as G but edge $(u, v) \in E$ has weight $w'(u, v) = w(u, v) + h(u) - h(v)$
- **Claim:** Shortest paths in G are also shortest paths in G'
- **Proof:**
  - Weight of path $\pi = (v_0, \ldots, v_k)$ in G is $w(\pi) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$
  - Weight of π in G' is: $\sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i) = w(\pi) + h(v_0) - h(v_k)$
  - (Sum of h's telescope, since there is a positive and negative $h(v_i)$ for each interior i)
  - Every path from v_0 to v_k changes by the same amount
  - So any shortest path will still be shortest □

Making Weights Non-negative

- Can we find a potential function such that G' has no negative edge weights?
- i.e., is there an h such that $w(u, v) + h(u) - h(v) \geq 0$ for every $(u, v) \in E$?
- Re-arrange this condition to $h(v) \leq h(u) + w(u, v)$, looks like **triangle inequality**!
- **Idea!** Condition would be satisfied if $h(v) = \delta(s, v)$ and $\delta(s, v)$ is finite for some s
- But graph may be disconnected, so may not exist any such vertex s... :(
- **Idea!** Add a new vertex s with a directed 0-weight edge to every $v \in V$! :)
- $\delta(s, v) \leq 0$ for all $v \in V$, since path exists a path of weight 0
- **Claim:** If $\delta(s, v) = -\infty$ for any $v \in V$, then the original graph has a negative-weight cycle
- **Proof:**
  - Adding s does not introduce new cycles (s has no incoming edges)
  - So if reweighted graph has a negative-weight cycle, so does the original graph □
- Alternatively, if $\delta(s, v)$ is finite for all $v \in V$:
  - $w'(u, v) = w(u, v) + h(u) - h(v) \geq 0$ for every $(u, v) \in E$ by triangle inequality!
  - New weights in G' are non-negative while preserving shortest paths!

Johnson's Algorithm

- Construct G_x from G by adding vertex x connected to each vertex $v \in V$ with 0-weight edge
- Compute $\delta_x(x, v)$ for every $v \in V$ (using Bellman-Ford)
- If $\delta_x(x, v) = -\infty$ for any $v \in V$:
  - Abort (since there is a negative-weight cycle in G)
- Else:
  - Reweight each edge $w'(u, v) = w(u, v) + \delta_x(x, u) - \delta_x(x, v)$ to form graph G'
  - For each $u \in V$:
    - Compute shortest-path distances $\delta'(u, v)$ to all v in G' (using Dijkstra)
    - Compute $\delta(u, v) = \delta'(u, v) - \delta_x(x, u) + \delta_x(x, v)$ for all $v \in V$

Correctness

- Already proved that transformation from G to G' preserves shortest paths
- Rest reduces to correctness of Bellman-Ford and Dijkstra
- Reducing from **Signed APSP** to **Non-negative APSP**
- Reductions save time! No induction today! :)

Running Time

- $O(|V| + |E|)$ time to construct G_x
- $O(|V||E|)$ time for Bellman-Ford
- $O(|V| + |E|)$ time to construct G'
- $O(|V| \cdot (|V| \log |V| + |E|))$ time for $|V|$ runs of Dijkstra
- $O(|V|^2)$ time to compute distances in G from distances in G'
- $O(|V|^2 \log |V| + |V||E|)$ time in total

6.006 Introduction to Algorithms

Spring 2020

Lecture 15: Recursive Algorithms

How to Solve an Algorithms Problem (Review)

- Reduce to a problem you already know (use data structure or algorithm)

Search Data Structures	Sort Algorithms	Graph Algorithms
Array	Insertion Sort	Breadth First Search
Linked List	Selection Sort	DAG Relaxation (DFS + Topo)
Dynamic Array	Merge Sort	Dijkstra
Sorted Array	Counting Sort	Bellman-Ford
Direct-Access Array	Radix Sort	Johnson
Hash Table	AVL Sort	
AVL Tree	Heap Sort	
Binary Heap		

- Design your own **recursive** algorithm
  - Constant-sized program to solve arbitrary input
  - Need looping or recursion, analyze by induction
  - Recursive function call: vertex in a graph, directed edge from $A \to B$ if B calls A
  - Dependency graph of recursive calls must be acyclic (if can terminate)
  - Classify based on shape of graph

Class	Graph
Brute Force	Star
Decrease & Conquer	Chain
Divide & Conquer	Tree
Dynamic Programming	DAG
Greedy/Incremental	Subgraph

  - Hard part is thinking inductively to construct recurrence on subproblems
  - How to solve a problem recursively (**SRT BOT**)
    1. **Subproblem** definition
    2. **Relate** subproblem solutions recursively
    3. **Topological order** on subproblems ($\Rightarrow$ subproblem DAG)
    4. **Base** cases of relation
    5. **Original** problem solution via subproblem(s)
    6. **Time** analysis

Merge Sort in SRT BOT Framework

- Merge sorting an array A of n elements can be expressed in SRT BOT as follows:
  - **S**ubproblems: $S(i, j) =$ sorted array on elements of $A[i : j]$ for $0 \le i \le j \le n$
  - **R**elation: $S(i, j) = \text{merge}(S(i, m), S(m, j))$ where $m = \lfloor (i + j)/2 \rfloor$
  - **T**opo. order: Increasing $j - i$
  - **B**ase cases: $S(i, i + 1) = [A[i]]$
  - **O**riginal: $S(0, n)$
  - **T**ime: $T(n) = 2\,T(n/2) + O(n) = O(n \lg n)$
- In this case, subproblem DAG is a tree (divide & conquer)

Fibonacci Numbers

- Suppose we want to compute the nth Fibonacci number F_n
- **S**ubproblems: $F(i) =$ the ith Fibonacci number F_i for $i \in \{0, 1, \ldots, n\}$
- **R**elation: $F(i) = F(i - 1) + F(i - 2)$ (definition of Fibonacci numbers)
- **T**opo. order: Increasing i
- **B**ase cases: $F(0) = 0$, $F(1) = 1$
- **O**riginal prob.: $F(n)$

```
1 def fib(n):
2     if n < 2: return n                  # base case
3     return fib(n - 1) + fib(n - 2)      # recurrence
```

- Divide and conquer implies a tree of **recursive calls** (draw tree)
- **T**ime: $T(n) = T(n-1) + T(n-2) + O(1) > 2T(n-2)$, $T(n) = \Omega(2^{n/2})$ exponential... :(
- Subproblem $F(k)$ computed more than once! ($F(n - k)$ times)
- Can we avoid this waste?

