University of Pittsburgh

ECE 2195: Special Topics – Computers Machine Learning

Logistic Regression

Mai Abdelhakim, PhD

ECE Department

Swanson School of Engineering

University of Pittsburgh

maia@pitt.edu

Classification

- Response is qualitative – Very common problem
- Examples:

  - Email spam detection system: spam or not spam
  - Medical system: Set of symptoms attributed to one of three possible medical conditions. Which of these conditions does the individual have
  - Handwritten digit recognition system: Is the digit 0, 1, 2,..or 9?

- Reponse Y can be represented by unordered set C

Can we use linear regression for binary classification?

- Assume two classes: We want to classify whether patient has stroke or drug overdose based on symptoms

- We can potentially use the dummy variable approach by defining

$$Y = \begin{cases} 0 & \text{if } \texttt{stroke}; \\ 1 & \text{if } \texttt{drug overdose}. \end{cases}$$

  - If the **predicted** Y > 0.5 (**not meaningful**), then classify as drug overdose, otherwise classify as stoke
    - Linear regression **may** work for two-level response

Can we still use linear regression for more than 2 classes?

- Assume three-level response (3 classes) - Three possible diagnosis: stroke, drug overdose and epileptic seizure
- Can linear regression be applied if we used the following encoding?

$$Y = \begin{cases} 1 & \text{if } \texttt{stroke}; \\ 2 & \text{if } \texttt{drug overdose}; \\ 3 & \text{if } \texttt{epileptic seizure}. \end{cases}$$

OR

$$Y = \begin{cases} 1 & \text{if } \texttt{epileptic seizure}; \\ 2 & \text{if } \texttt{stroke}; \\ 3 & \text{if } \texttt{drug overdose}. \end{cases}$$

  - Different coding result in different linear model and the encoding does not reflect a meaningful interpretation
    - Why error between stroke & seizure is higher than difference between stoke and overdose
  - **Linear regression is not appropriate here**
  - There is no general way to convert qualitative response into quantitative response

Logistic Regression

- **A classifier estimates the conditional probability** that X belongs to each possible class label (elements in C)

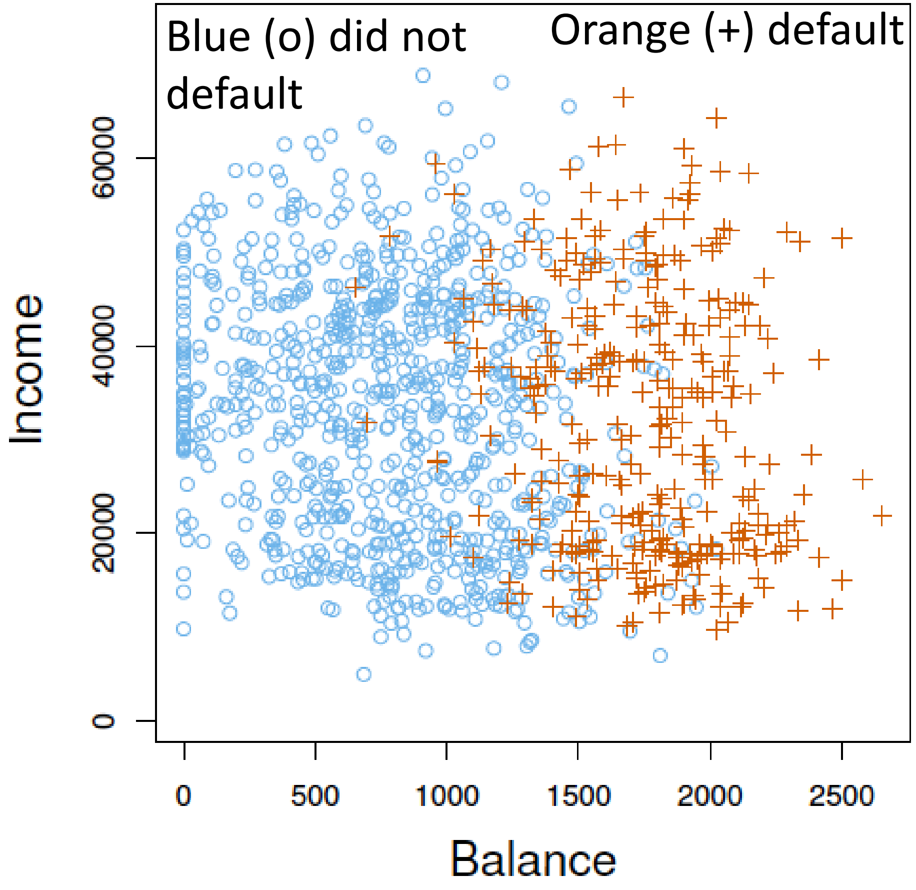
  - Recall optimal Bayes classifier
  - What is the probability that the email is spam (or not) given features of email

