Asset and Risk Management II

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Asset allocation: Black Litterman

The Problem with Active Asset Allocation

- The problem is that expected returns are difficult to estimate.
- How much can we trust our forecasts?
- How should one take into account the uncertainty about expected returns?
- These questions were addressed in the papers by Fisher Black and Robert Litterman: "Asset Allocation: Combining Investor Views with Market Equilibrium", Journal of Fixed Income, 1991 and "Global Portfolio Optimization", Financial Analysts Journal, Sept/Oct 1992.
- In these articles they present the following data:

“Optimal” Portfolios Based on Historical Data Do Not Make Sense!

- If you use historical data for expected future returns to calculate an optimal portfolio, with $\sigma_P = 10.7\%$, you will get portfolio weights of:

Exhibit 3

Optimal Portfolios Based on Historical Average Approach

(percent of portfolio value)

Unconstrained

	Germany	France	Japan	U.K.	U.S.	Canada	Australia
Currency exposure	-78,7	46,5	15,5	28,6		65,0	-5,2
Bonds	30,4	-40,7	40,4	-1,4	54,5	-95,7	-52,5
Equities	4,4	-4,4	15,5	13,3	44,0	-44,2	9,0

With constraints against shorting assets

	Germany	France	Japan	U.K.	U.S.	Canada	Australia
Currency exposure	-160,0	115,2	18,0	23,7		77,8	-13,8
Bonds	7,6	0,0	88,8	0,0	0,0	0,0	0,0
Equities	0,0	0,0	0,0	0,0	0,0	0,0	0,0

Black & Litterman Model

- Black & Litterman developed an approach to asset allocation that combines equilibrium and an uncertain view.
  - This was a response to clients' requests to incorporate specific "views" into the asset allocation of their portfolios.
  - It is central to asset allocation at Goldman Sachs Asset Management (GSAM) which has been run by Bob Litterman until 2009.
  - GSAM currently use an 'enhanced Black-Litterman approach' to calculate climate-aware expected returns.
- B&L use the estimated covariances.
- B&L combine these with "baseline" equilibrium expected returns.

Introducing the Views

- B&L then combine this with a "view" about expected returns or differences in expected returns and allow for some variance in the view.
- B&L's approach means that . . .
  - if the view has high variance (i.e., is uncertain), the portfolio will be close to the equilibrium portfolio (with no view we get the equilibrium weights);
  - if the view has low variance (i.e., is precise), the portfolio is more dramatically tilted away from equilibrium.

Black & Litterman Model - Step 1: Covariances from the Data

- Consider the asset allocation problem between bonds and stocks:

	Bonds	Stocks
Standard Deviation	8%	17%
Correlation	0.3	

- Variance-Covariance matrix

$$\begin{pmatrix} 0.00640 & 0.00408 \\ 0.00408 & 0.02890 \end{pmatrix}$$

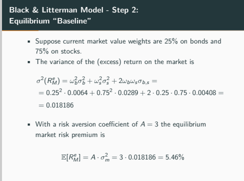
Black & Litterman Model - Step 2: Equilibrium "Baseline"

- Suppose current market value weights are 25% on bonds and 75% on stocks.
- The variance of the (excess) return on the market is

$$\sigma^2(R_M^e) = \omega_b^2\sigma_b^2 + \omega_s^2\sigma_s^2 + 2\omega_b\omega_s\sigma_{b,s} =$$
$$= 0.25^2 \cdot 0.0064 + 0.75^2 \cdot 0.0289 + 2 \cdot 0.25 \cdot 0.75 \cdot 0.00408 =$$
$$= 0.018186$$

