ECE 2521 Analysis of Stochastic Processes
Solutions to Problem Set 1

Comments from our Grader:

minor error (-1)

misunderstand the concept and critical error (-3)

Problem 1.1 Solution

1.6 a) In the first draw the outcome can be black (b) or white (w). If the first draw is black, then the second outcome can be b or w. However if the first draw is white, then the run only contains black balls so the second outcome must be b. Therefore $\mathcal{S} = \{bb, bw, wb\}$.

b) In this case all outcomes can be b or w. Therefore $\mathcal{S} = \{bb, bw, wb, ww\}$.

c) In part a) the outcome ww cannot occur so $f_{ww} = 0$. In part b) let N be a larger number of repetitions of the experiment. The number of times the first outcome is w is approximately $N/3$ since the run has one white ball and two black balls. Of these $N/3$ outcomes approximately $1/2$ are also white in the second draw. Thus $N/9$ if the outcome result is ww, and thus $f_{ww} = \frac{1}{9}$.

d) In the first experiment, the outcome of the first draw affects the probability of the outcomes in the second draw. In the second experiment, the outcome of the first draw does not affect the probability of the outcomes in the second draw.

Problem 1.2 Solution

1.9

$$\begin{aligned}
< X >_n &= \frac{1}{n}\sum_{j=1}^{n} X(j) \qquad n > 0 \\
&= \frac{n-1}{n}\frac{1}{n-1}\left\{\sum_{j=1}^{n-1} X(j) + X(n)\right\} \\
&= \left(1 - \frac{1}{n}\right) < X >_{n-1} + \frac{1}{n}X(n) \\
&= < X >_{n-1} + \frac{X(n) - < X >_{n-1}}{n}
\end{aligned}$$

Problem 1.3 Solution

2.1 (a) $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

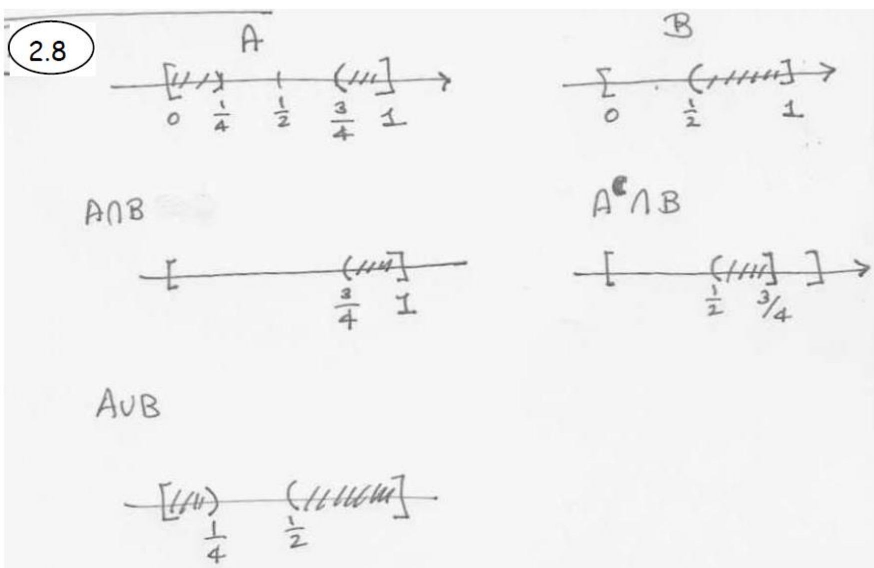
(b) $A = \{1, 2, 3, 4\}$ $B = \{2, 3, 4, 5, 6, 7, 8\}$ $D = \{1, 3, 5, 7, 9, 11\}$

(c) $A \cap B \cap D = \{3\}$ $A^c \cap B = \{5, 6, 7, 8\}$

$A \cup (B \cap D^c) = \{1, 2, 3, 4, 6, 8\}$

$(A \cup B) \cap D^c = \{2, 4, 6, 8\}$

Problem 1.4 Solution



Problem 1.5 Solution

2.14 **a)** $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^a) \cup (A^c \cap B^c \cap C)$

b) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

c) $A \cup B \cup C$

d) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$

e) $A^c \cap B^c \cap C^c$

Problem 1.6 Solution

2.30 **a)** Corollary 7 implies $P[A \cup B] \leq P[A] + P[B]$. (Eqn. 2.8). Applying this inequality twice, we have

$$P[(A \cup B) \cup C] \leq P[A \cup B] + P[C] \leq P[A] + P[B] + P[C]$$

b) Eqn. 2.8 implies the $n = 2$ case.
Suppose the result is true for n:

$$P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P[A_k] \qquad (*)$$

Then

$$\begin{aligned}
P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] \\
&= \leq P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}] \text{ by Eqn. 2.8} \\
&\leq \sum_{k=1}^{n} P[A_k] + P[A_{n+1}] \text{ by (*)} \\
&= \sum_{k=1}^{n+1} P[A_k]
\end{aligned}$$

which completes the induction argument.

