ECE 2521 Analysis of Stochastic Processes
Solutions to Problem Set 2

Problem 2.1 Solution

2.43 8, 9, or 10 characters long

- at least 1 special character from set of size 24
- numbers from size 10
- upper + lower case letters $26 \times 2 = 52$

} 62 choices

For length n:

- pick position of required special character + pick character
  n positions $\times$ 24 characters.
- pick number/letter/special character for remaining $n-1$ positions
  86^{n-1}

Total # passwords $= n \cdot 24 \cdot 86^{n-1}$

Length 8, 9, or 10 $= 8 \cdot 24 \cdot 86^7 + 9 \cdot 24 \cdot 86^8 + 10 \cdot 24 \cdot 86^9 = 6.24 \times 10^{19}$

Time to try all passwords $= 6.24 \times 10^{13}$ seconds $= 2(10^6)$ years

Problem 2.2 Solution

2.46 The order in which the 4 toppings are selected does not matter so we have sampling without ordering.

If toppings may not be repeated, Eqn. (2.22) gives

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas}$$

If toppings may be repeated, we have sampling with replacement and without ordering. The number of such arrangements is

$$\binom{14+4}{4} = 3060 \text{ possible deluxe pizzas.}$$

Problem 2.3 Solution

2.79

(a) $P[N=k] = P[N=k \mid \text{coin } 1]\, P[\text{coin } 1] + P[N=k \mid \text{coin } 2]\, P[\text{coin } 2]$

$$= \binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$k = 0, 1, 2, 3$

(b) $$P[\text{coin } 1 \mid N=k] = \frac{P[N=k \mid \text{coin } 1]\, P[\text{coin } 1]}{P[N=k]}$$

$$= \frac{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2}}{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}}$$

(c) Coin 1 is more probable if

$$\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} > \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{3-k} = 2^k \left(\frac{1}{2}\right)^{3-k} = \left(\frac{1}{8}\right) 4^k$$

$$0 > \ln \frac{1}{8} + k \ln 4$$

$$1.5 = \frac{\ln 8}{\ln 4} > k$$

Coin 1 more probable if $N = 0$ or 1
Coin 2 more probable otherwise.

(d) In general coin 1 is more probable if

$$\binom{n}{k} p_1^k (1-p_1)^{n-k} \frac{1}{2} > \binom{n}{k} p_2^k (1-p_2)^{n-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{n-k} = \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^n$$

$$T = \frac{n \ln\left(\frac{1-p_1}{1-p_2}\right)}{\ln\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)} > k$$

(e) If $p_2 = 1$ then $P[N=k \mid \text{coin } 2] = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$

We cannot determine coin with certainty only if all tosses are heads.

$P[\text{coin } 1 \mid m \text{ heads}] = (1-p_1)^m / [\, 1 + (1-p_1)^m \,]$

Problem 2.4 Solution

2.81 Let X denote the input and Y the output.

a)
$$\begin{aligned} P[Y=0] &= P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] \\ &\quad + P[Y=0|X=2]P[X=2] \\ &= (1-\varepsilon)\tfrac{1}{3} + \varepsilon\tfrac{1}{3} = \tfrac{1}{3} \end{aligned}$$

Similarly

$$P[Y=1] = \tfrac{1}{3}$$
$$P[Y=2] = \tfrac{1}{3}$$

b) Using Bayes' Rule

$$P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{3}\varepsilon}{\frac{1}{3}} = \varepsilon$$
$$P[X=1|Y=1] = \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{3}}{\frac{1}{3}} = 1-\varepsilon$$
$$P[X=2|Y=1] = 0$$

Problem 2.5 Solution

2.87 (a) $P[A\cup B] = P[A]+P[B]-P[A\cap B] = p_A + p_B - p_A p_B$

(b) $P[A\cup B] = P[A]+P[B] = p_A + p_B$

(c) $P[A\cup B\cup C] = P[A]+P[B]+P[C]-P[A\cap B]-P[A\cap C]-P[B\cap C] + P[A\cap B\cap C]$
$= p_A + p_B + p_C - p_A p_B - p_A p_C - p_B p_C + p_A p_B p_C$

