ECE 2521: Analysis of Stochastic Processes

Lecture 4

Department of Electrical and Computer Engineering
University of Pittsburgh

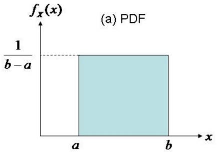
September, 22nd 2021

Azime Can-Cimino

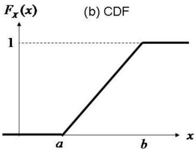
Uniform Random Variable

- Used for experiments that lead to outcomes that are equally likely to occur over any subinterval within the range of the RV
- $X \sim \text{Uniform}(a, b)$:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x < b, \\ 0 & \text{otherwise .} \end{cases}$$



$$F_X(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x < b, \\ 1 & \text{if } x \geq b, \end{cases}$$



Uniform Random Variable

Mean: $E[X] = \frac{a+b}{2}$

Proof: The mean is obtained by symmetry.

Variance: $\text{Var}[X] = \frac{(b-a)^2}{12}$

Proof: The second moment of the uniform random variable is

$$E[X^2] = \int_a^b x^2 \frac{1}{(b-a)} dx = \frac{b^3 - a^3}{3(a-b)}$$

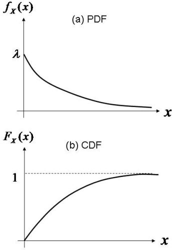
$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{(b-a)^2}{12}$$

Exponential Random Variable

- Continuous version of the discrete geometric RV
- Models the inter-arrival time for the Poisson process
    - The wait time for a bus
    - Time to emmision of a particle from a radioactive source
    - Time to failure of an equipment / lifetime of an equipment
- $X \sim \text{Exponential}(\lambda)$, λ is the rate at which events occur:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise .} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-\lambda x} & \text{if } x \geq 0, \end{cases}$$



Exponential Random Variable

Mean: $E[X] = \frac{1}{\lambda}$

$$\begin{aligned} E[X] &= \int_0^\infty x\lambda e^{-\lambda x} dx = (-xe^{-\lambda x})\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda} \end{aligned}$$

Variance: $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2\lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx \\ &= 0 + \frac{2}{\lambda} E[X] = \frac{2}{\lambda^2} \end{aligned}$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Exponential Random Variable - Example

A car battery has an average life of 3 years.

(a) What is the probability that it will last for more than 4 years?

(b) What is the probability that it will fail within the second and third years?

(c) If after 2 years, the battery is still in good condition. What is the probability that it will last another y more years?

Memoryless Property of Exponential RV

- Let X represent the unconditional life of the battery
- Let A be the event that the battery lasts longer than 2 years
- Let Y represent the additional life of the battery

$$
\begin{aligned}
\text{Prob}(Y > y|A) &= \text{Prob}(X > 2+y|X > 2) \\
&= \frac{\text{Prob}(X > 2+y \text{ and } X > 2)}{\text{Prob}(X > 2)} \\
&= \frac{\text{Prob}(X > 2+y)}{\text{Prob}(X > 2)} = \frac{1 - F_X(2+y)}{1 - F_X(2)} \\
&= e^{-\lambda y} = \text{Prob}(Y > y)
\end{aligned}
$$

- The exponential RV renews itself (memoryless) because this probability is the same as that obtained when the observation started at X=0. All that matters is the time lapse since the begining of the observation.

Erlang Random Variable

- Continuous version of the discrete Pascal RV
- Models the total arrival time for n arrivals of the Poisson process
  - The wait time for the nth bus
  - Time to emision of the nth particle from a radioactive source
- $X \sim \text{Erlang}(n, \lambda)$:

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & \text{if } x \geq 0, \\ 0 & \text{otherwise }. \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!} & \text{if } x \geq 0, \end{cases}$$

Erlang Random Variable

Mean: $E[X] = \frac{n}{\lambda}$

Variance: $\text{Var}[X] = \frac{n}{\lambda^2}$

Note: Since the Erlang RV models the total arrival time of n independent arrivals where each inter-arrival time is an independent identically distributed exponential RV, the mean and variance of the Erlang RV is simply n times that of the exponential RV.

