ECE 2521 Analysis of Stochastic Processes
Solutions to Problem Set 3

Problem 3.1 Solution

3.2

(a) $S = \{1, 2, 3, 4, 5, 6\}$ $\quad p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

where $p_j = P[\{j\}]$

(b)

S	S_X
1	$\to$ 0
2	$\to$ 1
3	$\to$ 1
4	$\to$ 2
5	$\to$ 2
6	$\to$ 3

(c)

$P[X=0] = p_1 = \frac{1}{6}$

$P[X=1] = p_2 + p_3 = \frac{2}{6}$

$P[X=2] = p_4 + p_5 = \frac{2}{6}$

$P[X=6] = p_6 = \frac{1}{6}$

Problem 3.2 Solution

3.9

(a) Let m be number of tails $0 \le m \le n$

then number of heads is $n-m$ and the diffrence is

$$Y = n - m - m = n - 2m \qquad 0 \le m \le n$$

$$\therefore \; S_Y = \{-n, -n+2, \ldots, n-2, n\}$$

(b) $P[Y=0] = P[n = 2m] = P[m = \frac{n}{2}]$ for n even.

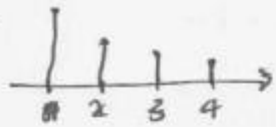
$P[Y=k] = P[n - 2m = k] = P[m = \frac{n-k}{2}]$ for $n-k$ even

Problem 3.3 Solution

3.12

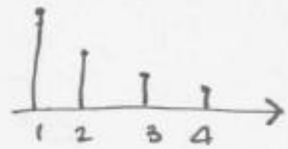
(a) $1 = p_1 + p_2 + p_3 + p_4 = p_1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12} p_1 \quad p_1 = \frac{12}{25}$

$p_1 = \frac{12}{25} \quad p_2 = \frac{6}{25} \quad p_3 = \frac{4}{25} \quad p_4 = \frac{3}{25}$



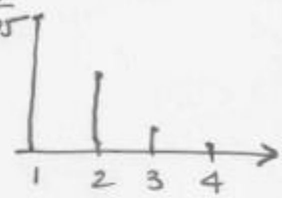
(b) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8} p_1$

$p_1 = \frac{8}{15} \quad p_2 = \frac{4}{15} \quad p_3 = \frac{2}{15} \quad p_4 = \frac{1}{15}$



(c) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{64}\right) = \frac{105}{64} p_1$

$p_1 = \frac{64}{105} \quad p_2 = \frac{32}{105} \quad p_3 = \frac{8}{105} \quad p_4 = \frac{1}{105}$



pmf decays more steeply in each example

(d) $1 = p_1 \sum_{i=1}^{\infty} \frac{1}{i}$ does not converge so this pmf does not extend to $\{1, 2, \ldots\}$

$1 = p_1 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = p_1 \frac{1}{1 - \frac{1}{2}} \Rightarrow p_1 = \frac{1}{2}.$

this extends to the geometric pmf.

$1 = p_1\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{1+2} + \left(\frac{1}{2}\right)^{1+2+3} + \ldots\right)$

$= p_1 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j(j-1)/2}$ this is a subseries of the geometric series so it converges

Problem 3.4 Solution

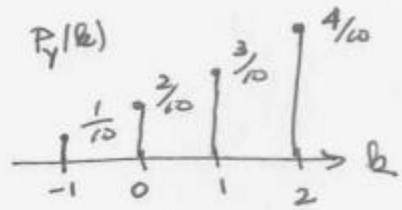
3.17

(a) $Y = 0+2 = 2$ with prob. $4/10$

$Y = -1+2 = 1$ " $3/10$

$Y = -2+2 = 0$ " $2/10$

$Y = -3+2 = -1$ " $1/10$



(b) $P[Y=2] = 4/10$

(c) $P[Y>0] = P[Y=2] + P[Y=1] = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

Problem 3.5 Solution

3.28

$$E[Y] = -1\cdot\frac{1}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{3}{10} + 2\cdot\frac{4}{10} = \frac{10}{10} = 1$$

$$E[Y^2] = 1\cdot\frac{1}{10} + 1\cdot\frac{3}{10} + 4\cdot\frac{4}{10} = \frac{20}{10} = 2$$

$$VAR[Y] = 2 - 1^2 = 1.$$

Problem 3.6 Solution

3.31 $P[X=k] = \binom{n}{k}\left(\frac{1}{2}\right)^n$

$$E[aX^2+bX] = aE[X^2] + bE[X]$$

$$E[X] = \sum_{j=0}^{n} j\binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n \sum_{j=0}^{n} j \frac{n!}{j!\,(n-j)!}$$

