ECE 2521 Analysis of Stochastic Processes
Solutions to Homework 6

Problem 1 Solution

6.7 a)

$$\begin{aligned}
1 &= \int_0^1\int_0^1\int_0^1 k(x+y+z)\,dx\,dy\,dz \\
&= k\int_0^1\int_0^1\left(\frac{1}{2}+y+z\right)dy\,dz \\
&= k\int_0^1\left(\left(\frac{1}{2}+z\right)+\frac{1}{2}\right)dz \\
&= k\left(1+\frac{1}{2}\right) \Rightarrow k=\frac{2}{3}
\end{aligned}$$

b) $f_{XY}(x,y) = \frac{2}{3}\int_0^1 (x+y+z)dz = \frac{2}{3}\left[x+y+\frac{1}{2}\right]$

$$f_Z(z|x,y) = \frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)} = \frac{x+y+z}{x+y+\frac{1}{2}}$$

c) $f_X(x) = \frac{2}{3}\int_0^1 (x+y+\frac{1}{2})\,dy = \frac{2}{3}\left[xy\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 + \frac{1}{2}y\Big|_0^1\right] = \frac{2}{3}\left[x+1\right]$

Problem 2 Solution

6.22

a) $$\underline{Z} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \qquad |A| = 1$$

$$\begin{aligned}
X_1 &= \;= U \\
X_2 &= V - X_1 = V - U \\
X_3 &= W - X_1 - X_2 = W - V
\end{aligned}$$

$$f_{\underline{Z}}(u,v,w) = \frac{f_{\underline{X}}(\underline{x})}{|A|}\bigg|_{\underline{x}=A^{-1}\underline{u}} = f_{\underline{X}}(u, v-u, w-v)$$

b)

$$\begin{aligned}
f_{\underline{Z}}(u,v,w) &= \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{u^2}{2}} e^{-(v-u)^2/2} e^{-(w-v)^2/2} \\
&= \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{2}[2u^2+2v^2+w^2-2uv-2vw]} \\
&= \frac{1}{(\sqrt{2\pi})^3} e^{-[u^2+v^2+\frac{1}{2}w^2-uv-vw]}
\end{aligned}$$

(6.22) (c)

$$f(v,w) = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-[u^2-uv]}\, du}\; e^{-[v^2+\frac{1}{2}w^2-vw]}$$

$$= e^{\frac{1}{4}v^2}\int_{-\infty}^{\infty} \frac{1/2}{\sqrt{2\pi\frac{1}{2}}} e^{-[u-\frac{1}{2}v]^2}$$

$$f(v,w) = \frac{1}{4\pi} e^{-[\frac{3}{4}v^2+\frac{1}{2}w^2-vw]}$$

Problem 3 Solution

(6.35)

Note symmetry in x, y, and z so number of calculations can be reduced drastically.

$$E[X] = \frac{2}{3}\int_0^1 x(x+1)dx = \frac{2}{3}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = \frac{5}{9}$$

$$= E[Y] = E[Z]$$

$$E[X^2] = \frac{2}{3}\int_0^1 x^2(x+1)\,dx = \frac{2}{3}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 = \frac{7}{18}$$

$$VAR[X] = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$E[XY] = \frac{2}{3}\int_0^1\int_0^1 xy\left(x+y+\frac{1}{2}\right)dx\,dy$$

$$= \frac{2}{3}\int_0^1 x\left(\underbrace{\int_0^1\left(yx+y^2+\frac{1}{2}y\right)dy}_{\frac{1}{2}x+\frac{1}{3}+\frac{1}{4}}\right)$$

$$= \frac{2}{3}\int_0^1\left(\frac{1}{2}x^2+\frac{7}{12}x\right)dx$$

$$= \frac{2}{3}\left[\frac{1}{2}\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{2}\right] = \frac{11}{36}$$

$$E[XY] - E[X]E[Y] = \frac{11}{36} - \left(\frac{5}{9}\right)^2 = -\frac{1}{324}$$

$$\underline{m}_{\underline{X}} = \begin{bmatrix} \frac{5}{9} \\ \frac{5}{9} \\ \frac{5}{9} \end{bmatrix} \qquad K_{\underline{X}} = \begin{bmatrix} \frac{13}{162} & -\frac{1}{324} & -\frac{1}{324} \\ -\frac{1}{324} & \frac{13}{162} & -\frac{1}{324} \\ -\frac{1}{324} & -\frac{1}{324} & \frac{13}{162} \end{bmatrix}$$