Gamma Random Variable

- Generalization of Erlang and Exponential RVs.
- $X \sim \text{Gamma}(\alpha, \lambda)$:

$$f_X(x) = \begin{cases} \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x \geq 0, \\ 0 & \text{otherwise} . \end{cases}$$

  where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x$ for $z > 0$.
- Exponential distribution with parameter λ if $\alpha = 1$
- Erlang distribution with parameters n and λ if $\alpha = n$
- By varying the parameters α and λ, it is possible to fit the Gamma pdf to many types of experimental data

Note: In general there is no closed form for the CDF

  - $\Gamma(1/2) = \sqrt{\pi}$
  - $\Gamma(z+1) = z\Gamma(z)$ for $z > 0$.
  - $\Gamma(m+1) = m!$ for m a nonnegative integer.

Beta Random Variable

- $X \sim \text{Beta}(a, b)$ with $a > 0$ and $b > 0$:

$$f_X(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)} & \text{if } x \in (0,1), \\ 0 & \text{otherwise} . \end{cases}$$

where $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \mathrm{d}x$ for $z > 0$.

- If $a = b = 1$, then X is a standard uniform random variable
- Useful to model a variety of behaviors for random variables that range over finite intervals

Note: $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Mean: $E[X] = \frac{a}{a+b}$

Variance: $\text{Var}[X] = \frac{ab}{(a+b)^2(a+b+1)}$

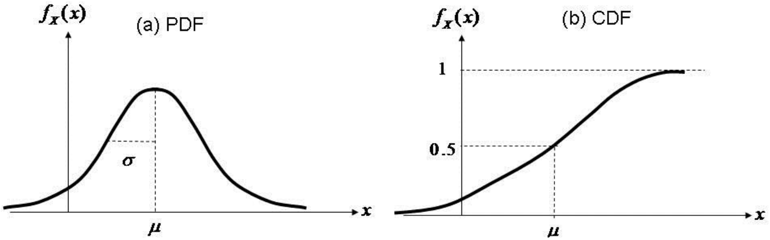
Gaussian Random Variable

- Important role in a broad range of applications because it models the additive effect of many independent factors in a variety of engineering, physical and statistical contexts
- The central limit theorem shows mathematically that the sum of a large number of independent and identically distributed (not necessarily Gaussian) random variables has an approximately Gaussian CDF, regardless of the CDF of the individual random variables
- In many scientific and engineering applications, the Gaussian random variable is used to model noise and unpredictable distortions of signal
- Also called the *normal* random variable because of its prevalence

Gaussian Random Variable

- $X \sim \text{Gaussian}(\mu, \sigma)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



Gaussian Random Variable

- The PDF for a Gaussian RV has a bell shape where the center of the bell or mean is $x = \mu$ and the width of the bell or standard deviation is σ
- If σ is small, the bell is narrow with a high pointy peak
- If σ is large, the bell is wide with a low flat peak
- The height of the peak is $1/\sigma\sqrt{2\pi}$
- The area under the bell shape is 1: $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- The CDF of the Gaussian RV cannot be expressed analytically because the integral of the Gaussian PDF between non-infinite limits cannot be expressed analytically; it is calculated by numerical integration and tabulated

Linear transformation of a Gaussian Random Variable

- Given that X is a Gaussian RV with mean μ and variance σ^2
- If Y is a linear function of X:

$$Y = aX + b$$

then Y is also Gaussian with mean:

$$E[Y] = a\mu + b$$

and variance

$$\mathrm{Var}[Y] = a^2\sigma^2$$

Standard Normal Gaussian Random Variable and its CDF

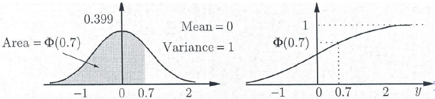
- Z is a standard normal random variable if it is Gaussian with $E[Z] = 0$ and $\text{Var}[Z] = 1$:

$$Z \sim \text{Gaussian}(\mu = 0, \sigma = 1)$$

- The CDF of the standard normal Gaussian RV is tabulated:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$$