(d) $P[A\cup B\cup C] = p_A + p_B + p_C$

Problem 2.6 Solution

2.100

(a) P[1 of n terminals transmits] $= n(1-p)^{n-1}p$

(b) Take derivative with respect to p:

$$0 = -n(n-1)(1-p)^{n-2}p + n(1-p)^{n-1}$$

$$\Rightarrow (n-1)p = (1-p) \Rightarrow np = 1-p+p \Rightarrow p = \frac{1}{n}$$

(c) $P_{success} = n\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} = \left(1-\frac{1}{n}\right)^{n-1} \to e^{-1} = \frac{1}{e}$ as $n \to \infty$.

$= 0.3678$

Problem 2.7 Solution

2.102

a) $P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$

b) Type 1 errors occur with problem $p\alpha$ and do not occur with problem $1-p\alpha$

$$P[k_1 \text{ type 1 errors}] = \binom{n}{k_1}(p\alpha)^{k_1}(1-p\alpha)^{n-k_1}$$

c) $P[k_2 \text{ type 2 errors}] = \binom{n}{k_2}(p(1-\alpha))^{k_2}(1-p(1-\alpha))^{n-k_2}$

d) Three outcomes: type 1 error, type 2 error, no error

$$P[k_1, k_2, n-k_1-k_2] = \frac{n!}{k_1!k_2!(n-k_1-k_2)!}(p\alpha)^{k_1}(p(1-\alpha))^{k_2}(1-p)^{n-k_1-k_2}$$

Problem 2.8 Solution
Total number of ways in which 8 cards can be drawn from 52 cards is $^{52}\mathbf{C_8}$.

(a) Let A be the event that the 8 cards include exactly 4 queens. To find the count for event A, let us split the deck of 52 cards into 2 small decks D1 and D2. D1 has the 4 queens and D2 has the remaining 48 cards.
Number of ways in which 4 queens can be drawn from D1 is $^{4}\mathbf{C_4}$.
Number of ways in which the other 4 cards can be drawn from D2 is $^{48}\mathbf{C_4}$.
➔ Probability of event A = P(A) = $\mathbf{(^{4}C_4) \times (^{48}C_4) / (^{52}C_8)}$

$$= \left(\frac{4!}{4!0!}\right)\left(\frac{48!}{4!44!}\right)\left(\frac{8!44!}{52!}\right)$$

$$= \left(\frac{48! \cdot 8!}{52! \cdot 4!}\right)$$

(b) Let B be the event that exactly 2 kings are drawn. To find the count for event B, let us split the deck of 52 cards into 2 small decks D1 and D2. D1 has the 4 kings and D2 has the remaining 48 cards.
Number of ways in which 2 kings can be drawn from D1 is $^{4}\mathbf{C_2}$.
Number of ways in which the remaining 6 cards can be drawn from D2 is $^{48}\mathbf{C_6}$.
➔ Probability of event B = P(B) = $\mathbf{(^{4}C_2) \times (^{48}C_6) / (^{52}C_8)}$

$$= \left(\frac{4!}{2!2!}\right)\left(\frac{48!}{6!42!}\right)\left(\frac{8!44!}{52!}\right)$$

$$= \left(\frac{336 \cdot 48! \cdot 44!}{52! \cdot 42!}\right)$$

(c) We need to find P(AUB).