$$= \left(\frac{1}{2}\right)^n \sum_{j=1}^{n} \frac{n!}{(j-1)!\,(n-j)!} \qquad \text{let } j' = j-1$$

$$= \left(\frac{1}{2}\right)^n n \sum_{j'=0}^{n-1} \frac{(n-1)!}{j'!\,(n-1-j')!} = n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1} \binom{n-1}{j'}$$

$$= n\left(\frac{1}{2}\right)^n 2^{n-1} = \frac{n}{2}.$$

$$E[X^2] = \sum_{j=0}^{n} j^2 \binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n n \sum_{j=1}^{n} j \frac{(n-1)!}{(j-1)!\,(n-j)!}$$

$$= n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1} (j'+1)\binom{n-1}{j'}$$

$$= n\left(\frac{1}{2}\right)\Bigg[\underbrace{\sum_{j'=0}^{n-1} j' \binom{n-1}{j'}\left(\frac{1}{2}\right)^{n-1}}_{\substack{(n-1)\frac{1}{2} \\ \text{expected value of} \\ \text{binomial}}} + \underbrace{\sum_{j'=0}^{n-1} \binom{n-1}{j'}\left(\frac{1}{2}\right)^{n-1}}_{\substack{1 \\ \text{binomial probs}}} \Bigg]$$

$$= \frac{n}{2}\left[\frac{n}{2}+1\right]$$

$$\therefore E[aX^2+bX] = a\frac{n}{2}\left(\frac{n}{2}+1\right) + b\frac{n}{2}$$ ✓ average reward.

3.31b

$$E[a^X] = \sum_{j=0}^{n} a^j \binom{n}{j}\left(\tfrac{1}{2}\right)^j = \sum_{j=0}^{n} \binom{n}{j}\left(\tfrac{a}{2}\right)^j$$

$$= \left(1+\tfrac{a}{2}\right)^n$$

Problem 3.7 Solution

3.42 Assume # of heads is k

then $E[Y|k] = n-2k$

$$\therefore \quad E[Y] = \sum_{k=0}^{n} E[Y|k]P[k] = \sum_{k=0}^{n} (n-2k)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n - 2E[X] = n-2np$$

$$= n(1-2p)$$

Similarly

$$E[Y^2|k] = (n-2k) = n^2 - 4kn + 4k^2$$

$$E[Y^2] = \sum_{k=0}^{n} (n^2-4kn+4k^2)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n^2 - 4nE[X] + 4E[X^2]$$

$$= n^2 - 4n^2p + 4(npq + (np)^2)$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2 - n^2\underbrace{(1-2p)^2}_{1-4p+4p^2}$$

$$= 4npq$$

Problem 3.8 Solution

3.46 $n=8$ $p=0.25$ Binomial random variable

(a) $P[N=0] = \binom{8}{0} p^0 (1-p)^{8-0} = (0.75)^8 = 0.100$

(b) $P[N=1] = \binom{8}{1} p(1-p)^7 = 8(0.25)(0.75)^7 = 0.267$

(c) $P[N>4] = \sum_{j=5}^{8} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.0273$

(d) $P[2<N<6] = \sum_{j=3}^{5} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.3172$

Problem 3.9 Solution

3.57 $\alpha_S = 48$ $\alpha_R = 24$ $\alpha_G = 12$ slice $= 1/12$

(a) $P[N_G=0] = \frac{(\alpha_G \frac{1}{12})^0}{0!} e^{-\alpha_G \frac{1}{12}} = e^{-1} = 0.368$

(b) $P[N_G=0, N_R \le 2] = P[N_G=0]\, P[N_R \le 2] = e^{-1} \sum_{k=0}^{2} \frac{(\alpha_R \frac{1}{12})^k}{k!} e^{-\alpha_R \frac{1}{12}}$

$= e^{-1}\left[e^{-2} + \frac{2}{1!}e^{-2} + \frac{4}{2!}e^{-2}\right]$

$= e^{-3}[1+2+2] = 5e^{-3} = 0.249$

(c) $P[N_G=0, N_R=0, N_S \ge 5] = P[N_G=0]\, P[N_R=0]\, P[N_S \ge 5]$

$= e^{-1} e^{-2} e^{-4} \sum_{k=0}^{5} \frac{4^k}{k!} = e^{-3}(0.785)$

$= 0.039$

It's hard to avoid the red and green thingies!