- The probabilities for any other Gaussian RV can be calculated by first transforming it to the standard normal RV, and then using the CDF of the standard normal RV **[DETOUR TO LAST YEAR NOTES]**



Calculating Probabilities for a General Gaussian RV

- If X is a Gaussian(μ, σ) random variable, then the CDF of X:

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

and the probability that X takes values in the interval $(a, b]$ is:

$$\text{Prob}(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Deriving distribution for the function of a continuous RV

Given a continuous random variable X with known PDF and function $Y = g(X)$:

- If we are interested in calculating expectations involving the new random variable Y, we do not need to derive the full probability model for Y, since the expectations of Y can be calculated from the definition of the function $Y = g(X)$ and the PDF of X using the expected value rule:
$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$
- If we would like to find the complete probability model for the new random variable $Y = g(X)$ that is a function of X, then the general procedure involves a two-step process:
  (1) Find the CDF $F_Y(y) = \text{Prob}(Y \leq y)$
  (2) Differentiate the CDF to obtain the PDF of Y, $f_Y(y) = \frac{dF_Y(y)}{dy}$

Example

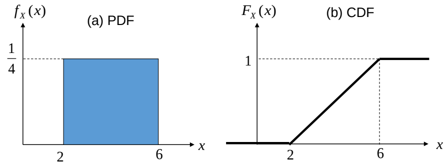
- Consider the random variable X with probability density function:

$$f_X(x) = \begin{cases} \frac{1}{4} & \text{if } 2 < x \leq 6, \\ 0 & \text{otherwise }, \end{cases}$$

and cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ \frac{1}{4}(x-2) & \text{if } 2 < x \leq 6, \\ 1 & \text{if } x > 6. \end{cases}$$

- Determine the PDFs of:
  (a) $U = 3X + 2$
  (b) $W = X^2$
  (c) $Q = (X-3)^2$



Example - Solution (a)

- First we determine the range and CDF for $U = 3X + 2$: $S_U = \{u | 8 < u \leq 20\}$
- The CDF for U is:

$$
\begin{aligned}
F_U(u) &= \text{Prob}(U \leq u) = \text{Prob}(3X + 2 \leq u) \\
&= \text{Prob}(X \leq \frac{u-2}{3}) = F_X(\frac{u-2}{3}) \\
&= \frac{1}{4}(\frac{u-2}{3} - 2) = \frac{1}{12}(u - 8)
\end{aligned}
$$

valid for $8 < u \leq 20$.

$$
F_U(u) = \begin{cases} 0 & \text{if } u \leq 8 \\ \frac{1}{12}(u-8) & \text{if } 8 < u \leq 20, \\ 1 & \text{if } u > 20. \end{cases}
$$

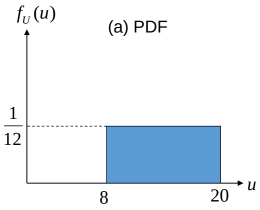
Example - Solution (a)

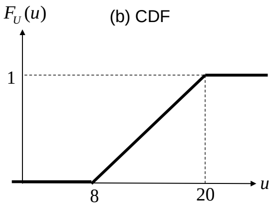
- The PDF for U is then:

$$f_U(u) = \frac{dF_U(u)}{du} = \frac{1}{12}$$

valid for $8 < u \leq 20$.

$$f_U(u) = \begin{cases} \frac{1}{12} & \text{if } 8 < u \leq 20, \\ 0 & \text{otherwise .} \end{cases}$$





Example - Solution (b)