Re-using Subproblem Solutions

- Draw subproblem dependencies as a DAG
- To solve, either:
  - **Top down:** record subproblem solutions in a memo and re-use (**recursion + memoization**)
  - **Bottom up:** solve subproblems in topological sort order (usually via loops)
- For Fibonacci, $n + 1$ subproblems (vertices) and $< 2n$ dependencies (edges)
- **T**ime to compute is then $O(n)$ additions

```python
# recursive solution (top down)
def fib(n):
    memo = {}
    def F(i):
        if i < 2: return i                  # base cases
        if i not in memo:                   # check memo
            memo[i] = F(i - 1) + F(i - 2)   # relation
        return memo[i]
    return F(n)                             # original
```

```python
# iterative solution (bottom up)
def fib(n):
    F = {}
    F[0], F[1] = 0, 1                       # base cases
    for i in range(2, n + 1):               # topological order
        F[i] = F[i - 1] + F[i - 2]          # relation
    return F[n]                             # original
```

- A subtlety is that Fibonacci numbers grow to $\Theta(n)$ bits long, potentially $\gg$ word size w
- Each addition costs $O(\lceil n/w \rceil)$ time
- So total cost is $O(n\lceil n/w \rceil) = O(n + n^2/w)$ time

Dynamic Programming

- Weird name coined by Richard Bellman
  - Wanted government funding, needed cool name to disguise doing mathematics!
  - Updating (dynamic) a plan or schedule (program)
- Existence of recursive solution implies decomposable subproblems[1]
- Recursive algorithm implies a graph of computation
- Dynamic programming if subproblem dependencies **overlap** (DAG, in-degree > 1)
- "Recurse but re-use" (Top down: record and lookup subproblem solutions)
- "Careful brute force" (Bottom up: do each subproblem in order)
- Often useful for **counting/optimization** problems: almost trivially correct recurrences

---

How to Solve a Problem Recursively (SRT BOT)

1. **Subproblem** definition subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often record partial state: add subproblems by incrementing some auxiliary variables
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

[1] This property often called **optimal substructure**. It is a property of recursion, not just dynamic programming

DAG Shortest Paths

- Recall the DAG SSSP problem: given a DAG G and vertex s, compute $\delta(s, v)$ for all $v \in V$
- **S**ubproblems: $\delta(s, v)$ for all $v \in V$
- **R**elation: $\delta(s, v) = \min\{\delta(s, u) + w(u, v) \mid u \in \text{Adj}^-(v)\} \cup \{\infty\}$
- **T**opo. order: Topological order of G
- **B**ase cases: $\delta(s, s) = 0$
- **O**riginal: All subproblems
- **T**ime: $\sum_{v \in V} O(1 + |\text{Adj}^-(v)|) = O(|V| + |E|)$
- DAG Relaxation computes the same min values as this dynamic program, just
  - step-by-step (if new value $<$ min, update min via edge relaxation), and
  - from the perspective of u and $\text{Adj}^+(u)$ instead of v and $\text{Adj}^-(v)$

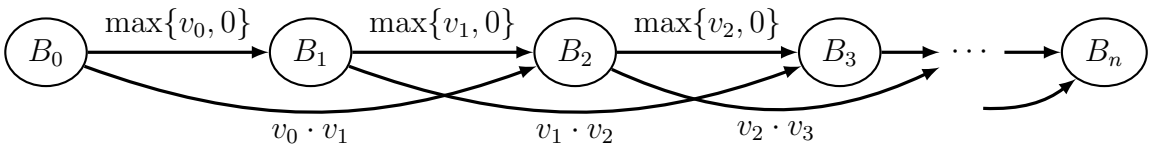
Bowling

- Given n pins labeled $0, 1, \ldots, n-1$
- Pin i has **value** v_i
- Ball of size similar to pin can hit either
  - 1 pin i, in which case we get v_i points
  - 2 adjacent pins i and $i+1$, in which case we get $v_i \cdot v_{i+1}$ points
- Once a pin is hit, it can't be hit again (removed)
- Problem: Throw zero or more balls to maximize total points
- Example: $[\,-1, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{9, 9}, \boxed{3}, \boxed{-3, -5}, \boxed{2, 2}\,]$

Bowling Algorithms

- Let's start with a more familiar divide-and-conquer algorithm:
  - **S**ubproblems: $B(i, j) =$ maximum score starting with just pins $i, i+1, \ldots, j-1$, for $0 \le i \le j \le n$
  - **R**elation:
    - $m = \lfloor (i+j)/2 \rfloor$
    - Either hit m and $m+1$ together, or don't
    - $B(i,j) = \max\{v_m \cdot v_{m+1} + B(i,m) + B(m+2,j), B(i,m+1) + B(m+1,j)\}$
  - **T**opo. order: Increasing $j - i$
  - **B**ase cases: $B(i,i) = 0$, $B(i,i+1) = \max\{v_i, 0\}$
  - **O**riginal: $B(0,n)$
  - **T**ime: $T(n) = 4\,T(n/2) + O(1) = O(n^2)$
- This algorithm works but isn't very fast, and doesn't generalize well (e.g., to allow for a bigger ball that hits three balls at once)

- Dynamic programming algorithm: use suffixes
  - **S**ubproblems: $B(i) =$ maximum score starting with just pins $i, i+1, \ldots, n-1$, for $0 \le i \le n$
  - **R**elation:
    - Locally brute-force what could happen with first pin (original pin i): skip pin, hit one pin, hit two pins
    - Reduce to smaller suffix and recurse, either $B(i+1)$ or $B(i+2)$
    - $B(i) = \max\{B(i+1), v_i + B(i+1), v_i \cdot v_{i+1} + B(i+2)\}$
  - **T**opo. order: Decreasing i (for $i = n, n-1, \ldots, 0$)
  - **B**ase cases: $B(n) = B(n+1) = 0$
  - **O**riginal: $B(0)$
  - **T**ime: (assuming memoization)
    - $\Theta(n)$ subproblems $\cdot\ \Theta(1)$ work in each
    - $\Theta(n)$ total time
- Fast and easy to generalize!
- Equivalent to maximum-weight path in Subproblem DAG:



Bowling Code

- Converting a SRT BOT specification into code is automatic/straightforward
- Here's the result for the Bowling Dynamic Program above:

```
# recursive solution (top down)
def bowl(v):
    memo = {}
    def B(i):
        if i >= len(v): return 0          # base cases
        if i not in memo:                 # check memo
            memo[i] = max(B(i+1),         # relation: skip pin i
                v[i] + B(i+1),            # OR bowl pin i separately
                v[i] * v[i+1] + B(i+2))   # OR bowl pins i and i+1 together
        return memo[i]
    return B(0)                           # original
```

```
# iterative solution (bottom up)
def bowl(v):
    B = {}
    B[len(v)] = 0                         # base cases
    B[len(v)+1] = 0
    for i in reversed(range(len(v))):     # topological order
        B[i] = max(B[i+1],                # relation: skip pin i
            v[i] + B(i+1),                # OR bowl pin i separately
            v[i] * v[i+1] + B(i+2))       # OR bowl pins i and i+1 together
    return B[0]                           # original
```

How to Relate Subproblem Solutions

- The general approach we're following to define a relation on subproblem solutions:
  - Identify a question about a subproblem solution that, if you knew the answer to, would reduce to "smaller" subproblem(s)
    - In case of bowling, the question is "how do we bowl the first couple of pins?"
  - Then locally brute-force the question by trying all possible answers, and taking the best
    - In case of bowling, we take the max because the problem asks to maximize
  - Alternatively, we can think of correctly guessing the answer to the question, and directly recursing; but then we actually check all possible guesses, and return the "best"
- The key for efficiency is for the question to have a small (polynomial) number of possible answers, so brute forcing is not too expensive
- Often (but not always) the nonrecursive work to compute the relation is equal to the number of answers we're trying

