ECE 2521 Analysis of Stochastic Processes
Solutions to Problem Set 5

Problem 5.1 Solution

4.84 $F_Y(y) = F_X\left(\frac{y-2}{3}\right)$

$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$

• X is Laplacian

$$F_Y(y) = \begin{cases} \frac{1}{2} e^{\alpha\left(\frac{y-2}{3}\right)} & y \le 2 \\ 1 - \frac{1}{2} e^{-\alpha\left(\frac{y-2}{3}\right)} & y > 2 \end{cases} \qquad f_Y(y) = \frac{1}{3}\frac{\alpha}{2} e^{-\alpha\left|\frac{y-2}{3}\right|}$$

• X is Gaussian

$$F_Y(y) = \Phi\left(\frac{\frac{y-2}{3} - m}{\sigma}\right) = \Phi\left(\frac{y-(2+3m)}{3\sigma}\right)$$

$$f_Y(y) = \frac{1}{3\sigma\sqrt{2\pi}} e^{-\left(\frac{y-2}{3} - m\right)^2/2\sigma^2} = \frac{1}{3\sigma\sqrt{2\pi}} e^{-(y-(2+3m))^2/2(3\sigma)^2}$$

• $X = b\cos(2\pi U)$

$$F_Y(y) = \begin{cases} 0 & y < -3b+2 \\ \frac{1}{\pi}\sin^{-1}\left(\frac{y-2}{3b}\right) + \frac{1}{\pi}\sin^{-1}\left(-\frac{1}{b}\right) & 2-3b \le y \le 3b+2 \\ 1 & y > 3b+2 \end{cases}$$

$$f_Y(y) = \frac{1}{3}\frac{1}{\pi b\sqrt{1-\left(\frac{y-2}{3b}\right)^2}} \qquad 2-3b \le y < 3b+2$$

Problem 5.2 Solution

4.86

$$f_X(x) = \sum_k f_U(u) \left| \frac{1}{n} x^{\frac{1}{n}-1} \right| \Big|_{x=x_k}$$

$$f_X(x) = f_U(\sqrt[n]{x}) \left(\frac{1}{n} x^{\frac{1}{n}-1} \right)$$

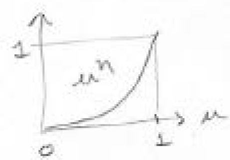
$$= \frac{1}{n} x^{\frac{1}{n}-1} \qquad 0 \le x \le 1$$

$$F_X(x) = \int_0^x \frac{1}{n} x^{\frac{1}{n}-1}\, dx = x^{\frac{1}{n}} \Big|_0^x = x^{\frac{1}{n}} \qquad 0 \le x \le 1$$

$$= \begin{cases} 0 & x < 0 \\ x^{1/n} & x \ge 0 \end{cases}$$

Alternatively we could start with the cdf:

$$F_X(x) = P[U^n \le x]$$
$$= P[U \le x^{1/n}]$$
$$= x^{1/n}$$



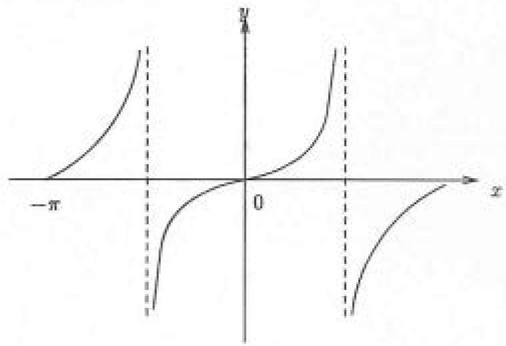
$0 \le x \le 1$

Problem 5.3 Solution

4.94 $Y = a\tan X$.

$$x = \tan^{-1}(y/a),\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\frac{dx}{dy} = \frac{1}{1 + (y/a)^2}\frac{1}{a} = \frac{a}{y^2 + a^2}$$



$$\begin{aligned} f_X(y) &= \sum_k f_X(x)\left|\frac{dx}{dy}\right|\Big|_{x=x_k} \\ &= 2 \cdot \frac{1}{2\pi}\frac{a}{y^2 + a^2} \\ &= \frac{a/\pi}{y^2 + a^2} \end{aligned}$$

Y is a Cauchy RV.

