ECE 2372 - Homework 1

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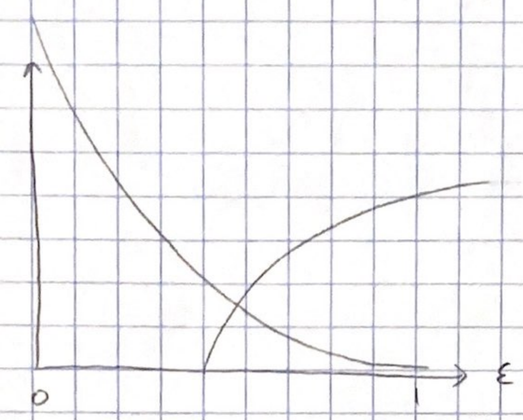
1.

a.

	m = 1	m = 1000	m = 1000000
p(heads) = 0.07	0.665	1	1
p(heads) = 0.075	0.458	1	1

$1-(1-\hat{p}_{tails})^m$

b. Hoeffding Inequality: $P[|\hat{R}_n(h^*) - R(h^*)| \geq \varepsilon] \leq 2me^{-2\varepsilon^2 n}$
$\leq 4e^{-20\varepsilon^2}$



Exact probability =

$P[\max_i |\hat{p}_i - p_i| > \varepsilon] =$
$1 - P[\max_i |\hat{p}_i - p_i| \leq \varepsilon] =$
$= 1 - 4e^{-20\varepsilon^2}$

2. pmf $g_k(x)$ $\quad f^*(x) = \arg\max_k \mu_k(x) = \arg\max_k \pi_k g_k(x)$

Maximize $1 - P(\text{Error}) = P[f(x) = Y]$

$$= \sum_{k=0}^{K-1} \pi_k \cdot P[f(x) = k \mid Y = k]$$

$$= \sum_{k=0}^{K-1} \pi_k \int_{\Gamma_k(f)} g_k(x)\, dx = \sum_{k=0}^{K-1} \mu_k(x)$$

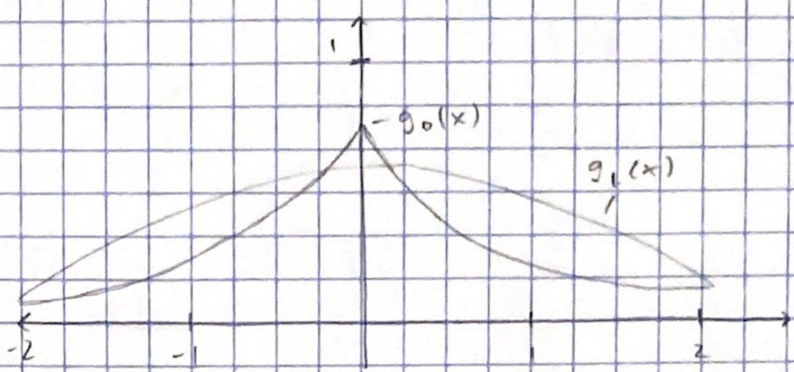
$x \in \Gamma_k(f) \longleftrightarrow \pi_k g_k(x)$ is maximized, also $\mu_k(x)$

$\therefore f^*(x) = \arg\max_k \mu_k(x) = \arg\max_k \pi_k g_k(x)$

3. $X | Y=0 \sim g_0(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|}$

$X | Y=1 \sim g_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

a.



b. $\pi_0 g_0(x) = \pi_1 g_1(x)$ at decision boundary

$\pi_0 = \pi_1$ so $g_0(x) = g_1(x)$ for boundary

$$\frac{g_1(x)}{g_0(x)} \underset{0}{\overset{1}{\gtrless}} 1 \quad \text{since } \pi_0 = \pi_1$$

Since $\pi_0 = \pi_1$, $f^*(x) = \arg\max g_k(x)$

= max between $g_0(x)$ and $g_1(x)$

c. $R(f^*) = P[X < \frac{1}{2} | Y=1] + P[X > \frac{1}{2} | Y=0]$

$= \frac{1}{2} \int_{-\infty}^{1/2} g_1(t)\, dt + \frac{1}{2} \int_{1/2}^{\infty} g_0(t)\, dt$

4. $\hat{f}(x) = \begin{cases} 1 & a^T x + b \geq 0 \\ 0 & \text{otherwise} \end{cases}$ Derive a and b

If $k=2$, then

$$\delta_0(x) \underset{1}{\overset{0}{\gtrless}} \delta_1(x)$$

$$x^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \log \pi_0 \underset{1}{\overset{0}{\gtrless}} x^T \Sigma^{-1} \mu_1 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \log \pi_1$$

$$\underbrace{x^T \Sigma^{-1} \mu_0}_{\bullet} \underbrace{- \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0}_{\square} + \log \pi_0 \underbrace{- x^T \Sigma^{-1} \mu_1}_{\bullet} + \underbrace{\frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1}_{\square} - \log \pi_1 \overset{1}{\underset{0}{\gtrless}} 0$$

$$x^T \Sigma^{-1} (\mu_0 - \mu_1) - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \log \left(\frac{\pi_0}{\pi_1} \right) \overset{1}{\underset{0}{\gtrless}} 0$$

if a is $\Sigma^{-1}(\mu_0 - \mu_1)$ and b is

$-\frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \log\left(\frac{\pi_0}{\pi_1}\right)$, the decision boundary becomes

$$a^T x + b \overset{1}{\underset{0}{\gtrless}} 0$$