- With a risk aversion coefficient of $A = 3$ the equilibrium market risk premium is

$$\mathbb{E}[R_M^e] = A \cdot \sigma_m^2 = 3 \cdot 0.018186 = 5.46\%$$



- Assume a mean-variance optimizing investor who is able to invest a fraction $(1 - x)$ of his wealth in a T-Bill and x in the market portfolio.
- The T-Bill has an excess return of zero and a variance of zero.
- The market has an expected excess return of r_m^e and a variance of σ_m^2.
- Now $\max_x xr_m^e - \frac{1}{2}Ax^2\sigma_m^2$
- So, $0 = r_m^e - Ax\sigma_m^2$
- But we know that $x = 1$ because we talk about the market
- Hence, $r_m^e = A\sigma_m^2$

Black & Litterman Model - Step 2: Equilibrium "Baseline"

- The baseline expected excess returns on the bonds and stocks are

$$\begin{aligned}
\mathbb{E}[R_b^e] &= \frac{Cov[R_b^e, R_M^e]}{\sigma_M^2}\mathbb{E}[R_M^e] = \frac{Cov[R_b^e, \omega_b R_b^e + \omega_s R_s^e]}{\sigma_M^2}\mathbb{E}[R_M^e] = \\
&= \frac{\omega_b Var[R_b^e] + \omega_s Cov[R_b^e, R_s^e]}{\sigma_M^2}\mathbb{E}[R_M^e] = \\
&= \frac{0.25 \cdot 0.0064 + 0.75 \cdot 0.00408}{0.018186} 5.46\% = 1.40\% \\
\mathbb{E}[R_s^e] &= 6.81\%
\end{aligned}$$

- Variance-covariance matrix of expected returns (with 100 observations):

$$\begin{pmatrix} 0.0000640 & 0.0000408 \\ 0.0000408 & 0.0002890 \end{pmatrix}$$

Black & Litterman Model - Steps 3 & 4: Views and Revised (Posterior) Expectations

- View:
  - Suppose you think the expected returns on bonds exceed those of stocks by 0.5% over the next month:

$$\mathbb{E}[R_b^e] - \mathbb{E}[R_s^e] = 0.5\%$$

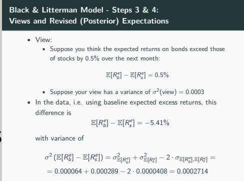
  - Suppose your view has a variance of $\sigma^2(\text{view}) = 0.0003$
- In the data, i.e. using baseline expected excess returns, this difference is

$$\mathbb{E}[R_b^e] - \mathbb{E}[R_s^e] = -5.41\%$$

with variance of

$$\sigma^2\left(\mathbb{E}[R_b^e] - \mathbb{E}[R_s^e]\right) = \sigma^2_{\mathbb{E}[R_b^e]} + \sigma^2_{\mathbb{E}[R_s^e]} - 2 \cdot \sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]} =$$

$$= 0.000064 + 0.000289 - 2 \cdot 0.0000408 = 0.0002714$$



Note: Stddev of view is 0.01732051.

This means that we are 90% sure that the actual outperformance of bonds vs. stocks will be between $0.5\% + -1.64 \cdot 1.732\%$, i.e. between 3.34% and -2.34%.

Black & Litterman Model - Steps 3 & 4: Views and Revised (Posterior) Expectations

- The difference between your view and the baseline expected excess returns is

$$D = 0.5\% - (-5.41\%) = 5.91\%$$

with variance

$$\sigma_D^2 = \sigma^2(\text{view}) + \sigma^2\left(\mathbb{E}[R_b^e] - \mathbb{E}[R_s^e]\right) =$$
$$= 0.0003 + 0.0002714 = 0.0005714$$

Black & Litterman Model: Revised (Posterior) Expectations and Portfolio Optimization

- Revised (Posterior) Expectations:

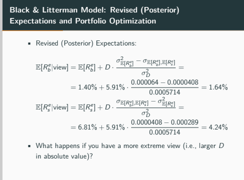
$$\mathbb{E}[R_b^e|\text{view}] = \mathbb{E}[R_b^e] + D \cdot \frac{\sigma^2_{\mathbb{E}[R_b^e]} - \sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]}}{\sigma_D^2} =$$

$$= 1.40\% + 5.91\% \cdot \frac{0.000064 - 0.0000408}{0.0005714} = 1.64\%$$

$$\mathbb{E}[R_s^e|\text{view}] = \mathbb{E}[R_s^e] + D \cdot \frac{\sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]} - \sigma^2_{\mathbb{E}[R_s^e]}}{\sigma_D^2} =$$

$$= 6.81\% + 5.91\% \cdot \frac{0.0000408 - 0.000289}{0.0005714} = 4.24\%$$

- What happens if you have a more extreme view (i.e., larger D in absolute value)?

