ECE 2521: Analysis of Stochastic Processes

Lecture 5

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Moment Generating Functions (MGF) or Transforms

- For a random variable X, the **moment generating function (MGF)** of X is:

$$\Phi_X(s) = E\left[e^{sX}\right]$$

- If X is a **continuous** random variable with PDF $f_X(x)$:

$$\Phi_X(s) = \int_{-\infty}^{\infty} e^{sX} f_X(x)dx$$

- Notice the correspondence to the Laplace transform of the PDF (with reversed sign in the exponent)
- If X is a **discrete** random variable with PMF $P_X(x_i)$:

$$\Phi_X(s) = \sum_{x_i \in S_X} e^{sx_i} P_X(x_i).$$

Characteristic Function

- If we replace $s = j\omega$, then we obtain the **characteristic function** of X:

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)dx$$

- Remarks:
  - $\Phi_X(\omega) = \mathcal{F}[f_X(x)]$, where $\mathcal{F}[\cdot]$ is the Fourier transform (with reversed sign in the exponent)
  - PDF can be computed using the characteristic function:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega)e^{-j\omega x}d\omega$$

  - $\Phi_X(\omega)$ is well defined for all density functions:

$$|\Phi_X(\omega)| \leq \int_{-\infty}^{\infty} |f_X(x)|dx = \int_{-\infty}^{\infty} f_X(x)dx = 1$$

Characteristic Function

- If X is a **discrete random variable** with PMF $P_X(x_k)$:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \sum_{x_k \in S_X} P_X(x_k)\delta(x - x_k)e^{j\omega x}dx = \sum_{x_k \in S_X} P_X(x_k)e^{j\omega x_k}$$

Note: We obtain the discrete-time Fourier transform (DTFT) of the PMF (with reversed sign in the exponent) which is a periodic function of ω with period 2π.

- Let $z = e^{j\omega}$:

$$\Phi_X(z) = \sum_{x_k \in S_X} P_X(x_k)z^{x_k}$$

Note: If x_k are integers, then $\Phi_X(z) = \sum_{x_k \in S_X} P_X(x_k)z^{x_k}$ is the Z-transform of the PMF (with reversed sign in the exponent).

Moment Generating Functions (MGF) or Transforms

Why and when to use MGFs (transforms)?

- Provides an alternative representation for PDF/PMF
- Is invertible: given the MGF it is possible to compute PDF/PMF
- Is a complete probability model of a random variable
- Some calculations that are difficult in the PDF/PMF domain, can be simple in the MGF domain (transform domain)
- Is often convenient for:
    - Finding the PDF of a sum of random variables
    - Calculation of moments
    - Analytical derivations and proving of theorems

Moment Generating Functions (MGF) or Transforms

Random Variable	PMF	MGF $\phi_X(s)$
Bernoulli (p)	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$	$1-p+pe^s$
Binomial (n,p)	$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$	$(1-p+pe^s)^n$
Geometric (p)	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{pe^s}{1-(1-p)e^s}$
Pascal (k,p)	$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$	$(\dfrac{pe^s}{1-(1-p)e^s})^k$
Poisson (α)	$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$e^{\alpha(e^s-1)}$
Disc. Uniform (k,l)	$P_X(x) = \begin{cases} \frac{1}{l-k+1} & x=k,k+1,\ldots,l \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{sk}-e^{s(l+1)}}{1-e^s}$

Moment Generating Functions (MGF) or Transforms

Random Variable	PDF	MGF $\phi_X(s)$
Constant (a)	$f_X(x) = \delta(x-a)$	e^{sa}
Uniform (a, b)	$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{bs} - e^{as}}{s(b-a)}$
Exponential (λ)	$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{\lambda}{\lambda - s}$
Erlang (n, λ)	$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$(\dfrac{\lambda}{\lambda - s})^n$
Gaussian (μ, σ)	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	$e^{s\mu + s^2\sigma^2/2}$

Example 1

Let

$$p_X(x) = \begin{cases} 1/2, & \text{if x=2} \\ 1/6, & \text{if x=3} \\ 1/3, & \text{if x=5} \end{cases}$$

Find the moment generating function (MGF) of X.

