Facial Angle Calculation

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Topics

❶ Explain and describe the concept of “facial angle" that the code, provided on GitHub, aims to calculate.

❷ Present the mathematical proof upon which the calculation is based.

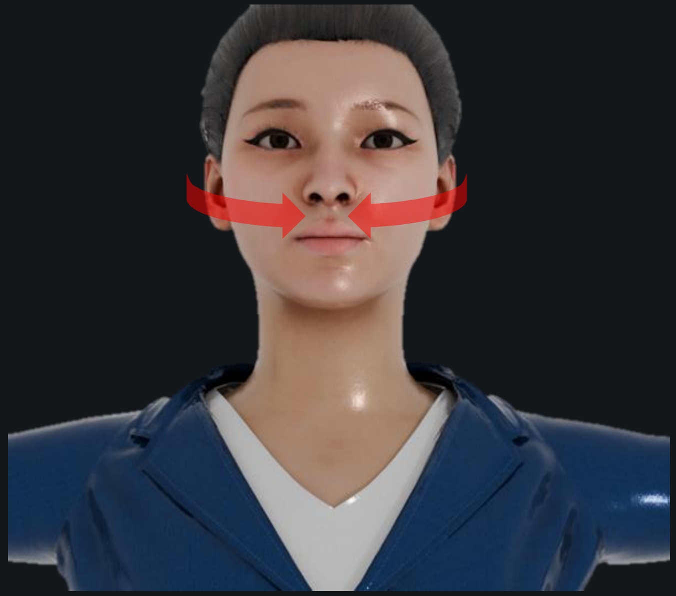
Facial angle - Definition

What is it “facial angle”?

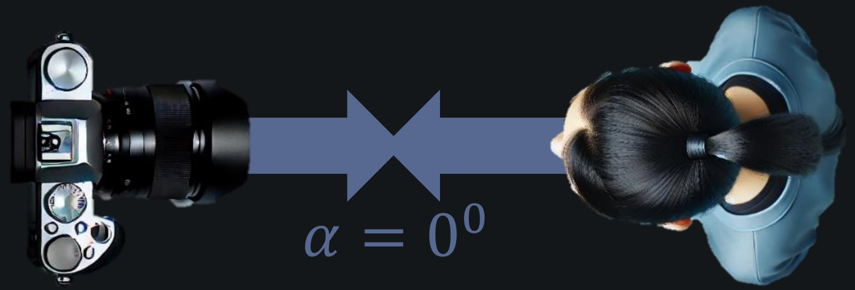
Shortly speaking, 'facial angle' in our context is defined as:

The angle between the camera's direction and the direction of the face

What is it "facial angle"?



Input Image

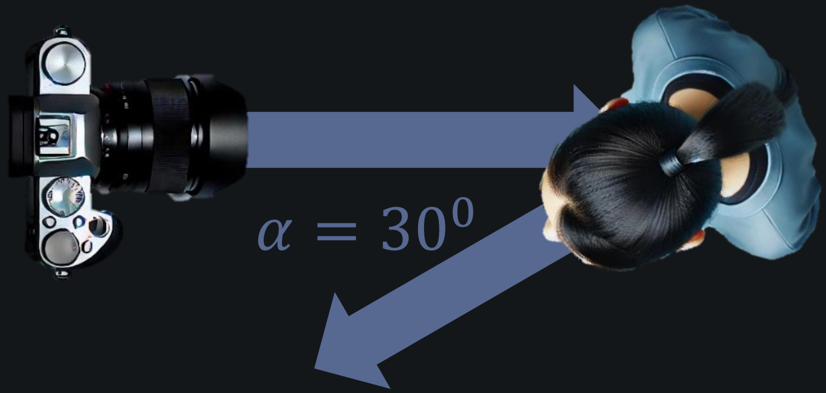


Top-Down View

In this situation, the face angle is said to be 0 degrees.

What is it "facial angle"?

Input Image



Top-Down View

In this situation, the face angle is said to be 30 degrees.

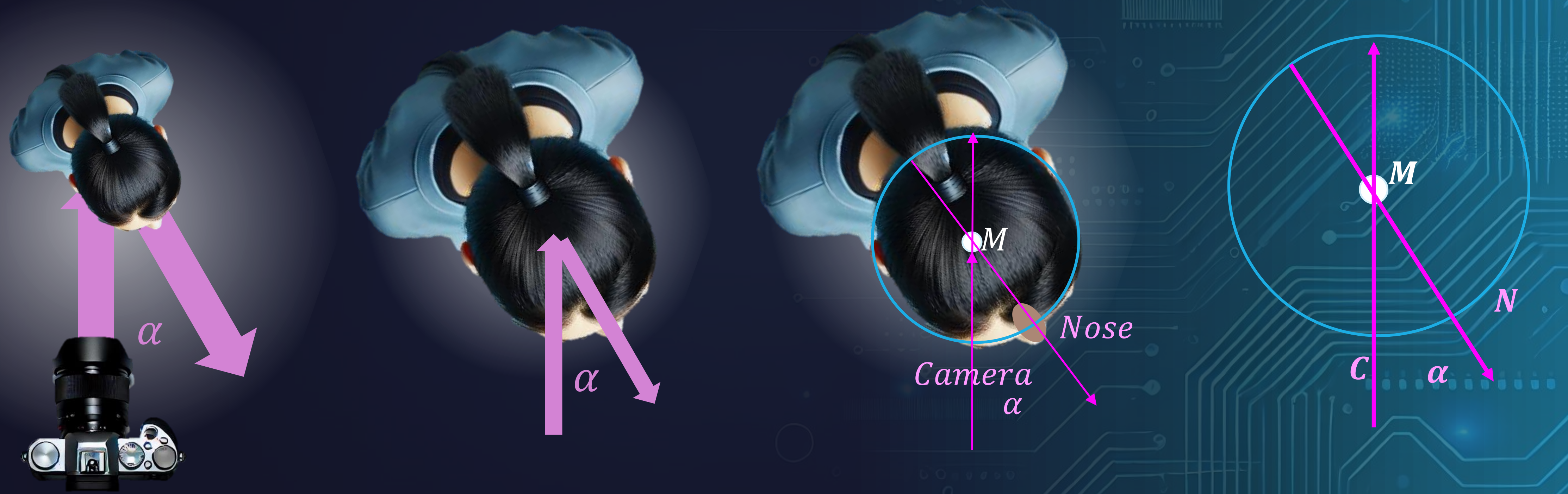
Concept & Proof

Concept & Proof

The next slides will represent two definitions that we will use in the proof:

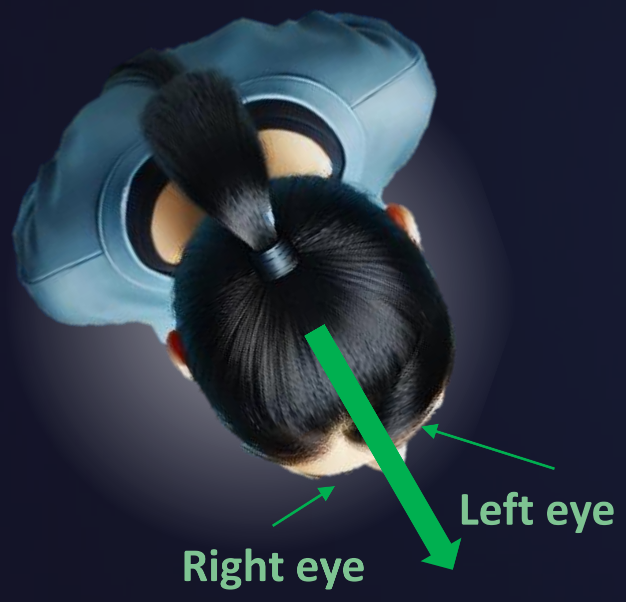
- Angles:
  - α
  - β
- The "Tet" ratio.

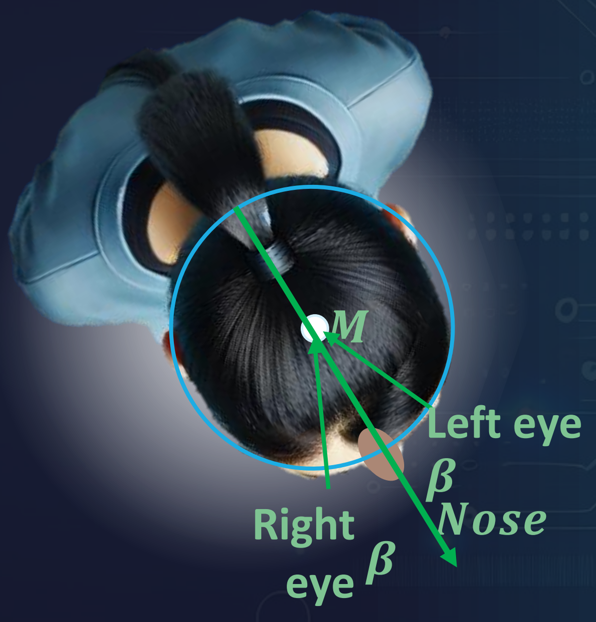
Definitions - α



As we said, α I s the angle between the nose and the camera.