- **The predicted class label is the one with the highest probability**

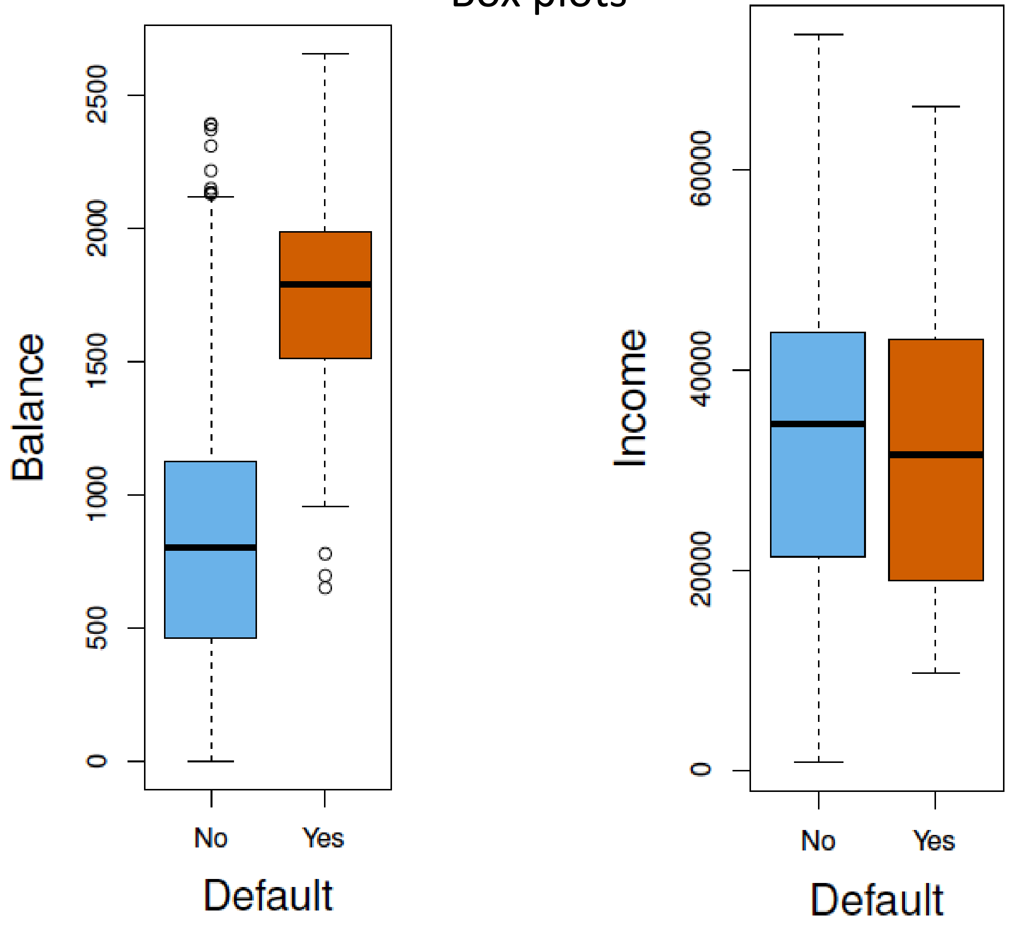
- **Logistic regression: models the probability** that the **response** Y **belongs to a particular category**

