ECE 1673: Linear Control Systems (4 Credits, Spring 2022)

Lecture 4: System Responses of First-Order Systems

January 20, 2022

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Outline of this lecture

- Review of last lecture
- General consideration on system responses
- Time responses of first-order systems

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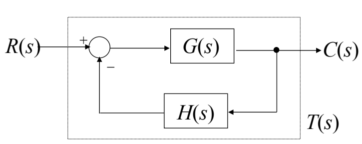
Review of last lecture

- Block diagrams and signal flow graphs
  - Finding system transfer functions involves solving simultaneously algebra equations



$$T(s) = \frac{\text{Gain of the feedforward path}}{1 - \text{Gain of the loop}}$$

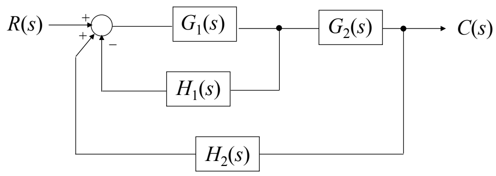
$$= \frac{G(s)}{1-(-1)G(s)H(s)} = \frac{G(s)}{1+G(s)H(s)}$$

(Negative feedback)

Review of last lecture

- Block diagrams and signal flow graphs

Exercise: Find out the transfer function of the following system:



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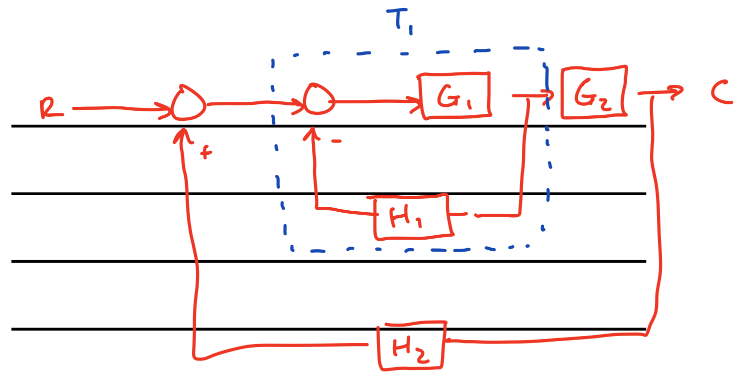
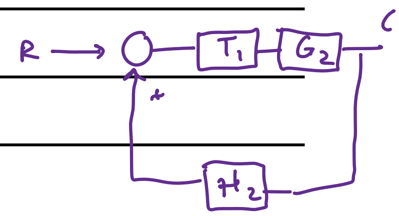


$$T_1 = \frac{G_1}{1+G_1H_1}$$



$$\frac{C(s)}{R(s)} = \frac{T_1 G_2}{1 - T_1 G_2 H_2} = \frac{\frac{G_1}{1+G_1H_1} G_2}{1 - \frac{G_1}{1+G_1H_1} G_2} = \boxed{\frac{G_1 G_2}{1+G_1H_1 - G_1G_2}}$$

General considerations on system responses

- Why do we emphasize first-order and second-order systems?
  - Higher-order systems can be considered to be sum of the responses of first- and second-order systems

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General considerations on system responses

- Why we emphasize first-order and second-order systems?
- Common input signals under investigation
  - Step function
  - Ramp function
  - Sinusoidal function (frequency response)

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Strictly proper: order of numerator is smaller than order of denominator

Time responses of first-order systems

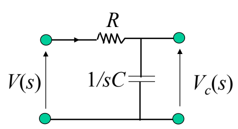
- First-order systems

dc gain

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

Time constant

  - An example:



$$G(s)=\frac{V_c(s)}{V(s)}=\frac{1/(Cs)}{R+1/(Cs)}=\frac{1}{RCs+1}$$

Question: What does this circuit often used for?

