Homework #7

Due 11/17

1. The joint PDF of X and Y is gives as:

$$f_{XY}(x,y) = \begin{cases} 3e^{-(x+2y)} & 0 \leq x \leq \infty, 0 \leq y \leq x \\ 0 & otherwise \end{cases}$$

   Give the expression for the MSE of the optimal nonlinear estimator. Note that optimal nonlinear estimator also obeys an orthogonality principle: the error is orthogonal to any nonlinear function of the data. Please show that the MSE is expressed as $MSE = E\{Y^2\} - E\{[\widehat{Y_{NL}}]^2\}$. Here $\widehat{Y_{NL}}$ denotes the nonlinear estimator.

   You can use the same procedure I used when driving the MSE for the linear estimator, and then use iterated expectation (condition on X) on the term $E\{Y\,\widehat{Y_{NL}}\}$. You may live your answer in terms of integrals, but each term in your answer need only be a single integral involving a marginal PDF as opposed to a double integral involving the joint PDF.

2. An elevator has a capacity of maximum 19 adults. Suppose each adults weigh has a mean of 170 lbs with a standard deviation of 15 lbs. Give an approximation of the probability that the total weight of the people, if there are a maximum number of them, is greater than 3340 lbs. Why are these people so heavy, I wonder too.

   Hint: Note that the total weight is a sum of 19 iid RVs. Central Limit theorem (which we haven't seen yet, don't panic) says that such a sum will have a Gaussian distribution.

3. We have seen that given a vector X of N jointly Gaussian random variables, any linear transformation to a set of M $(M \leq N)$ new variables, Y, will produce jointly Gaussian random variables.
   My random wish is to create a set of N jointly Gaussian random variables, Y, with a specified covariance matrix, C. I could start with a set of uncorrelated Gaussian random variables using a typical Gaussian random number generator, and then perform a linear transformation to produce a new set of Gaussian random variables with my desired covariance matrix. I heard that typical Gaussian random number generators create random variables with a zero mean and unit variance. But, I need your help. How should I select the transformation to produce my desired covariance matrix? Since you are at it, I also want a specific mean for my new random variable.

   If it would help you to help me, I can give some numbers. My desired random variable has a mean vector of μ =[1,0,3,-2] and desired covariance matrix of

$$C = \begin{bmatrix} 30 & -10 & -20 & 4 \\ -10 & 30 & 4 & -20 \\ -20 & 4 & 30 & -10 \\ 4 & -20 & -10 & 30 \end{bmatrix}$$

4. Now I am just dreaming the opposite of the problem you have just solved. That is given a vector Y with a mean μ and covariance matrix C, how can I create a vector of zero-mean, unit-variance and uncorrelated random variables? It is like washing random vector's sins away. Tell be about the transformation I need for this.

5. Suppose $X = [X_1, X_2, X_3, X_4]$ is a Gaussian random vector. I take the X_i to be uncorrelated with equal variance so that the covariance matrix is $C_{XX} = \sigma^2 I$. Here I is denoting the identity matrix and, σ^2 must be the individual variances. Please let $Z = X_1 X_2 + X_3 X_4$. (Oops that's nonlinear!) Find the PDF of Z.
   Hint: This is one where characteristic function gets handy.

1. orthogonality Principle:

$$E\{[y-\hat{y}_{NL}]\,G(x)\}=0$$

$$f_x(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^x 3e^{-(x+2y)}\,dy=\frac{3}{2}e^{-x}\left(1-e^{-2x}\right)u(x)$$

$$f_{y|x}(y|x)=\frac{f(x,y)}{f_x(x)}=\frac{3e^{-(x+2y)}}{\frac{3}{2}e^{-x}(1-e^{-2x})}=\frac{2e^{-2y}}{1-e^{-2x}}$$

2. mean $= 19\cdot 170$

$= 3230$

variance $= 19\cdot 15^2 = 4275$

$P(W>3340)=$

$$1-\Phi\left(\frac{3340-3230}{\sqrt{4275}}\right)=1-\Phi(1.682)=0.04633$$