ECE 1673: Linear Control Systems (4 Credits, Spring 2022)

Lecture 5: System Responses of Second-Order Systems

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Lab 2 and Homework 2

- Lab 2
  - Due 2/15 (Tuesday)
- Homework 2
  - Due 2/3 (Thursday)

Outline of this lecture

- Time responses of second-order systems
- Time response specifications in design
- Frequency response of systems

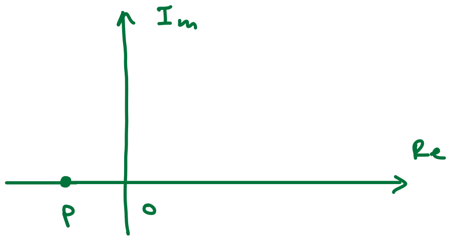
Review of first-order systems

Nominal form: $\frac{K}{\tau s + 1}$

DC Gain: $\frac{\text{output}}{\text{input}}$ in SS , $T(0)$

Time Constant : τ

$\frac{1}{s-p}$, $p < 0$



Relationship between p and τ

$$\frac{1}{s-p} = \frac{1/-p}{\frac{1}{-p}s + 1}$$

$$\tau = -\frac{1}{p} = \frac{1}{|p|}$$

Place pole further from origin to make system faster

Zero input response, zero state response
(no input) (no initial condition)

Transfer function $\xrightarrow{ILT}$ Diff. eq $\xrightarrow{\text{LT with IC}}$ Input due to IC

$C(s) = T(s) \cdot R(s)$

Forced response (steady-state response) and transient response (natural response)

$\boxed{\frac{K}{\tau s + 1}}$ $Y(s) \; \frac{K}{s} - \frac{K}{s + 1/\tau}$

$y(t) = K \cdot 1(t) - Ke^{-t/\tau} \, 1(t)$

Forced resp. (input)

Transient resp. (system)

Second Order System

Nominal Form: $\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\zeta, \omega_n > 0$

ζ: damping ratio

ω_n: natural frequency

Find out unit-step response

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

(LT of unit step is $1/s$)

(i) Case: $\zeta > 1$ $\longrightarrow$ overdamped – no oscillation

Poles of $Y(s)$: $p_0 = 0$ (input)

$$p_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

This is a real number since $\zeta > 1$

Both poles will be real and negative

$$Y(s) = \frac{\omega_n^2}{(s-p_1)(s-p_2)} \frac{1}{s}$$

$$= \frac{K_0}{s} + \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2}$$

Both of these will go to 0 as $t \to \infty$

$$K_0 = s \cdot Y(s)\Big|_{s=0} = s \cdot T(s) \cdot \frac{1}{s}\Big|_{s=0}$$

$$= T(0) = \text{DC Gain} = 1$$

Two first-order systems added together

Time constant of $\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n}$ $(\zeta > 1)$

τ_1 due to $s - p_1$: $\dfrac{1}{|p_1|}$

τ_2 due to $s - p_2$: $\dfrac{1}{|p_2|}$

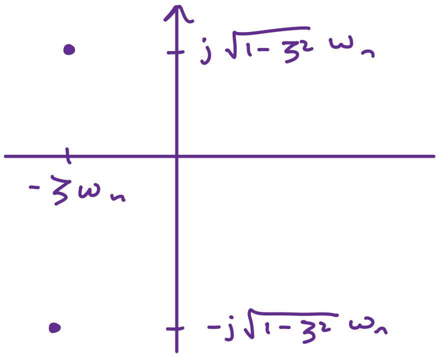
p_2 is slower (the bottleneck)

$$\tau = \max\{\tau_1, \tau_2\} = \frac{1}{\min\{|p_1|, |p_2|\}}$$

(ii) Case: $\zeta < 1$, underdamped

$$p_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

$$= -\zeta\omega_n \pm j\left(\sqrt{1-\zeta^2}\right)\omega_n$$



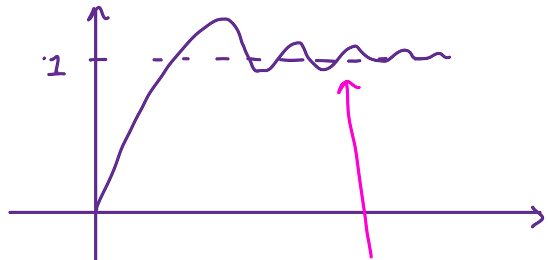
$$Y(s) = \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\left(\sqrt{1-\zeta^2}\;\omega_n\right)^2}\cdot\frac{1}{s}$$

$$= \frac{k_0}{s} + \frac{k_1\,(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\left[\left(\sqrt{1-\zeta^2}\right)\omega_n\right]^2}$$

$$+ \frac{k_2\,\omega_n}{(s+\zeta\omega_n)^2+\left[\left(\sqrt{1-\zeta^2}\right)\omega_n\right]^2}$$