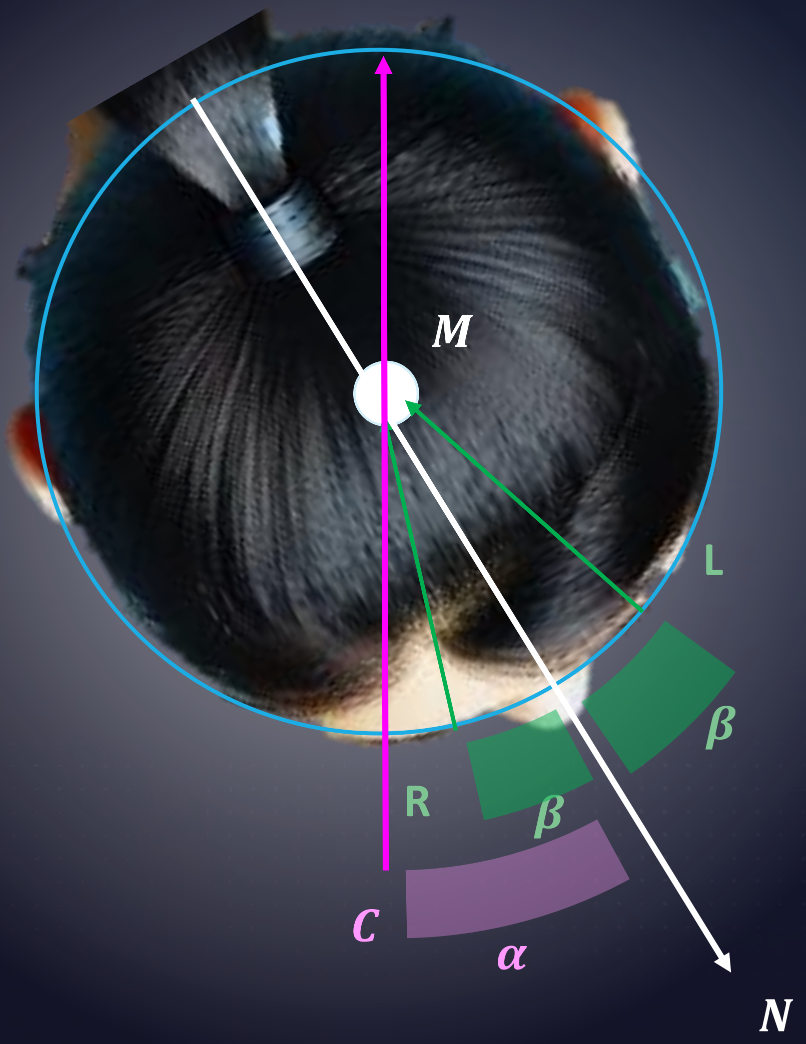
Definitions - β

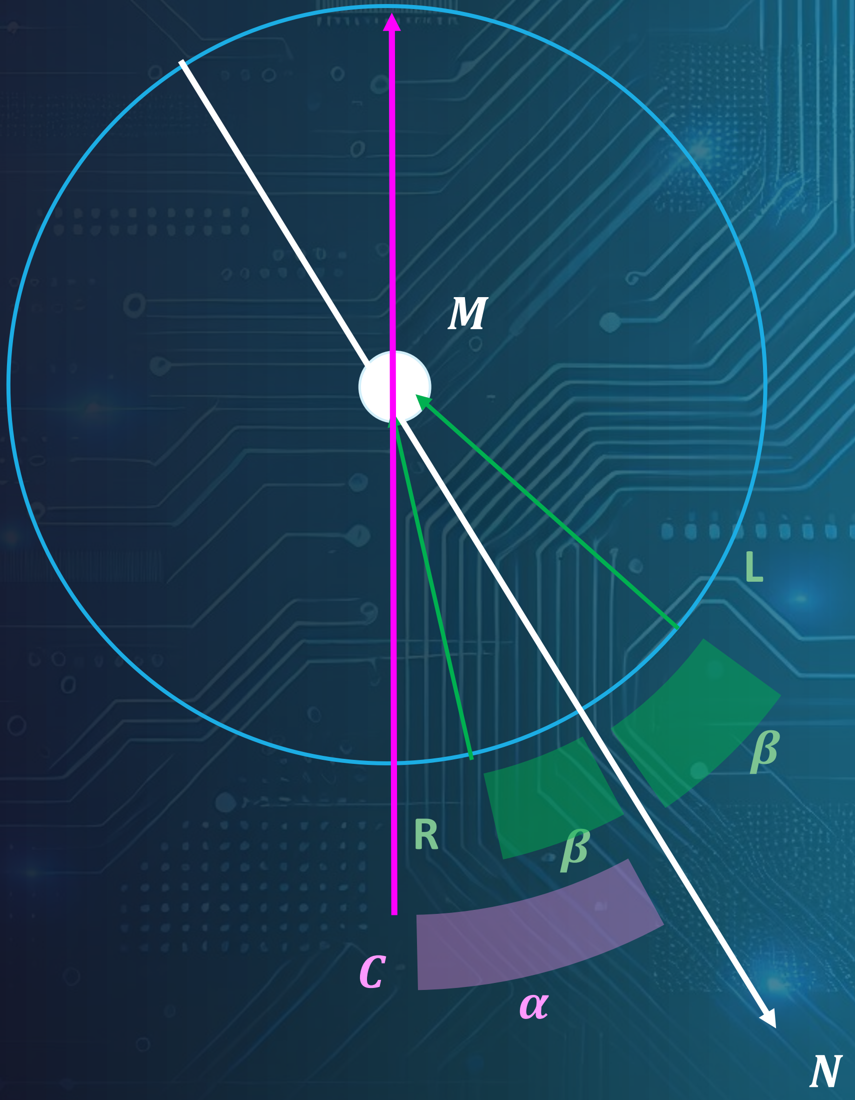




β is the angle between the nose and the eyes. Of course , the beta angle – as defined here – in not the same for all the people. But from experiments I conducted, it is similar enough for us to use it.

Angles - Conclusion:

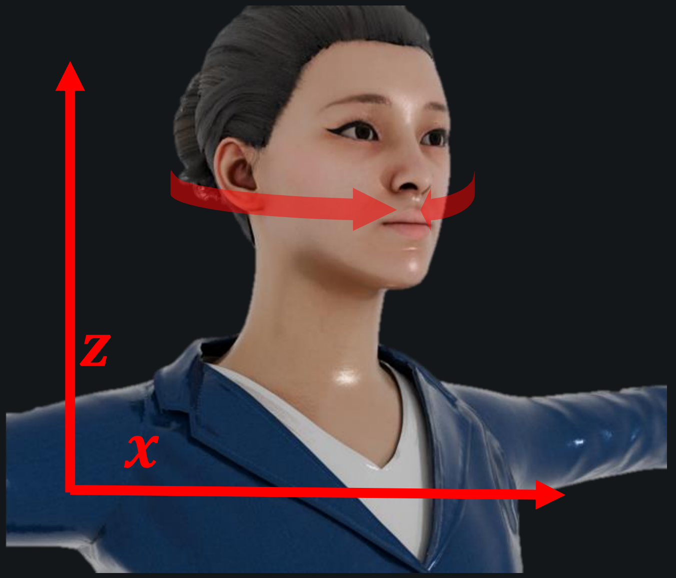




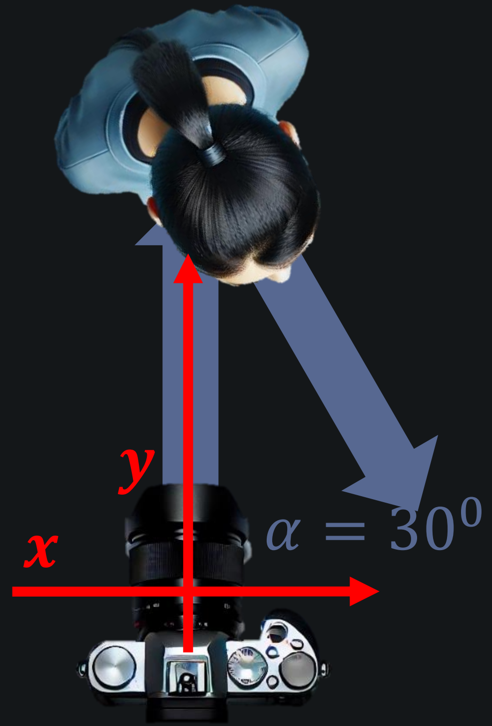
Definitions- the Tet ratio

In order to introduce the concept of Tet ratio, the following slides will illustrate the X, Y, and Z axes on the images, providing context for the underlying principles.

