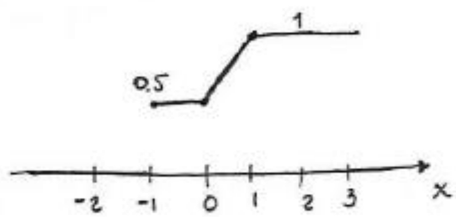
ECE 2521 Analysis of Stochastic Processes
Solutions to Problem Set 4

Problem 4.1 Solution

4.12

a)



Mixed type random variable

b) $P[X \le -1] = 0.5$

$P[X = -1] = 0.5$

$P[X < 0.5] = P[X \le 0.5] - P[X = 0.5]$

$= \frac{1 + 0.5}{2} - 0$

$= 0.75$

$P[-0.5 < X < 0.5] = P[X \le 0.5] - P[X = 0.5] - P[X \le -0.5]$

$= \frac{1 + 0.5}{2} - 0 - 0.5$

$= 0.75 - 0.5$

$= 0.25$

$P[X > -1] = 1 - P[X \le -1]$

$= 1 - 0.5$

$= 0.5$

$P[X \le 2] = 1$

$P[X > 3] = 1 - P[X \le 3]$

$= 1 - 1$

$= 0$

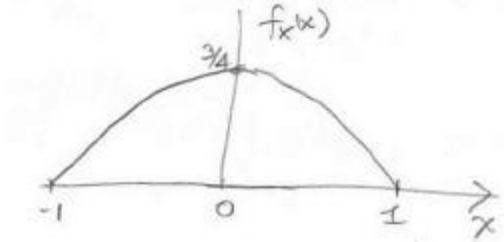
Problem 4.2 Solution

4.17

$$1 = c\int_{-1}^{1}(1-x^2)\,dx = c\left[x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} \right] = c\left[2 - \tfrac{1}{3}2\right] = \tfrac{4}{3}c$$

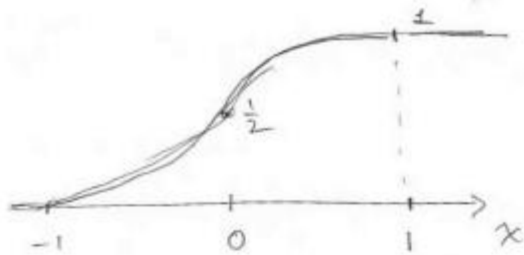
$$\Rightarrow c = \frac{3}{4}$$

$$f_X(x) = \frac{3}{4}(1-x^2) \qquad -1 \le x \le 1$$



$$F_X(x) = \frac{3}{4}\int_{-1}^{x}(1-y^2)\,dy = \frac{3}{4}\left[y\Big|_{-1}^{x} - \frac{y^3}{3}\Big|_{-1}^{x} \right]$$

$$= \frac{3}{4}\left[(x+1) - \frac{1}{3}(x^3+1)\right]$$



$$P[X=0] = F_X(0) = 0$$

$$P[0 < X < 0.5] =$$

$$= \frac{3}{4}\left[\left(\tfrac{1}{2}+1\right) - \tfrac{1}{3}\left(\tfrac{1}{8}+1\right)\right]$$

$$- \frac{3}{4}\left[1 - \tfrac{1}{3}\right]$$

$$= \frac{11}{32}$$

$$P\left[\,|X - \tfrac{1}{2}| < \tfrac{1}{4}\right] = P\left[\tfrac{1}{4} < X < \tfrac{3}{4}\right]$$

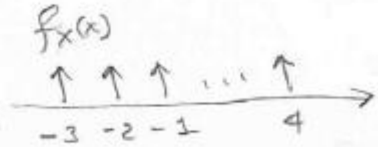
$$= \frac{3}{4}\left[\left(\tfrac{3}{4}+1\right) - \tfrac{1}{3}\left(\left(\tfrac{3}{4}\right)^3+1\right)\right] - \frac{3}{4}\left[\left(\tfrac{1}{4}+1\right) - \tfrac{1}{3}\left(\left(\tfrac{1}{4}\right)^3+1\right)\right]$$