6.006 Introduction to Algorithms

Spring 2020

Lecture 16: Dyn. Prog. Subproblems

Dynamic Programming Review

- Recursion where subproblem dependencies **overlap**, forming DAG
- "Recurse but re-use" (Top down: record and lookup subproblem solutions)
- "Careful brute force" (Bottom up: do each subproblem in order)

Dynamic Programming Steps (SRT BOT)

1. **Subproblem** definition subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often multiply possible subsets across multiple inputs
   - Often record partial state: add subproblems by incrementing some auxiliary variables
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
   - Identify a question about a subproblem solution that, if you knew the answer to, reduces the subproblem to smaller subproblem(s)
   - Locally brute-force all possible answers to the question
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

Longest Common Subsequence (LCS)

- Given two strings A and B, find a longest (not necessarily contiguous) subsequence of A that is also a subsequence of B.
- Example: $A =$ `hieroglyphology`, $B =$ `michaelangelo`
- Solution: `hello` or `heglo` or `iello` or `ieglo`, all length 5
- Maximization problem on length of subsequence

1. **Subproblems**
   - $\boxed{x(i,j) = \text{length of longest common subsequence of suffixes } A[i:] \text{ and } B[j:]}$
   - For $0 \le i \le |A|$ and $0 \le j \le |B|$
2. **Relate**
   - Either first characters match or they don't
   - If first characters match, some longest common subsequence will use them
   - (if no LCS uses first matched pair, using it will only improve solution)
   - (if an LCS uses first in $A[i]$ and not first in $B[j]$, matching $B[j]$ is also optimal)
   - If they do not match, they cannot both be in a longest common subsequence
   - **Guess** whether $A[i]$ or $B[j]$ is not in LCS
   - $x(i,j) = \begin{cases} x(i+1,j+1)+1 & \text{if } A[i] = B[j] \\ \max\{x(i+1,j), x(i,j+1)\} & \text{otherwise} \end{cases}$
   - (draw subset of all rectangular grid dependencies)
3. **Topological order**
   - Subproblems $x(i,j)$ depend only on strictly larger i or j or both
   - Simplest order to state: Decreasing $i + j$
   - Nice order for bottom-up code: Decreasing i, then decreasing j
4. **Base**
   - $x(i, |B|) = x(|A|, j) = 0$ (one string is empty)
5. **Original problem**
   - Length of longest common subsequence of A and B is $x(0,0)$
   - Store parent pointers to reconstruct subsequence
   - If the parent pointer increases both indices, add that character to LCS

6. **Time**
    - # subproblems: $(|A| + 1) \cdot (|B| + 1)$
    - work per subproblem: $O(1)$
    - $O(|A| \cdot |B|)$ running time

```
def lcs(A, B):
    a, b = len(A), len(B)
    x = [[0] * (b + 1) for _ in range(a + 1)]
    for i in reversed(range(a)):
        for j in reversed(range(b)):
            if A[i] == B[j]:
                x[i][j] = x[i + 1][j + 1] + 1
            else:
                x[i][j] = max(x[i + 1][j], x[i][j + 1])
    return x[0][0]
```

Longest Increasing Subsequence (LIS)

- Given a string A, find a longest (not necessarily contiguous) subsequence of A that strictly increases (lexicographically).
- Example: $A =$ `carbohydrate`
- Solution: `abort`, of length 5
- Maximization problem on length of subsequence
- Attempted solution:
  - Natural subproblems are prefixes or suffixes of A, say suffix $A[i:]$
  - Natural question about LIS of $A[i:]$: is $A[i]$ in the LIS? (2 possible answers)
  - But then how do we recurse on $A[i+1:]$ and guarantee increasing subsequence?
  - Fix: add **constraint** to subproblems to give enough structure to achieve increasing property

1. **Subproblems**
   - $\boxed{x(i) = \text{length of longest increasing subsequence of suffix } A[i:] \text{ that includes } A[i]}$
   - For $0 \leq i \leq |A|$
2. **Relate**
   - We're told that $A[i]$ is in LIS (first element)
   - Next question: what is the *second* element of LIS?
     - Could be any $A[j]$ where $j > i$ and $A[j] > A[i]$ (so increasing)
     - Or $A[i]$ might be the *last* element of LIS
   - $x(i) = \max\{1 + x(j) \mid i < j < |A|, A[j] > A[i]\} \cup \{1\}$
3. **Topological order**
   - Decreasing i
4. **Base**
   - No base case necessary, because we consider the possibility that $A[i]$ is last
5. **Original problem**
   - What is the first element of LIS? **Guess!**
   - Length of LIS of A is $\max\{x(i) \mid 0 \leq i < |A|\}$
   - Store parent pointers to reconstruct subsequence

6. **Time**
    - # subproblems: $|A|$
    - work per subproblem: $O(|A|)$
    - $O(|A|^2)$ running time
    - Exercise: speed up to $O(|A| \log |A|)$ by doing only $O(\log |A|)$ work per subproblem, via AVL tree augmentation

```
def lis(A):
    a = len(A)
    x = [1] * a
    for i in reversed(range(a)):
        for j in range(i, a):
            if A[j] > A[i]:
                x[i] = max(x[i], 1 + x[j])
    return max(x)
```

Alternating Coin Game

- Given sequence of n coins of value $v_0, v_1, \ldots, v_{n-1}$
- Two players ("me" and "you") take turns
- In a turn, take first or last coin among remaining coins
- My goal is to maximize total value of my taken coins, where I go first
- First solution exploits that this is a **zero-sum game**: I take all coins you don't

1. **Subproblems**
   - Choose subproblems that correspond to the state of the game
   - For every contiguous subsequence of coins from i to j, $0 \leq i \leq j < n$
   - $\boxed{x(i,j) = \text{maximum total value I can take starting from coins of values } v_i, \ldots, v_j}$
2. **Relate**
   - I must choose either coin i or coin j **(Guess!)**
   - Then it's your turn, so you'll get value $x(i+1,j)$ or $x(i,j-1)$, respectively
   - To figure out how much value I get, subtract this from total coin values
   - $x(i,j) = \max\{v_i + \sum_{k=i+1}^{j} v_k - x(i+1,j), v_j + \sum_{k=i}^{j-1} v_k - x(i,j-1)\}$
3. **Topological order**
   - Increasing $j-i$
4. **Base**
   - $x(i,i) = v_i$
5. **Original problem**
   - $x(0,n-1)$
   - Store parent pointers to reconstruct strategy
6. **Time**
   - # subproblems: $\Theta(n^2)$
   - work per subproblem: $\Theta(n)$ to compute sums
   - $\Theta(n^3)$ running time
   - Exercise: speed up to $\Theta(n^2)$ time by precomputing all sums $\sum_{k=i}^{j} v_k$ in $\Theta(n^2)$ time, via dynamic programming (!)