Example: Default Data Set

- Predict whether an individual will default on his/her credit card payment
  - Two features: balance on the credit card and income
- Dataset contains information of n=10,000 individuals



Box plots



Logistic regression models the conditional probability

- Consider that we take a single feature: balance

- Logistic regression models the probability of default given the balance:

$$\Pr(\texttt{default} = \texttt{Yes} | \texttt{balance})$$

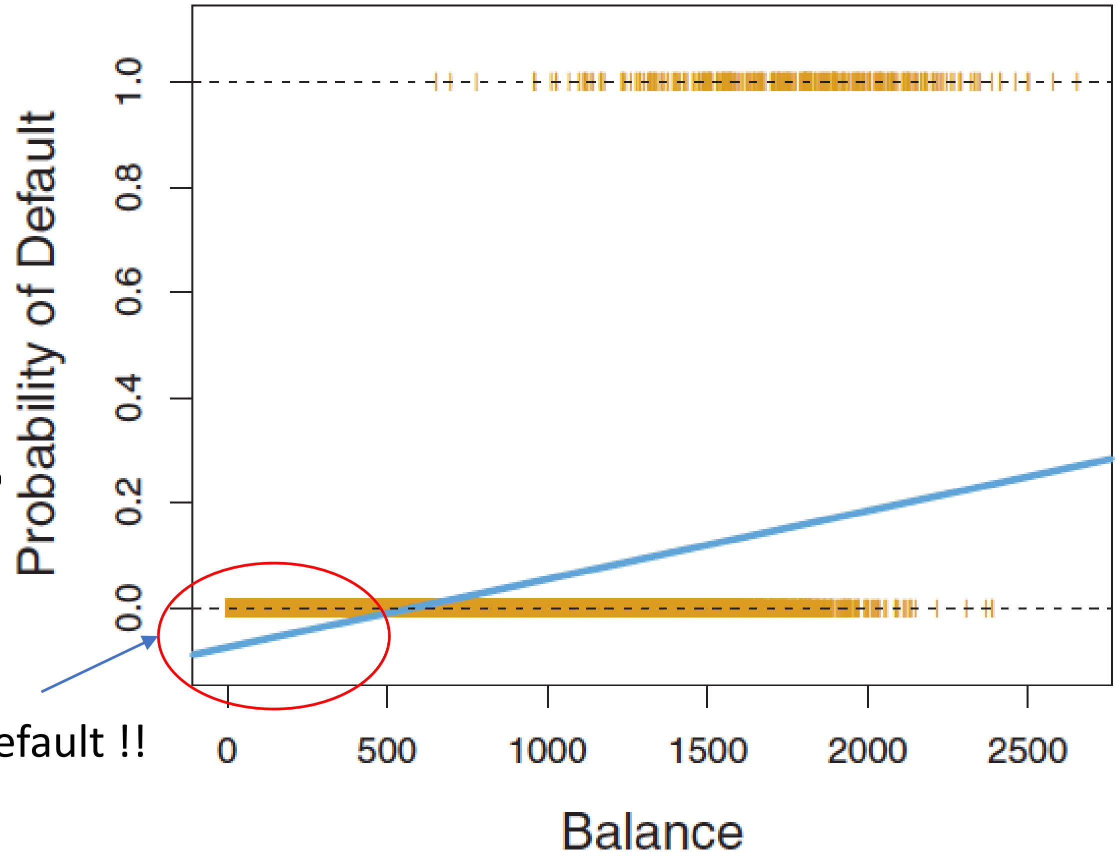
  - Range?
    - This conditional probability is in the range [0,1]

Linear Model for Probability?