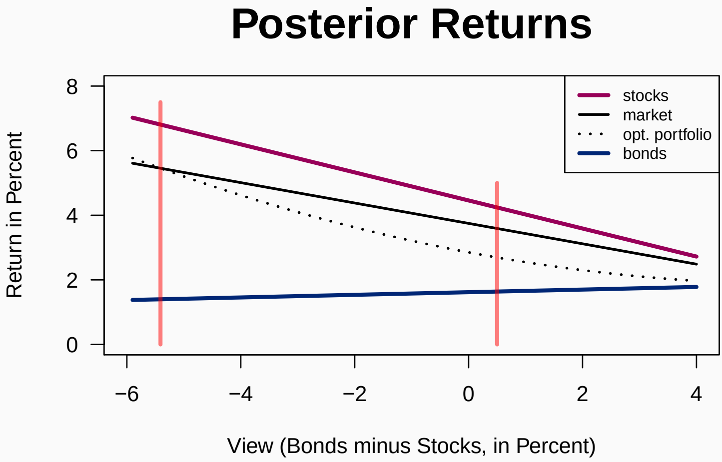


- How to interpret $\frac{\sigma^2_{\mathbb{E}[R_b^e]} - \sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]}}{\sigma_D^2}$? The numerator is the covariance of bonds' expected returns with exp. returns on the view portfolio $(b - s)$.
- $\sigma^2_{\mathbb{E}[R_b^e]} - \sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]} = \sigma(\mathbb{E}[R_b^e], \mathbb{E}[R_b^e - R_s^e]) = \sigma(\mathbb{E}[R_b^e], \mathbb{E}[R_b^e] - \mathbb{E}[R_s^e])$. This is the first element (relating to bonds) of the covariance matrix of equilibrium mean returns applied to the views portfolio, i.e. $\tau\Sigma P'$.
- If this is small, we have lot of confidence in the equilibrium return for bonds. We do not want to adjust the equilibrium returns a lot.
- What does *small* mean? Compare it to total variance of D, which is the sum of variance of equilibrium returns on the views portfolio and the variance of the view.
- Note that $\left(\sigma^2_{\mathbb{E}[R_b^e]} - \sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]}\right) - \left(\sigma_{\mathbb{E}[R_b^e],\mathbb{E}[R_s^e]} - \sigma^2_{\mathbb{E}[R_s^e]}\right) = \sigma^2\left(\mathbb{E}[R_b^e] - \mathbb{E}[R_s^e]\right)$

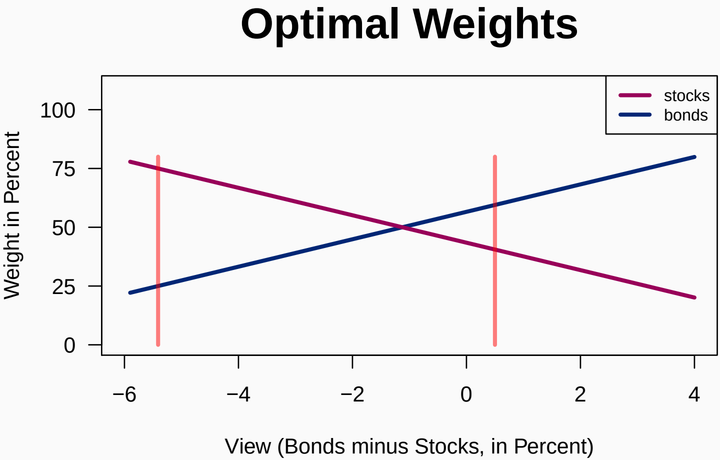
Black & Litterman Model: Revised (Posterior) Expectations and Portfolio Optimization