$$= 0.2734$$

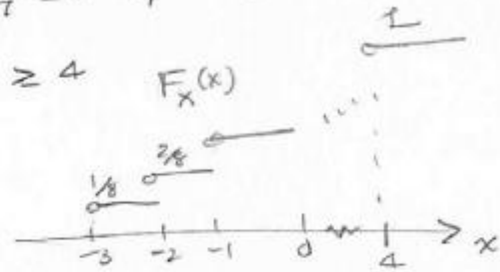
Problem 4.3 Solution

4.27

(a) $f_X(x) = \sum_{j=-3}^{4} \frac{1}{8}\,\delta(x-j)$



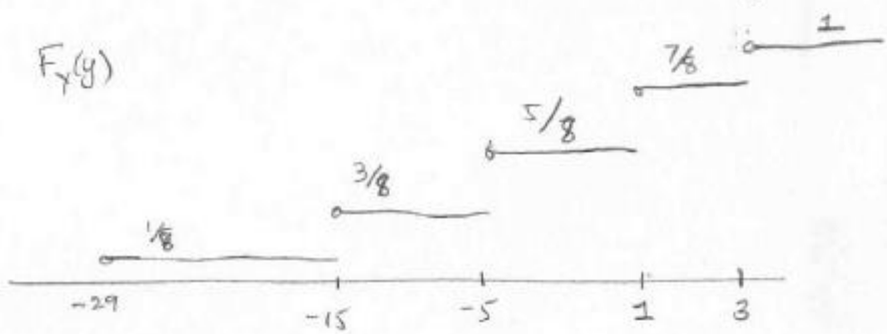
$$F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{j+3}{8} & \frac{j+3}{7} \le x < \frac{j+4}{7} \quad j = -3, \dots, 3 \\ 1 & x \ge 4 \end{cases}$$



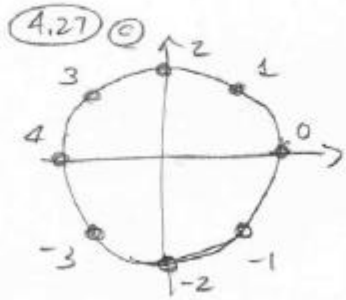
(b)

X	-3	-2	-1	0	1	2	3	4
$Y = -2X^2 + 3$	-15	-5	1	3	1	-5	-15	-29

$$f_Y(y) = \frac{1}{8}\delta(y+29) + \frac{2}{8}\delta(y+15) + \frac{2}{8}\delta(y+5) + \frac{2}{8}\delta(y-1) + \frac{1}{8}\delta(y-3)$$



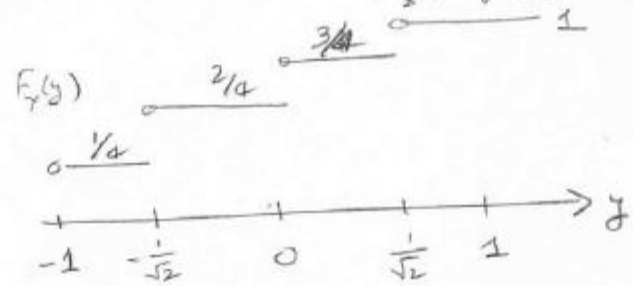
4.27 (c)



x	-3	-2	-1	0	1	2	3	4
$\cos\frac{\pi x}{8}$	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1

$$f_Y(y) = \frac{2}{8}\delta(y+1) + \frac{2}{8}\delta\left(y+\frac{1}{\sqrt{2}}\right) + \frac{2}{8}\delta\left(y-\frac{1}{\sqrt{2}}\right) + \frac{2}{8}\delta(y-1)$$

$F_Y(y)$

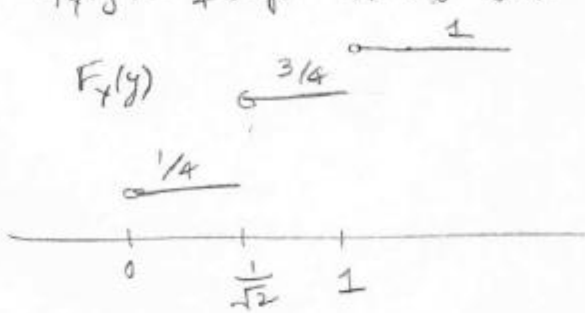


4.27 d

$$Y = \cos^2\frac{\pi X}{8} = \begin{cases} \frac{1}{2} & x = -3, -1, 1, 3 \\ 1 & x = 0, 4 \\ 0 & x = 2, -2 \end{cases}$$

$$f_Y(y) = \frac{1}{4}\delta(y) + \frac{2}{4}\delta\left(y-\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\delta(y-1)$$