- Determine the range and CDF for $W = X^2$:
  $S_W = \{w | 4 < w \leq 36\}$
- The CDF for W is

$$
\begin{aligned}
F_W(w) &= \text{Prob}(W \leq w) = \text{Prob}(X^2 \leq w) \\
&= \text{Prob}(X \leq \sqrt{w}) = F_X(\sqrt{w}) = \frac{1}{4}(\sqrt{w} - 2)
\end{aligned}
$$

valid for $4 < w \leq 36$.

$$
F_W(w) = \begin{cases} 0 & \text{if } w \leq 4 \\ \frac{1}{4}(\sqrt{w} - 2) & \text{if } 4 < w \leq 36, \\ 1 & \text{if } w > 36. \end{cases}
$$

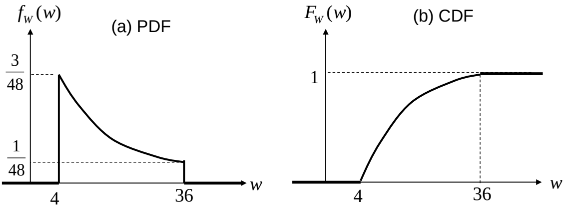
Example - Solution (b)

- The probability density function for W is then:

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{1}{8\sqrt{w}}$$

valid for $4 < w \leq 36$.

$$f_W(w) = \begin{cases} \frac{1}{8\sqrt{w}} & \text{if } 4 < w \leq 36, \\ 0 & \text{otherwise .} \end{cases}$$



Example - Solution (c)

- Determine the range and CDF for $Q = (X-3)^2$:

$$S_Q = \{q | 0 < q \leq 9\}$$

Note: The function Q is non-monotonic over the range of X

- The CDF of Q may be discontinuous at $q = 1$ because the transformation from X to Q is a two-to-one function for $0 < q \leq 1$ and $2 < X < 4$, while it is a one-to-one function for $1 \leq q \leq 9$ and $4 \leq X \leq 6$.

Example - Solution (c)

First consider $0 < q \leq 1$, the CDF is:

$$\begin{aligned}
F_Q(q) &= \text{Prob}(Q \leq q) = \text{Prob}((X-3)^2 \leq q) \\
&= \text{Prob}(-\sqrt{q} \leq X - 3 \leq \sqrt{q}) \\
&= \text{Prob}(3 - \sqrt{q} \leq X \leq 3 + \sqrt{q}) \\
&= F_X(3+\sqrt{q}) - F_X(3-\sqrt{q}) \\
&= \frac{1}{4}(1+\sqrt{q}) - \frac{1}{4}(1-\sqrt{q}) = \frac{\sqrt{q}}{2}
\end{aligned}$$

The corresponding probability density function for Q is:

$$f_Q(q) = \frac{dF_Q(q)}{dq} = \frac{1}{4\sqrt{q}}$$

Example - Solution (c)

Next consider $1 \leq q \leq 9$, the CDF is:

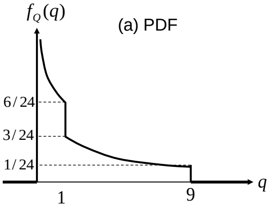
$$
\begin{aligned}
F_Q(q) &= \text{Prob}(Q \leq q) \\
&= \text{Prob}(X \leq 3 + \sqrt{q}) \\
&= F_X(3 + \sqrt{q}) \\
&= \frac{1}{4}(1 + \sqrt{q})
\end{aligned}
$$