Problem 4 Solution

6.57

a) $K_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$K_Y = AA^T = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad K_Y(y_1, y_2, y_3) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

b) $\det(K_Y) = 4$

$K_Y^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

$$f_Y(y_1, y_2, y_3) = \frac{e^{-\frac{1}{2}\left(\frac{1}{4}\right)\left[3x_1^2 + 4x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3 + 2x_1x_3\right]}}{(2\pi)^{3/2}\,(2)}$$

c) $$f(y_1, y_2) = \frac{e^{-\frac{1}{2}\left[2y_1^2 + 2y_2^2 - 2y_1y_2\right]\left(\frac{1}{3}\right)}}{2\pi\sqrt{3}}$$

$K' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \det(K') = 3$ $\quad K'^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$f(y_1, y_3) = \frac{e^{-\frac{1}{2}\left(\frac{1}{2}\right)\left[x^2 + z^2\right)}}{2\pi\,(2)}$$

$K' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \det(K') = 4$

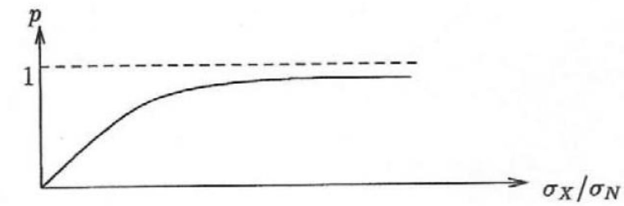
d) $P = \begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix}$ $\quad \Lambda = K_Z$

$A = P^T$

Problem 5 Solution

6.72 a) $COV[XY] = E[XY] = E[X(X+N)] = VAR[X] = \sigma_X^2$

$$\rho = \frac{COV[XY]}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sigma_X(\sigma_X^2+\sigma_N^2)^{1/2}} = \left(\frac{1}{1+\sigma_N^2/\sigma_X^2}\right)^{1/2}$$



b)

$$\begin{aligned}
\hat{X} &= \rho\frac{\sigma_X}{\sigma_Y}Y = \rho^2 Y \\
&= \frac{COV[XY]}{\sigma_Y^2}Y \\
&= \frac{\sigma_X^2}{\sigma_Y^2}Y \\
MSE &= E[(X-\hat{X})^2] \\
&= E[(X-\rho^2 Y)^2] \\
&= VAR[X^2] + \rho^4 VAR[Y] - 2\rho^2 COV[X,Y] \\
&= \sigma_X^2 + \rho^2\sigma_X^2 - 2\rho^2\sigma_X^2 \\
&= \sigma_X^2\ (1-\rho^2)
\end{aligned}$$

6.72 From Example 6.26 the MAP estimator is the same as the MMSE estimator

On the other hand, the ML receiver is given by

$$\hat{X}_{ML} = \frac{\sigma_X}{\rho\sigma_Y}(Y - m_Y) + m_X = \frac{\sigma_X}{\rho\sigma_Y}Y$$

$$= \frac{\sigma_X\sqrt{1+\sigma_N^2/\sigma_X^2}}{\sqrt{\sigma_X^2+\sigma_N^2}}Y = Y$$

Thus the ML estimator gives a different estimate.