Problem 5.4 Solution

4.103

$$
\begin{aligned}
\phi_X(w) &= \int_{-\infty}^{\infty} f_X(x)e^{jwx}dx \\
&= \int_{-\infty}^{0} \frac{\alpha}{2}e^{\alpha x}e^{jwx}dx + \int_{0}^{\infty} \frac{\alpha}{2}e^{-\alpha x}e^{jwx}dx \\
&= \frac{\alpha}{2}\frac{1}{\alpha + jw} + \frac{\alpha}{2}\frac{1}{\alpha - jw} \\
&= \frac{\alpha^2}{\alpha^2 + w^2}
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \frac{1}{j}\frac{d\phi_X(w)}{dw}|_{w=0} \\
&= \frac{1}{j} \cdot \frac{\alpha^2 \cdot 2w}{-(\alpha^2 + w^2)^2}|_{w=0} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \frac{1}{j^2}\frac{d\phi_X(w)}{dw^2}|_{w=0} \\
&= -\frac{\alpha^2}{j^2}\frac{2(\alpha^2 + w^2)^2 - 2w \cdot 2(\alpha^2 + w^2) \cdot 2w}{(\alpha^2 + w^2)^4}|_{w=0} \\
&= \frac{2}{\alpha^2} \\
VAR[X] &= E[X^2] - E^2[X] = \frac{2}{\alpha^2}
\end{aligned}
$$

Problem 5.5 Solution

4.107

~~91~~ We take the inverse transform of $e^{-|\omega|}$ to show that it corresponds to a Cauchy pdf:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|}e^{j-\omega x}d(x) &= \frac{1}{2}\int_{-\infty}^{0} e^{\omega}e^{-j\omega x}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega}e^{-j\omega}d\omega \\
&= \frac{1}{2\pi}\left[\frac{e^{\omega(1-jx)}}{1-jx}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[\frac{e^{-\omega(1+jx)}}{-(1+jx)}\right]_{0}^{\infty} \\
&= \frac{1}{2\pi}\left[\frac{1}{1-jx} + \frac{1}{1+jx}\right] = \frac{1}{\pi(1+x^2)} \quad \checkmark
\end{aligned}
$$

Problem 5.6 Solution

5.2

(a)

M \ S	0	1	2	
0	00	01	02	$\frac{1}{4}$
1	10	11	12	$\frac{1}{2}$
2	20	21	22	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

S_{XY}

X \ Y	0	1	2	3	4
−2			02		
−1		01		12	
0	00		11		22
1		10		21	
2			20		

(b)

	fair coin	for biased coin:
$P[X=-2, Y=2] = P[\{02\}]$	$= \frac{1}{16}$	$9/64$
$P[X=-1, Y=1] = P[\{01\}]$	$= \frac{1}{8}$	$6/64$
$P[X=-1, Y=3] = P[\{12\}]$	$= \frac{1}{8}$	$18/64$
$P[X=0, Y=0] = P[\{00\}]$	$= \frac{1}{16}$	$1/64$
$P[X=0, Y=2] = P[\{11\}]$	$= \frac{1}{4}$	$12/64$
$P[X=0, Y=4] = P[\{22\}]$	$= \frac{1}{16}$	$9/64$
$P[X=1, Y=1] = P[\{10\}]$	$= \frac{1}{8}$	$2/64$
$P[X=1, Y=3] = P[\{21\}]$	$= \frac{1}{8}$	$6/64$
$P[X=2, Y=2] = P[\{20\}]$	$= \frac{1}{16}$	$1/64$

(c)

$$P[X+Y=1] = 0$$

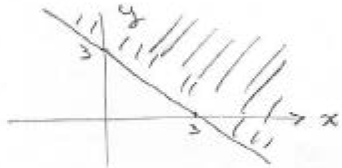
$$P[X+Y=2] = P[(X,Y) \in \{(-1,3), (0,2), (1,1)\}] = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

Problem 5.7 Solution

5.8

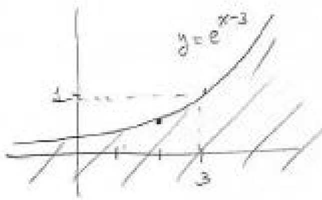
a) $\{X+Y>3\} = \{Y>3-X\}$

Not product form



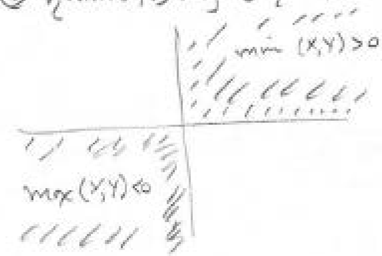
b) $\{e^X > Ye^3\}$

$= \{Y < e^{X-3}\}$



Not product form

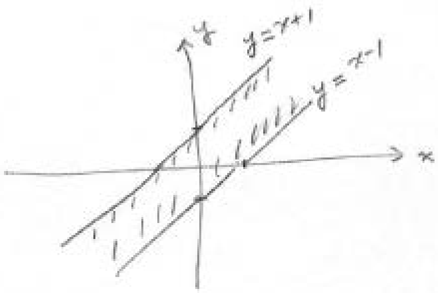
c) $\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}$