- Assume two possible classes, and denote $f(X) = \Pr(Y = 1|X)$
- Now we can assume that this probability is the response we want to predict
- Can we use the linear regression model?

$$f(X) = \beta_0 + \beta_1 X$$

- Linear regression may result in values outside range [0-1]
  - Not estimate of probability, thus is not used



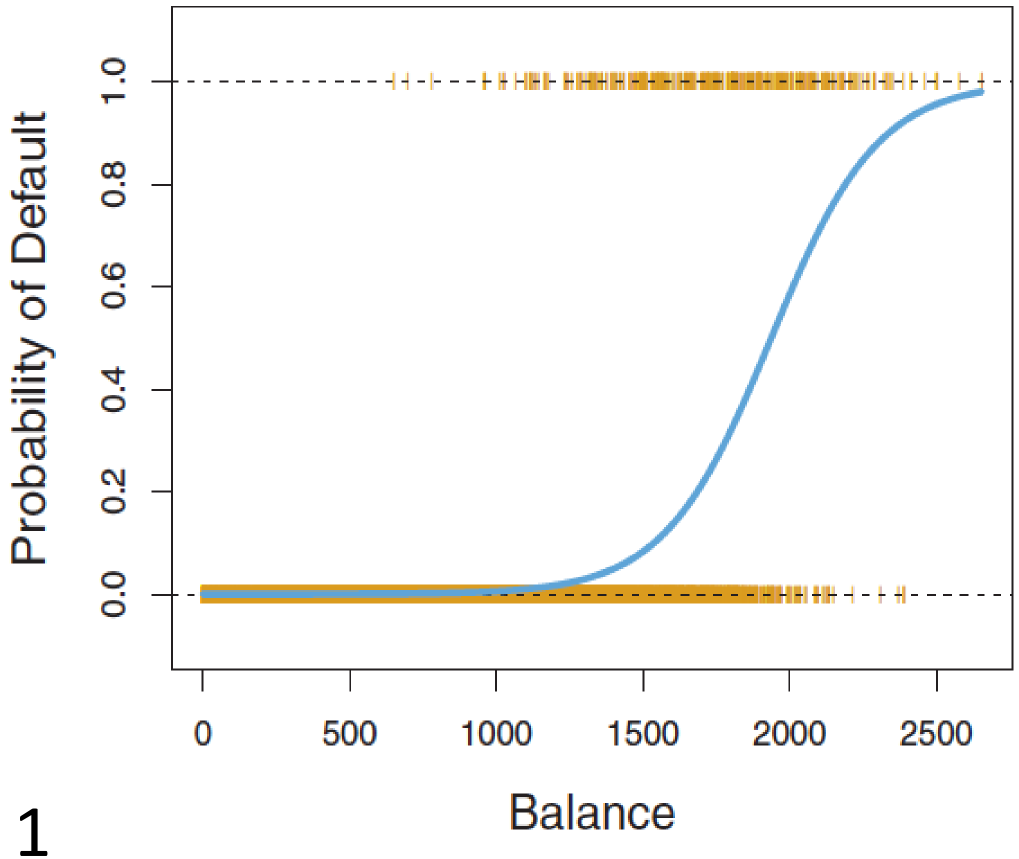
Predict negative probability of default !!

Logistic Model

Logistic – Sigmoid function f(a)=1/(1+exp(-a))

- To solve this problem the logistic regression uses the form (*e = 2.71828* is a mathematical constant):

$$f(X) = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}$$



- *f (X) is always in the range [0,1]*
- Large exp. power (∞): logistic regression output 1
- Small exp. power (−∞): logistic regression output 0

Logit term is linear - hence the term logistic regression

- We can rearrange the terms, and get logit $\log(f(x)/(1 - f(X))$:

$$\frac{f(X)}{1 - f(X)} = e^{\beta_0 + \beta_1 X}$$

$$\log\left(\frac{f(X)}{1 - f(X)}\right) = \beta_0 + \beta_1 X$$

This log is base e, i.e. ln

Logistic regression has a **logit** that is **linear** with X

What is the decision boundary?

Estimating Coefficients – Maximum Likelihood

- It is more common to use maximum likelihood to estimate coefficients
- Likelihood gives probability of the observed zeros and ones in the training data.

$$\ell(\beta_0, \beta_1) = \prod_{i:y_i=1} f(x_i) \prod_{i':y_{i'}=0} (1 - f(x_{i'}))$$

Note that $f(x_i)$is the probability $\Pr(Yi = 1|xi)$..
Therefore, 1- $f(x_i)$ = $\Pr(Yi = 0|xi)$.