- Second solution uses **subproblem expansion**: add subproblems for when you move next

1. **Subproblems**
   - Choose subproblems that correspond to the full state of the game
   - Contiguous subsequence of coins from i to j, and which player p goes next
   - $x(i, j, p)$ = maximum total value I can take when player $p \in \{\text{me}, \text{you}\}$ starts from coins of values $v_i, \ldots, v_j$
2. **Relate**
   - Player p must choose either coin i or coin j **(Guess!)**
   - If $p =$ me, then I get the value; otherwise, I get nothing
   - Then it's the other player's turn
   - $x(i, j, \text{me}) = \max\{v_i + x(i+1, j, \text{you}), v_j + x(i, j \quad 1, \text{you})\}$
   - $x(i, j, \text{you}) = \min\{x(i+1, j, \text{me}), x(i, j \quad 1, \text{me})\}$
3. **Topological order**
   - Increasing $j \quad i$
4. **Base**
   - $x(i, i, \text{me}) = v_i$
   - $x(i, i, \text{you}) = 0$
5. **Original problem**
   - $x(0, n \quad 1, \text{me})$
   - Store parent pointers to reconstruct strategy
6. **Time**
   - # subproblems: $\Theta(n^2)$
   - work per subproblem: $\Theta(1)$
   - $\Theta(n^2)$ running time

Subproblem Constraints and Expansion

- We've now seen two examples of constraining or expanding subproblems
- If you find yourself lacking information to check the desired conditions of the problem, or lack the natural subproblem to recurse on, try subproblem constraint/expansion!
- More subproblems and constraints give the relation more to work with, so can make DP more feasible
- Usually a trade-off between number of subproblems and branching/complexity of relation
- More examples next lecture

6.006 Introduction to Algorithms

Spring 2020

Lecture 17: Dyn. Prog. III

Dynamic Programming Steps (SRT BOT)

1. **Subproblem** definition subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often multiply possible subsets across multiple inputs
   - Often record partial state: add subproblems by incrementing some auxiliary variables
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
   - Identify a question about a subproblem solution that, if you knew the answer to, reduces the subproblem to smaller subproblem(s)
   - Locally brute-force all possible answers to the question
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

---

Recall: DAG Shortest Paths [L15]

- **S**ubproblems: $\delta(s, v)$ for all $v \in V$
- **R**elation: $\delta(s, v) = \min\{\delta(s, u) + w(u, v) \mid u \in \text{Adj}^-(v)\} \cup \{\infty\}$
- **T**opo. order: Topological order of G

Single-Source Shortest Paths Revisited

1. **Subproblems**
   - Expand subproblems to add information to make acyclic!
     (an example we've already seen of subproblem expansion)
   - $\boxed{\delta_k(s,v) = \text{weight of shortest path from } s \text{ to } v \text{ using } \textit{at most } k \text{ edges}}$
   - For $v \in V$ and $0 \leq k \leq |V|$
2. **Relate**
   - Guess last edge (u, v) on shortest path from s to v
   - $\delta_k(s,v) = \min\{\delta_{k-1}(s,u) + w(u,v) \mid (u,v) \in E\} \cup \{\delta_{k-1}(s,v)\}$
3. **Topological order**
   - Increasing k: subproblems depend on subproblems only with strictly smaller k
4. **Base**
   - $\delta_0(s,s) = 0$ and $\delta_0(s,v) = \infty$ for $v \neq s$ (no edges)
   - (draw subproblem graph)
5. **Original problem**
   - If has finite shortest path, then $\delta(s,v) = \delta_{|V|-1}(s,v)$
   - Otherwise some $\delta_{|V|}(s,v) < \delta_{|V|-1}(s,v)$, so path contains a negative-weight cycle
   - Can keep track of parent pointers to subproblem that minimized recurrence
6. **Time**
   - # subproblems: $|V| \times (|V| + 1)$
   - Work for subproblem $\delta_k(s,v)$: $O(\deg_{\text{in}}(v))$

$$\sum_{k=0}^{|V|} \sum_{v \in V} O(\deg_{\text{in}}(v)) = \sum_{k=0}^{|V|} O(|E|) = O(|V| \cdot |E|)$$

This is just **Bellman-Ford**! (computed in a slightly different order)

All-Pairs Shortest Paths: Floyd–Warshall

- Could define subproblems $\delta_k(u, v) =$ minimum weight of path from u to v using at most k edges, as in Bellman–Ford
- Resulting running time is $|V|$ times Bellman–Ford, i.e., $O(|V|^2 \cdot |E|) = O(|V|^4)$
- Know a better algorithm from L14: Johnson achieves $O(|V|^2 \log |V| + |V| \cdot |E|) = O(|V|^3)$
- Can achieve $\Theta(|V|^3)$ running time (matching Johnson for dense graphs) with a simple dynamic program, called **Floyd–Warshall**
- Number vertices so that $V = \{1, 2, \ldots, |V|\}$

1. **Subproblems**
   - $d(u, v, k) =$ minimum weight of a path from u to v that only uses vertices from $\{1, 2, \ldots, k\} \cup \{u, v\}$
   - For $u, v \in V$ and $1 \leq k \leq |V|$
2. **Relate**
   - $x(u, v, k) = \min\{x(u, k, k-1) + x(k, v, k-1), x(u, v, k-1)\}$
   - Only constant branching! No longer guessing previous vertex/edge
3. **Topological order**
   - Increasing k: relation depends only on smaller k
4. **Base**
   - $x(u, u, 0) = 0$
   - $x(u, v, 0) = w(u, v)$ if $(u, v) \in E$
   - $x(u, v, 0) = \infty$ if none of the above
5. **Original problem**
   - $x(u, v, |V|)$ for all $u, v \in V$
6. **Time**
   - $O(|V|^3)$ subproblems
   - Each $O(1)$ work
   - $O(|V|^3)$ in total
   - Constant number of dependencies per subproblem brings the factor of $O(|E|)$ in the running time down to $O(|V|)$.

Arithmetic Parenthesization

- Input: arithmetic expression $a_0 *_1 a_1 *_2 a_2 \cdots *_{n-1} a_{n-1}$
  where each a_i is an integer and each $*_i \in \{+, \times\}$
- Output: Where to place parentheses to maximize the evaluated expression
- Example: $7 + 4 \times 3 + 5 \to ((7) + (4)) \times ((3) + (5)) = 88$
- Allow **negative** integers!
- Example: $7 + (-4) \times 3 + (-5) \to ((7) + ((-4) \times ((3) + (-5)))) = 15$

