ECE 2521: Analysis of Stochastic Processes

Lecture 3

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September, 15^{th} *2021*

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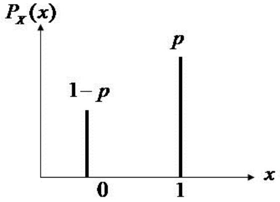
Bernoulli Random Variable

- Used for experiments with two outcomes of interest:
  - The success or failure of an experiment
  - Classifying a phone line as free or busy
  - Assesing a person as sick or healthy from a certain illness
  - Determining whether a bit is received correctly or in error in a noisy communication channel
- $X \sim \text{Bernoulli}(p)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} 1-p & \text{if } x = 0, \\ p & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$



Bernoulli Random Variable

- Mean of X:

$$E[X] = \sum_x x p_X(x) = 0 \cdot p_X(0) + 1 \cdot p_X(1) = 0(1-p) + 1(p) = p$$

- Second moment of X:

$$E[X^2] = \sum_x x^2 p_X(x) = 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) = 0(1-p) + 1(p) = p$$

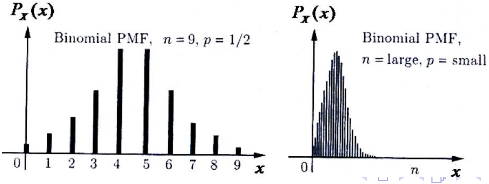
- Variance of X:

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Binomial Random Variable

- Used for experiments that involve n independent trials of the Bernoulli experiment:
    - Transmission of n bits with x received in error
    - Monitor n phone lines and find that x of them are busy
- $X \sim \text{Binomial}(n, p)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x = \{0, 1, \cdots, n\} \\ 0 & \text{otherwise.} \end{cases}$$



Binomial Random Variable

- Mean: $E[X] = np$
- Variance: $\text{Var}[X] = np(1-p)$

Note: Since the Binomial experiment involves n *independent* trials of the Bernoulli experiment, the mean and variance of the Binomial random variable is n times that of the Bernoulli random variable.

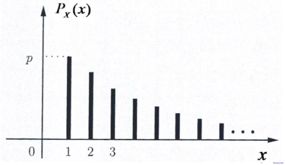
Example

Gary Kasparov plays chess against $n = 10$ opponents. Probability that Gary losses or draws (not win) against any opponent is $p = 0.1$. What is the probability that he beats 9 of the opponents?

Geometric Random Variable

- Used for an infinite sequence of Benoulli trials and provides the probability of obtaining the first success in the xth trial
  - Test of integrated circuits with a probability p that each circuit is rejected. Let the random variable X be the number of tests up to and including the first test that discovers a reject.
- $X \sim \text{Geometric}(p)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} p(1-p)^{x-1} & \text{if } x = 1, 2, 3, \cdots \\ 0 & \text{otherwise.} \end{cases}$$



Geometric Random Variable

- Mean of the geometric random variable X is

$$\begin{aligned} E[X] &= \sum_{x=1}^{\infty} x P_X(x) = \sum_{x=1}^{\infty} x p(1-p)^{x-1} \\ &= \frac{p}{(1-p)} \sum_{x=1}^{\infty} x(1-p)^x = \frac{p}{(1-p)} \frac{1-p}{[1-(1-p)]^2} = \frac{1}{p} \end{aligned}$$

where the second last equality is obtained by applying the following mathetical property: $\sum_{x=1}^{\infty} x q^x = \frac{q}{(1-q)^2}$.

- Variance:

$$\mathrm{Var}[X] = \frac{1-p}{p^2}$$

Example

In an accelerator, a particle has probability 0.01 of hitting a target material.

(a) What is the probability that the first particle to hit the target is the 100th?

(b) What is the probability that the target will be hit by any of the first 100 particles.

Hint Let X be the RV representing the number of attempts needed to first hit the target: $X \sim \text{Geometric}(p = 0.01)$

Pascal Random Variable (Negative Binomial Distribution)

- For an infinite sequence of Benoulli trials each with probability p, the Pascal random variable is the number of trials up to and including the kth success
    - We keep testing circuits until we find k rejects. Let random variable X be the number of tests required to find k rejects. Then, the PMF of X is Pascal.
- To derive the Pascal PMF, we consider the scenario of finding the kth success in the xth trial: there are exactly $(k-1)$ successes in the previous $(x-1)$ trials, and the kth success occurs in the xth trial. The probability of this event is:

$$\begin{aligned} P(X=x) &= P((k-1) \text{ successes in } (x-1) \text{ attempts, success on attempt } x) \\ &= \binom{x-1}{k-1} p^{k-1}(1-p)^{(x-1)-(k-1)} \cdot p \end{aligned}$$