- Get coefficients that maximizes the likelihood, then use them for predictions

Note

- Maximizing the likelihood function is equivalent to minimizing the cost function $J(\beta)$ defined as

$$J(\beta) = -\sum_{i=1}^{n}[y_i log(P(y_i = 1|x)) + (1 - y_i)log(1 - P(y_i = 1|x))]$$

Called cross-entropy function

proof!

Non-linear optimization problem. Can be solved iteratively (for e.g. stochastic gradient descent)

No closed form solution due to the non-linear sigmoid function

Coefficient statistics

- Similar aspects to linear regression: accuracy of coefficient estimate and p-value
- Here we have Z-statistics (instead of t-statistics as it have normal distribution):

$$\hat{\beta}_1 / SE(\hat{\beta}_1)$$

Credit card Default data using Balance

	Coefficient	Std. Error	Z-statistic	P-value
Intercept	-10.6513	0.3612	-29.5	< 0.0001
balance	0.0055	0.0002	24.9	< 0.0001

What is the probability of default for someone with balance $1000 ?

Predictions and estimating coefficients

- After estimating coefficients, we can make predictions.
- Example: estimated probability of default for someone with balance $1000 is

$$\hat{f}(X) = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 X}} = \frac{e^{-10.6513+0.0055\times 1000}}{1+e^{-10.6513+0.0055\times 1000}} = 0.006$$

Coefficient values are in the previous table

Multiple Features

Y is 1 or zero, and is predicted using multiple features

Denote $p(X)$ as the probability $\Pr(Y = 1|X)$

$$\Pr(Y = 1|X) = f(X) = \frac{e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}{1 + e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}$$

$$\log\left(\frac{f(X)}{1 - f(X)}\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

Example

	Coefficient	Std. Error	Z-statistic	P-value
`Intercept`	-10.8690	0.4923	-22.08	< 0.0001
`balance`	0.0057	0.0002	24.74	< 0.0001
`income`	0.0030	0.0082	0.37	0.7115
`student[Yes]`	-0.6468	0.2362	-2.74	0.0062

- **Student** with card **balance** of \$1,500 and an **income** of \$40,000 (units of income in data is in \$1000) has an estimated probability of default:

$$\hat{f}(X) = \frac{e^{-10.869+0.00574\times 1{,}500+0.003\times 40-0.6468\times 1}}{1+e^{-10.869+0.00574\times 1{,}500+0.003\times 40-0.6468\times 1}} = 0.058$$