The MSE for the ML estimator is

$$\begin{aligned}
MSE_{ML} &= E[(X-\hat{X}_{ML})^2] \\
&= E[(X-Y)^2] \\
&= E[N^2] \\
&= \sigma_N^2
\end{aligned}$$

In comparison to the MAP estimator MSE we have

$$MSE_{MAP} = \sigma_x^2(1-\rho^2) = \sigma_x^2\left(1-\frac{1}{1+\frac{\sigma_N^2}{\sigma_x^2}}\right) = \sigma_N^2 \underbrace{\frac{\sigma_x^2}{\sigma_x^2+\sigma_N^2}}_{<1}$$

$$\therefore \quad MSE_{MAP} < MSE_{ML}$$

Problem 6 Solution

$$\mathcal{E}[S_n] = \mathcal{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathcal{E}[X_i] = n\mu .$$

$$VAR(S_n) = \underbrace{\sum_{k=1}^{n} VAR(X_k)}_{\substack{\text{sum of diag.}\\ \text{elements of}\\ \text{covariance matrix } K}} + \underbrace{\sum_{j=1}^{n}\sum_{k=1}^{n}{}_{j\neq k}\; COV(X_j, X_k)}_{\substack{\text{sum of off-diag.}\\ \text{element of } K}}$$

.

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho^2\sigma^2 & \ldots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \ldots & \rho^{n-2}\sigma^2 \\ \vdots & & & & \\ \rho^{n-1}\sigma^2 & & \ldots & & \sigma^2 \end{bmatrix}$$

$$\begin{aligned} VAR(S_n) &= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1}\sum_{k=0}^{j-1} \rho^k \\ &= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1} \frac{1-\rho^j}{1-\rho} \\ &= n\sigma^2 + 2\rho\sigma^2 \left[\frac{n-1}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)\frac{1-\rho^{n-1}}{1-\rho}\right] \end{aligned}$$

Problem 7.7 Solution (Optional)

6.73 $f(x,y,z) = \frac{2}{3}(x+y+z) \qquad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$

$$f(x,y) = \frac{2}{3}\left[x+y+\frac{1}{2}\right] \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

$$f(x) = \frac{2}{3}\left[x+1\right] \qquad 0 \le x \le 1$$

(a) $\Rightarrow E[X] = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{2}{3}\left[\frac{1}{3}+\frac{1}{2}\right] = \frac{2}{3}\,\frac{5}{6} = \frac{5}{9} = E[Y] = E[Z]$

$$E[X^2] = \frac{2}{3}\int_0^1 x^2(x+1)\,dx = \frac{2}{3}\left[\frac{1}{4}+\frac{1}{3}\right] = \frac{2}{3}\,\frac{7}{12} = \frac{7}{18}$$

$$VAR[X] = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{7}{18} - \frac{25}{81} = \frac{63-50}{162} = \frac{13}{162} = VAR[Y] = VAR[Z]$$

$$E[XY] = \frac{2}{3}\int_0^1\int_0^1 xy\left(x+y+\frac{1}{2}\right)dx\,dy = \frac{2}{3}\left[\int_0^1\int_0^1\left(x^2y + xy^2 + \frac{1}{2}xy\right)dx\,dy\right]$$

$$= \frac{2}{3}\int_0^1 dx\left[x^2\frac{1}{2} + x\frac{1}{3} + \frac{1}{2}x\frac{1}{2}\right] = \frac{2}{3}\left[\underbrace{\frac{1}{2}\frac{1}{3}+\frac{1}{3}\frac{1}{2}+\frac{1}{4}\frac{1}{2}}_{\frac{4+4+3}{24}}\right]$$

$$= \frac{11}{36}$$

$$COV(X,Y) = \frac{11}{36} - \left(\frac{5}{9}\right)^2 = \frac{99-100}{324} = \frac{-1}{324}$$ almost uncorrelated

$$= COV(X,Z) = COV(Y,Z)$$

The optimum linear estimator for Y given X and Z is:

$$\hat{X} = (a_1, a_2)\begin{bmatrix} X - m_X \\ Z - m_Z \end{bmatrix} + m_Y$$

where from Eqn 6.63a

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} VAR(X) & COV(X,Z) \\ COV(X,Z) & VAR(Z) \end{bmatrix}^{-1}}_{K_{XZ}^{-1}} \underbrace{\begin{bmatrix} COV(Y,X) \\ COV(Y,Z) \end{bmatrix}}_{E\left[(Y-m_Y)\begin{bmatrix} X-m_X \\ Z-m_Z \end{bmatrix}\right]}$$