Solution:

$$\Phi_X(s) = E\left[e^{sX}\right] = \frac{1}{2}e^{2s} + \frac{1}{6}e^{3s} + \frac{1}{3}e^{5s}$$

Example 2

Find the moment generating function of an exponential random variable, with parameter λ:

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

Solution:

$$\begin{aligned}
\Phi_X(s) = E\left[e^{sX}\right] &= \lambda \int_0^\infty e^{sx} e^{-\lambda x} dx \\
&= \lambda \int_0^\infty e^{(s-\lambda)x} dx \\
&= \lambda \frac{e^{(s-\lambda)x}}{s-\lambda}\Big|_0^\infty \quad (\text{if } s < \lambda) \\
&= \frac{\lambda}{\lambda - s}
\end{aligned}$$

MGF of a linear function of a random variable

- Consider a random variable X with its MGF: $\Phi_X(s)$
- Let $Y = aX + b$ be a linear function of X (a, b are contants)
- The MGF of Y can be defined as following:

$$\Phi_Y(s) = e^{sb}\Phi_X(sa)$$

Proof: MGF of $Y = aX + b$ can be derived as a function of $\Phi_X(s)$:

$$\begin{aligned}\Phi_Y(s) &= E\left[e^{sY}\right] = E\left[e^{s(aX+b)}\right] = E\left[e^{saX}e^{sb}\right] = e^{sb}E[e^{saX}] \\ &= e^{sb}\Phi_X(sa)\end{aligned}$$

Example

Find the moment generating function of a Gaussian random variable, with mean μ and variance σ^2.

- First, let us find the MGF of a standard normal RV $X \sim \text{Gaussian}(0,1)$
- The PDF of a standard normal is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- The associated MGF is:

$$\begin{aligned}
\Phi_X(s) &= E\left[e^{sX}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{sx}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2+sx}\,dx \\
&= e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2+sx-s^2/2}\,dx = e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2}\,dx = e^{s^2/2}
\end{aligned}$$

Example

- Then for any general Gaussian RV $Y \sim \text{Gaussian}(\mu, \sigma)$ with PDF:

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}$$

- Y is obtained from the standard normal via a linear transformation:

$$Y = \sigma X + \mu$$

- The MGF of a Gaussian RV $Y \sim \text{Gaussian}(\mu, \sigma)$ is then obtained as:

$$\Phi_Y(s) = e^{sb}\Phi_X(sa) = e^{\mu s} e^{s^2\sigma^2/2} = e^{\sigma^2 s^2/2+\mu s}$$

Moment Generating Properties

- Recall that for a random variable X the nth moment is $E\left[X^n\right]$
- The first, second, and nth moments of a random variable X can be obtained from its MGF $\Phi_X(s)$:

(i) Mean (first moment):

$$E[X] = \left.\frac{d\Phi_X(s)}{ds}\right|_{s=0}$$

$$\frac{d}{ds}\Phi_X(s) = \frac{d}{ds}E[e^{sX}] = E[\frac{d}{ds}e^{sX}] = E[Xe^{sX}]$$

Setting $s = 0$:

$$E[X] = E[Xe^{0X}] = \left.\frac{d}{ds}\Phi_X(s)\right|_{s=0}$$

Moment Generating Properties

(ii) Second moment:

$$E[X^2] = \frac{d^2\Phi_X(s)}{ds^2}\bigg|_{s=0}$$

$$\frac{d^2}{ds^2}\Phi_X(s) \quad = \quad \frac{d^2}{ds^2}E[e^{sX}] = E[\frac{d^2}{ds^2}e^{sX}] = E[X^2 e^{sX}]$$

Setting $s = 0$:

$$E[X^2] = E[X^2 e^{0X}] = \frac{d^2}{ds^2}\Phi_X(s)\bigg|_{s=0}$$

(iii) More generally, nth moment:

$$E[X^n] = \frac{d^n\Phi_X(s)}{ds^n}\bigg|_{s=0}$$

Example 1

$$p_X(x) = \begin{cases} 1/4, & \text{if x=5} \\ 1/3, & \text{if x=6} \\ 5/12, & \text{if x=8} \end{cases}$$

Find the moment generating function (MGF) and the expected value of X.

$$\Phi_X(s) = E\left[e^{sX}\right] = \frac{1}{4}e^{5s} + \frac{1}{3}e^{6s} + \frac{5}{12}e^{8s}$$

$$\begin{aligned} E[X] &= \left.\frac{d}{ds}\Phi_X(s)\right|_{s=0} = \left.\frac{1}{4}\cdot 5e^{5s} + \frac{1}{3}\cdot 6e^{6s} + \frac{5}{12}\cdot 8e^{8s}\right|_{s=0} \\ &= \frac{5}{4} + \frac{6}{3} + \frac{40}{12} = \frac{79}{12} \end{aligned}$$

Example 2

Find the mean and variance of an exponential (λ) random variable using its MGF.

