ECE/ME 2646: Linear System Theory

Lecture 3: LTI Systems; Rational Transfer Functions and State-space Equations; Linearization

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Linear time-invariant systems

- Definition
  - A system is said to be time invariant if for every state-input-output pair

$$\left.\begin{array}{l}\mathbf{x}(t_0)=\mathbf{x}_0\\ \mathbf{u}(t),\ t\ge t_0\end{array}\right\}\rightarrow \mathbf{y}(t),\ t\ge t_0$$

  and any T, we have

$$\left.\begin{array}{l}\mathbf{x}(t_0+T)=\mathbf{x}_0\\ \mathbf{u}(t-T),\ t\ge t_0+T\end{array}\right\}\rightarrow \mathbf{y}(t-T),\ t\ge t_0+T$$

  - The above definition means that if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T

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  - The above definition means that if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T
  - Examples of time-variant systems: burning rocket and brain

Linear time-invariant systems

- Definition
- Input-output description

For a causal, SISO, LTI system relaxed at 0:

$$\boxed{g(t,\tau) = g(t+T,\tau+T) = g(t-\tau,0) \equiv g(t-\tau)}$$

$$\boxed{y(t) = \int_0^t g(t,\tau)u(\tau)d\tau} \rightarrow \boxed{y(t) = \int_0^t g(t-\tau)u(\tau)d\tau}$$

Convolution integral

Linear time-invariant systems

- Definition
- Input-output description

Question: If an LTI system is causal, what is the value of $g(t)$ for $t < 0$?

Linear time-invariant systems

- Definition
- Input-output description

The condition for an LTI system to be causal is $g(t) = 0$ for $t < 0$

Question: The impulse response of an ideal lowpass filter is given by

$$g(t) = 2\omega \frac{\sin 2\omega(t-t_0)}{2\omega(t-t_0)}.$$

Is the ideal lowpass filter causal?

Linear time-invariant systems

- Definition
- Input-output description
- Transfer-function matrix

For a causal, MIMO, LTI system (with p input and q output) relaxed at 0:

$$\boxed{\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)}$$

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix}$$

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s: $\boxed{\hat{g}(s) = N(s)/D(s)}$
  - $\hat{g}(s)$ is proper, if deg (degree of) $D(s) \geq \deg N(s)$
  - $\hat{g}(s)$ is strictly proper if $\deg D(s) > \deg N(s)$
  - $\hat{g}(s)$ is biproper $\deg D(s) = \deg N(s)$
  - $\hat{g}(s)$ is improper if $\deg D(s) < \deg N(s)$

Question: Improper rational transfer functions rarely arise in practice, since they will amplify high-frequency noise. Why?

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s: $\boxed{\hat{g}(s) = N(s)/D(s)}$
- Poles and zeros
  - A real or complex number λ is called a pole if $D(\lambda) = 0$, and λ is called a zero if $N(\lambda) = 0$
  - If $N(s)$ and $D(s)$ have no common factors of degree 1 or higher, they are called coprime

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s:

$$\boxed{\hat{g}(s) = N(s)/D(s)}$$

- Poles and zeros
- State-space equations
  - Deriving transfer-function matrix from state-space equation

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}} \xrightarrow{\text{Laplace transform}} \boxed{\begin{aligned}s\hat{\mathbf{x}}(s) - \mathbf{x}(0) &= \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}\hat{\mathbf{x}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}}$$

For zero initial state

$$\boxed{\hat{\mathbf{y}}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\,\hat{\mathbf{u}}(s)} \rightarrow \boxed{\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

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Linearization

- Many physical systems can be described by nonlinear differential equations

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t)\\ \mathbf{y}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\end{aligned}}$$

- Some nonlinear equations can be approximated by linear equations (but how?)

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Linearization

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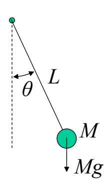
- Some nonlinear equations can be approximated by linear equations



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

For small value of θ

$$L\frac{d^2\theta(t)}{dt^2} + g\theta(t) = 0$$

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$\sin\theta \rightarrow \theta$

Can't do anything to derivs.