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Input: V(s) Output: $V_c(s)$

Circuit used for low pass filter

Time responses of first-order systems

- First-order systems

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

  - An example
  - Initial conditions

$$\left(s+\frac{1}{\tau}\right)C(s)=\frac{K}{\tau}R(s) \longrightarrow \frac{dc(t)}{dt}+\frac{1}{\tau}c(t)=\frac{K}{\tau}r(t)$$

With zero initial condition

$$C(s)=\frac{c(0)}{s+(1/\tau)}+\frac{K}{\tau s+1}R(s) \longleftarrow sC(s)-c(0)+\frac{1}{\tau}C(s)=\frac{K}{\tau}R(s)$$

$$=\frac{K}{\tau s+1}\left(R(s)+c(0)\frac{\tau}{K}\right)$$

With initial term

Initial condition is equivalent to an input of impulse function

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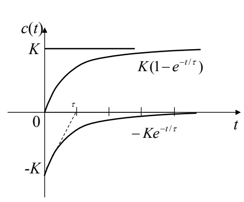
Time responses of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$R(s)=1/s,$

$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau},$$

$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$



The limit of $c(t)$ as t goes to infinity is called the final value, or steady-state value of the response.

The parameter τ is called time constant; we may consider an exponential term to be zero after four time constants.

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Ex: $\frac{Y(s)}{R(s)} = \frac{b_0}{s^2 + a_1 s + a_0}$ (Going off of Slide 8)

$Y(t) = 1(t)$

$y(0^-) = 3, \quad \dot{y}(0^-) = 2$

How to find out zero state response (zero initial condition)?

$Y_{zs}(s) = \frac{b_0}{s^2 + a_1 s + a_0} \cdot \frac{1}{s}$

LT of input

Add these two together

How to find out zero input response?

$Y_{ZI}(s) =$

Step 1: Find the differential equation

$(s^2 + a_1 s + a_0) Y(s) = b_0 R(s)$ $\longrightarrow$ Do ILT

$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 r(t)$

Redo LT including initial condition

$s^2 y(s) - \dot{y}(0^-) - s y(0^-) + a_1 [s y(s) - y(0^-)] + a_0 y(s) = b_0 r(s)$

Set $r(t) = 0$ zero-input

$y(s)(s^2 + a_1 s + a_0) = \dot{y}(0^-) + s y(0^-) + a_1 y(0^-)$

$y(s) = \frac{2 + 3s + 3a_1}{s^2 + a_1 s + a_0}$

Do ILT (PF expansion) to get zero input response

Find out step response of $\dfrac{K}{\tau s+1}$ Slide 9

$$\frac{C}{R} = \frac{K}{\tau s + 1} \qquad C = \frac{K}{\tau s+1} \cdot \frac{1}{s} = \frac{K/\tau}{s + 1/\tau} \cdot \frac{1}{s}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+1/\tau} \qquad \begin{array}{l} p_1 = 0 \\ p_2 = -1/\tau \end{array}$$

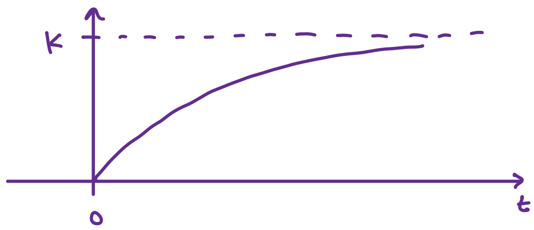
$$K_1 = (s-p_1)\left(\frac{K/\tau}{s+1/\tau} \cdot \frac{1}{s}\right)\Bigg|_{s=p_1=0}$$

$$= K$$

$$K_2 = (s-p_2)\left(\frac{K/\tau}{s+1/\tau} \cdot \frac{1}{s}\right)\Bigg|_{s=-1/\tau} = -K$$

$$C(s) = \frac{K}{s} - \frac{K}{s+1/\tau}$$

$$c(t) = K \cdot 1(t) - Ke^{-t/\tau} 1(t)$$

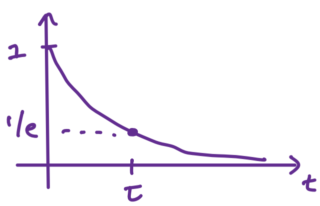


$t \to \infty$; $c(t) \to K$

DC Gain = $\dfrac{\text{Output}}{\text{Input}}$ in steady state = $\dfrac{K}{1(t)} = \dfrac{K}{1} = K$