Axises



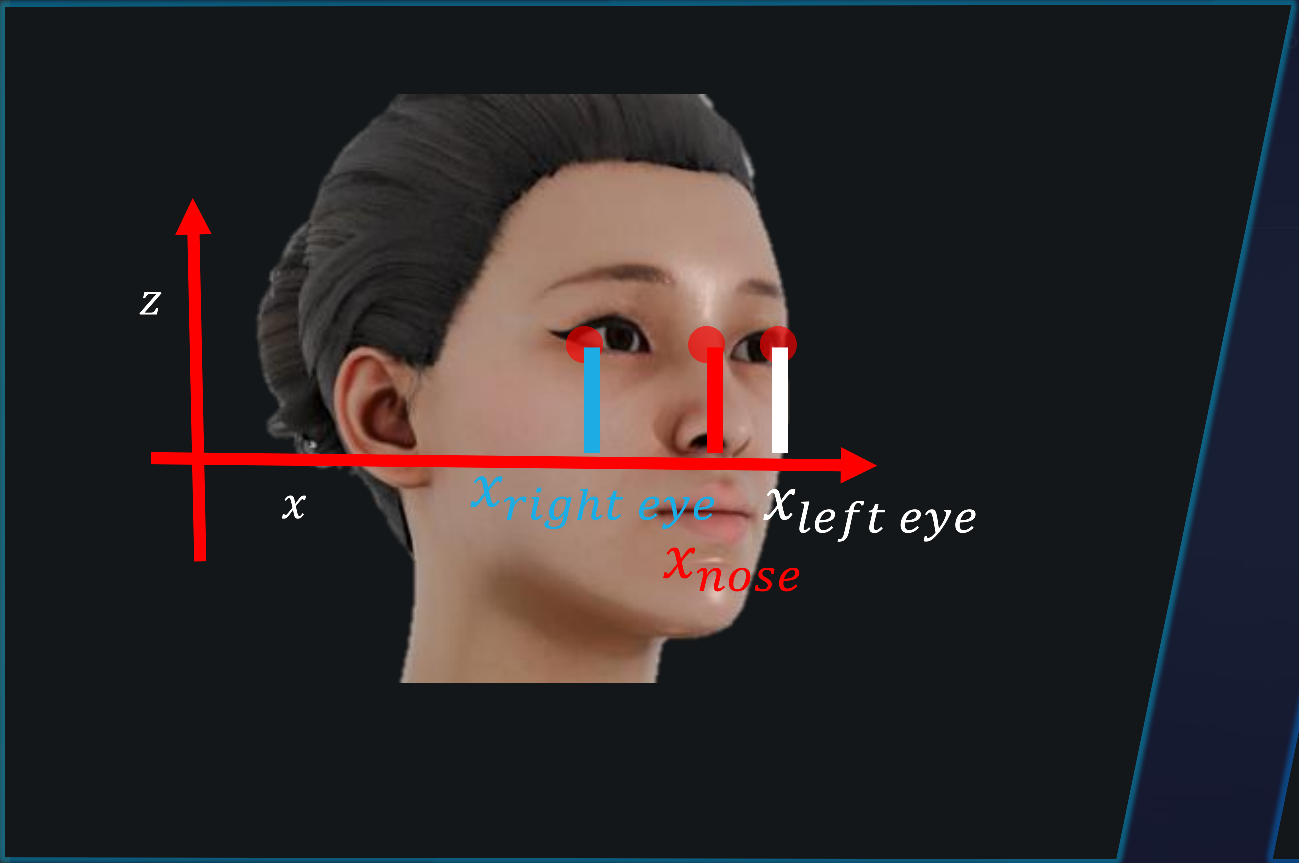
Input Image



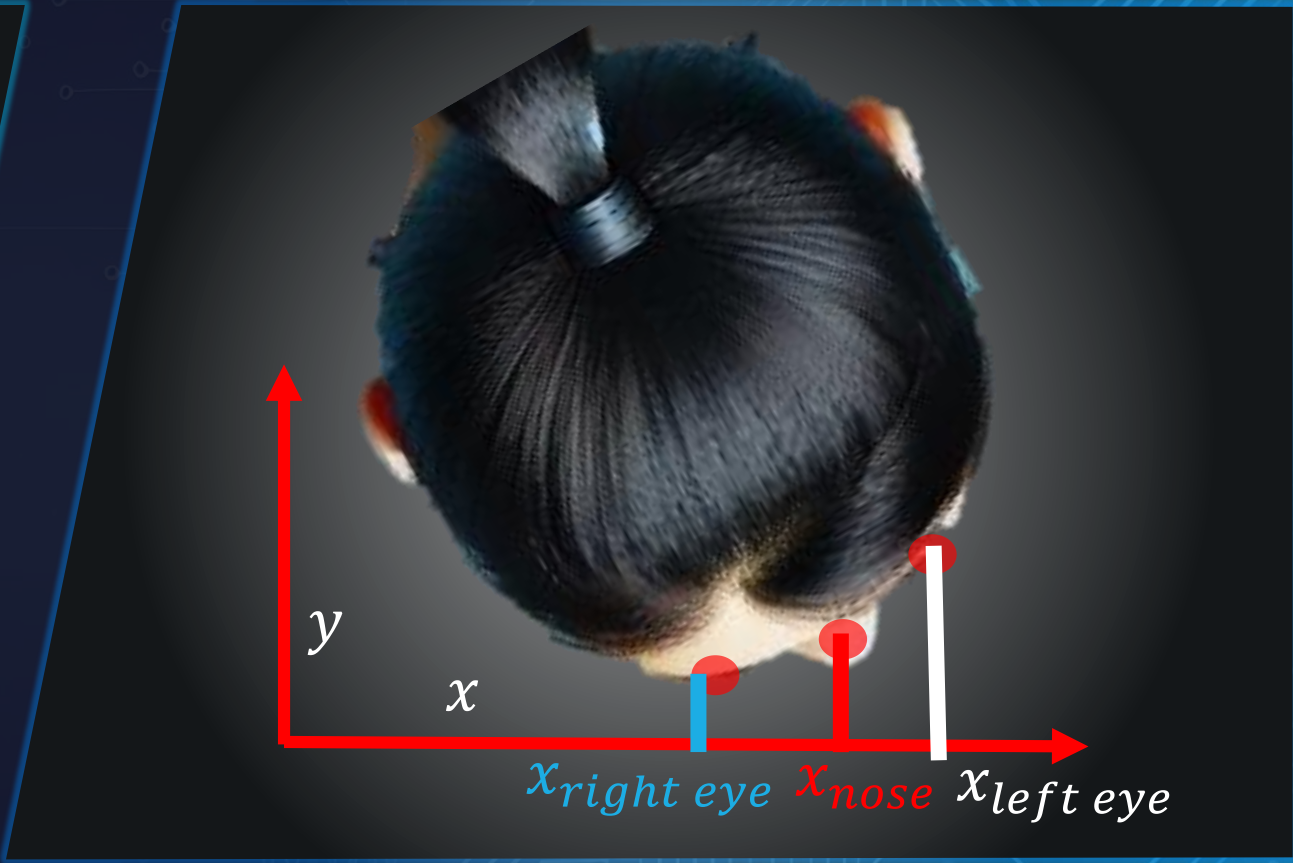
Top-Down View

Z-axis is defined as the "height" axis of the subject

Input Image

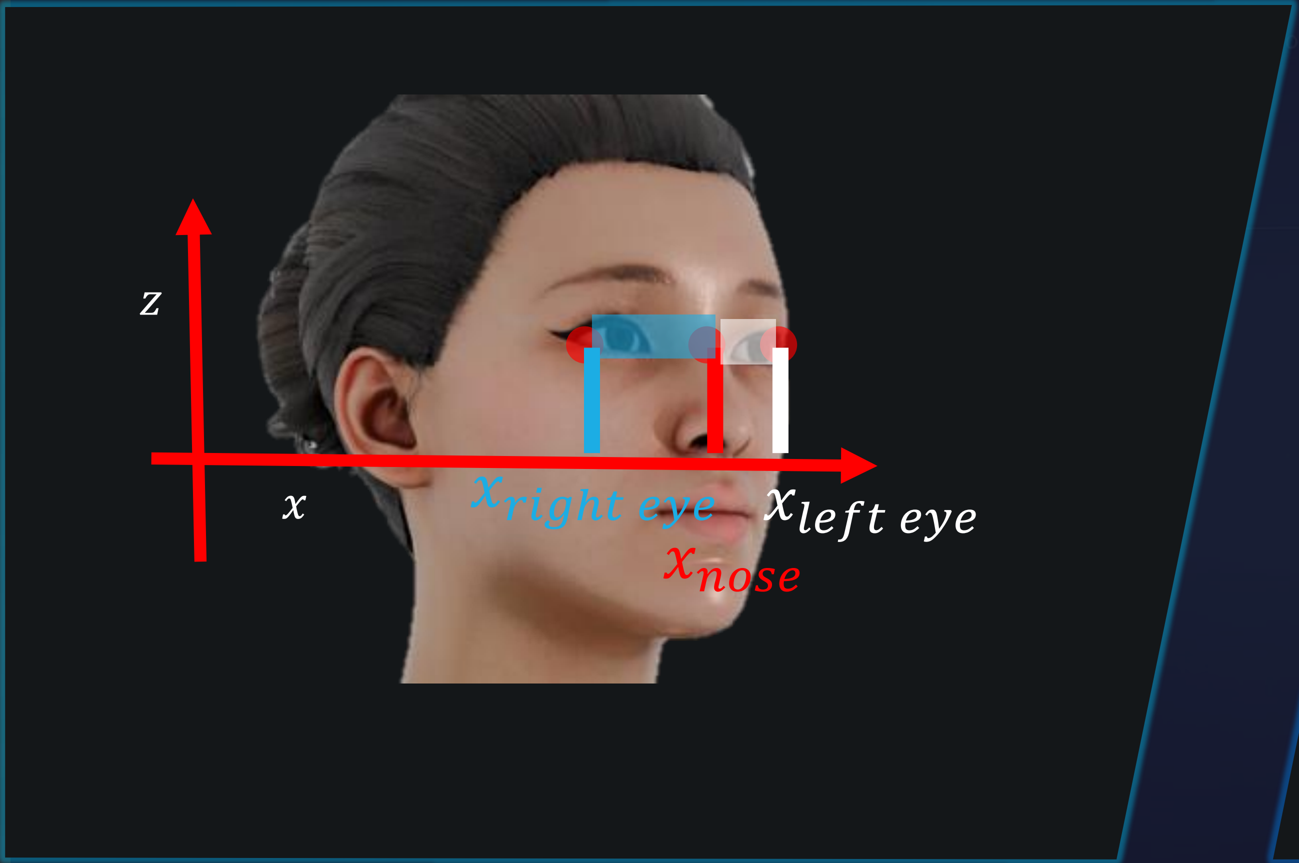


Top-Down View

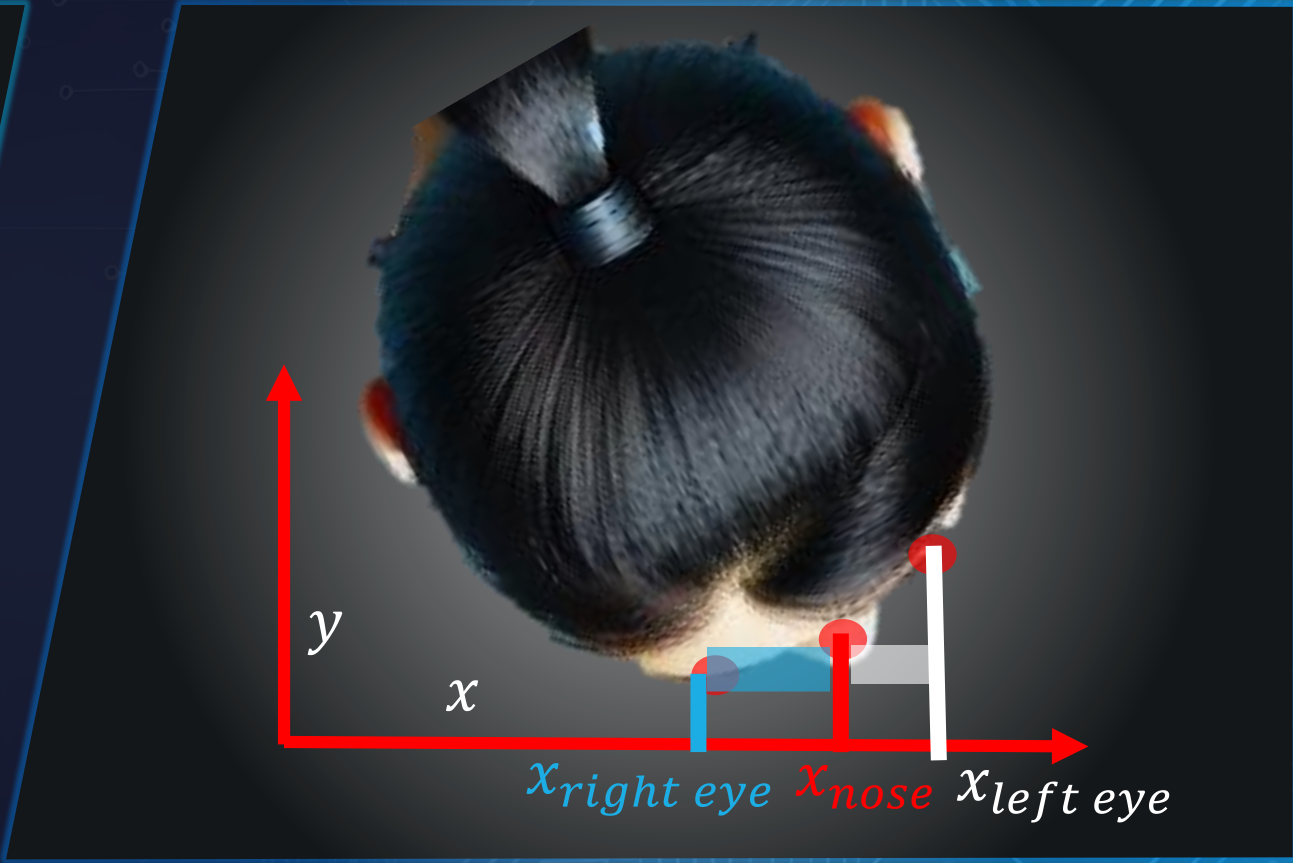


In these images, we can see a correlation between the X values of the eyes and nose, both in the top-down