1. **Subproblems**
   - Sufficient to maximize each subarray? No! $(-3) \times (-3) = 9 > (-2) \times (-2) = 4$
   - $\boxed{x(i, j, \text{opt}) = \text{opt value obtainable by parenthesizing } a_i *_{i+1} \cdots *_{j-1} a_{j-1}}$
   - For $0 \le i < j \le n$ and $\text{opt} \in \{\min, \max\}$
2. **Relate**
   - Guess location of outermost parentheses / last operation evaluated
   - $x(i, j, \text{opt}) = \text{opt}\,\{x(i, k, \text{opt}') *_k x(k, j, \text{opt}'')) \mid i < k < j; \text{opt}', \text{opt}'' \in \{\min, \max\}\}$
3. **Topological order**
   - Increasing $j - i$: subproblem $x(i, j, \text{opt})$ depends only on strictly smaller $j - i$
4. **Base**
   - $x(i, i + 1, \text{opt}) = a_i$, only one number, no operations left!
5. **Original problem**
   - $X(0, n, \max)$
   - Store parent pointers (two!) to find parenthesization (forms binary tree!)
6. **Time**
   - # subproblems: less than $n \cdot n \cdot 2 = O(n^2)$
   - work per subproblem $O(n) \cdot 2 \cdot 2 = O(n)$
   - $O(n^3)$ running time

Piano Fingering

- Given sequence $t_0, t_1, \ldots, t_{n-1}$ of n **single** notes to play with right hand (will generalize to multiple notes and hands later)
- Performer has right-hand fingers $1, 2, \ldots, F$ ($F = 5$ for most humans)
- Given metric $d(t, f, t', f')$ of **difficulty** of transitioning from note t with finger f to note t' with finger f'
  - Typically a sum of penalties for various difficulties, e.g.:
  - $1 < f < f'$ and $t > t'$ is uncomfortable
  - Legato (smooth) play requires $t \neq t'$ (else infinite penalty)
  - Weak-finger rule: prefer to avoid $f' \in \{4, 5\}$
  - $\{f, f'\} = \{3, 4\}$ is annoying
- Goal: Assign fingers to notes to minimize total difficulty
- First attempt:

1. **Subproblems**
   - $x(i) =$ minimum total difficulty for playing notes $t_i, t_{i+1}, \ldots, t_{n-1}$
2. **Relate**
   - Guess first finger: assignment f for t_i
   - $x(i) = \min\{x(i+1) + d(t_i, f, t_{i+1}, ?) \mid 1 \leq f \leq F\}$
   - Not enough information to fill in $?$
   - Need to know which finger at the start of $x(i+1)$
   - But different starting fingers could hurt/help both $x(i+1)$ and $d(t_i, f, t_{i+1}, ?)$
   - Need a table mapping start fingers to optimal solutions for $x(i+1)$
   - I.e., need to expand subproblems with start condition

- Solution:

1. **Subproblems**
   - $x(i, f) =$ minimum total difficulty for playing notes $t_i, t_{i+1}, \ldots, t_{n-1}$ starting with finger f on note t_i
   - For $0 \leq i < n$ and $1 \leq f \leq F$
2. **Relate**
   - Guess next finger: assignment f' for t_{i+1}
   - $x(i, f) = \min\{x(i+1, f') + d(t_i, f, t_{i+1}, f') \mid 1 \leq f' \leq F\}$
3. **Topological order**
   - Decreasing i (any f order)
4. **Base**
   - $x(n-1, f) = 0$ (no transitions)
5. **Original problem**
   - $\min\{x(0, f) \mid 1 \leq f \leq F\}$
6. **Time**
   - $\Theta(n \cdot F)$ subproblems
   - $\Theta(F)$ work per subproblem
   - $\Theta(n \cdot F^2)$
   - No dependence on the number of different notes!

Guitar Fingering

- Up to $S =$ number of strings different ways to play the same note
- Redefine "finger" to be tuple (finger playing note, string playing note)
- Throughout algorithm, F gets replaced by $F \cdot S$
- Running time is thus $\Theta(n \cdot F^2 \cdot S^2)$

Multiple Notes at Once

- Now suppose t_i is a set of notes to play at time i
- Given a bigger transition difficulty function $d(t, f, t', f')$
- Goal: fingering $f_i : t_i \to \{1, 2, \ldots, F\}$ specifying how to finger each note (including which string for guitar) to minimize $\sum_{i=1}^{n-1} d(t_{i-1}, f_{i-1}, t_i, f_i)$
- At most T^F choices for each fingering f_i, where $T = \max_i |t_i|$
  - $T \leq F = 10$ for normal piano (but there are exceptions)
  - $T \leq S$ for guitar
- $\Theta(n \cdot T^F)$ subproblems
- $\Theta(T^F)$ work per subproblem
- $\Theta(n \cdot T^{2F})$ time
- $\Theta(n)$ time for $T, F \leq 10$

Video Game Appliactions

- Guitar Hero / Rock Band
  - $F = 4$ (and 5 different notes)
- Dance Dance Revolution
  - $F = 2$ feet
  - $T = 2$ (at most two notes at once)
  - Exercise: handle sustained notes, using "where each foot is" (on an arrow or in the middle) as added state for suffix subproblems

6.006 Introduction to Algorithms

Spring 2020

Lecture 18: Pseudopolynomial

Dynamic Programming Steps (SRT BOT)

1. **Subproblem** definition subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often multiply possible subsets across multiple inputs
   - Often record partial state: add subproblems by incrementing some auxiliary variables
   - Often smaller integers than a given integer (**today's focus**)
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
   - Identify a question about a subproblem solution that, if you knew the answer to, reduces the subproblem to smaller subproblem(s)
   - Locally brute-force all possible answers to the question
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

Rod Cutting

- Given a rod of length L and value $v(\ell)$ of rod of length ℓ for all $\ell \in \{1, 2, \ldots, L\}$
- Goal: Cut the rod to maximize the value of cut rod pieces
- Example: $L = 7$, $v \underset{\ell =}{} [\underset{0}{0}, \underset{1}{1}, \underset{2}{10}, \underset{3}{13}, \underset{4}{18}, \underset{5}{20}, \underset{6}{31}, \underset{7}{32}]$
- Maybe greedily take most valuable per unit length?
- Nope! $\arg\max_\ell v[\ell]/\ell = 6$, and partitioning $[6, 1]$ yields 32 which is not optimal!
- Solution: $v[2] + v[2] + v[3] = 10 + 10 + 13 = 33$
- Maximization problem on value of partition

1. **Subproblems**
   - $\boxed{x(\ell)\text{: maximum value obtainable by cutting rod of length } \ell}$
   - For $\ell \in \{0, 1, \ldots, L\}$
2. **Relate**
   - First piece has some length p **(Guess!)**
   - $x(\ell) = \max\{v(p) + x(\ell \quad p) \mid p \in \{1, \ldots, \ell\}\}$
   - (draw dependency graph)
3. **Topological order**
   - Increasing ℓ: Subproblems $x(\ell)$ depend only on strictly smaller ℓ, so acyclic
4. **Base**
   - $x(0) = 0$ (length-zero rod has no value!)
5. **Original problem**
   - Maximum value obtainable by cutting rod of length L is $x(L)$
   - Store choices to reconstruct cuts
   - If current rod length ℓ and optimal choice is ℓ', remainder is piece $p = \ell \quad \ell'$
   - (maximum-weight path in subproblem DAG!)
6. **Time**
   - # subproblems: $L + 1$
   - work per subproblem: $O(\ell) = O(L)$
   - $O(L^2)$ running time

Is This Polynomial Time?