- What happens if you have a more precise view (i.e., smaller σ^2(view))?
  - Note: σ^2(view) enters only in σ_D^2
- Step 5: Portfolio optimization using revised (posterior) expectations. Solution

Sensitivities: Returns



Sensitivities: Weights



Generalization of B&L

1. Matrix version of Black & Litterman in 5 steps
2. Examples

Black & Litterman: Step 1

- Obtain a sample of past excess returns of the universe of n assets, and estimate their variance-covariance matrix, denoted by $\mathbf{\Sigma}$. The universe of assets could either be all the assets in the market portfolio (if we are talking about a representative investor), or alternatively the investor's long-term strategic asset allocation (i.e. the assets that the investor wishes to hold over the long run).

Black & Litterman: Step 2

- The baseline equilibrium returns are obtained by inverting the investor's original optimization problem

$$\max_{\boldsymbol{w}} U(r_p) = \max_{\boldsymbol{w}} \left(r_f + \boldsymbol{w}'\boldsymbol{\Pi} - \frac{1}{2} A \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \right) ,$$

  - where $\boldsymbol{w}$ represents the vector of portfolio weights,
  - $\boldsymbol{\Pi}$ is the vector of excess returns expected by the investor
  - and A represents the investor's risk aversion
- The first order condition is $\boldsymbol{w}^* = \frac{1}{A}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}$
- Solving for the excess returns expected by the investor (baseline expectations) yields $\boldsymbol{\Pi} = A\boldsymbol{\Sigma}\boldsymbol{w}^*$

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Note: We will denote the vector of „true“ expected returns by $\boldsymbol{\mu}$.

Black & Litterman: Step 2 (ctd)

- Note that we believe that the market‘s excess return expectation is subject to error (=noise).
- It is plausible to assume that the noise in the expectations is related to the covariance matrix of the returns.
- I.e. the expected excess return of a highly risky asset class is likely to be subject to more noise than the expected excess return of a less risky asset class.
- Note, this is related to the concept of "standard error". If the investor estimates his expectation from 100 historical observations, then his expectations would be distributed according to $\tau\mathbf{\Sigma}$, where τ equals $\frac{1}{100} = 0.01$.

Black & Litterman: Step 3
The Portfolio Manager's View

- In the Black Litterman model, the manager's views are interpreted as a "noisy signal".
- They can be expressed in the following way:

$$\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{v} + \boldsymbol{\epsilon}\,,$$

  - where the matrix $\boldsymbol{P}$ contains one line for each "view" (=forecast). Potential relative forecasts are defined as linear combinations.
  - The vector $\boldsymbol{v}$ contains the forecasted values and $\boldsymbol{\epsilon}$ is an error term which is assumed to be normally distributed.

Black & Litterman: Step 3

- Let $\boldsymbol{\Omega}$ denote the variance-covariance matrix of the forecast error terms (i.e. the $\boldsymbol{\epsilon}$'s).
- It is a challenge to consistently specify $\boldsymbol{\Omega}$. In practice, one of the following approaches is used:
  - Estimate $\boldsymbol{\Omega}$ using historical forecast errors from the view-generating model(s).
  - Use a diagonal $\boldsymbol{\Omega}$. This corresponds to the assumption that views represent independent draws from the future distribution of returns (as in the original Black and Litterman paper).