τ (time constant) is scaling the t

$e^{-t/\tau}$



τ small : system converging faster (faster response)

4τ is settling time for system

Time responses of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

Pole is 0

Pole is $-1/\tau$

$$R(s)=1/s,$$

$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau}$$

$$c(t)=K-Ke^{-t/\tau},\quad t>0$$

Forced response or steady-state response

Natural response or transient response

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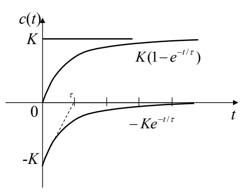
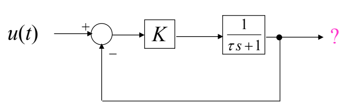
Time responses of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$

– An example: realizing fast step response with a simple feedback controller





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Time responses of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response
- System dc gain
  – The system dc gain is the steady-state gain to a constant input for the case the output has a final value, and it is equal to the system transfer function evaluated at $s = 0$ (why?)

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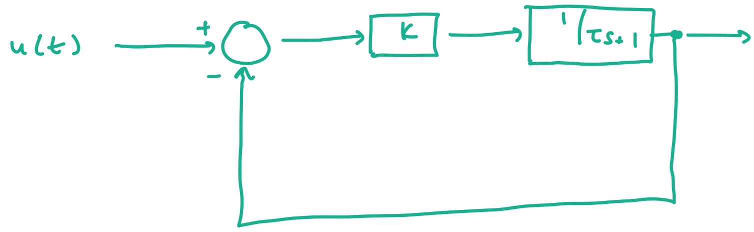
Slide 11 Example



$$T(s) = \frac{K/\tau s+1}{1 + K/\tau s+1} = \frac{K}{\tau s+1+K} = \frac{K/(1+K)}{\frac{\tau}{(1+K)}s + 1}$$

DC Gain $= \frac{K}{1+K}$ Time Constant $= \frac{\tau}{1+K}$

High gain of control → DC gain closer to 1

↓

Time constant smaller

→ Faster system

DC gain of a stable $T(s)$ (Slide 12)
Steady state value of the unit step response

$\frac{C}{R} = T(s) \qquad r(t) = 1(t)$

$\lim_{t\to\infty} c(t) = \text{DC Gain}$

$c(s) = T(s)R(s) = T(s)\cdot\frac{1}{s}$ since $R(s)$ is unit step function

$\lim_{t\to\infty} c(t) = \lim_{s\to 0} s\, c(s) = \lim_{s\to 0} s\cdot T(s)\cdot\frac{1}{s}$

$= T(0)$

Given $T(s)$, we know:

1. System is LTI
2. Impulse response
3. Any response under 0 initial condition
4. Differential equation
5. DC Gain ($T(0)$)

First Order System Examples
RC circuit

Time responses of first-order systems

- First-order systems
- Step response
- System dc gain

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$$

- Ramp response

$$R(s) = 1/s^2,$$

$$C(s) = \frac{1}{s^2}\frac{K/\tau}{s+1/\tau} = \frac{K}{s^2} - \frac{K\tau}{s} + \frac{K\tau}{s+1/\tau},$$

$$c(t) = Kt - K\tau + K\tau\, e^{-t/\tau}, \quad t > 0$$

Steady-state response

$$c_{ss}(t) = Kt - K\tau$$

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References

- C. L. Phillips and J. Parr. Feedback Control Systems, 5th Edition, Prentice Hall, 2011.

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