$F_Y(y)$



Problem 4.4 Solution

4.30

a) $F_X(x|C) = \dfrac{P[\{X \le x\} \cap \{X > 0\}]}{P[X>0]} = \dfrac{P[0 < X \le x]}{P[X>0]} \qquad x \geq 0$

$$= \begin{cases} 0 & x \le 0 \\ \dfrac{F_X(x) - F_X(0)}{1 - F_X(0)} & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x \le 0 \\ \dfrac{-\frac{1}{4}e^{-2x} + \frac{1}{4}}{1 - (1 - \frac{1}{4})} = 1 - e^{-2x} & x \geq 0 \end{cases}$$

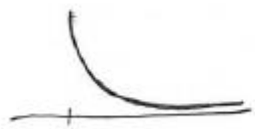
b) $F_X(x|C) = \dfrac{P[\{X \le x\} \cap \{X = 0\}]}{P[X=0]}$

$$= \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Problem 4.5 Solution

4.34

a) $f_X(x) = \begin{cases} 0 & x < x_m \\ \dfrac{\alpha x_m^\alpha}{x^{\alpha-1}} & x \geq x_m \end{cases}$



b) $F_X(x \mid X > t) = \dfrac{P[\{X \leq x\} \cap \{X > t\}]}{P[X > t]} = \dfrac{P[t < X \leq x]}{P[X > t]}$

$= \begin{cases} 0 & x \leq t \\ \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & x \geq t \end{cases}$

if $t \geq x_m$ $\quad F_X(x \mid X > t) = \dfrac{1 - \frac{x_m^\alpha}{x^\alpha} - 1 + \frac{x_m^\alpha}{t^\alpha}}{1 - \left(1 - \frac{x_m^\alpha}{t^\alpha}\right)} = t^\alpha\left(\frac{1}{t^\alpha} - \frac{1}{x^\alpha}\right) = 1 - \left(\frac{t}{x}\right)^\alpha \quad x \geq t$

if $t < x_m$ $\quad F_X(x \mid X > t) = 1 - \left(\dfrac{x_m}{x}\right)^\alpha \quad x \geq x_m$

$f_X(x \mid X > t) = \dfrac{f_X(x)}{1 - F_X(t)} \quad x \geq t$

if $t \geq x_m$ $\quad f_X(x \mid X > t) = \dfrac{\alpha \frac{x_m^\alpha}{x^{\alpha-1}}}{\frac{x_m^\alpha}{t^\alpha}} = \alpha t \left(\dfrac{t}{x}\right)^{\alpha-1} \quad x \geq t$

if $t < x_m$ $\quad f_X(x \mid X > t) = \alpha \dfrac{x_m^\alpha}{x^{\alpha-1}} \quad x \geq x_m$

c) $\dfrac{P[\{X > t + x\} \cap \{X > t\}]}{P[X > t]} \xrightarrow[t \to \infty]{} 1$ The longer you wait the longer you are likely to wait more!

Problem 4.6 Solution

4.38

a) $F_Y(x) = F_Y(x|B_0)P[B_0] + F_Y(x|B_1)P[B_1]$

$= P[Y \le x \mid X=-1](1-p) + P[Y \le x \mid X=1]p$

$= P[X+N \le x \mid X=-1](1-p) + P[X+N \le x \mid X=1]p$

$= P[N \le x+1](1-p) + P[N \le x-1]p$

$= F_N(x+1)(1-p) + F_N(x-1)p$

$f_Y(x) = \frac{d}{dx}F_Y(x)$

$= (1-p) f_N(x+1) + p f_N(x-1)$

$f_Y(x|B_0) = f_N(x+1) = \frac{\alpha}{2}e^{-\alpha|x+1|}$

$f_Y(x|B_1) = f_N(x-1) = \frac{\alpha}{2}e^{-\alpha|x-1|}$

$f_Y(x) = \frac{1}{2}\left[\frac{\alpha}{2}e^{-\alpha|x+1|} + \frac{\alpha}{2}e^{-\alpha|x-1|}\right] = \frac{1}{4}\alpha\left[e^{-\alpha|x+1|} + e^{-\alpha|x-1|}\right]$