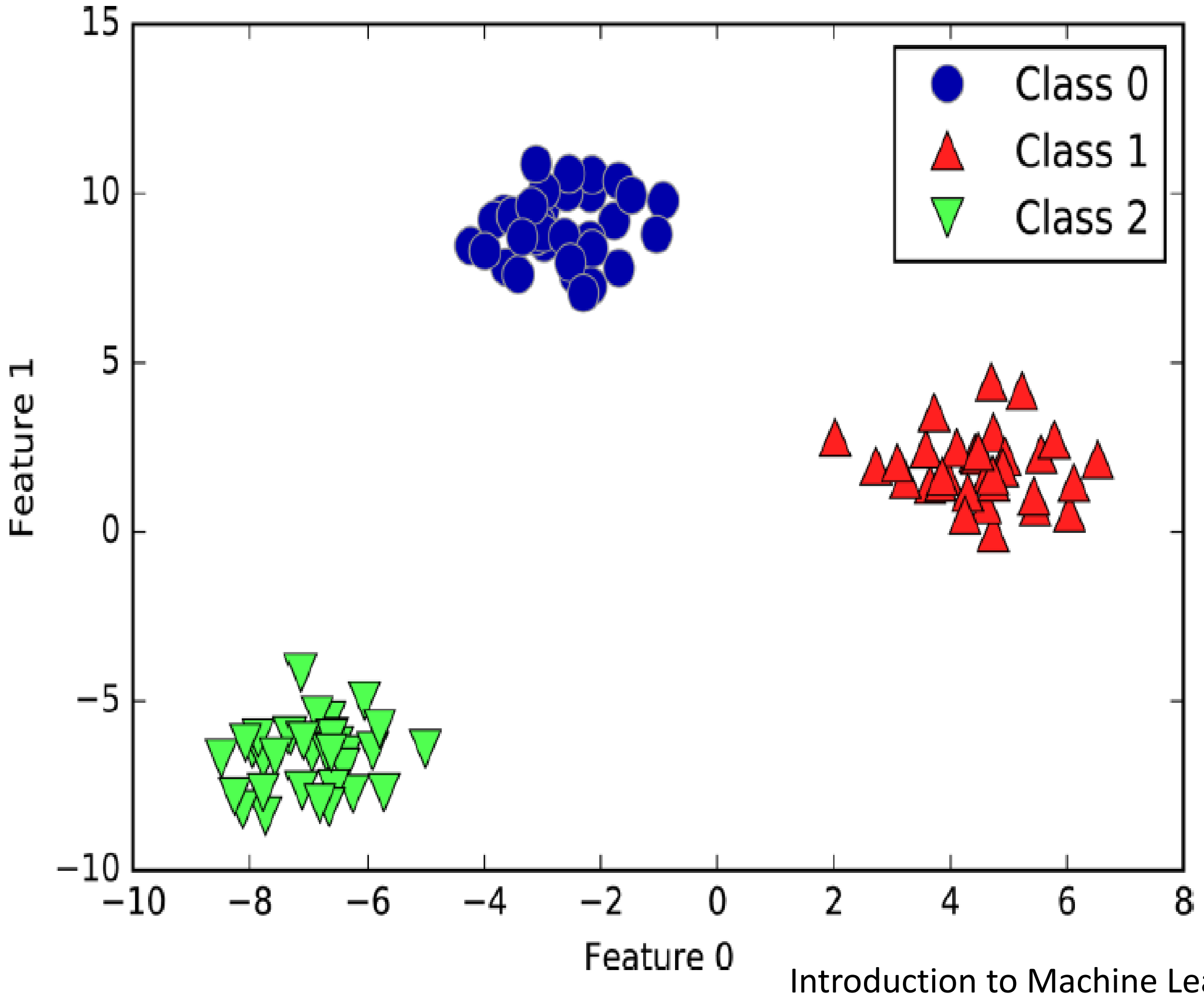
Multiple Classes

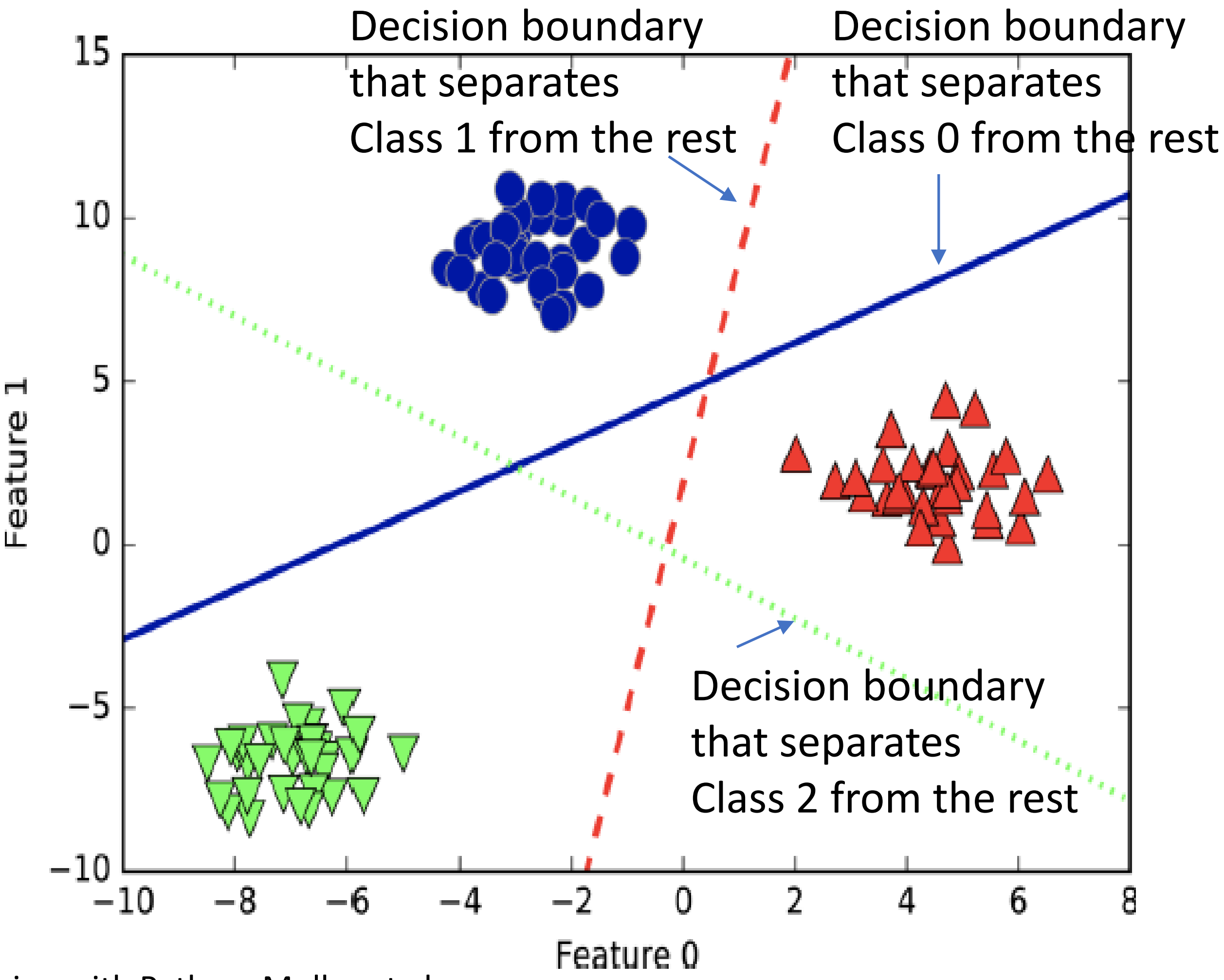
- One vs all
  - Predict whether response is from: class 1 or not, class 2 or not,..
- Train classifier for each class j to predict y=j or not
- Pick class that has max $P(y=j|X)$

- We'll talk about other methods that are more popular for multiclass classification