$$y(t) = k_0 1(t) + k_1 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)$$
$$+ k_2 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)$$

$$\tau = \frac{1}{\zeta\,\omega_n}$$

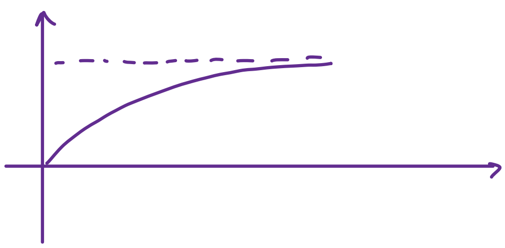


Damped frequency:
$\omega_n\sqrt{1-\zeta^2}$

(iii) Case $\zeta = 1$: Critically damped

Two poles are equal to each other

$p_1 = p_2 = -\zeta\omega_n$



Converges faster than overdamped because they share natural frequency

Time constant $\tau = \dfrac{1}{\zeta\omega_n}$

Time responses of second-order systems

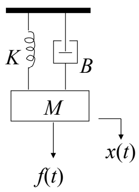
- Second-order systems

Natural frequency

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s^2+a_1 s+a_0}=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$$

Damping ratio

  – An example:



$$M\frac{d^2x}{dt^2}=f(t)-B\frac{dx}{dt}-Kx$$

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}$$

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Time responses of second-order systems

- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response
  – Case 1: $\zeta<1$ (underdamped), including $\zeta=0$ (undamped)

$$\boxed{c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),}\quad \text{where } \beta=\sqrt{1-\varsigma^2}$$
$$\text{and } \theta=\tan^{-1}(\beta/\varsigma)$$

  – Case 2: $\zeta>1$ (overdamped)

$$\boxed{c(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2},}$$
$$\text{where } \tau_{1,2}=1/\left(\varsigma\omega_n\pm\omega_n\sqrt{\varsigma^2-1}\right)$$

  – Case 3: $\zeta=1$ (critically damped)

$$\boxed{c(t)=1+k_1e^{-t/\tau}+k_2te^{-t/\tau},}\quad \text{where } \tau=1/\omega_n$$

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Time responses of second-order systems

- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response

Case 1: $\zeta<1$ (underdamped)

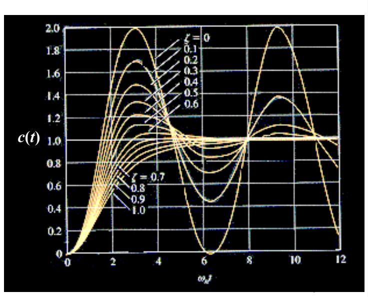
$$\boxed{c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),}$$

Case 2: $\zeta>1$ (overdamped)

$$\boxed{c(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2},}$$

Case 3: $\zeta=1$ (critically damped)

$$\boxed{c(t)=1+k_1e^{-t/\tau}+k_2te^{-t/\tau},}$$



Public door - if system is designed well, it should be overdamped (closing slowly, don't want overshoot)

Examples of underdamped, overdamped, and critically damped responses:
(1) Heavy public doors with dashpots
(2) An example of sound

Sound - underdamped b/c sound is oscillation

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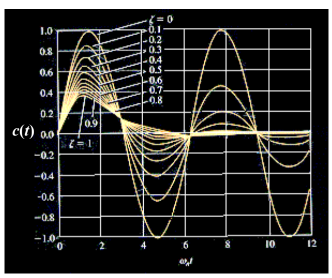
Time responses of second-order systems

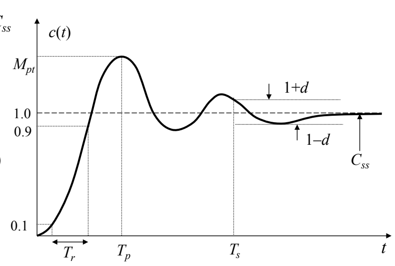
- Second-order systems $G(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
- Step response
  - Case 1
  - Case 2
  - Case 3
  - Initial condition and impulse response

The initial condition excitation of higher-order systems cannot be modeled as simply as that of the first-order system; however, the impulse response of any system does give an indication of the nature of the initial-condition response, and thus the transient response



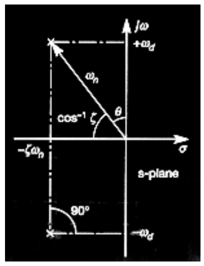
Time response specifications in design

- Some parameters
  - Rise time, T_r
  - Peak value of the step response, M_{pt} ; time to reach it, T_p (how to calculate T_p?)
  - Steady state value, C_{ss}
  - Percent overshoot, $\dfrac{M_{pt}-C_{ss}}{C_{ss}}\times 100$
  - Settling time, T_s (how to calculate T_s?)