- **(Strongly) polynomial time** means that the running time is bounded above by a constant-degree polynomial in the **input size** measured in words
- In Rod Cutting, input size is $L + 1$ words (one integer L and L integers in v)
- $O(L^2)$ is a constant-degree polynomial in $L + 1$, so YES: (strongly) polynomial time

```
# recursive
x = {}
def cut_rod(l, v):
    if l < 1:   return 0                              # base case
    if l not in x:                                    # check memo
        for piece in range(1, l + 1):                 # try piece
            x_ = v[piece] + cut_rod(l - piece, v)     # recurrence
            if (l not in x) or (x[l] < x_):           # update memo
                x[l] = x_
    return x[l]
```

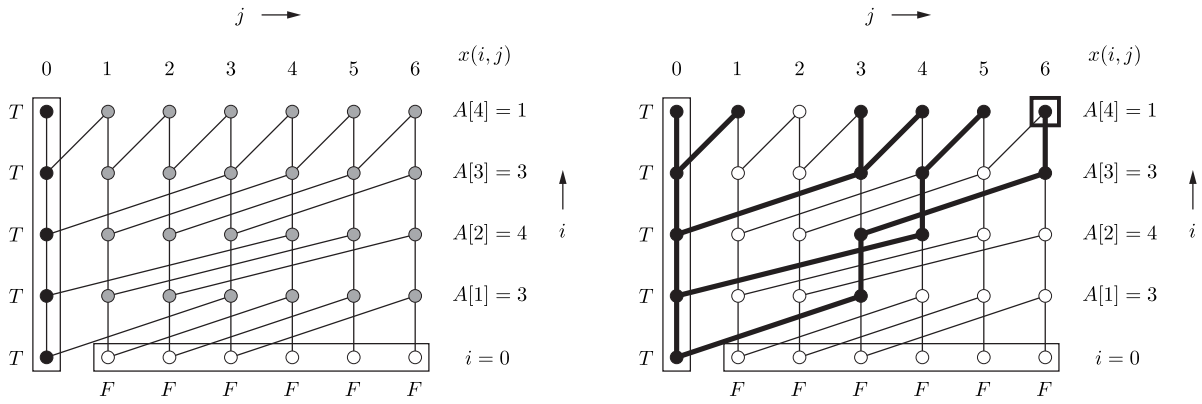
```
# iterative
def cut_rod(L, v):
    x = [0] * (L + 1)                                 # base case
    for l in range(L + 1):                            # topological order
        for piece in range(1, l + 1):                 # try piece
            x_ = v[piece] + x[l - piece]              # recurrence
            if x[l] < x_:                             # update memo
                x[l] = x_
    return x[L]
```

```
# iterative with parent pointers
def cut_rod_pieces(L, v):
    x = [0] * (L + 1)                                 # base case
    parent = [None] * (L + 1)                         # parent pointers
    for l in range(1, L + 1):                         # topological order
        for piece in range(1, l + 1):                 # try piece
            x_ = v[piece] + x[l - piece]              # recurrence
            if x[l] < x_:                             # update memo
                x[l] = x_
                parent[l] = l - piece                 # update parent
    l, pieces = L, []
    while parent[l] is not None:                      # walk back through parents
        piece = l - parent[l]
        pieces.append(piece)
        l = parent[l]
    return pieces
```

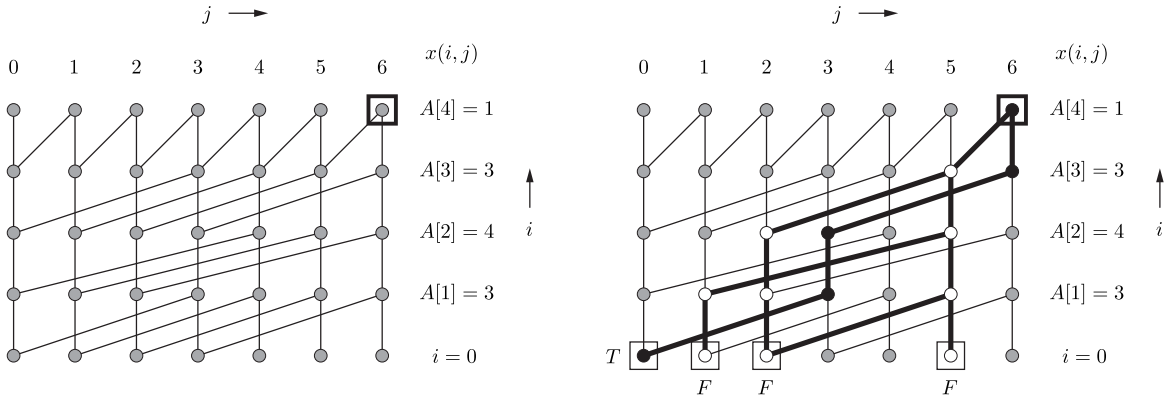
Subset Sum

- Input: Sequence of n positive integers $A = \{a_0, a_1, \ldots, a_n\}$
- Output: Is there a subset of A that sums exactly to T? (i.e., $\exists A' \subseteq A$ s.t. $\sum_{a \in A'} a = T$?)
- Example: $A = (1, 3, 4, 12, 19, 21, 22)$, $T = 47$ allows $A' = \{3, 4, 19, 21\}$
- Optimization problem? Decision problem! Answer is YES or NO, TRUE or FALSE
- In example, answer is YES. However, answer is NO for some T, e.g., $2, 6, 9, 10, 11, \ldots$

1. **Subproblems**
   - $\boxed{x(i,t) = \text{does any subset of } A[i:] \text{ sum to } t?}$
   - For $i \in \{0, 1, \ldots, n\}, t \in \{0, 1, \ldots, T\}$
2. **Relate**
   - Idea: Is first item a_i in a valid subset A'? (Guess!)
   - If yes, then try to sum to t a_i 0 using remaining items
   - If no, then try to sum to t using remaining items
   - $x(i,t) = \text{OR} \begin{cases} x(i+1, t \quad A[i]) & \text{if } t \quad A[i] \\ x(i+1, t) & \text{always} \end{cases}$
3. **Topological order**
   - Subproblems $x(i,t)$ only depend on strictly larger i, so acyclic
   - Solve in order of decreasing i
4. **Base**
   - $x(i, 0) =$ YES for $i \in \{0, \ldots, n\}$ (space packed exactly!)
   - $x(0, t) =$ NO for $j \in \{1, \ldots, T\}$ (no more items available to pack)
5. **Original problem**
   - Original problem given by $x(0, T)$
   - Example: $A = (3, 4, 3, 1), T = 6$ solution: $A' = (3, 3)$
   - Bottom up: Solve all subproblems (Example has 35)



- Top down: Solve only **reachable** subproblems (Example, only 14!)



6. **Time**

- # subproblems: $O(nT)$, $O(1)$ work per subproblem, $O(nT)$ time

Is This Polynomial?