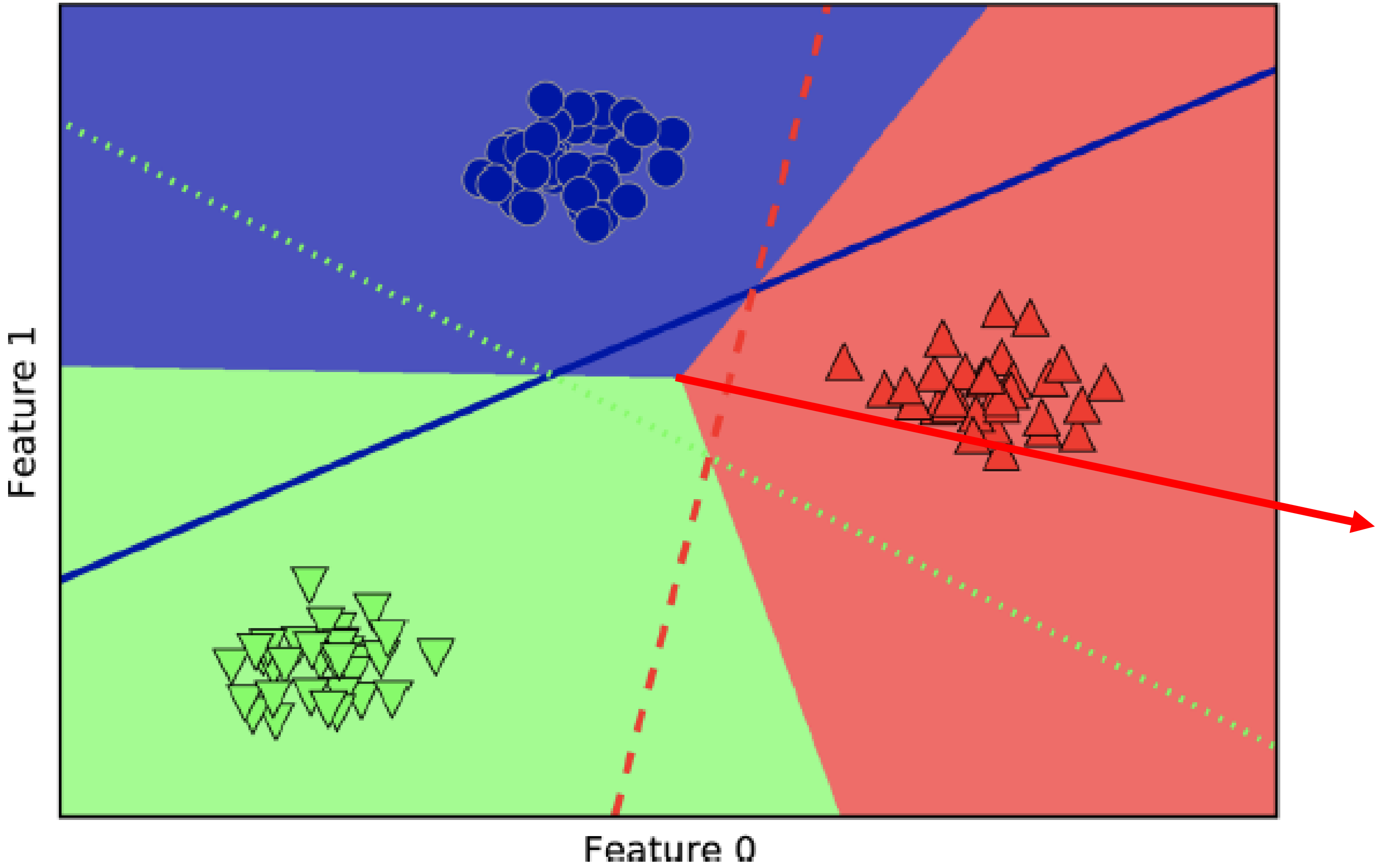
One vs. All for Multiclass Classification

- Assume two features and three classes shown in the Figure
- One can build three binary classifiers





Introduction to Machine Learning with Python, Muller et al.



The region (triangle) where the classification is "rest" by all three binary classifiers: decision is made to assign to the class with the closest boundary

Introduction to Machine Learning with Python, Muller et al

Logistic Regression in Python

From sklearn.linear_model import LogisticRegression

LogRegModel= LogisticRegression( )

- Use .fit and .score as before.

Regularization in Logistics Regression

- Regularization can be applied to logistic regression
- Same principles as before:
  - Ridge: All coefficients shrink towards zero
  - Lasso: Shrinks coefficients, and some coefficients will be forced to be zero (feature selection)

Logistic Regression in Python - Ridge

From **sklearn.linear_model** import **LogisticRegression**
LogRegModel= LogisticRegression(C=100)

Regularization strength, Ridge
Large C => less regularization

- By **default,** logistic regression in scikit-learn implements **Ridge regularization** to the classification problem, with strength defined by parameters C.
  - By **default, C=1** in LogisticRegression( )
  - **Large C** means **less regularization** strength
    - very large C means is close to the no regularization case
  - **Small C** means **more regularization** and coefficients will be close to zero
  - Note that **C is opposite to alpha or λ** in the regression functions

Logistic Regression in Python - Lasso

- We can implement Lasso regularization (also called L1 regularization) which limits the model to few features

  LogRegModel= LogisticRegression(C=0.1, penalty=”ℓ1”)

Finding Class probabilities in Python

- predict_proba: gives the probability of each of the classes given the observation
  - First row is: [probability that first observation is in class 1, probability that the first observation is in class 2,..]
- In python:

```
FittedLogRegModel1= LogisticRegression().fit(X_train,Y_train)
Probabilities=FittedLogRegModel1.predict_proba(X_test)
```