Linearization

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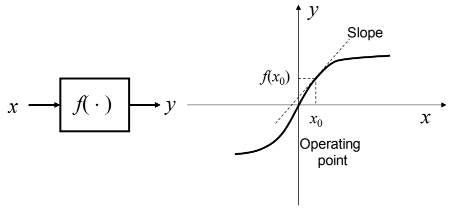

Linearization

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- Some nonlinear equations can be approximated by linear equations
  - For some input $\mathbf{u}_0(t)$ and some initial state, $\mathbf{x}_0(t)$ is the solution of the above equations

$$\boxed{\dot{\mathbf{x}}_0(t) = \mathbf{h}(\mathbf{x}_0(t),\mathbf{u}_0(t),t)}$$

  - Suppose $\mathbf{x}(t) = \mathbf{x}_0(t) + \bar{\mathbf{x}}(t)$ for slightly perturbed input $\mathbf{u}(t) = \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)$ and initial state

Linearization

- Many physical systems can be described by nonlinear differential equations
- Some nonlinear equations can be approximated by linear equations

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t),\mathbf{u}(t),t)\\ \dot{\mathbf{x}}_0(t) &= \mathbf{h}(\mathbf{x}_0(t),\mathbf{u}_0(t),t)\\ \mathbf{x}(t) &= \mathbf{x}_0(t) + \bar{\mathbf{x}}(t)\\ \mathbf{u}(t) &= \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{\mathbf{x}}_0(t) + \dot{\bar{\mathbf{x}}}(t) &= \mathbf{h}(\mathbf{x}_0(t) + \dot{\bar{\mathbf{x}}}(t), \mathbf{u}_0(t) + \dot{\bar{\mathbf{u}}}(t), t)\\ &= \mathbf{h}(\mathbf{x}_0(t),\mathbf{u}_0(t),t) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\bar{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\bar{\mathbf{u}} + \cdots\end{aligned}}$$

$$\mathbf{h} = [h_1 \cdots h_n]', \ \mathbf{x} = [x_1 \cdots x_n]', \ \mathbf{u} = [u_1 \cdots u_p]'$$

$$\mathbf{A}(t) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{bmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 & \cdots & \partial h_1/\partial x_n \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 & \cdots & \partial h_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial h_n/\partial x_1 & \partial h_n/\partial x_2 & \cdots & \partial h_n/\partial x_n \end{bmatrix}$$

Jacobians

$$\mathbf{B}(t) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{u}} = \begin{bmatrix} \partial h_1/\partial u_1 & \partial h_1/\partial u_2 & \cdots & \partial h_1/\partial u_p \\ \partial h_2/\partial u_1 & \partial h_2/\partial u_2 & \cdots & \partial h_2/\partial u_p \\ \vdots & \vdots & & \vdots \\ \partial h_n/\partial u_1 & \partial h_n/\partial u_2 & \cdots & \partial h_n/\partial u_p \end{bmatrix}$$

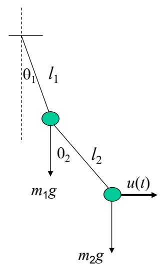
$$\boxed{\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}(t)\bar{\mathbf{x}}(t) + \mathbf{B}(t)\bar{\mathbf{u}}}$$

Linearization

- Many physical systems can be described by nonlinear differential equations
- Some nonlinear equations can be approximated by linear equations
- An example

$$x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2$$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) \\
&\quad + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\cdot u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{g}{l_2}\sin x_3 + \frac{1}{m_2 l_2}(\cos x_3)\, u
\end{aligned}$$



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Linearization

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- Some nonlinear equations can be approximated by linear equations
- An example

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\cdot u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{g}{l_2}\sin x_3 + \frac{1}{m_2 l_2}(\cos x_3)\, u
\end{aligned}$$

$$\Downarrow$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(m_1+m_2)g/(m_1 l_1) & 0 & m_2 g/m_1 l_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 l_2 \end{bmatrix} u$$

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