Lets find P(A∩B) first. To find the count for the event A∩B, let us split the deck of 52 cards into 3 small decks D1, D2 and D3. D1 has the 4 queens, D2 has the 2 kings and D3 has the remaining 44 cards.
Number of ways in which 4 queens can be drawn from D1 is $^{4}\mathbf{C_4}$.
Number of ways in which 2 kings can be drawn from D2 is $^{4}\mathbf{C_2}$.
Number of ways in which the remaining 2 cards can be drawn from D3 is $^{44}\mathbf{C_2}$.
➔ Probability of event A∩B = P(A∩B) = $\mathbf{(^{4}C_4) \times (^{4}C_2) \times (^{44}C_2) / (^{52}C_8)}$

$$= \left(\frac{4!}{4!0!}\right)\left(\frac{4!}{2!2!}\right)\left(\frac{44!}{2!42!}\right)\left(\frac{8!44!}{52!}\right)$$

$$= \left(\frac{3 \cdot 44! \cdot 44! \cdot 8!}{52! \cdot 42!}\right)$$

Now, P(AUB) = P(A) + P(B) – P(A∩B)
➔ P(AUB) = $\mathbf{[(^{4}C_4) \times (^{48}C_4) + (^{4}C_2) \times (^{48}C_6) - (^{4}C_4) \times (^{4}C_2) \times (^{44}C_2)] / {}^{52}C_8}$
~ 0.09825

Problem 2.9 Solution

Solution to Problem 1.49. (a) Since the cars are all distinct, there are $20!$ ways to line them up.

(b) To find the probability that the cars will be parked so that they alternate, we count the number of "favorable" outcomes, and divide by the total number of possible outcomes found in part (a). We count in the following manner. We first arrange the US cars in an ordered sequence (permutation). We can do this in $10!$ ways, since there are 10 distinct cars. Similarly, arrange the foreign cars in an ordered sequence, which can also be done in $10!$ ways. Finally, interleave the two sequences. This can be done in two different ways, since we can let the first car be either US-made or foreign. Thus, we have a total of $2 \cdot 10! \cdot 10!$ possibilities, and the desired probability is

$$\frac{2 \cdot 10! \cdot 10!}{20!}.$$

Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose the foreign cars are indistinguishable, and also that the US cars are indistinguishable. Out of the 20 available spaces, we need to choose 10 spaces in which to place the US cars, and thus there are $\binom{20}{10}$ possible outcomes. Out of these outcomes, there are only two in which the cars alternate, depending on whether we start with a US or a foreign car. Thus, the desired probability is $2/\binom{20}{10}$, which coincides with our earlier answer.

Problem 2.10 Solution

In this problem, there is a tendency to reason that since the opposite face is either heads or tails, the desired probability is $1/2$. This is, however, wrong, because given that heads came up, it is more likely that the two-headed coin was chosen. The correct reasoning is to calculate the conditional probability

$$\begin{aligned} p &= \mathbf{P}(\text{two-headed coin was chosen} \mid \text{heads came up}) \\ &= \frac{\mathbf{P}(\text{two-headed coin was chosen and heads came up})}{\mathbf{P}(\text{heads came up})}. \end{aligned}$$

We have

$$\mathbf{P}(\text{two-headed coin was chosen and heads came up}) = \frac{1}{3},$$

$$\mathbf{P}(\text{heads came up}) = \frac{1}{2},$$

so by taking the ratio of the above two probabilities, we obtain $p = 2/3$. Thus, the probability that the opposite face is tails is $1 - p = 1/3$.

Problem 2.11 Solution

For system (a), the components 1-3 are parallel to components 4-5.

The failure probability of system (a) is

=Prob(series subsystem 1-3 fail)*Prob(series subsystem 4-5 fail)

$$
\begin{aligned}
&= \left[1-(1-q_1)(1-q_2)(1-q_3)\right]\left[1-(1-q_4)(1-q_5)\right] \\
&= [1-(1-0.1)(1-0.05)^2][1-(1-0.15)^2] \\
&= 0.0521
\end{aligned}
$$

For system (b), the failure probability is

$$
\begin{aligned}
&= \left[1-(1-q_1)(1-q_4 q_5)\right]\left[1-(1-q_2)(1-q_3)(1-q_6 q_7)\right] \\
&= [1-(1-0.1)(1-0.15^2)][1-(1-0.05)^2(1-0.2^2)] \\
&= 0.0161
\end{aligned}
$$