Not product form but union of product form

d) $\{|X-Y| \geq 1\} = \{-1 \leq X-Y \leq 1\}$

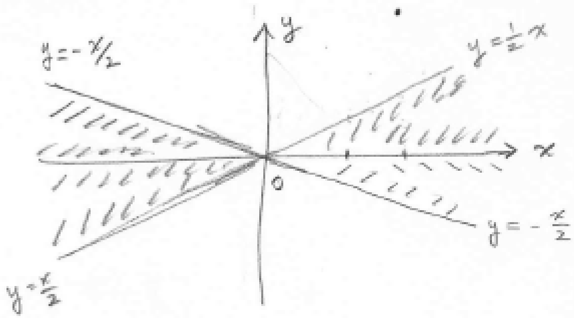
$= \{Y-1 \leq X \leq Y+1\}$



Not product form

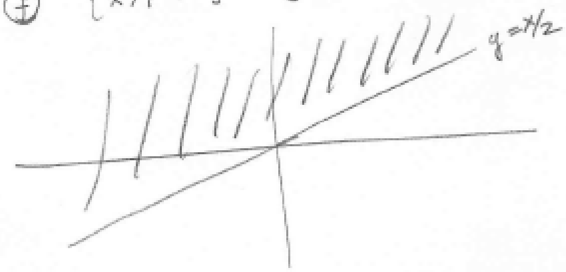
(e) $\{|X/Y| > 2\}$

$-\frac{x}{y} > 2$ $\Leftrightarrow -x/2 > y$	$x/y > 2$ $\Leftrightarrow \frac{1}{2}x > y$
$x/y > 2$ $\Leftrightarrow \frac{x}{2} < y$	$-x/y > 2$ $\Leftrightarrow -\frac{x}{2} < y$



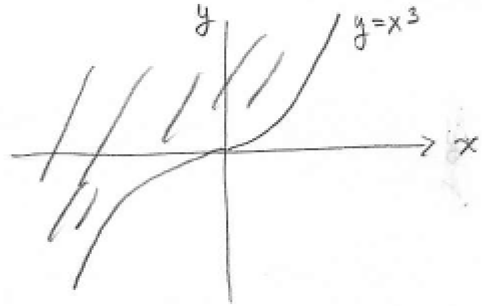
Not product form.

(f) $\{X/Y < 2\} = \{X/2 < Y\}$



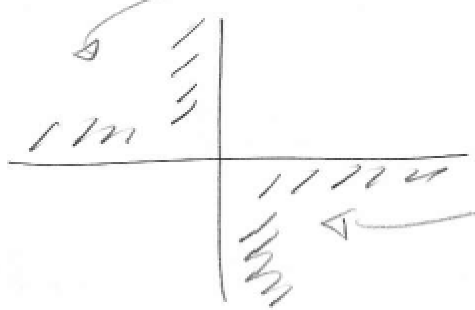
Not product form.

(g) $\{X^3 > Y\}$



Not product form

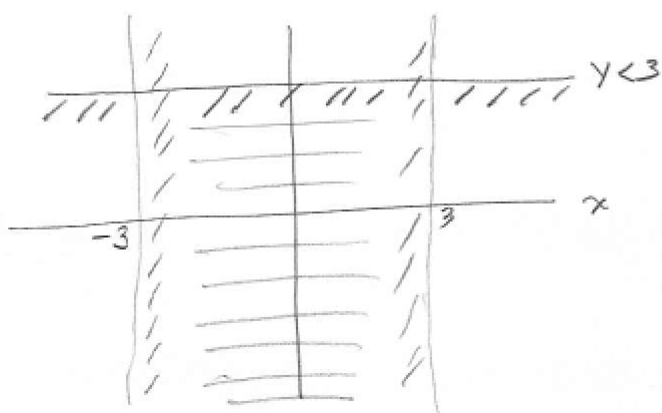
(h) $\{XY < 0\} = \{X < 0, Y > 0\} \cup \{X > 0, Y < 0\}$