view and the input image. Although the X values are different due to the varying scale, the ratio between the lengths remains identical.

Input Image



Top-Down View



So: the ratio $\frac{x_{left\ eye} - x_{nose}}{x_{nose} - x_{right\ eye}}$ in the picture [1] is equal to $\frac{x_{left\ eye} - x_{nose}}{x_{nose} - x_{right\ eye}}$ in reality [2].

We denote this ratio by Tet, in short - T.

Concept – the claim

As we saw above, given an image, we can calculate Tet. I estimated Beta through experiments conducted with images generated by BLENDER.

Claim:

Given Tet and Beta, we can estimate Alpha, by using the formula:

$$\alpha = \boldsymbol{tan}^{-1}\left(\cot\left(\frac{\beta}{2}\right) * \frac{1-T}{T+1}\right)$$

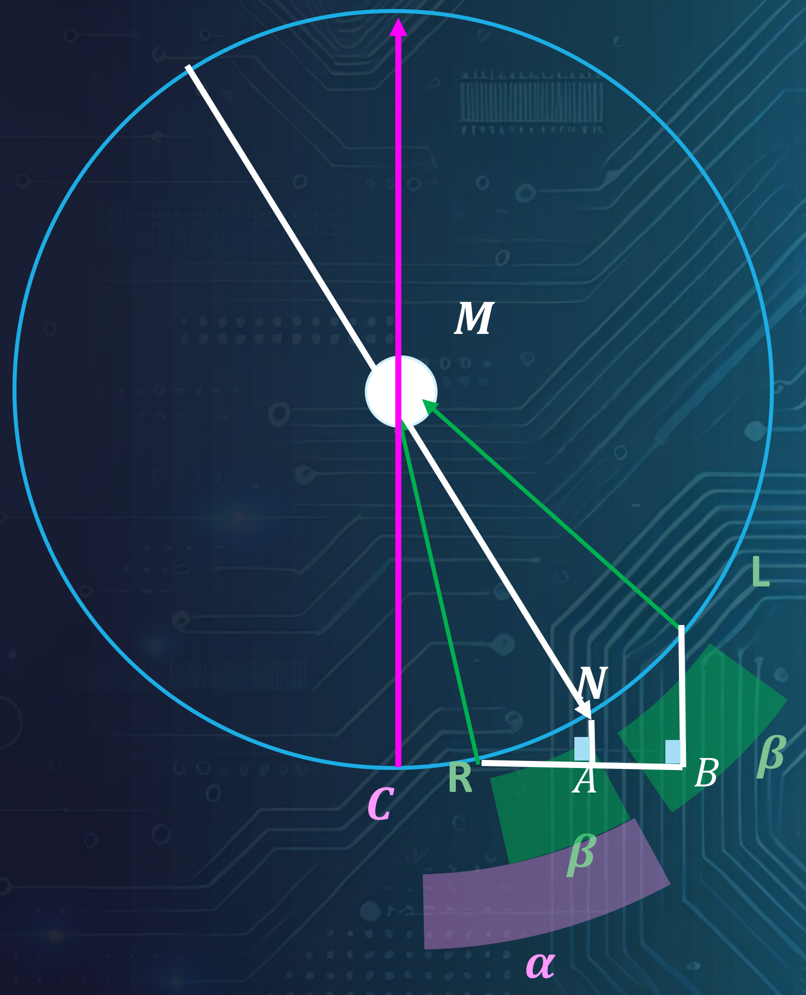
Concept – proof (1/7)