- Input size is $n + 1$: one integer T and n integers in A
- Is $O(nT)$ bounded above by a polynomial in $n + 1$? NO, not necessarily
- On w-bit word RAM, $T \leq 2^w$ and $w \quad \lg(n + 1)$, but we don't have an upper bound on w
- E.g., $w = n$ is not unreasonable, but then running time is $O(n2^n)$, which is **exponential**

Pseudopolynomial

- Algorithm has **pseudopolynomial time**: running time is bounded above by a constant-degree polynomial in input size and input integers
- Such algorithms are polynomial in the case that integers are polynomially bounded in input size, i.e., $n^{O(1)}$ (same case that Radix Sort runs in $O(n)$ time)
- Counting sort $O(n + u)$, radix sort $O(n \log_n u)$, direct-access array build $O(n + u)$, and Fibonacci $O(n)$ are all pseudopolynomial algorithms we've seen already
- Radix sort is actually **weakly polynomial** (a notion in between strongly polynomial and pseudopolynomial): bounded above by a constant-degree polynomial in the input size measured in bits, i.e., in the logarithm of the input integers
- Contrast with Rod Cutting, which was polynomial
  - Had pseudopolynomial dependence on L
  - But luckily had $\quad L$ input integers too
  - If only given subset of sellable rod lengths (Knapsack Problem, which generalizes Rod Cutting and Subset Sum — see recitation), then algorithm would have been only pseudopolynomial

Complexity

- Is Subset Sum solvable in polynomial time when integers are not polynomially bounded?
- No if P $\neq$ NP. What does that mean? Next lecture!

Main Features of Dynamic Programs

- Review of examples from lecture
- **Subproblems:**
  - **Prefix/suffixes:** Bowling, LCS, LIS, Floyd–Warshall, Rod Cutting (coincidentally, really Integer subproblems), Subset Sum
  - **Substrings:** Alternating Coin Game, Arithmetic Parenthesization
  - **Multiple sequences:** LCS
  - **Integers:** Fibonacci, Rod Cutting, Subset Sum
    - **Pseudopolynomial:** Fibonacci, Subset Sum
  - **Vertices:** DAG shortest paths, Bellman–Ford, Floyd–Warshall
- **Subproblem constraints/expansion:**
  - **Nonexpansive constraint:** LIS (include first item)
  - $\mathbf{2}\times$ **expansion:** Alternating Coin Game (who goes first?), Arithmetic Parenthesization (min/max)
  - $\boldsymbol{\Theta(1)}\times$ **expansion:** Piano Fingering (first finger assignment)
  - $\boldsymbol{\Theta(n)}\times$ **expansion:** Bellman–Ford (# edges)
- **Relation:**
  - **Branching** = # dependant subproblems in each subproblem
  - $\boldsymbol{\Theta(1)}$ **branching:** Fibonacci, Bowling, LCS, Alternating Coin Game, Floyd–Warshall, Subset Sum
  - $\boldsymbol{\Theta(\mathrm{degree})}$ **branching** (source of $|E|$ in running time): DAG shortest paths, Bellman–Ford
  - $\boldsymbol{\Theta(n)}$ **branching:** LIS, Arithmetic Parenthesization, Rod Cutting
  - **Combine multiple solutions (not path in subproblem DAG):** Fibonacci, Floyd–Warshall, Arithmetic Parenthesization
- **Original problem:**
  - **Combine multiple subproblems:** DAG shortest paths, Bellman–Ford, Floyd–Warshall, LIS, Piano Fingering

6.006 Introduction to Algorithms

Spring 2020

Lecture 19: Complexity

Decision Problems

- **Decision problem**: assignment of inputs to YES (1) or NO (0)
- Inputs are either **NO inputs** or **YES inputs**

Problem	Decision
s-t Shortest Path	Does a given G contain a path from s to t with weight at most d?
Negative Cycle	Does a given G contain a negative weight cycle?
Longest Simple Path	Does a given G contain a **simple** path with weight at least d?
Subset Sum	Does a given set of integers A contain a subset with sum S?
Tetris	Can you survive a given sequence of pieces in given board?
Chess	Can a player force a win from a given board?
Halting problem	Does a given computer program terminate for a given input?

- **Algorithm/Program**: constant-length code (working on a word-RAM with $\Omega(\log n)$-bit words) to solve a problem, i.e., it produces correct output for every input and the length of the code is independent of the instance size
- Problem is **decidable** if there exists a program to solve the problem in finite time

Decidability

- Program is finite (constant) string of bits, i.e., a nonnegative integer $\in \mathbb{N}$.
  Problem is function $p : \mathbb{N} \to \{0, 1\}$, i.e., infinite string of bits.
- (# of programs $|\mathbb{N}|$, countably infinite) $\ll$ (# of problems $|\mathbb{R}|$, uncountably infinite)
- (Proof by Cantor's diagonalization argument, probably covered in 6.042)
- Proves that most decision problems not solvable by any program (undecidable)
- E.g., the Halting problem is undecidable (many awesome proofs in 6.045)
- Fortunately most problems we think of are algorithmic in structure and are decidable

Decidable Decision Problems

R	problems decidable in finite time	('R' comes from recursive languages)
EXP	problems decidable in exponential time $2^{n^{O(1)}}$	(most problems we think of are here)
P	problems decidable in polynomial time $n^{O(1)}$	(efficient algorithms, the focus of this class)

- These sets are distinct, i.e., **P** $\subsetneq$ **EXP** $\subsetneq$ **R** (via time hierarchy theorems, see 6.045)
- E.g., Chess is in **EXP** $\setminus$ **P**

Nondeterministic Polynomial Time (NP)

- **P** is the set of decision problems for which there is an algorithm A such that, for every input I of size n, A on I runs in poly(n) time and solves I correctly
- **NP** is the set of decision problems for which there is a **verification** algorithm V that takes as input an input I of the problem and a **certificate** bit string of length polynomial in the size of I, so that:
  - V always runs in time polynomial in the size of I;
  - if I is a YES input, then there is some certificate c so that V outputs YES on input (I, c); and
  - if I is a NO input, then no matter what certificate c we choose, V always output NO on input (I, c).
- You can think of the certificate as a **proof** that I is a YES input.
  If I is actually a NO input, then no proof should work.

Problem	Certificate	Verifier
s-t Shortest Path	A path P from s to t	Adds the weights on P and checks whether $\leq d$
Negative Cycle	A cycle C	Adds the weights on C and checks whether < 0
Longest Simple Path	A path P	Checks whether P is a **simple** path with weight $\geq d$
Subset Sum	A set of items A'	Checks whether $A' \in A$ has sum S
Tetris	Sequence of moves	Checks that the moves allow survival

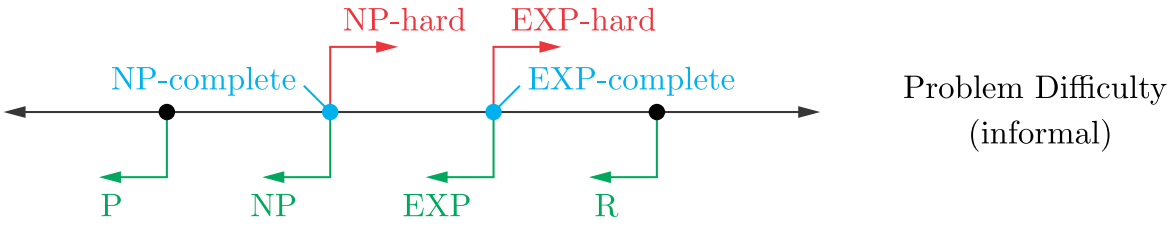
- **P** $\subseteq$ **NP**: The verifier V just solves the instance ignoring any certificate
- **NP** $\subseteq$ **EXP**: Try all possible certificates! At most $2^{n^{O(1)}}$ of them, run verifier V on all
- **Open:** Does **P** $=$ **NP**? **NP** $=$ **EXP**?
- Most people think **P** $\subsetneq$ **NP** ($\subsetneq$ **EXP**), i.e., generating solutions harder than checking
- If you prove either way, people will give you lots of money ($1M Millennium Prize)
- Why do we care? If can show a problem is hardest problem in **NP**,
  then problem cannot be solved in polynomial time if **P** $\neq$ **NP**
- How do we relate difficulty of problems? Reductions!