Not product form but union of product form

(i) $\max\{(|X|, Y) < 3\} = \max\{(-X, X, Y) < 3\}$

$= \{-X < 3, X < 3, Y < 3\}$



product form

Problem 5.8 Solution

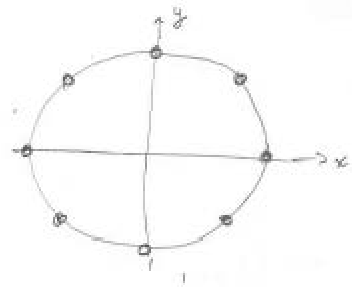
5.12

(a) Mapping S to S_{XY}

Θ	0	1	2	3	4	5	6	7
X,Y	$(r,0)$	$(\frac{r}{\sqrt{2}},\frac{r}{\sqrt{2}})$	$(0,r)$	$(\frac{-r}{\sqrt{2}},\frac{r}{\sqrt{2}})$	$(-r,0)$	$(\frac{-r}{\sqrt{2}},\frac{-r}{\sqrt{2}})$	$(0,-r)$	$(\frac{r}{\sqrt{2}},\frac{-r}{\sqrt{2}})$

(b) Joint PMF

X \ Y	$-r$	$\frac{-r}{\sqrt{2}}$	0	$\frac{r}{\sqrt{2}}$	r
$-r$	0	0	1/8	0	0
$\frac{-r}{\sqrt{2}}$	0	1/8	0	1/8	0
0	1/8	0	0	0	1/8
$\frac{r}{\sqrt{2}}$	0	1/8	0	1/8	0
r	0	0	1/8	0	0



(c) $p_X(x) = \sum_k p_{XY}(x,k)$

$p_X(r) = 1/8$ $\qquad$ $p_Y(r) = 1/8$

$p_X(\frac{r}{\sqrt{2}}) = 1/4$ $\qquad$ $p_Y(\frac{r}{\sqrt{2}}) = 1/4$

$p_X(0) = 1/4$ $\qquad$ $p_Y(0) = 1/4$

$p_X(\frac{-r}{\sqrt{2}}) = 1/4$ $\qquad$ $p_Y(\frac{-r}{\sqrt{2}}) = 1/4$

$p_X(-r) = 1/8$ $\qquad$ $p_Y(-r) = 1/8$

(d) $P[A] = p_X(0) = 1/4$

$P[B] = 1 - p_Y(1) = 7/8$

$P[C] = p_{XY}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}) = 1/8$

$P[D] = p_{XY}(-r,0) = 1/8$

Problem 5.9 Solution

5.19

(g) $P[\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}]$

$$= P[\{\min(X,Y)>0\}] + P[\{\max(X,Y)<0\}]$$

$$= P[X>0, Y>0] + P[X<0, Y<0]$$

$$= F_{XY}(0^-, 0^-) + 1 - P[\{X>0\}^c \cup \{Y>0\}^c]$$

$$= F_{XY}(0^-, 0^-) + 1 - P[X \le 0] - P[Y \le 0] + P[X \le 0, Y \le 0]$$

$$= F_{XY}(0^-, 0^-) + 1 - F_{XY}(0, \infty) - F_{XY}(\infty, 0) + F_{XY}(0, 0)$$

(h) $P[\{X<0, Y>0\} \cup \{X>0, Y<0\}]$

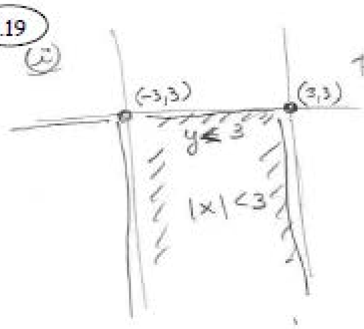
$$= P[\{X<0, Y>0\}] + P[\{X>0, Y<0\}]$$

$$= P[\{X<0\}] - P[X<0, Y \le 0] + P[\{Y<0\}] - P[X \le 0, Y<0]$$

$$= F_{XY}(0^-, \infty) - F_{XY}(0^-, 0) + F_{XY}(\infty, 0^-) - F_{XY}(0, 0^-)$$

5.19

(i)



$$P[-3 < X < 3, Y < 3] = F_{XY}(3^-, 3^-) - F_{XY}(-3, 3^-)$$

Problem 5.10 Solution

5.27

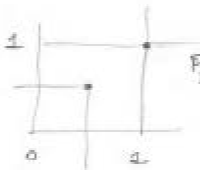
$f_{XY}(x,y) = k x(1-x) y \qquad 0<x<1,\ 0<y<1$

(a)

$$1 = k\int_0^1\int_0^1 x(1-x)\,y\,dx\,dy = k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 = k\underbrace{\left[\frac{1}{2}-\frac{1}{3}\right]}_{\frac{3-2}{6}}\left[\frac{1}{2}\right]$$