Pascal Random Variable (Negative Binomial Distribution)

- $X \sim \text{Pascal}(k, p)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{(x-1)-(k-1)} & \text{if } x = k, k+1, k+2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

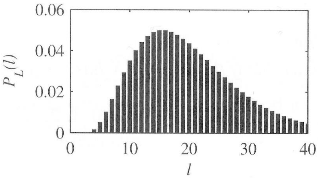
- Mean: $E[X] = \frac{k}{p}$
- Variance: $\text{Var}[X] = \frac{k(1-p)}{p^2}$

Example

Each circuit has a probability $p = 0.2$ of being defective. We seek four defective circuits. Let RV L represent the number of tests necessary to find the four circuits. Then, the PMF of L is:

$$L \sim \text{Pascal}(k = 4, p = 0.2)$$

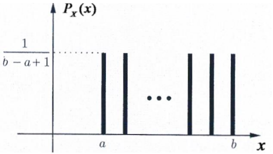
$$p_L(l) = \begin{cases} \binom{l-1}{3}(0.2)^4(0.8)^{l-4} & \text{if } l = 4, 5, \cdots \\ 0 & \text{otherwise.} \end{cases}$$



Discrete Uniform Random Variable

- Used for experiments with finite outcomes that have equal probabilities
  - Multiple channels of a router with equally likely usage for each channel
  - Roll of a fair six sided die
- $X \sim \text{Uniform}(a, b)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} \frac{1}{b-a+1} & \text{if } x = a, a+1, \cdots, b \\ 0 & \text{otherwise.} \end{cases}$$



Discrete Uniform Random Variable

- Mean: $E[X] = \frac{(a+b)}{2}$, by symmetry since X is uniformly distributed between a and b
- Variance:

  First consider the special case, $a = 1$ and $b = n$:

$$E[X^2] = \sum_{x=1}^{n} x^2 P_X(x) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{6}(n+1)(2n+1)$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)^2 = \frac{n^2-1}{12}$$

For the general case, let $n = b - a + 1$, then

$$\text{Var}[X] = \frac{(b-a+1)^2 - 1}{12} = \frac{(b-a)(b-a+2)}{12}$$

Poisson Random Variable

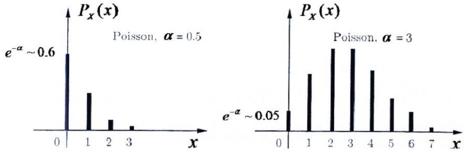
- Used to describe phenomenon that occur randomly in time. While the time of each occurrence is completely random, there is a known average number of occurrences per unit time.
- $X \sim \text{Poisson}(\alpha)$, if its PMF has the following form:

$$p_X(x) = \begin{cases} \frac{\alpha^x}{x!} e^{-\alpha} & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

  where the parameter $\alpha > 0$.

Note: When dealing with problems involving time t with an arrival rate of λ for an event, the Poisson probability of x arrivals in time t is found by setting the Poisson parameter $\alpha = \lambda t$.

Poisson Random Variable



$$p_X(x) = \begin{cases} \frac{\alpha^x}{x!} e^{-\alpha} & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

- Examples:
  - Arrival of information requests at a world wide web server
  - Emission of particles from a radioactive source
  - Number of ongoing calls in a given time interval
- Mean: $E[X] = \alpha$
- Variance: $\text{Var}[X] = \alpha$

Example

Calls arrive at a telephone switching office with an average rate of $\lambda = 0.25$ calls per second.

(a) What is the probability of 4 calls arriving in a 2 second interval?

(b) What is the probability of 4 calls arriving in a 20 second interval?

(c) What is the mean and standard deviation of the number of calls arriving in 2 seconds?

(d) What is the mean and standard deviation of the number of calls arriving in 20 seconds?

Example - Solution

Let J be the RV representing the number of calls that arrive in t seconds. Then, J follows a Poisson distribution.