b) $P[Y<0|B_1] = P[X+N<0|X=1] = P[N<-1]$

$= \frac{\alpha}{2}e^{-\alpha|-1|} = \frac{\alpha}{2}e^{-\alpha}$

$P[Y \ge 0|B_0] = P[X+N \ge 0|X=-1] = P[N \ge 1]$

$= \frac{\alpha}{2}e^{-\alpha}$

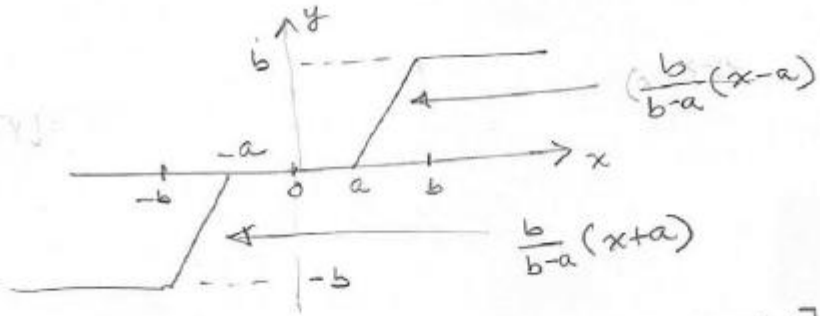
c) $P_E = P[Y<0|B_1]P[B_1] + P[Y \ge 0|B_0]P[B_0]$

$= 0.5\frac{\alpha}{2}e^{-\alpha} + 0.5 \quad \frac{\alpha}{2}e^{-\alpha} \quad = \frac{\alpha}{2}e^{-\alpha}$

Problem 4.7 Solution

4.55

(a)



$$E[Y] = -b\,P[X \le -b] + b\,P[X \ge b] + 0 \cdot P[-a \le X \le a] + \int_{-b}^{a} \frac{b}{b-a}(x+a) f_X(x)\,dx + \int_{a}^{b} \frac{b}{b-a}(x-a) f_X(x)\,dx$$

$$E[Y^2] = b^2\,P[X \le b] + b^2\,P[X \ge b] + \int_{-b}^{-a} \frac{b^2}{(b-a)^2}(x+a)^2 f_X(x)\,dx + \int_{a}^{b} \frac{b^2}{(b-a)^2}(x-a)^2 f_X(x)\,dx$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

(b) $f_X(x) = \frac{1}{2} e^{-|x|} dx \quad -\infty < x < \infty, \quad a = 1, \; b = 2$

$$E[Y] = -2 \underbrace{P[X \le -2]}_{\frac{1}{2}e^{-2}} + 2 \underbrace{P[X \ge 2]}_{\frac{1}{2}e^{2}} + \underbrace{\int_{-2}^{1} 2(x+1)\, e^{x} dx + \int_{1}^{2} 2(x-1)\, e^{-x} dx}_{=0}$$

$$E[Y^2] = 4 \cdot \frac{1}{2} e^{-2} + 4 \cdot \frac{1}{2} e^{-2} + 4\int_{-2}^{-1} (x+1)^2 e^{x} dx + 4\int_{1}^{2} (x-1)^2 e^{-x} dx$$

$$= 4e^{-2} + 8\int_{1}^{2} (x-1)^2 e^{-x} dx$$

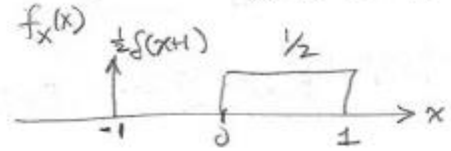
(4.55) $\int_1^2 \underbrace{(x-1)}_{y}^2 e^{-x}\,dx = \int_0^1 y^2 e^{-y-1}\,dy = e^{-1}\left[e^{-y}(x^2+2x+2)\right]_0^1$

$= e^{-1}\left[5e^{-1} - 2\right] = 5e^{-2} - 2e^{-1}$

$\therefore\ VAR[Y] = 4e^{-2} + 40e^{-2} - 16e^{-1} = 44e^{-2} - 16e^{-1}$

(c) $a = 1/2 \quad b = 3/2 \qquad P[X \le -\frac{3}{2}] = 0 = P[X \ge \frac{3}{2}]$

$f_X(x)$



$E[Y] = \int_{-3/2}^{-1/2} \frac{3}{2}\left(x+\frac{1}{2}\right)\frac{1}{2}\delta(x+1)\,dx + \int_{1/2}^{1} \frac{3}{2}\left(x-\frac{1}{2}\right)\cdot\frac{dx}{2}$

