ECE 2372 - Homework 4 Avery Peiffer

1. $y = x^2$

x uniformly on $[-1, 1]$

$D = \{(x_1, x_1^2), (x_2, x_2^2)\}$

(a) $h(x) = b \rightarrow$ constant line

$$h_D(x) = \frac{x_1^2 + x_2^2}{2}$$

$$\bar{h}(x) = E[h_D(x)] = \frac{1}{2} E[x_1^2 + x_2^2]$$

$$= \frac{1}{2} E[x_1^2] + \frac{1}{2} E[x_2^2]$$

Know $E[x^2] = Var[x] - E[x]^2$

Since it's a uniform dist.:

$$E[x] = \frac{1}{2}(1 + -1) = 0$$

$$Var[x] = \frac{1}{12}(1 - -1)^2 = \frac{1}{3}$$

$$E[x_1^2] = 0 - \frac{1}{3} = \frac{1}{3}$$

Likewise, $E[x_2^2] = \frac{1}{3}$

$$\bar{h}(x) = \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{3}$$

(b) $E[(\bar{h}(x) - x^2)^2]$

$$= E\left[\left(\frac{1}{3} - x^2\right)^2\right] = E\left[x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right]$$

$$= E[x^4] - \frac{2}{3}E[x^2] + \frac{1}{9}$$

$\rightarrow E[x^2] = 1/3$ from A

$$\rightarrow E[x^n] = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \rightarrow \frac{(1)^5 - (-1)^5}{(5)(1 + +1)} = \frac{1}{5}$$

Overall, $= \frac{1}{5} - \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{9} = \frac{9}{45} - \frac{10}{45} + \frac{5}{45} = \frac{4}{45}$

(c) $E_x[E_d[(h_D(X) - \bar{h}(X))^2]]$

$= E_x\left[E_d\left[\left(h_D(x) - \frac{1}{3}\right)^2\right]\right]$ $\qquad E_d[h_D(x)] = \bar{h}(x) = \frac{1}{3}$

$$= E_x\left[E_d\left[h_D(x)^2 - \frac{2}{3}h_D(x) + \frac{1}{9}\right]\right]$$

$$= E_x\left[E_d\left[h_D(x)^2\right] - \frac{2}{3} E_d\left[h_D(x)\right] + \frac{1}{9} \right]$$

$$= E_x\left[E_d\left[h_D(x)^2\right] - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{9} \right]$$

$$= E_x\left[\int_{-1}^{1} \int_{-1}^{1} \left(\frac{x_1^2 + x_2^2}{2}\right)^2 dx_1\, dx_2 - \frac{1}{9} \right]$$

$$= E_x\left[\frac{28}{45} - \frac{5}{45}\right] = \frac{23}{45}$$

2. x on $[-1, 1]$ uniformly $\qquad (x_1, y_1)$

$y = \sin(\pi x) \qquad (x_2, y_2)$

$$\hat{\theta} = (A^T A + \Gamma^T \Gamma)^{-1} A^T y$$

$$\begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \Bigg(\underbrace{\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}}_{\begin{bmatrix} 2 & x_1 + x_2 \\ x_1 + x_2 & x_1^2 + x_2^2 \end{bmatrix}} + \Gamma^T \Gamma \Bigg)^{-1} \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underbrace{\Gamma = \begin{bmatrix} 1 & -x_1 \\ 1 & -x_2 \end{bmatrix} \qquad \Gamma^T = \begin{bmatrix} 1 & 1 \\ -x_1 & -x_2 \end{bmatrix}}_{\begin{bmatrix} 2 & -x_1 - x_2 \\ -x_1 - x_2 & x_1^2 + x_2^2 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & x_1 + x_2 \\ x_1 + x_2 & x_1^2 + x_2^2 \end{bmatrix} + \begin{bmatrix} 2 & -(x_1 + x_2) \\ -(x_1 + x_2) & x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2x_1^2 + 2x_2^2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 2 & x_1+x_2 \\ x_1+x_2 & x_1^2+x_2^2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2(x_1^2+x_2^2) \end{bmatrix}\right) \neq 0$$

$$\det\left(\begin{bmatrix} 6 & x_1+x_2 \\ x_1+x_2 & 3(x_1^2+x_2^2) \end{bmatrix}\right) = 0$$

$$18(x_1^2+x_2^2) - (x_1+x_2)^2 = 0$$
$$18x_1^2 + 18x_2^2 - x_1^2 - 2x_1x_2 - x_2^2 = 0$$
$$17x_1^2 + 17x_2^2 - 2x_1x_2 = 0$$
$$= (x_1 - x_2)^2$$

b) Set Γ to zero, so $\hat{\theta}$ is now $(A^TA)^{-1}A^Ty$

c) $\text{bias} = E_x[(\bar{h}(x) - \sin(\pi x))^2]$

$\bar{h}(x) = b \rightarrow \bar{h}(x) = 0$

$\bar{h}(x) = ax+b \rightarrow \bar{h}(x) = 4/5$

$\bar{h}(x) = b$

$\text{bias} = E_x[(\sin \pi x)^2] = 0.5$

$\text{variance} = E_x[E_D[(h_D(x) - \bar{h}(x))^2]] = 0.26$

$\bar{h}(x) = ax+b$

bias = 0.21

variance = 0.24

(d) $\Gamma \rightarrow$ Identity matrix

A becomes less sensitive to noise?