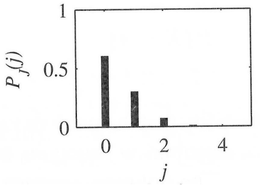
(a) For the $t = 2$ second interval, $\alpha = \lambda t = (0.25)(2) = 0.5$, and the probability of receiving 4 calls is
$\text{Prob}(J = 4) = \frac{\alpha^j}{j!} e^{-\alpha} = \frac{0.5^4}{4!} e^{-0.5} = 0.0016$

(b) For the $t = 20$ second interval, $\alpha = \lambda t = (0.25)(20) = 5$, and the probability of receiving 4 calls is
$\text{Prob}(J = 4) = \frac{\alpha^j}{j!} e^{-\alpha} = \frac{5^4}{4!} e^{-5} = 0.18$

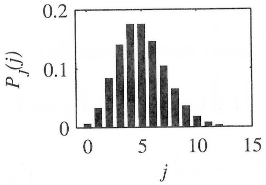
(c) For the 2 second interval, the mean number of calls received is $\alpha = 0.5$ and the standard deviation is $\sqrt{\alpha} = \sqrt{0.5} = 0.7$.

(d) For the 20 second interval, the mean number of calls received is 5 and the standard deviation is $\sqrt{5} = 2.2$.

Example - Solution (continued)



$$P_J(j) = \begin{cases} (0.5)^j e^{-0.5}/j! & j = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$



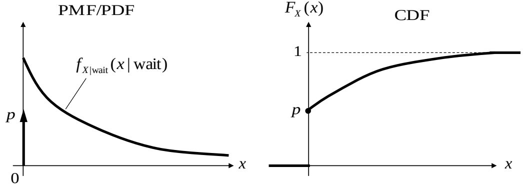
$$P_J(j) = \begin{cases} 5^j e^{-5}/j! & j = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Types of Random Variables

- A **discrete RV** is characterized by a *probability mass function* (PMF)
- A **continuous RV** has a range that is a continuous interval and is characterized by *probability density function* (PDF)
  - Arrival time of a particle $(0 \le t < \infty)$
  - Temperature in Farenheight in the park $(-10 \le \theta \le 98)$
  - Voltage across a resistor $(-\infty < v < \infty)$
  - Phase angle of a sinusoidal radio wave $(0 \le \phi < 2\pi)$
- A **mixed RV** falls in neither categories: it takes some discrete values and other values continuously over some interval

Mixed Random Variables Example

- When you go to the bank, if some teller is free (with probability p), you don't need to wait.
- If none of the tellers are free, you wait according to some PDF.
- The waiting time in this case is a mixed random variable and can be characterized by a *cumulative distribution function* (CDF)



Cumulative Distribution Function (CDF)

- The cumulative distribution function (CDF) of a general random variable (discrete, continuous, or mixed) is

$$F_X(x) = Prob(X \leq x) = \begin{cases} \sum_{u \leq x} p_X(u) & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{x} f_X(u)du & \text{if } X \text{ is continuous.} \end{cases}$$

- The CDF provides the probability that the random variable X is smaller than or equal to the value x

Cumulative Distribution Function (CDF)

- For any random variable X, the CDF satisfies
  (a) $F_X(-\infty) = 0$
  (b) $F_X(\infty) = 1$
  (c) $\text{Prob}(x_1 < x \leq x_2) = F_X(x_2) - F_X(x_1)$
  (d) CDFs are monotonically non-decreasing:
  $x_1 \leq x_2 \Rightarrow F_X(x1) \leq F_X(x_2)$
  (e) $P(X > x) = 1 - F_X(x)$

Note: The CDF of a discrete variable X is discontinuous (jumps) at values of x with nonzero probability. On the other hand, continuous random variables are continuous functions of x.

Note: For a discrete RV, the CDF yields the PMF by differencing:
$p_X(u) = F_X(u) - F_X(u-1) = \text{Prob}(X \leq u) - \text{Prob}(X \leq u-1)$

Example

The discrete random variable Y has PMF

$$p_Y(y) = \begin{cases} 1/5 & \text{if } y = 3, \\ 2/5 & \text{if } y = 5, \\ 2/5 & \text{if } y = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Find and sketch the CDF of Y.

Probability Density Function (PDF)

- Continuous RVs take values from a continuous range
- Probability of an event is determined by the interval within the range occupied by the event: as the interval shrinks, the amount of probability associated with the interval decreases
- In the limit the interval shrinks to a point: the **point has zero probability**
- Continuous random variables are characterized by a **probability density function** (PDF)

Probability Density Function (PDF)

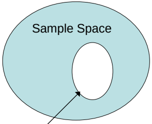
- The CDF yields the PDF by differentiation:

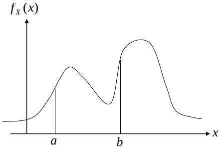
$$f_X(x) = \frac{dF_X(x)}{dx}$$

- The PDF $f_X(x)$ of a continuous random variable X provides the probability of observing X over an interval $(a, b]$:

$$\text{Prob}(a < X \leq b) = \int_a^b f_X(x)dx$$

- Satisfies the axioms:
  (1) Nonnegativity: $f_X(x) \geq 0$ for all x
  (2) Normalization: $\int_{-\infty}^{\infty} f_X(x)dx = 1$





Example

Alice walks to school and her travel time varies between 15 mins and 45 mins. She is twice as likely to take less than half hour for her travel than longer than half hour. Let random variable X be her travel time.