Black & Litterman: Step 4

- The prior distribution of expected returns, p is given by:

$$p(\boldsymbol{\mu}) \propto exp\left\{-\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{\Pi})'\left(\tau\boldsymbol{\Sigma}\right)^{-1}(\boldsymbol{\mu}-\boldsymbol{\Pi})\right\}$$

- From the definition of the forecasting signal it follows that the likelihood of expected returns, L, is given by

$$L(\boldsymbol{\mu}) \propto exp\left\{-\frac{1}{2}(\boldsymbol{P}\boldsymbol{\mu}-\boldsymbol{v})'\boldsymbol{\Omega}^{-1}(\boldsymbol{P}\boldsymbol{\mu}-\boldsymbol{v})\right\}$$

Black & Litterman: Step 4

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Probability density function (PDF) of the normal distribution:

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) ,$$

where μ is the mean and σ is the standard deviation of the distribution.
PDF of a multivariate normal dist. with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$:

$$f(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) ,$$

where $\mathbf{x}$ is the vector of random variables, $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\Sigma}$ is the covariance matrix, k is the dimensionality of the vector $\mathbf{x}$, and $|\boldsymbol{\Sigma}|$ is the determinant of the covariance matrix.

Black & Litterman: Step 4

- Bayes‘ Theorem implies that if the prior and the signal are normal (as defined on the previous slide), then the posterior is also normal with:
  - Variance-covariance

$$\bar{\boldsymbol{\Sigma}} = \left[(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}$$

  - Mean

$$\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\Sigma}}\left[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}\right]$$

  - One can also update the covariance matrix of the assets as

$$\boldsymbol{\Sigma}_{\boldsymbol{BL}} = \boldsymbol{\Sigma} + \bar{\boldsymbol{\Sigma}}$$

Black & Litterman: Step 5

- Find the optimal asset allocation by applying Markowitz Optimization based on the posterior means

$$\max_{\boldsymbol{w}} U(r_p) = \max_{\boldsymbol{w}} \left(r_f + \boldsymbol{w}'\bar{\boldsymbol{\mu}} - \frac{1}{2} A \boldsymbol{w}' \boldsymbol{\Sigma}_{\boldsymbol{BL}} \boldsymbol{w} \right)$$

with solution

$$\boldsymbol{w}^* = \frac{1}{A} \boldsymbol{\Sigma}_{\boldsymbol{BL}}^{-\boldsymbol{1}} \bar{\boldsymbol{\mu}}$$

- In practice, the original $\boldsymbol{\Sigma}$ is often used for the optimization step.

Black & Litterman Model

- Assume that there are no views ($\boldsymbol{P} = 0$) then we get

$$\bar{\boldsymbol{\mu}} = \boldsymbol{\Pi}$$

- The same is true if our views are "diffuse", i.e. measured with infinite error: $\boldsymbol{\Omega} \rightarrow$ infinity. BL returns are identical to equilibrium returns.
- Assume that the views are estimated without errors, i.e. $\boldsymbol{\Omega}^{-1} = \infty$, and $\boldsymbol{P}$ has a full rank and is positive definite, then we get: BL returns are identical to view return

$$\bar{\boldsymbol{\mu}} = \boldsymbol{P}^{-1}\boldsymbol{v}$$

- The same is true if our baseline expectations (i.e. our priors) are "diffuse", i.e. measured with infinite error, i.e. $\boldsymbol{\Sigma} \rightarrow \infty$.

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Regarding $P = 0$:

Technically, if we have P, it needs to have at least one row - in this case containing only zeros.

Black & Litterman Model: Application

- How are the views expressed in the B&L model?
- Remember, views are expressed as:

$$\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{v} + \boldsymbol{\epsilon}$$

  - Relative views: e.g. expected return of asset class A will outperform the return of asset class B by v_1
  - Absolute views: e.g. expected return of asset class C will be v_2

$$\mu_A - \mu_B = v_1 + \epsilon_1$$

$$\mu_C = v_2 + \epsilon_2$$

Black & Litterman Model: Application

- In terms of matrix equation we get (assuming 3 asset classes):

$$\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{v} + \boldsymbol{\epsilon}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}$$

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

How to Estimate Forecasting Accuracy?