(c) $P[\bigcap_{k=1}^{n} A_k] = 1 - P[(\bigcap_{k=1}^{n} A_k)^c] = 1 - P[\bigcup_{k=1}^{n} A_k^c]$

$\geq 1 - \sum_{k=1}^{n} P[A_k^c]$ using the result of part b.

Problem 1.7 Solution

2.34

Assume that the probability of any subinterval I of $[-1,2]$ is proportional to its length, then

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1,2]$ then we must have that

$$1 = P[S] = P[[-1,2]] = k \text{ length } ([-1,2]) = 3k \Rightarrow k = \frac{1}{3}.$$

a) $P[A] = \frac{1}{3} \text{ length } ([-1,0)) = \frac{1}{3}(1) = \frac{1}{3}$

$P[B] = \frac{1}{3} \text{ length } ((-0.5,1)) = \frac{1}{3}\frac{3}{2} = \frac{1}{2}$

$P[C] = \frac{1}{3} \text{ length } ((0.75,2)) = \frac{1}{3}\frac{5}{4} = \frac{5}{12}$

$P[A \cap B]$ No overlap = 0

$P[A \cap C] = P[\emptyset] = 0$

b) $P[A \cup B] = P[S]$ ~~#### ~~ $\frac{1}{3} \text{ length } ([-1,1)) = \frac{2}{3}$

$$P[A \cup C] = \frac{1}{3} \text{ length } (A \cup C)$$

$$= \frac{1}{3}\left(1 + \frac{5}{4}\right) = \frac{3}{4}$$

$$P[A \cup B \cup C] = P[S] = 1$$

Now use axioms and corollaries:

$$\begin{aligned}
P[A \cup B] &= P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5} \\
&= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \quad \checkmark \\
P[A \cup C] &= P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] = \frac{1}{3} + \frac{5}{18} = \frac{3}{4} \quad \checkmark \quad \text{by Cor. 5} \\
P[A \cup B \cup C] &= P[A] + P[B] + P[C] \\
&\quad - P[A \cap B] - P[A \cap C] - P[B \cap C] \\
&\quad + P[A \cap B \cap C] \quad \text{by Eq. (2.7)} \\
&= \frac{1}{3} + \frac{1}{2} + \frac{5}{18} - \frac{1}{6} - 0 - \frac{1}{12} + 0 \\
&= 1 \quad \checkmark
\end{aligned}$$

Problem 1.8 Solution

2.119 $\mathcal{F} = \{\phi, A, A^c, S\}$

(i) $\phi \in \mathcal{F}$ ✓

(ii) if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$?

$$A \cup A^c = S \in \mathcal{F}$$

and any other union of events in $\mathcal{F}$ yields an event in $\mathcal{F}$ ✓

(iii) if $B \in \mathcal{F}$ then $B^c \in \mathcal{F}$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$A^c \in \mathcal{F} \Rightarrow A \in \mathcal{F}$$

and similarly for other events in $\mathcal{F}$ ✓

$\therefore$ $\mathcal{F}$ is a field.

Problem 1.9 Solution

Total number of components in the circuit is N=120+75+15+80+25+10=325

Event	What it means?	Probability
A	Component is a resistor	$P(A)$ = (120+80)/325 ~ 0.615
B	Component is a capacitor	$P(B)$ = (75+25)/325 ~ 0.307
C	Component is an inductor	$P(C)$ = (15+10)/325 ~ 0.077
D	Component has high tolerance	$P(D)$ = (120+75+15)/325 ~ 0.646
E	Component has low tolerance	$P(E)$ = (80+25+10)/325 ~ 0.354
$A \cap D$	Component is a high tolerance resistor	$P(A \cap D)$ = 120/325 ~ 0.369
$C \cap E$	Component is a low tolerance inductor	$P(C \cap E)$ = 10/325 ~ 0.031
$A \cup B$	Component is a resistor or a capacitor	$P(A \cup B)$ = (120+80+75+25)/325 ~ 0.923
$(A \cup B) \cap D$	Component is a high tolerance resistor or a high tolerance capacitor	$P((A \cup B) \cap D)$ =(120+75)/325 ~ 0.6

Problem 1.10 Solution

P1.10 [MATLAB Problem] A pair of "fair" dice are rolled and the values of their up-faces are added to obtain an outcome of this random experiment.