(6.73)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{324} \begin{bmatrix} 26 & -1 \\ -1 & 26 \end{bmatrix} \right) \begin{bmatrix} -1/324 \\ -1/324 \end{bmatrix} = \frac{324}{705} \begin{bmatrix} 26 & 1 \\ 1 & 26 \end{bmatrix} \begin{bmatrix} -\frac{1}{324} \\ -\frac{1}{324} \end{bmatrix}$$

$$= - \begin{bmatrix} \frac{27}{705} \\ \frac{27}{705} \end{bmatrix}$$

$$\bar{X}_{LMMSE} = -\frac{27}{705} [1, 1] \begin{bmatrix} X - \frac{5}{9} \\ Z - \frac{5}{9} \end{bmatrix} + \frac{5}{9}$$

$$= -\frac{27}{705}\left(X - \frac{5}{9}\right) - \frac{27}{705}\left(Z - \frac{5}{9}\right) + \frac{5}{9}$$

$$= -\frac{27}{705}(X+Z) + \frac{5}{9}\left(\frac{651}{705}\right)$$

fm Eqn 6.63b

$$MSE_{LMSE} = \sigma_X^2 - a^T \begin{bmatrix} COV(YX) \\ COV(YZ) \end{bmatrix} = \frac{13}{162} - \frac{27}{705}(1,1) \begin{bmatrix} -\frac{1}{324} \\ -\frac{1}{324} \end{bmatrix}$$

$$= \frac{13}{162} - \frac{54}{324(705)} = \frac{13}{162} - \frac{1}{235(6)} = 0.0795$$

(b)

$$f(y \mid x, z) = \frac{\frac{2}{3}(x+y+z)}{\frac{2}{3}(x+z+\frac{1}{2})} \qquad 0 < y < 1$$

$$E[Y \mid x, z] = \frac{1}{x+z+\frac{1}{2}} \int_0^1 y(x+y+z)\,dy = \frac{x\frac{1}{2} + \frac{1}{3} + z\frac{1}{2}}{x+z+\frac{1}{2}} = \frac{\frac{1}{2}(x+z)+\frac{1}{3}}{x+z+\frac{1}{2}}$$

$$\hat{Y}_{MMSE} = \frac{\frac{1}{2}(X+Z) + \frac{1}{3}}{X+Z+\frac{1}{2}}$$

(b) Continued —

6.73 Five equations following Eqn 6.59

$$MSE_{MMSE} = \int_0^1\int_0^1 dx\,dz\; E[(Y-\hat{Y})^2|x,z)\, f_{XZ}(x,z)$$

$$E[(Y-\hat{Y})^2|x,z] = \int_0^1 (y - E[Y|x,z])^2 \;\frac{(x+y+z)}{x+z+\frac{1}{2}}\, dy$$

$$= E[Y^2|x,z] - 2E^2[Y|x,z] + E[Y|x,z]^2$$

$$E[Y^2|x,z] = \int_0^1 y^2\, \frac{x+y+z}{x+z+\frac{1}{2}}\, dy = \frac{\frac{1}{3}x+\frac{1}{4}+\frac{1}{3}z}{x+z+\frac{1}{2}}$$

$$E[(Y-\hat{Y})^2|x,z] = \frac{\frac{1}{3}(x+z)+\frac{1}{4}}{x+z+\frac{1}{2}} - \left(\frac{\frac{1}{2}(x+z)+\frac{1}{3}}{x+z+\frac{1}{2}}\right)^2$$

$$= \frac{\frac{1}{12}\left[(x+z)^2+(x+z)+\frac{1}{6}\right]}{\left((x+z)+\frac{1}{2}\right)^2}$$