$= \frac{3}{4}\left(-1+\frac{1}{2}\right) + \frac{3}{4}\int_0^{1/2} x'\,dx' = -\frac{3}{8} + \frac{3}{4}\frac{1}{2}\left(\frac{1}{2}\right)^2 = -\frac{9}{32}$

$E[Y^2] = \int_{-3/2}^{-1/2} \frac{9}{4}\left(x+\frac{1}{2}\right)^2 \frac{1}{2}\delta(x+1)\,dx + \frac{9}{4}\int_{1/2}^{1} \left(x-\frac{1}{2}\right)^2 \frac{dx}{2}$

$= \frac{9}{4}\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + \frac{9}{8}\int_0^{1/2} x'^2\,dx' = \frac{21}{64}$

$VAR(Y) = \frac{21}{64} - \left(\frac{9}{32}\right)^2 = \frac{255}{1024}$

(d)

$$Y = \begin{cases} \frac{1}{2} & u^3 > \frac{1}{2} \\ 2\left(u^3 - \frac{1}{4}\right) & \frac{1}{4} < u^3 < \frac{1}{2} \\ 0 & -\frac{1}{4} < u^3 < \frac{1}{4} \\ 2\left(u^3 + \frac{1}{4}\right) & -\frac{1}{2} < u^3 < -\frac{1}{4} \\ -\frac{1}{2} & u^3 < -\frac{1}{2} \end{cases}$$

4.55d – continued –

$$E[Y] = \underbrace{\tfrac{1}{2}P[U^3 > \tfrac{1}{2}] - \tfrac{1}{2}P[U^3 < -\tfrac{1}{2}]}_{=0}$$

$$\underbrace{+ \underbrace{\int_{-1/2}^{-1/4} 2(u^3 + \tfrac{1}{4})\frac{du}{2}}_{\text{let } u' = -u} + \int_{1/4}^{1/2} 2(u^3 - \tfrac{1}{4})\frac{du}{2}}_{=0}$$

$$E[Y] = 0$$

$$VAR[Y] = E[Y^2] = \left(\tfrac{1}{2}\right)^2 P[U^3 > \tfrac{1}{2}] + \left(-\tfrac{1}{2}\right)^2 P[U^3 < -\tfrac{1}{2}]$$

$$+ \int_{-1/2}^{-1/4} 4(u^3 + \tfrac{1}{4})^2 \frac{du}{2} + \int_{1/4}^{1/2} 4(u^3 - \tfrac{1}{4})^2 \frac{du}{2}$$

$$= 2\left(\tfrac{1}{2}\right)^2 \frac{1 - \left(\frac{1}{2}\right)^{1/3}}{2} + \cancel{2} \cdot 4 \int_{1/4}^{1/2} \underbrace{(u^3 - \tfrac{1}{4})^2}_{u^6 - \frac{1}{2}u^3 + \frac{1}{16}} \frac{du}{\cancel{2}}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{1/3}}{1/4} + 4\underbrace{\int_{1/4}^{1/2} u^6\,du}_{\left.\frac{u^7}{7}\right|_{1/4}^{1/2}} - 2\underbrace{\int_{1/4}^{1/2} u^3\,du}_{\left.\frac{u^4}{4}\right|_{1/4}^{1/2}} + \frac{1}{4}\underbrace{\int_{1/4}^{1/2} du}_{1/4}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{1/3}}{1/4} + \frac{4}{7}\left[\left(\tfrac{1}{2}\right)^7 - \left(\tfrac{1}{4}\right)^7\right] - \frac{1}{2}\left[\left(\tfrac{1}{2}\right)^4 - \left(\tfrac{1}{4}\right)^4\right] + \frac{1}{16}$$

$$= 4(.7937) + \frac{4}{7}\left(\tfrac{1}{2}\right)^7\left[1 - \left(\tfrac{1}{2}\right)^7\right] - \frac{1}{2}\left(\tfrac{1}{2}\right)^4\left[1 - \left(\tfrac{1}{2}\right)^4\right] + \frac{1}{16}$$