(a) Consider all possible pairs which will result in a minimum value of sum equal to 2 $(1+1)$ to the maximum value of 12 $(6+6)$. Then the sample-space events and the associated probabilities are given by the following table:

sum:	2	3	4	$\cdots$	7	$\cdots$	10	11	12
Possibilities:	1	2	3	$\cdots$	6	$\cdots$	3	2	1
Probability:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\cdots$	$\frac{6}{36}$	$\cdots$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b) The MATLAB function Dice2Sum:

```
function [Ns,s] = Dice2Sum(N)
% Simulates Sum of Two Independent Dice up-Faces
% [Ns,s] = Dice2Sum(N)
% Pre:
%       N = Number of trails
% Post:
%       s = Sample space
%      Ns = Count
%
s = 2:12;                     % Sample space
D1 = randint(N,1,[1,6]);      % Throw of the first dice
D2 = randint(N,1,[1,6]);      % Throw of the second dice
S = D1+D2;                    % Add two face values
% The next three lines determine the count in an efficient manner
  S = sort([S;s']); S = diff(S);
  I = find(S==1); I = [0;I;N];
  Ns = diff(I)-1;
```

Estimates of probabilities using $N = 100000$ and its bar-graph is computed using the following MATLAB script:

```
clc;
%% (b) Verification of the probabilities of the Sample space in (a)
N = 100000; [Ns,s] = Dice2Sum(N);
p = Ns/N; % Probabilities using the relative frequencies
% Bar Plot
if exist('Hf_Hw1P08'); close(Hf_Hw1P08); end
Hf_Hw1P08 = figure('paperunits','inches','paperposition',[0,0,5,3],...
    'number','off','name','Hw1P08');
bar(s,p,'b'); axis([1,13,0,0.18]);
xlabel('Event','fontsize',10); ylabel('Probability','fontsize',10);
set(gca,'ytick',[0:6]/36, 'ygrid','on');
set(gca,'yticklabel',...
   ['0/36';'1/36';'2/36';'3/36';'4/36';'5/36';'6/36'])
title('Bar Graph for "Sum of Two Dice" Problem','fontsize',12);
```

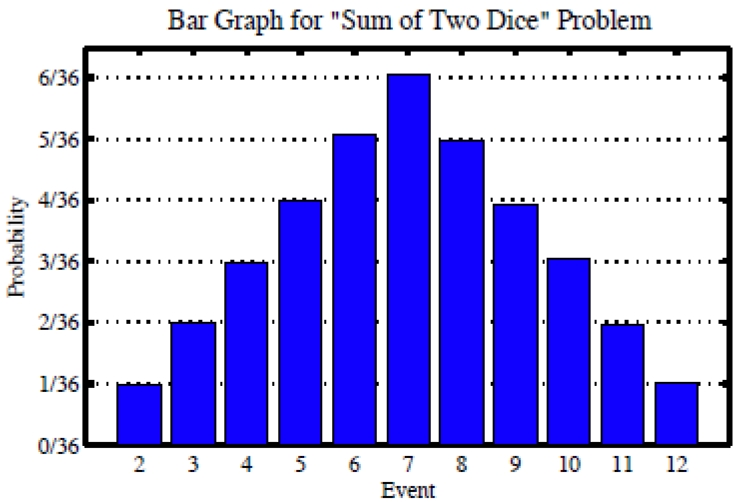


Figure 2: Bar-graph of probabilities in Problem-8.

The bar-graph is shown in Figure 2.

(c) First, the event $A \cap B = \{3, 5, 6, 7, 9, 11, 12\}$. Since, each outcome is disjoint, we have

$$\Pr(A \cap B) = \Pr(3) + \Pr(5) + \Pr(6) + \Pr(7) + \Pr(9) + \Pr(11) + \Pr(12)$$
$$= \frac{2}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} + \frac{1}{36} = \frac{2}{3}$$

Matlab verification:

```
%% (c) Probability of (A.Union.B) = {3,5,6,7,9,11,12}
P_AUB = sum(p([3,5,6,7,9,11,12]-1));
disp(sprintf('\n Probability of AUB = %g\n',P_AUB))
```

```
  Probability of AUB = 0.66666
```

as expected.