The MGF of an exponential (λ) random variable X (*from previous example*):

$$\Phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$E[X] = \frac{d}{ds}\Phi_X(s)\bigg|_{s=0} = \frac{\lambda}{(\lambda - s)^2}\bigg|_{s=0} = \frac{1}{\lambda}$$

$$E\left[X^2\right] = \frac{d^2}{ds^2}\Phi_X(s)\bigg|_{s=0} = \frac{2\lambda}{(\lambda - s)^3}\bigg|_{s=0} = \frac{2}{\lambda^2}$$

$$\mathrm{Var}[X] = E\left[X^2\right] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Example 3

The MGF of a random variable X is give by:

$$\Phi_X(s) = \frac{1}{5} + \frac{2}{5}e^{2s} + \frac{2}{5}e^{12s}$$

Find the PMF of X.

Solution:

Since the MGF is a sum of terms of the form e^{sx} we can compare with the general formula $\Phi_X(s) = \sum_x e^{sx} p_X(x)$ and then by pattern matching:

$$P_X(x) = \begin{cases} 1/5, & \text{if } x = 0 \\ 2/5, & \text{if } x = 2, 4 \\ 0, & \text{otherwise} \end{cases}$$

Example 4

The MGF of a random variable X is given by:

$$\Phi_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

where p is a constant $0 < p \leq 1$. Find the distribution of X.

Solution:

Recall the identity,

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots \quad \text{when } |\alpha| < 1,$$

Now let $\alpha = (1-p)e^s$ in the above example.

$$\begin{aligned}\Phi_X(s) = \frac{pe^s}{1-(1-p)e^s} &= pe^s\left(1 + (1-p)e^s + (1-p)^2e^{2s} + (1-p)^3e^{3s} + \cdots\right) \\ &= pe^s + p(1-p)e^{2s} + p(1-p)^2e^{3s} + p(1-p)^3e^{4s} + \cdots\end{aligned}$$

Now the probabilities $P(X=k)$ are found by reading the coefficients of e^{ks}, i.e,

$$P(X=k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

Which is the **geometric** (p) distribution.

Pairs of Random Variables

- Many experiments produce outcomes that are used to compute two or more random variables
- The probability model for such an experiment contains the properties of the individual random variables and it also contains the relationships among the random variables

Example

- Consider a radio transmitter or sonar that emits signal X and the corresponding signal Y that arrives at the receiver
- Noise and distortion along the propagation path prevent us from observing X directly
- We use Y to infer X through their joint distributions

Joint Probability Mass Function for Two Discrete RVs

- The joint PMF of discrete random variables X and Y is:

$$p_{X,Y}(x,y) = \text{Prob}[X = x, Y = y]$$

- Provides the probability of observing the event $\{X = x, Y = y\}$
- Found by summing the probabilities of all outcomes of the experiment that lead to that event
- Complete probability model for a pair of discrete RVs
- Can be represented by a list, a matrix, or a graph
- Satisfies the axioms of probability:
  (1) Normalization: $\sum_x \sum_y p_{X,Y}(x,y) = 1$
  (2) Nonnegativity: $p_{X,Y}(x,y) \geq 0$

Marginal Probability Mass Function

- The marginal probability mass functions for random variables X and Y can be obtained from their joint PMF by:

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x, y)$$

$$p_Y(y) = \sum_{x \in S_X} p_{X,Y}(x, y)$$

Note: The terminology *marginal* comes from the matrix representation of the joint PMF since the marginal PMFs of X and Y can be obtained by adding rows or columns of the joint PMF.

Example

- Consider two rolls of a fair four-sided die with outcomes (A_i, B_j)
- Define random variable X and Y as follows:

$$X : \text{Sum of the the rolls}, (A_i + B_j)$$

$$Y : \text{Twice the absolute difference} : 2|A_i - B_j|$$

- Find the following:
  (1) Joint PMF for X and Y, $p_{X,Y}(x, y)$.
  (2) The marginal PMFs for X and Y, $p_X(x)$ and $p_Y(y)$.