(a) Find and plot the probability density function for X.

(b) What is the probability that Alice takes less than 25 mins to travel?

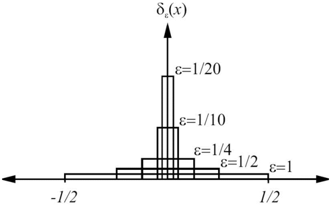
(c) Find and plot the CDF.

Delta (Unit Impulse) Function

- $\delta(x)$ is a useful mathematical tool that unites the analysis of discrete and continuous random variables
- Allows us to use the same formula to describe calculations with different types of random variables

$$d_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{if } -\epsilon/2 \leq x < \epsilon/2, \\ 0 & \text{otherwise} . \end{cases}$$

The delta (unit impulse) function is: $\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x)$.



Note: For each ϵ, the area under the curve of $d_\epsilon(x)$ equals 1.

Delta (Unit Impulse) Function

$$d_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{if } -\epsilon/2 \leq x < \epsilon/2, \\ 0 & \text{otherwise .} \end{cases}$$

- **Properties:**
  - *Normalization:* The area under the delta function is 1

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

  - *Sifting Property:* For any continuous function $g(x)$:

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0)$$

Unit Step Function

- The unit step function is:

$$u(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

- Relationship between the delta function and step function:
  - *Integration*:

$$\int_{-\infty}^{x} \delta(v) dv = u(x)$$

  - *Differentiation*:

$$\delta(x) = \frac{du(x)}{dx}$$

Unit Step Function - Application to Random Variables

- Express the CDF for a discrete random variable X using step functions: $F_X(x) = \sum_{x_i \in S_X} p(x_i) u(x - x_i)$.
- Express the PDF for a discrete random variable X by making use of delta functions: $f_X(x) = \sum_{x_i \in S_X} p(x_i) \delta(x - x_i)$.
- Express the mean of a discrete random variable X as an integral:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} p(x_i) \delta(x - x_i) dx \\ &= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x p(x_i) \delta(x - x_i) dx = \sum_{x_i \in S_X} x_i p(x_i) \end{aligned}$$

Example

The discrete random variable Y with the following PMF:

$$p_Y(y) = \begin{cases} 1/5 & \text{if } y = 3, \\ 2/5 & \text{if } y = 5, \\ 2/5 & \text{if } y = 7, \\ 0 & \text{otherwise}, \end{cases}$$

and the following CDF:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 3, \\ 1/5 & \text{if } 3 \le y < 5, \\ 3/5 & \text{if } 5 \le y < 7, \\ 1 & \text{if } y \ge 7, \end{cases}$$

can be expressed more compactly.

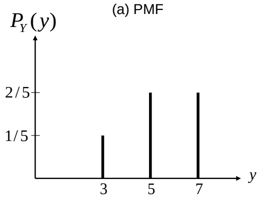
Example (continued)

For instance, its CDF can be expressed using the step function as

$$F_Y(y) = \frac{1}{5}u(y-3) + \frac{2}{5}u(y-5) + \frac{2}{5}u(y-7)$$

It's corresponding PDF can be expressed using the delta function as:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{5}\delta(y-3) + \frac{2}{5}\delta(y-5) + \frac{2}{5}\delta(y-7)$$



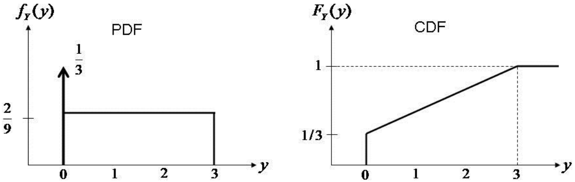
Application to Mixed Random Variables

A mixed random variable X has a probability density function that contains both impulses and non-zero finite values.

Example: Someone dials the telephone and we record the duration of the call.