$$= 3.212$$

Problem 4.8 Solution

4.56

a) $E[Y] = 3E[X]+2$

$VAR[Y] = VAR[3X+2] = VAR[3X] = 9\,VAR[X]$

b) Laplacian R.V. $E[X]=0$

$VAR[X] = \frac{2}{\alpha^2}$

$E[Y] = 2$

$VAR[Y] = 9\left(\frac{2}{\alpha^2}\right) = \frac{18}{\alpha^2}$

c) Caussian R.V. $E[X]=m$

$VAR[X]=\sigma^2$

$E[Y] = 3m+2$

$VAR[Y] = 9\sigma^2$

d) $E[X] = b\int_0^1 \cos(2\pi u)\,du = -b\sin(2\pi u)\Big|_0^1 = 0$

$VAR[X] = b^2\int_0^1 \cos^2(2\pi u)\,du$

$= b^2\int_0^1 \frac{1}{2}du + \frac{b^2}{2}\int_0^1 \cos 4\pi u\,du$

$= b^2\frac{1}{2} + b^2\left(\frac{1}{4\pi}\right)\left(-\sin 4\pi u\right)\Big|_0^1$

$= \frac{b^2}{2}$

$E[Y]=2$

$VAR[Y] = \frac{9b^2}{2}$

Problem 4.9 Solution

4.63

a) $P[X>4] = 1 - F_X(4) = 1 - \Phi\left(\frac{4-5}{4}\right) = 1 - \Phi\left(-\frac{1}{4}\right) = \Phi\left(\frac{1}{4}\right) = 0.598$

$P[X \geq 7] = 1 - F_X(7) = 1 - \Phi\left(\frac{7-5}{4}\right) = 1 - \Phi\left(\frac{1}{2}\right) = 0.308$

$P[6.72 < X < 10.16] = \Phi\left(\frac{10.16-5}{4}\right) - \Phi\left(\frac{6.72-5}{4}\right) = \Phi(1.29) - \Phi(0.43) = 0.235$

$P[2 < X < 7] = \Phi\left(\frac{7-5}{4}\right) - \Phi\left(\frac{2-5}{4}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{3}{4}\right) = 0.465$

$P[6 \leq X \leq 8] = \Phi\left(\frac{8-5}{4}\right) - \Phi\left(\frac{6-5}{4}\right) = \Phi\left(\frac{3}{4}\right) - \Phi\left(\frac{1}{4}\right) = 0.175$

b) $P[X<a] = 0.8869$

$\Phi\left(\frac{a-5}{4}\right) = 0.8869 = 1 - Q(x)$

$Q(x) = 0.1131 \rightarrow x = 1.2 = \frac{a-5}{4} \rightarrow a = 9.8$

c) $P[X>b] = 1 - \Phi\left(\frac{b-5}{4}\right) = 0.11131$

$Q(x) = 0.11131 \rightarrow x = 1.2 = \frac{b-5}{4} \rightarrow b = 9.8$

d) $P[13 < X \leq c] = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) - \Phi\left(\frac{13-5}{4}\right) = \Phi\left(\frac{c-5}{4}\right) - \Phi(2) = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) = 0.0123 + 0.9772 = 0.9895$

$Q\left(\frac{c-5}{4}\right) = 0.0105 \rightarrow x = 2.3 = \frac{c-5}{4} \rightarrow c = 14.2$

Problem 4.10 Solution

4.70) Gamma RV

(a) $$E[X] = \int_0^\infty x \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \int_0^\infty \frac{\lambda(\lambda x)^{\alpha} e^{-\lambda x}}{\lambda\,\Gamma(\alpha)} dx$$

$$= \int_0^\infty \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)/\Gamma(\alpha)}{\lambda} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx}_{1}$$

$$= \frac{\alpha}{\lambda}$$

(b) $$E[X^2] = \int_0^\infty x^2 \frac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha+1}}{\Gamma(\alpha+2)} e^{-\lambda x} dx}_{1}$$

$$= \frac{(\alpha+1)(\alpha)}{\lambda^2}$$

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$$

$$VAR[X] = \frac{(\alpha+1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha}{\lambda^2}$$

(c) m-Erlang $\alpha = m$

$$E[X] = \frac{m}{\lambda} \quad VAR[X] = \frac{m}{\lambda^2}$$

(d) chi-square $\alpha = k/2 \quad \lambda = \frac{1}{2}$

$$E[X] = \frac{k}{2}\frac{1}{1/2} = k$$

$$VAR[X] = \frac{k}{2}\frac{1}{1/4} = 2k$$

Problem 4.11 Solution

4.73

$$P[X \geq 15] = \sum_{k=0}^{4-1} \frac{\left(\frac{1}{3} 15\right)^k}{k!} e^{-\frac{15}{3}}$$

$$= \sum_{k=0}^{3} \frac{5^k}{k!} e^{-5}$$

$$= 0.2650$$