- Suppose the view is that asset class C will return 4%.
- We have a 90% confidence that the true expected return varies around the mean between $[3.5\%, 4.5\%]$.
- For a normally distributed variable, a 90% percent confidence interval can be stated as $[\mathbb{E}(r) - 1.64\sigma, \mathbb{E}(r) + 1.64\sigma]$.
- Hence, $0.01 = 0.045 - 0.035 = 2 \cdot 1.64\sigma$.
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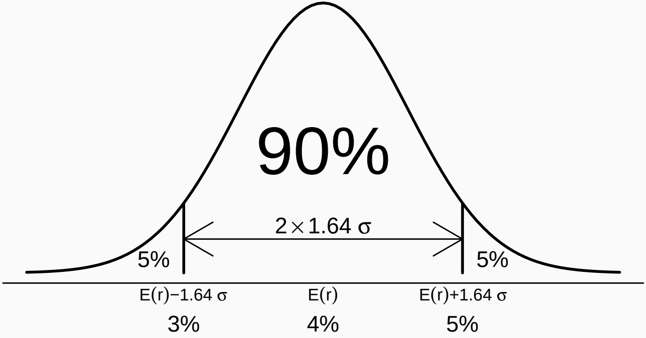
- qnorm$(0.95) = 1.644854$
- $2s = 0.045 - 0.035 = 0.01 = (\mu + 1.645\sigma) - (\mu - 1.645\sigma)$
- $0.01 = (2 * 1.644854\sigma)$
- $\sigma \approx 0.003$
- For interval between 3% and 5%
- $0.02 = (2 * 1.644854\sigma)$
- $\sigma \approx 0.006$

How to Estimate Forecasting Accuracy?

- Data as before, but:
- What if the interval is between 3% and 5%?



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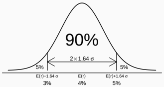

- $0.02 = (2 * 1.644854\sigma)$
- $\sigma \approx 0.006$

Black & Litterman Model: Example (1)

- Example: 3 asset classes A, B, C. Market portfolio consists of $1/3$ in each asset, risk aversion coefficient $A = 1.5$
- Covariance and precision matrices

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \quad \boldsymbol{\Omega} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

- Assume that $\tau = 0.1$

Black & Litterman Model: Example (2)

- Views about the returns of asset A, B, C (i.e. only absolute views are specified)

$$\boldsymbol{P\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0 \\ 0 \end{pmatrix}$$

- The implied baseline equilibrium returns are

$$\boldsymbol{\Pi} = \begin{pmatrix} \Pi_A \\ \Pi_B \\ \Pi_C \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}$$

- Can we show that these are also consistent with the CAPM?

Black & Litterman Model: Example (2)

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- Can we show that these are also consistent with the CAPM?

- $$\sigma_{A,M} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{0.1}{3}$$

- $$\sigma_M^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{0.3}{9} = \frac{0.1}{3}$$

- $$\beta_A = \frac{\sigma_{A,M}}{\sigma_M^2} = 1$$

- $$\mathbb{E}[r_A^e] = 1 \cdot \mathbb{E}[r_M^e] = \frac{1}{3}0.05 + \frac{1}{3}0.05 + \frac{1}{3}0.05 = 0.05$$

Black & Litterman Model: Example (3)

- The implied posterior equilibrium returns are

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.035 \\ 0.025 \\ 0.025 \end{pmatrix}$$

- Interpret!

Black & Litterman Model: Example (4)
Case 1: Precise views

- Precise views - e.g. divide $\boldsymbol{\Omega}$ by $10,000$

$$\mathbb{E}[\boldsymbol{r}] = \boldsymbol{P}^{-1}\boldsymbol{v}$$

- This results in

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0 \\ 0 \end{pmatrix}$$

- Result: If there is no uncertainty in the views, the views equal the posteriors.