Time response specifications in design

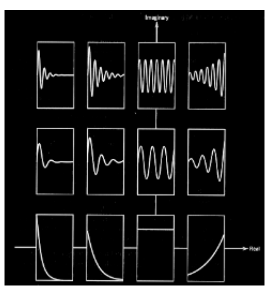
- Some parameters
- Time response and pole locations
  - The settling time is inversely related to the real part of the poles (the speed of response is increased by moving the poles to the left in the s-plane) $\boxed{T_s = k\tau = \dfrac{k}{\varsigma\omega_n}}$
  - Decreasing the angle $\cos^{-1}\zeta$ (increasing ζ) reduces the percent overshoot



k=4

Time response specifications in design

- Some parameters
- Time response and pole locations
  - The settling time is inversely related to the real part of the poles (the speed of response is increased by moving the poles to the left in the s-plane)
  - Decreasing the angle $\cos^{-1}\zeta$ (increasing ζ) reduces the percent overshoot



This picture shows how changing pole locations in the *s*-plane affects responses

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t,\quad R(s) = \frac{As}{s^2+\omega_1^2},$$

Assume that $\lim_{t\to\infty} c_g(t) = 0$

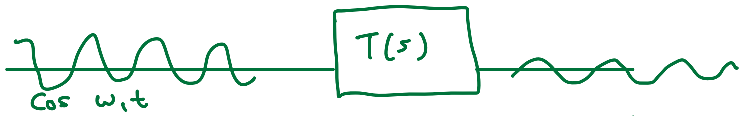
$$C(s) = G(s)R(s) = \frac{k_1}{s-j\omega_1} + \frac{k_2}{s+j\omega_1} + C_g(s)$$

$$k_1 = \frac{1}{2}AG(j\omega_1),\ k_2 = \frac{1}{2}AG(-j\omega_1),\quad \boxed{G(j\omega_1) = |G(j\omega_1)|e^{j\phi(\omega_1)}}$$

$$c_{ss}(t) = k_1 e^{j\omega_1 t} + k_2 e^{-j\omega_1 t} = A|G(j\omega_1)|\frac{e^{j(\omega_1 t+\phi(\omega_1))} + e^{-j(\omega_1 t+\phi(\omega_1))}}{2}$$

$$= A|G(j\omega_1)|\cos(\omega_1 t + \phi(\omega_1))$$

Forced response



$\cos \omega_1 t$

$A \cos(\omega_1 t + \phi)$ in SS

$$A = |T(j\omega_1)|$$

$$\phi = \angle T(j\omega_1) \quad \text{(angle of it)}$$

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t, \quad G(j\omega_1) = |G(j\omega_1)|e^{j\phi(\omega_1)}$$

$$\boxed{c_{ss}(t) = A|G(j\omega_1)|\cos(\omega_1 t + \phi(\omega_1))}$$

  – The steady-state gain of a system for a sinusoidal input is the magnitude of the transfer function evaluation at $s = j\omega_1$, and the phase shift of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
  – The steady-state gain of a system for a sinusoidal input is the *magnitude* of the transfer function evaluation at $s = j\omega_1$, and the *phase shift* of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$
  – $G(j\omega)$ is defined as the frequency response function

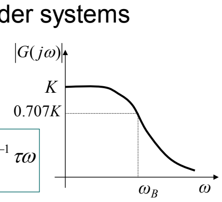
$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$$

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems

$$G(s) = \frac{K}{\tau s + 1}$$

$$\boxed{|G(j\omega)| = \frac{K}{(1+\tau^2\omega^2)^{1/2}}, \quad \phi(\omega) = -\tan^{-1}\tau\omega}$$



  – System bandwidth, ω_B: The frequency at which the gain is equal to $1/\mathrm{sqrt}(2)$ (approximately 0.707) times the gain at very low frequencies

Low pass filter

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$G(s)=\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}=\frac{1}{(s/\omega_n)^2+2\varsigma(s/\omega_n)+1}$$

$$G(j\omega)=\frac{1}{[1-(\omega/\omega_n)^2]+j2\varsigma(\omega/\omega_n)}$$

$$\boxed{|G(j\omega)|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$

Question: What will happen if $\zeta=0$ and $\omega=\omega_n$?

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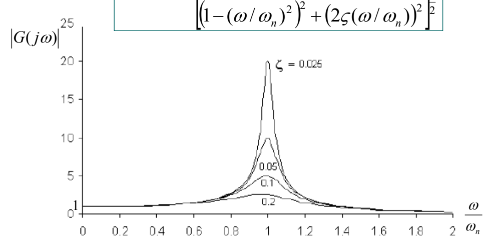
DC Gain = 1 (when $\omega = 0$)

= 0 ($\omega \to \infty$)

Frequency responses of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$\boxed{|G(j\omega)|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$



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$\zeta = 0$: DC Gain $\to \infty$

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References

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- http://www.ketchum.org/bridgecollapse.html
- http://www.scar.utoronto.ca/~pat/fun/movies1d.html

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