$$E[(Y-\hat{Y})^2] = \frac{1}{12}\int_0^1\int_0^1 \frac{(x+z)^2+(x+z)+\frac{1}{6}}{\left((x+z)+\frac{1}{2}\right)^2}\;\frac{2}{3}\left((x+z)+\frac{1}{2}\right) dx\,dz$$

$$= \frac{1}{18}\int_0^1\int_0^1 \frac{(x+z)^2+(x+z)+\frac{1}{6}}{(x+z)+\frac{1}{2}}\, dx\,dz$$

$$= \frac{1}{18}\int_0^1\int_0^1 \left((x+z)+\frac{1}{2} - \frac{1/12}{(x+z)+\frac{1}{2}}\right) dx\,dz$$

$$= \frac{1}{18}\int_0^1 dx\left[x+\frac{1}{2}+\frac{1}{2} - \frac{1}{12}\underbrace{\ln(z+x+\tfrac{1}{2})\Big|_0^1}_{\ln(x+\frac{3}{2})-\ln(x+\frac{1}{2})}\right]$$

(b) continued —

6.73

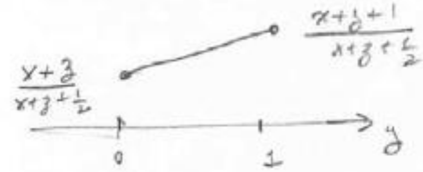
$$MSE_{MMSE} = \frac{1}{18}\left[\frac{1}{2}+1-\frac{1}{12}\left\{\underbrace{\left[\left(x+\tfrac{3}{2}\right)\ln\left(x+\tfrac{3}{2}\right)-x\right]_0^1}_{\left(\frac{5}{2}\ln\frac{5}{2}-1-\frac{3}{2}\ln\frac{3}{2}\right)}-\underbrace{\left[\left(x+\tfrac{1}{2}\right)\ln\left(x+\tfrac{1}{2}\right)-x\right]_0^1}_{\left(\frac{3}{2}\ln\frac{3}{2}-1-\frac{1}{2}\ln\frac{1}{2}\right)}\right\}\right]$$

$$\underbrace{\left(\tfrac{5}{2}\ln\tfrac{5}{2}-1-\tfrac{3}{2}\ln\tfrac{3}{2}\right)-\left(\tfrac{3}{2}\ln\tfrac{3}{2}-1-\tfrac{1}{2}\ln\tfrac{1}{2}\right)}_{\frac{5}{2}\ln\frac{5}{2}-2\frac{3}{2}\ln\frac{3}{2}+\frac{1}{2}\ln\frac{1}{2}}$$

$$= \frac{1}{18}\left[\frac{3}{2}-\frac{1}{12}\left(\frac{5}{2}\ln\frac{5}{2}-3\ln\frac{3}{2}+\frac{1}{2}\ln\frac{1}{2}\right)\right]$$

$$= 0.08187$$ larger than LMMSE (need to recheck both).

(c) MAP Estimator

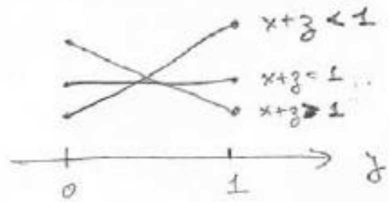


$$f(y|x,z) = \frac{(x+y+z)}{(x+z+\frac{1}{2})} \qquad 0<y<1$$

$$\Rightarrow \hat{Y}_{MAP} = 1 \qquad MSE_{MAP} = E[(Y-1)^2] = E[Y^2]-2E[Y]+1$$

$$= \frac{7}{18} - 2\left(\frac{5}{9}\right)+1 = .277$$

ML Estimator



$$f(x,z|y) = \frac{\frac{2}{3}(x+y+z)}{\frac{2}{3}(y+1)}$$

$$\therefore \quad \hat{Y}_{ML} = \begin{cases} 1 & x+z<1 \\ 0 & x+z>1 \end{cases}$$