Problem 3.10 Solution

3.91

58 a)

$$
\begin{aligned}
& P[\text{signal present}|X=k] \\
= & \frac{P[\text{signal present}, X=k]}{P[X=k|\text{signal present}]P[\text{present}] + P[X=k|\text{signal absent}]P[\text{absent}]} \\
= & \frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p}{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p + \frac{\lambda_0^k}{k!}e^{-\lambda_0}(1-p)} \\
= & \frac{\lambda_1^k e^{-\lambda_1}p}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}
\end{aligned}
$$

Similarly,

$$
P[\text{signal absent}|X=k] = \frac{\lambda_0^k e^{-\lambda_0}(1-p)}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}
$$

b) Decide signal present if $P[\text{signal present}|\text{X=k}] > P[\text{signal absent}|\text{X=k}]$, i.e.,

$$
\lambda_1^k e^{-\lambda_1}p > \lambda_0^k e^{-\lambda_0}(1-p)
$$

$$
\left(\frac{\lambda_1}{\lambda_0}\right)^k > \frac{1-p}{p}e^{\lambda_1-\lambda_0} \qquad (\lambda_1 > \lambda_0)
$$

$$
k > \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}
$$

The threshold T is

$$
T = \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}
$$

c)

$$
\begin{aligned}
P_e = & P[X < T|\text{signal present}]P[\text{present}] + P[X > T|\text{signal absent}]P[\text{absent}] \\
= & p\sum_{k=o}^{\lfloor T \rfloor} \frac{e^{-\lambda_1}\lambda_1^k}{k!} + (1-p)\sum_{k=\lceil T \rceil}^{\infty} \frac{e^{-\lambda_0}\lambda_0^k}{k!}
\end{aligned}
$$

Problem 3.11 Solution

```
n=10;                   % Number of coin flips per experiment.
m=100;                  % Number of times to repeat experiment.
p = 0.3; q = 1-p;       % Probability of Tail in a loaded coin
X=rand(n,m) <= p;       % Simulate loaded coin flipping.
Y=sum(X);               % Calculate number of tails per experiment.
```

```
Rel_Freq=hist(Y,[0:n])/m; % Compute relative frequencies.
for k=0:n
PMF(k+1)=nchoosek(n,k)*p^k*q^(n-k); % Compute actual PMF.
end
% Plot Results
plot([0:n],Rel_Freq,'o',[0:n],PMF,'*')
legend('Relative Frequency','True PMF')
xlabel('k'); ylabel('P_X(k)')
title('Comparison of estimated and true PMF for Example 2.26')
```

Figure 1 shows the plot of the actual and computed PMF for $n = 10$ and $p = 0.3$.

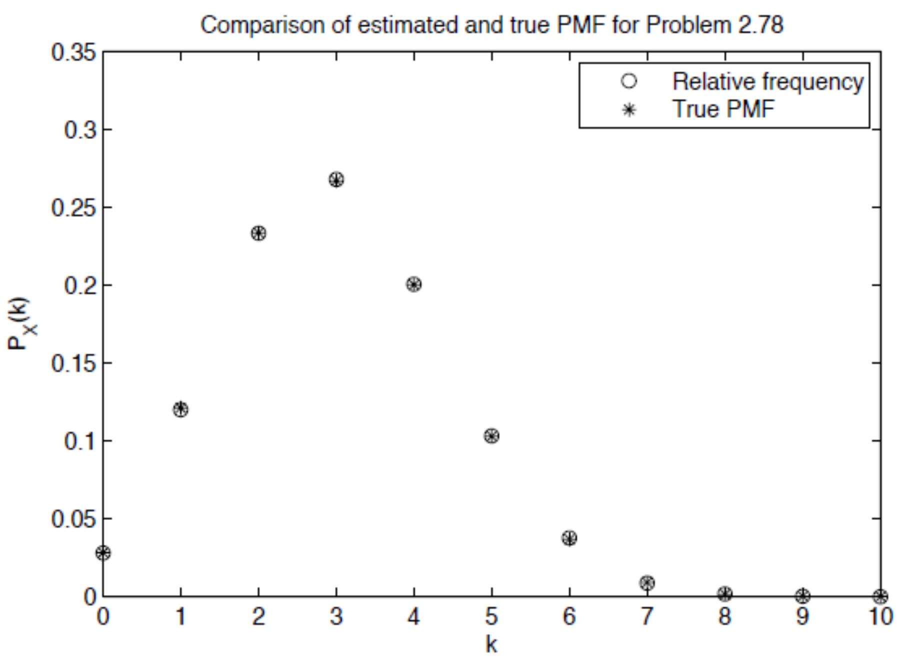


Figure 1: PMF plots in Problem P1.10