$\Rightarrow k = 12$

(b)



$0<x<1 \quad 0<y<1$

$$F_{XY}(x,y) = 12\int_0^x\int_0^y (x'-x'^2)\,y'\,dx'\,dy'$$

$$= 12\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\left(\frac{y^2}{2}\right)$$

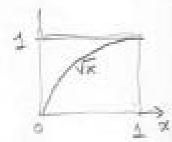
$$= (3x^2-2x^3)\,y^2$$

(c)

$$f_X(x) = \int_0^1 f_{XY}(x,y')\,dy' = 12x(1-x)\int_0^1 y\,dy = 6x(1-x)$$

$$f_Y(y) = \int_0^1 f_{XY}(x',y)\,dx' = 12y\int_0^1 (x-x^2)\,dx = 12y\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1$$

$$= 2y$$



(d)

$$P[Y<\sqrt{X}] = 12\int_0^1 dx\int_0^{\sqrt{x}} (x-x^2)\,y\,dy$$

$$= 12\int_0^1 dx\,(x-x^2)\underbrace{\left[\frac{y^2}{2}\right]_0^{\sqrt{x}}}_{x}$$

$$= 6\int_0^1 dx\,(x^2-x^3) = 6\underbrace{\left[\frac{x^3}{3}-\frac{x^4}{4}\right]}_{\frac{4-3}{12}} = \frac{1}{2}$$

$$P[X<Y] = 12\int_0^1 dy\int_0^y (x-x^2)\,y\,dx$$

$$= 12\int_0^1 y\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^y dy = 12\int_0^1\left(\frac{y^3}{2}-\frac{y^4}{3}\right)dy$$

$$= 12\left[\frac{1}{2}\frac{y^4}{4}-\frac{1}{3}\frac{y^5}{5}\right]_0^1 = 12\left[\underbrace{\frac{1}{2}\frac{1}{4}}_{\frac{1}{8}}-\underbrace{\frac{1}{3}\frac{1}{5}}_{\frac{1}{15}}\right] = \frac{7}{10}$$

Problem 5.11 Solution

For the transformation from arbitrary to uniform, we want

$$f_Y(y) = \left.\frac{f_X(x)}{\left|\frac{dg}{dx}\right|}\right|_{x=g^{-1}(y)} = 1.$$

One obvious way to achieve this would be to select the transformation such that

$$\frac{dg}{dx} = f_X(x) \Rightarrow g(x) = F_X(x).$$

Hence $Y = F_X(x)$ transforms $X \sim f_X(x)$ to $Y \sim \text{uniform}(0,1)$. For the transformation from uniform to abitrary, just use this result in reverse. $Y = F_Y^{-1}(X)$, will transform $X \sim \text{uniform}(0,1)$ to $Y \sim f_Y(y)$.

Problem 5.12 Solution

From the previous problem, the transformations should be chosen according to $Y = F_Y^{-1}(X)$.

(a) Exponential Distribution

$$f_Y(y) = b\,e^{-by}\,u(y) \Rightarrow F_Y(y) = 1 - \exp(-b\,y).$$

$$X = 1 - \exp(-bY) \Rightarrow Y = -\frac{1}{b}\ln(1-X).$$

Note that since X is uniform over $(0,1)$, $1-X$ will be as well. Hence $Y = -\frac{1}{b}\ln(X)$ will work also.

(b)Rayleigh Distribution

$$f_Y(y) = \frac{y}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right)u(y) \Rightarrow F_Y(y) = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right).$$

$$X = 1 - \exp\left(-\frac{Y^2}{2\sigma^2}\right) \Rightarrow Y = \sqrt{-2\sigma^2\ln(1-X)} \text{ or } \sqrt{-2\sigma^2\ln(X)}.$$

(c)Cauchy Distribution

$$f_Y(y) = \frac{b/\pi}{b^2+y^2} \Rightarrow F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{y}{b}\right).$$

$$X = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{Y}{b}\right). \Rightarrow Y = b\tan(\pi x - \pi/2) = -b\cot(\pi x).$$

Problem 5.13 Solution

Solution: MATLAB script:

```
>> N = 10^7;                          % Number of independent random samples
>> X = randn(1,N);                    % Uniform random variable X
>> NA = sum(X > 1.5);                 % Cardinality of A = X>1.5
>> NB = sum(X > 2);                   % Cardinality of B = X>2
>> PBA = NB/NA,                       % Conditional probability of B given A
PBA = 0.3422
>> PBA_exact = qfunc(2)/qfunc(1.5), % Exact conditional probability
PBA_exact = 0.3405
```