Claim - given

- $\sphericalangle RMN = \sphericalangle LMN = \beta$
- $T = \dfrac{AB}{AR}$
- $(\mathrm{BR} \perp BL \;\; and \; \mathrm{BR} \perp N_F A)$

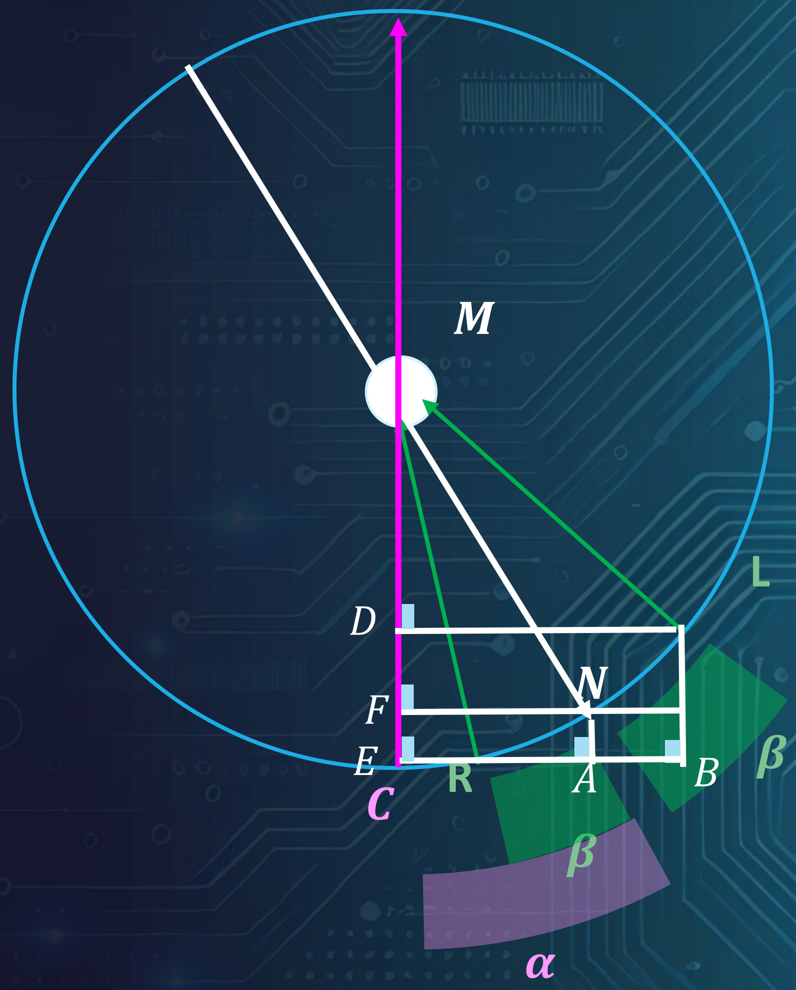
We need to calculate:

$$\sphericalangle \mathrm{CMN} = \alpha$$



Concept – proof (2/7)

Let's construct an auxiliary line: we draw a height from L to MC, marking the intersection point as D. We extend AB, and mark its intersection with MC as E. It's easy to prove that EA is perpendicular to MC - this follows directly from the way we defined the axes. From point N, we extend a line parallel to AE, intersecting MC at a 90^0 angle.



Concept – proof (3/7)

In the first step, we will express T in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

ΔMCR:

$$\sphericalangle CMR = \sphericalangle CMN - \sphericalangle RMN = \alpha - \beta$$

$$\frac{ER}{MR} = \sin(\sphericalangle CMR) = \sin(\alpha - \beta) \rightarrow \mathbf{ER = R * sin(\boldsymbol{\alpha} - \boldsymbol{\beta})}$$

ΔMFN:

$$\frac{FN}{MN} = \sin(\sphericalangle FMN) = \sin(\alpha) \rightarrow \mathbf{FN = R * sin(\boldsymbol{\alpha})}$$

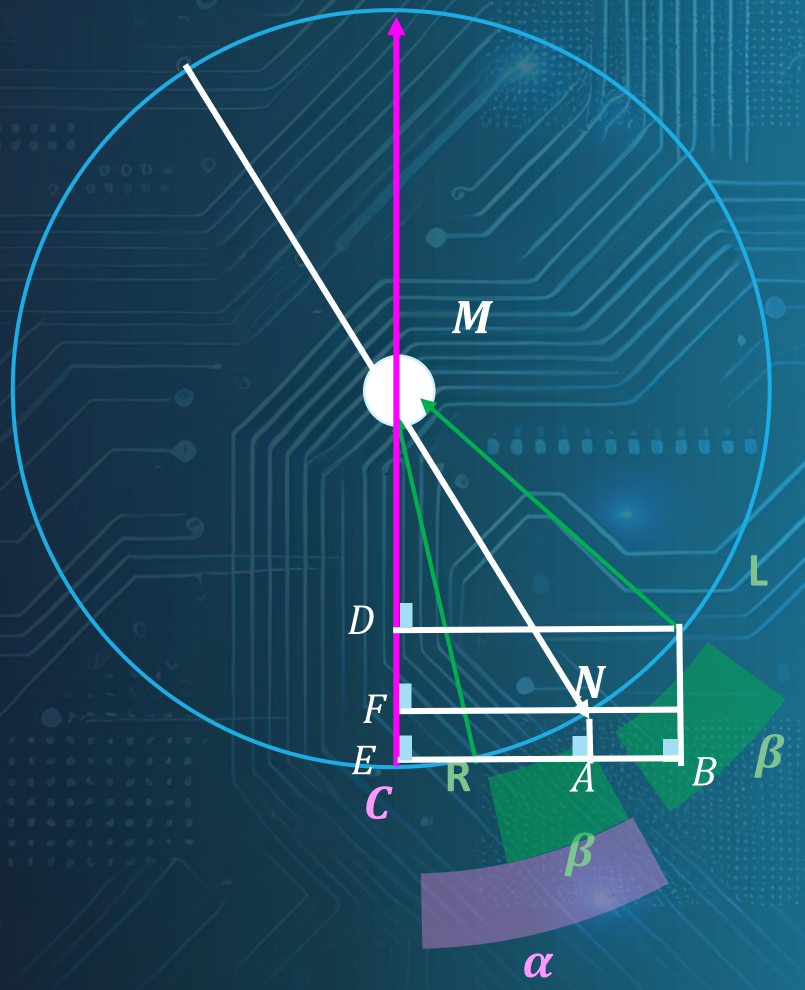
ΔMLD:

$$\sphericalangle DML = \sphericalangle DMN + \sphericalangle NML = \alpha + \beta$$

$$\frac{DL}{ML} = \sin(\sphericalangle DML) = \sin(\alpha + \beta) \rightarrow \boldsymbol{DL} = \mathbf{R * sin(\boldsymbol{\alpha} + \boldsymbol{\beta})}$$

DL = AE because ALDC is a rectangle.

FN = BE because FNBE is a rectangle.



Concept – proof (4/7)

So:

$$\frac{AB}{BR} = \frac{AE - BE}{BE - ER} = \frac{DL - BE}{FN - ER} = \frac{\mathrm{R} * \sin(\alpha + \beta) - \mathrm{R} * \sin(\alpha)}{\mathrm{R} * \sin(\alpha) - \mathrm{R} * \sin(\alpha - \beta)} = \frac{\sin(\alpha + \beta) - \sin(\alpha)}{\sin(\alpha) - \sin(\alpha - \beta)}$$

now we will use the formula: $\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right) cos\left(\frac{x+y}{2}\right)$

$$T = \frac{AB}{BR} = \frac{\sin(\alpha + \beta) - \sin(\alpha)}{\sin(\alpha) - \sin(\alpha - \beta)} = \frac{2\sin\left(\frac{\alpha + \beta - \alpha}{2}\right)\cos\left(\frac{\alpha + \beta + \alpha}{2}\right)}{2\sin\left(\frac{\alpha - (\alpha - \beta)}{2}\right)\cos\left(\frac{\alpha + (\alpha - \beta)}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\beta}{2}\right)\cos\left(\alpha + \frac{\beta}{2}\right)}{2\sin\left(\frac{\beta}{2}\right)\cos\left(\alpha - \frac{\beta}{2}\right)} = \frac{\cos\left(\alpha + \frac{\beta}{2}\right)}{\cos\left(\alpha - \frac{\beta}{2}\right)}$$

Concept – proof (5/7)

Now, we want to express $\boldsymbol{\alpha}$ in terms of T and $\boldsymbol{\beta}$.

$$T = \frac{\cos\left(\alpha + \frac{\beta}{2}\right)}{\cos\left(\alpha - \frac{\beta}{2}\right)}$$

$$\rightarrow T * \cos\left(\alpha - \frac{\beta}{2}\right) = \cos\left(\alpha + \frac{\beta}{2}\right)$$

We will use the formula:

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\rightarrow T * \left(\cos(\alpha)\, cos\left(\frac{\beta}{2}\right) + \sin(\alpha)\sin\left(\frac{\beta}{2}\right)\right) = \cos(\alpha)\, cos\left(\frac{\beta}{2}\right) - \sin(\alpha)\sin\left(\frac{\beta}{2}\right)$$

$$\rightarrow T * \cos(\alpha)\, cos\left(\frac{\beta}{2}\right) + T * \sin(\alpha)\sin\left(\frac{\beta}{2}\right) = \cos(\alpha)\, cos\left(\frac{\beta}{2}\right) - \sin(\alpha)\sin\left(\frac{\beta}{2}\right)$$

Concept – proof (6/7)

Now divide both side with $cos\left(\frac{\beta}{2}\right)$:

$$\rightarrow T * \cos(\alpha) + T * \sin(\alpha)\frac{\sin\left(\frac{\beta}{2}\right)}{cos\left(\frac{\beta}{2}\right)} = \cos(\alpha) - \sin(\alpha)\frac{\sin\left(\frac{\beta}{2}\right)}{cos\left(\frac{\beta}{2}\right)}$$

$$\rightarrow T * \cos(\alpha) + T * \sin(\alpha)\tan\left(\frac{\beta}{2}\right) = \cos(\alpha) - \sin(\alpha)\tan\left(\frac{\beta}{2}\right)$$

Now divide both sides with $cos(\alpha)$:

$$\rightarrow T + T * \frac{\sin(\alpha)}{\cos(\alpha)}\tan\left(\frac{\beta}{2}\right) = 1 - \frac{\sin(\alpha)}{\cos(\alpha)}\tan\left(\frac{\beta}{2}\right)$$

$$\rightarrow T + T * \tan(\alpha)\tan\left(\frac{\beta}{2}\right) = 1 - \tan(\alpha)\tan\left(\frac{\beta}{2}\right)$$

$$\rightarrow T * \tan(\alpha)\tan\left(\frac{\beta}{2}\right) + \tan(\alpha)\tan\left(\frac{\beta}{2}\right) = 1 - T$$

Concept – proof (7/7)

$$\rightarrow \tan(\alpha)\left(T * \tan\left(\frac{\beta}{2}\right) + \tan\left(\frac{\beta}{2}\right)\right) = 1 - T$$

$$\rightarrow \tan(\alpha) = \frac{1-T}{T * \tan\left(\frac{\beta}{2}\right) + \tan\left(\frac{\beta}{2}\right)} = \frac{1-T}{\tan\left(\frac{\beta}{2}\right)(T+1)} = \cot\left(\frac{\beta}{2}\right) * \frac{1-T}{T+1}$$

$$\rightarrow \alpha = \boldsymbol{tan^{-1}}\left(\cot\left(\frac{\beta}{2}\right) * \frac{1-T}{T+1}\right).$$

In the code, T represented by Q.

```
float tan_a = (*Q - 1) / ((*Q + 1)) * 1.3923;
*yaw = atan(tan_a) * 180.0 / 3.141;
```