Reductions

- Suppose you want to solve problem A
- One way to solve is to convert A into a problem B you know how to solve
- Solve using an algorithm for B and use it to compute solution to A
- This is called a **reduction** from problem A to problem B ($A \rightarrow B$)
- Because B can be used to solve A, B is **at least as hard** as A ($A \leq B$)
- General algorithmic strategy: reduce to a problem you know how to solve

A	Conversion	B
Unweighted Shortest Path	Give equal weights	Weighted Shortest Path
Integer-weighted Shortest Path	Subdivide edges	Unweighted Shortest Path
Longest Path	Negate weights	Shortest Path

- Problem A is **NP-hard** if every problem in **NP** is polynomially reducible to A
- i.e., A is at least as hard as (can be used to solve) every problem in **NP** ($X \leq A$ for $X \in$ **NP**)
- **NP-complete = NP** $\cap$ **NP-hard**
- All **NP-complete** problems are equivalent, i.e., reducible to each other
- First **NP-complete** problem? Every decision problem reducible to satisfying a logical circuit, a problem called "Circuit SAT".
- Longest Simple Path and Tetris are **NP-complete**, so if any problem is in **NP** $\setminus$ **P**, these are
- Chess is **EXP-complete**: in **EXP** and reducible from every problem in **EXP** (so $\notin$ **P**)



Examples of NP-complete Problems

- Subset Sum from L18 ("weakly NP-complete" which is what allows a pseudopolynomial-time algorithm, but no polynomial algorithm unless **P** = **NP**)
- 3-Partition: given n integers, can you divide them into triples of equal sum? ("strongly NP-complete": no pseudopolynomial-time algorithm unless **P** = **NP**)
- Rectangle Packing: given n rectangles and a target rectangle whose area is the sum of the n rectangle areas, pack without overlap
  - Reduction from 3-Partition to Rectangle Packing: transform integer a_i into $1 \times a_i$ rectangle; set target rectangle to $n/3 \times \left(\sum_i a_i\right)/3$
- Jigsaw puzzles: given n pieces with possibly ambiguous tabs/pockets, fit the pieces together
  - Reduction from Rectangle Packing: use uniquely matching tabs/pockets to force building rectangles and rectangular boundary; use one ambiguous tab/pocket for all other boundaries
- Longest common subsequence of n strings
- Longest simple path in a graph
- Traveling Salesman Problem: shortest path that visits all vertices of a given graph (or decision version: is minimum weight $\leq d$)
- Shortest path amidst obstacles in 3D
- 3-coloring given graph (but 2-coloring $\in$ **P**)
- Largest clique in a given graph
- SAT: given a Boolean formula (made with AND, OR, NOT), is it every true?
  E.g., x AND NOT x is a NO input
- Minesweeper, Sudoku, and most puzzles
- Super Mario Bros., Legend of Zelda, Pokémon, and most video games are **NP-hard** (many are harder)

6.006 Introduction to Algorithms

Spring 2020

Lecture 20: Course Review

6.006: Introduction to Algorithms

- Goals:
  1. Solve hard computational problems (with **non-constant-sized inputs**)
  2. Argue an algorithm is **correct** (Induction, Recursion)
  3. Argue an algorithm is **"good"** (Asymptotics, Model of Computation)

  – (effectively communicate all three above, to human or computer)
- Do there always exist "good" algorithms?
  - Most problems are not solvable efficiently, but many we think of are!
  - **Polynomial** means polynomial in size of input
  - **Pseudopolynomial** means polynomial in size of input AND size of numbers in input
  - NP: **Nondeterministic Polynomial** time, polynomially checkable certificates
  - NP-hard: set of problems that can be used to solve any problem in NP in poly-time
  - NP-complete: intersection of NP-hard and NP

How to solve an algorithms problem?

- Reduce to a **problem** you know how to solve
  - Search/Sort (Q1)
    - Search: Extrinsic (Sequence) and Intrinsic (Set) Data Structures
    - Sort: Comparison Model, Stability, In-place
  - Graphs (Q2)
    - Reachability, Connected Components, Cycle Detection, Topological Sort
    - Single-Source / All-Pairs Shortest Paths
- Design a new recursive algorithm
  - Brute Force
  - Divide & Conquer
  - Dynamic Programming (Q3)
  - Greedy/Incremental

Next Steps

- (U) 6.046: Design & Analysis of Algorithms
- (G) 6.851: Advanced Data Structures
- (G) 6.854: Advanced Algorithms

6.046

- Extension of 6.006
  - **Data Structures:** Union-Find, Amortization via potential analysis
  - **Graphs:** Minimum Spanning Trees, Network Flows/Cuts
  - **Algorithm Design (Paradigms):** Divide & Conquer, Dynamic Programming, Greedy
  - **Complexity:** Reductions
- Relax Problem (change definition of correct/efficient)
  - **Randomized Algorithms**
    - 6.006 mostly deterministic (hashing)
    - Las Vegas: always correct, probably fast (like hashing)
    - Monte Carlo: always fast, probably correct
    - Can generally get faster randomized algorithms on structured data
  - **Numerical Algorithms/Continuous Optimization**
    - 6.006 only deals with integers
    - Approximate real numbers! Pay time for precision
  - **Approximation Algorithms**
    - Input optimization problem (min/max over weighted outputs)
    - Many optimization problems NP-hard
    - How close can we get to an optimal solution in polynomial time?
- Change Model of Computation
  - Cache Models (memory hierarchy cost model)
  - Quantum Computer (exploiting quantum properties)
  - Parallel Processors (use multiple CPUs instead of just one)
    - Multicore, large shared memory
    - Distributed cores, message passing

Future Courses

Model

- Computation / Complexity (6.045, 6.840, 6.841)
- Randomness (6.842)
- Quantum (6.845)
- Distributed / message passing (6.852)
- Multicore / shared memory (6.816, 6.846)
- Graph and Matrix (6.890)
- Constant Factors / Performance (6.172)

Application

- Biology (6.047)
- Game Theory (6.853)
- Cryptography (6.875)
- Vision (6.819)
- Graphics (6.837)
- Geometry (6.850)
- Folding (6.849)

6.006 Introduction to Algorithms

Spring 2020