- In a simple model, $\frac{1}{3}$ of the calls have duration of 0 minutes because the line is busy or no one answers.
- The remaining $\frac{2}{3}$ of calls have duration that is uniformly distributed between 0 and 3 minutes.
- Let Y denote the duration of the call.
- Find the CDF $F_Y(y)$ and the PDF $f_Y(y)$ of the call.

Application to Mixed Random Variables



$$f_Y(y) = \begin{cases} \frac{1}{3}\delta(y) + \frac{2}{9} & \text{if } 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ \frac{1}{3} + \frac{2}{9}y & \text{if } 0 \le y < 3, \\ 1 & \text{if } y \ge 3. \end{cases}$$

Application to Mixed Random Variables

- The cumulative distribution function of random variable X is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1, \\ (x+1)/4 & \text{if } -1 \leq x < 1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

- Sketch the CDF and find the following:
  (1) Prob$[X \leq 1]$
  (2) Prob$[X < 1]$
  (3) Prob$[X = 1]$
  (4) PDF $f_X(x)$.

Expected Value or Mean

- Expected value (mean) of a continuous random variable X is:

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Expected value of a function $g(X)$ of a continuous random variable X is:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Note: If the function is linear, then

$$E[aX + b] = aE[X] + b$$

Proof :

$$E[aX + b] = \int_{-\infty}^{\infty} (ax + b) f_X(x) dx = a \int_{-\infty}^{\infty} x f_X(x) dx + b \int_{-\infty}^{\infty} f_X(x) dx = aE[X] + b$$

Variance and Standard Deviation

- For any random variable X:

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx$$

or alternatively:

$$\mathrm{Var}[X] = E[X^2] - \mu_X^2$$

where, the second moment:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

Example 1

Find the mean, variance and standard deviation of Alice's travel time to school from previous example.

$$f_X(x) = \begin{cases} \frac{2}{45} & \text{if } 15 \leq x \leq 30, \\ \frac{1}{45} & \text{if } 30 \leq x \leq 45, \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{15}^{30} x \frac{2}{45} dx + \int_{30}^{45} x \frac{1}{45} dx = 27.5$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{15}^{30} x^2 \frac{2}{45} dx + \int_{30}^{45} x^2 \frac{1}{45} dx = 825$$

$$\mathrm{Var}[X] = E[X^2] - \mu_X^2 = 825 - (27.5)^2 = 68.75$$

$$\sigma_X = \sqrt{\mathrm{Var}[X]} = \sqrt{68.75} = 8.29$$

Example 2

The probability density function of the random variable Y is:

$$f_Y(y) = \begin{cases} ay^2 & \text{if } -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the PDF and determine the following:

(1) the value of a

(2) probabilty that Y is positive

(3) the mean of Y

(4) its variance and standard deviation.

Conditioning a Continuous Random Variable

- For a continuous random variable X and conditioning event B, with $\text{Prob}(B) > 0$, the **conditional probability density function** (PDF) of X given B is

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{\text{Prob}(B)} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

- The conditional PDF satisfies the axioms of probability:
  (1) *Non-negativity*: $f_{X|B}(x) \geq 0$.
  (2) *Normalization*: $\int_{-\infty}^{\infty} f_{X|B}(x)dx = 1$

Conditional Expected Value for a Continuous RV

- The **conditional expected value of random variable** X given condition B is:

$$E[X|B] = \mu_{x|B} = \int_{-\infty}^{\infty} x f_{X|B}(x) dx$$

- The **conditional expected value of a function** of a random variable $Y = g(X)$ given condition B is:

$$E[Y|B] = E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x) dx$$

- The **conditional variance of random variable** X given condition B is then found using:

$$\mathrm{Var}[X|B] = E[(X - \mu_{x|B})^2] = E[X^2|B] - \mu_{x|B}^2$$

Total Probability Theorem for Conditional PDF

- Given events B_1, B_2, ..., B_m form a partition of the sample space, we can obtain the unconditional PDF for X from the conditional PDFs using:

$$f_X(x) = \sum_{i=1}^{m} f_{X|B_i}(x) P(B_i)$$

Example

The probability density function of random variable Y is:

$$f_Y(y) = \begin{cases} 1/10 & \text{if } 0 \leq y \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Find the following:

(1) Prob$(Y \leq 6)$

(2) The conditional PDF $f_{Y|Y\leq 6}(y)$

(3) Prob$(Y > 8)$

(4) The conditional PDF $f_{Y|Y>8}(y)$

(5) $E[Y|Y \leq 6]$

(6) $E[Y|Y > 8]$