Black & Litterman Model: Example (5)
Case 2: Large Uncertainty in the Views

- There is a large uncertainty in the views, e.g.:

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \quad \boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Result: The posteriors are almost identical to the baseline expectations (=priors)!

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.0497 \\ 0.0495 \\ 0.0495 \end{pmatrix}$$

Black & Litterman Model: Example (6)
Case 3: Correlation among Assets

- What are the consequences of correlations among the assets?
- We assume high positive correlations
- Covariance and precision matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.1 & 0.09 & 0.09 \\ 0.09 & 0.1 & 0.09 \\ 0.09 & 0.09 & 0.1 \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

Variance matrix assets — Variance matrix estimation error

Black & Litterman Model: Example (7)

- Absolute views only about the returns of A, B and C

$$\boldsymbol{P\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0 \\ 0 \end{pmatrix}$$

- The implied equilibrium returns are (note: the new baseline expectations, i.e. priors, are 0.14 for each asset class)

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.0430 \\ 0.0411 \\ 0.0411 \end{pmatrix}$$

- Explain

Black & Litterman Model: Example (8)
Case 4: High Precision of View and Correlation

- We continue to assume correlations among asset classes
- But we change the variances of the estimation error
- Covariance and precision matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.1 & 0.09 & 0.09 \\ 0.09 & 0.1 & 0.09 \\ 0.09 & 0.09 & 0.1 \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} 0.0001 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

Variance matrix assets — Variance estimation error

Black & Litterman Model: Example (9)

- Results

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.0207 \\ 0.0255 \\ 0.0255 \end{pmatrix}$$

- The Black Litterman returns have the following general characteristics
  - High precision (confidence) translates into a return that is close to the view
  - Returns of assets B and C are adjusted because of the high correlation with asset A.

Black & Litterman Model: Example (10)

- We introduce a relative view of assets A and B.

$$\boldsymbol{P\mu} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0 \\ 0 \end{pmatrix}$$

- The implied equilibrium returns are, for the cases with and without correlations, respectively:

$$\begin{pmatrix} \Pi_A \\ \Pi_B \\ \Pi_C \end{pmatrix} = \begin{pmatrix} 0.14 \\ 0.14 \\ 0.14 \end{pmatrix} \qquad \begin{pmatrix} \Pi_A \\ \Pi_B \\ \Pi_C \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}$$

Black & Litterman Model: Example (10)

- With correlated assets we get

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.142 \\ 0.138 \\ 0.140 \end{pmatrix}$$

- Without correlations we get

$$\begin{pmatrix} \bar{\mu}_A \\ \bar{\mu}_B \\ \bar{\mu}_C \end{pmatrix} = \begin{pmatrix} 0.0567 \\ 0.0433 \\ 0.0500 \end{pmatrix}$$

Conclusions

- Active portfolio management
  - "Naive" application of the Markowitz portfolio problem often results in unreasonable portfolios.
- Additional structure and equilibrium considerations allow for better results.
- The Black & Litterman model allows combination of equilibrium with a view
  - The amount of "tilt" depends in an intuitive way on the precision of the view.
  - The model predictions are well-behaved and reasonable.
  - In addition to the specifics of the model, it is useful as a general way to think about asset management.

Black & Litterman Model: Portfolio Optimization

Example: Solution

We have $A = 3$, $\mathbb{E}[R_b^e|\text{view}] = 1.64\%$, $\mathbb{E}[R_e^e|\text{view}] = 4.24\%$, and keep Σ unchanged. The optimal weights are given by $\mathbf{w}^* = \frac{1}{A}\Sigma^{-1}\mathbb{E}\mathbf{R^e}$:

$$\mathbf{w}^* = \frac{1}{3} \cdot \begin{pmatrix} 0.0064 & 0.00408 \\ 0.00408 & 0.0289 \end{pmatrix}^{-1} \begin{pmatrix} 0.0164 \\ 0.0424 \end{pmatrix} = \begin{pmatrix} 0.595 \\ 0.405 \end{pmatrix}$$

Back