Example - Solution

First Roll, A_i

Second Roll, B_j	1	2	3	4
1	X=2, Y=0	X=3, Y=2	X=4, Y=4	X=5, Y=6
2	X=3, Y=2	X=4, Y=0	X=5, Y=2	X=6, Y=4
3	X=4, Y=4	X=5, Y=2	X=6, Y=0	X=7, Y=2
4	X=5, Y=6	X=6, Y=4	X=7, Y=2	X=8, Y=0

Tabular Method for $p_{X,Y}(x,y)$

	X=2	X=3	X=4	X=5	X=6	X=7	X=8
Y=0	1/16	0	1/16	0	1/16	0	1/16
Y=2	0	2/16	0	2/16	0	2/16	0
Y=4	0	0	2/16	0	2/16	0	0
Y=6	0	0	0	2/16	0	0	0

Example - Solution

- The joint PMF for X and Y:

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{16} & \text{if } (x,y) = \{(2,0),(4,0),(6,0),(8,0)\} \\ \frac{2}{16} & \text{if } (x,y) = \{(3,2),(5,2),(7,2),(4,4),(6,4)(5,6)\} \\ 0 & \text{otherwise.} \end{cases}$$

- The marginal PMFs for X and Y:

$$p_X(x) = \begin{cases} \frac{1}{16} & \text{if } x = 2,8 \\ \frac{2}{16} & \text{if } x = 3,7 \\ \frac{3}{16} & \text{if } x = 4,6 \\ \frac{4}{16} & \text{if } x = 5 \\ 0 & \text{otherwise.} \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{4}{16} & \text{if } y = 0,4 \\ \frac{6}{16} & \text{if } y = 2 \\ \frac{2}{16} & \text{if } y = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Joint PDF for Two Continuous RVs

- The PDF of two continuous random variables X and Y allows us to calculate the probability of observing the random variables over some area:

$$\text{Prob}(a < X \leq b, c < Y \leq d) = \int_c^d \int_a^b f_{X,Y}(x,y)dxdy$$

- Complete probability model for a pair of continuous RVs
- Satisfies the axioms of probability:
  (1) Normalization: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
  (2) Nonnegativity: $f_{X,Y}(x,y) \geq 0$

Marginal Probability Density Function

- The marginal PDFs for random variables X and Y can be obtained from their joint PDF by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Example

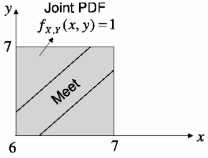
- Romeo and Juliet have a date at the park for 6 pm
- Each is expected to arrive with a delay time that is uniformly distributed between 0 and 1 hour
- The person to arrive first will only wait 15 mins
- What is the probability that they will meet?

Example - Solution

- Let random variables X and Y represent the arrival times of Romeo and Juliet, respectively
- X and Y are independent, so their joint PDF is given by:

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y) \\ &= \begin{cases} 1 & \text{if } 6 \le x \le 7, 6 \le y \le 7 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

- Joint PDF for random variables X and Y representing the arrival time in hours for Romeo and Juliet respectively:



- Probability that they will meet: $\text{Prob}(\text{Meet}) = 1 - (3/4)^2 = 7/16$.

Joint Cumulative Distribution Function

- For any pair of general random variables, X and Y, their joint CDF is defined by: $F_{X,Y}(x,y) = \text{Prob}(X \leq x, Y \leq y)$
- Satisfies the following properties:
  (a) $0 \leq F_{X,Y}(x,y) \leq 1$
  (b) $F_X(x) = F_{X,Y}(x,\infty)$
  (c) $F_Y(y) = F_{X,Y}(\infty,y)$
  (d) $F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = F_{X,Y}(-\infty,-\infty) = 0$
  (e) If $x \leq x_1$ and $y \leq y_1$, then $F_{X,Y}(x,y) \leq F_{X,Y}(x_1,y_1)$.
  (f) $F_{X,Y}(\infty,\infty) = 1$
- For two continuous random variables, X and Y, the following relates their joint PDF and joint CDF:

$$F_{XY} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)dxdy \Leftrightarrow f_{X,Y} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Joint Cumulative Distribution Function

- Marginal CDF:

$$\begin{aligned} F_X(x) &= F_{X,Y}(x,\infty) = \text{Prob}[X \le x, Y < \infty] \\ F_Y(y) &= F_{X,Y}(\infty,y) = \text{Prob}[X < \infty, Y \le y] \end{aligned}$$

- Probability calculation on one variable:

$$\text{Prob}[x_1 \le X \le x_2, Y \le y] = F_{X,Y}(x_2,y) - F_{X,Y}(x_1,y)$$

- Probability calculation on both variables:

$$\text{Prob}[x_1 \le X \le x_2, y_1 \le Y \le y_2]$$

$$= F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$$