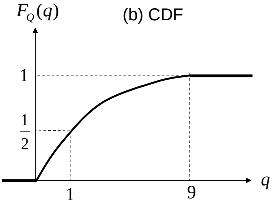
The corresponding PDF for Q is:

$$f_Q(q) = \frac{dF_Q(q)}{dq} = \frac{1}{8\sqrt{q}}$$

Example - Solution (c)

$$F_Q(q) = \begin{cases} 0 & \text{if } q \le 0 \\ \frac{\sqrt{q}}{2} & \text{if } 0 < q \le 1, \\ \frac{\sqrt{q}}{4} + \frac{1}{4} & \text{if } 1 < q \le 9, \\ 1 & \text{if } q > 9. \end{cases} \qquad f_Q(q) = \begin{cases} \frac{1}{4\sqrt{q}} & \text{if } 0 < q \le 1, \\ \frac{1}{8\sqrt{q}} & \text{if } 1 < q \le 9, \\ 0 & \text{otherwise .} \end{cases}$$





Applying a Formula

- The PDF of the derived random variable $Y = g(X)$ can be obtained directly from the PDF of the original random variable X if $g(X)$ is a monotonic function with an inverse (i.e. $X = g^{-1}(Y) = h(Y)$)
- Start from the CDF of Y: **[DETOUR TO LAST YEAR NOTES]**

$$\begin{aligned}
F_Y(y) &= \text{Prob}(Y \le y) = \text{Prob}(g(X) \le y) \\
&= \begin{cases} \text{Prob}(X \le h(y)) & \text{if } g(X) \text{ is monotonically increasing} \\ \text{Prob}(X \ge h(y)) & \text{if } g(X) \text{ is monotonically decreasing} \end{cases} \\
&= \begin{cases} F_X(h(y)) & \text{if } g(X) \text{ is monotonically increasing} \\ 1 - F_X(h(y)) & \text{if } g(X) \text{ is monotonically decreasing.} \end{cases}
\end{aligned}$$

- Differentiate the CDF to obtain the PDF for Y:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(h(y))}{dx}\left|\frac{dx}{dy}\right| = f_X(h(y))\left|\frac{dx}{dy}\right|$$

Formula for a Linear Function

- If $Y = g(X) = aX + b$ is a linear function of X, then the PDF of Y can be obtained from

$$f_Y(y) = \left|\frac{1}{a}\right| f_X\left(\frac{y-b}{a}\right)$$

Note: Multiplying a random variable by a constant a, stretches ($|a| > 1$) or shrinks ($|a| < 1$) the original PDF

Note: Adding a constant to a random variable simply shifts the CDF or PDF by that constant

Fundamental Theorem

- To find $f_Y(y)$ for a specific y, we solve the equation $y = g(x)$
- Denote its real roots by x_n, such that $y = g(x_1) = \cdots = g(x_n)$

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{|g'(x_n)|}$$

From Previous Example

(a) $U = 3X + 2$. The relationship between X and U is linear:

$$f_U(u) = \frac{1}{3} \cdot f_X(\frac{u-2}{3}) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

valid for $8 < u \leq 20$.

(c) $0 < q \leq 1$. Two roots: $x_1 = 3 - \sqrt{q}$ and $x_2 = 3 + \sqrt{q}$:

$$f_Q(q) = f_X(x1)|\frac{dx_1}{dq}| + f_X(x_2)|\frac{dx_2}{dq}| = \frac{1}{4} \cdot \frac{1}{|-2\sqrt{q}|} + \frac{1}{4} \cdot \frac{1}{|2\sqrt{q}|} = \frac{1}{4\sqrt{q}}$$

(c) $1 \leq q \leq 9$. Only one root: $x = 3 + \sqrt{q}$:

$$f_Q(q) = f_X(x)|\frac{dx}{dq}| = \frac{1}{4} \cdot \frac{1}{|2\sqrt{q}|} = \frac{1}{8\sqrt{q}}$$

Example

- The random variable X has PDF given by:

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$$

  for $x > 0$.
- Let random variable Y be defined by $Y = X^2$.
- What is the PDF of Y?
- Since the relationship between X and Y is one to one over the region defined, we can apply the general formula:
- In this case, $X = h(Y) = \sqrt{Y}$, and $\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$:

$$f_Y(y) = f_X(h(y))\left|\frac{dx}{dy}\right| = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$$