ECE/ME 2646: Linear System Theory

Lecture 4: Review of Linear Algebra

Zhi-Hong Mao
Professor of ECE and Bioengineering
University of Pittsburgh, Pittsburgh, PA

Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix

If two rows (columns) are switched, the determinant changes the sign.

If a whole row (column) is scaled by a number k, the determinant is scaled by a number k.

If an entire row or an entire column is 0, the determinant is 0.

Question1: Does $\det k\mathbf{A}$ equal $k \det \mathbf{A}$?

Question2: Does the operation det constitute a linear system?

Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix or block triangular matrix

$$\det \begin{bmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & A_3 & * \\ 0 & 0 & 0 & A_4 \end{bmatrix} = ?$$

If $\mathbf{A}$ is a square block upper triangular matrix, in which the diagonal blocks are square, then the determinant of $\mathbf{A}$ equals the product of the determinants of its diagonal blocks.

Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix
  - Elementary operations that preserve the determinant
  - Determinant of product of matrices

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$
$$(\text{or } |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|)$$

Question: How about $\det(\mathbf{A}+\mathbf{B}) = \det \mathbf{A} + \det \mathbf{B}$?

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Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix
  - Elementary operations that preserve the determinant
  - Determinant of product of matrices
  - Laplace expansion

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} c_{ij},$$

where a_{ij} denotes the entry at the i-th row and j-th column of $\mathbf{A}$, the number c_{ij} is the cofactor corresponding to a_{ij} and equals $(-1)^{i+j} \det M_{ij}$, and M_{ij} is the $(n-1)$ by $(n-1)$ submatrix of $\mathbf{A}$ by deleting its i-th row and j-th column.

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Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix
  - Elementary operations that preserve the determinant
  - Determinant of product of matrices
  - Laplace expansion

Exercise:

$$\begin{vmatrix} 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ -1 & 1 & 3 & 1 \\ 2 & 2 & 0 & 0 \end{vmatrix}$$

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Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix
  - Elementary operations that preserve the determinant
  - Determinant of product of matrices
  - Laplace expansion
  - Inverse of square matrices

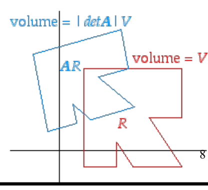
$$\mathbf{A}^{-1} = \frac{\operatorname{Adj}\mathbf{A}}{\det \mathbf{A}} = \frac{1}{\det \mathbf{A}}[c_{ij}]',$$

where c_{ij} is the cofactor corresponding to a_{ij} (the entry at the i-th row and j-th column of $\mathbf{A}$) and equals $(-1)^{i+j}\det M_{ij}$, and M_{ij} is the $(n-1)$ by $(n-1)$ submatrix of $\mathbf{A}$ by deleting its i-th row and j-th column.

Review of linear algebra

- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix
  - Elementary operations that preserve the determinant
  - Determinant of product of matrices
  - Laplace expansion
  - Inverse of square matrices
  - Geometry of determinant
    - Determinant indicates the ratio for the change of volume under the transformation corresponding to the matrix

Let $\mathbf{A}$ be an n by n matrix. Let R be a region in $\mathbf{R}^n$, and $\mathbf{A}R$ be the image under the transformation given by $\mathbf{A}$. Then (volume of $\mathbf{A}R$) = $|\det \mathbf{A}|$ (volume of R).



Review of linear algebra

- Determinant

Remark: For an n by n matrix $\mathbf{A}$, the following statements are equivalent to each other:
(1) $\mathbf{A}$ is nonsingular;
(2) $\mathbf{A}$ is invertible;
(3) $\mathbf{A}$ has full rank;
(4) all the columns (rows) of $\mathbf{A}$ are linearly independent;
(5) $\det \mathbf{A}$ is nonzero;
(6) $\mathbf{Ax} = \mathbf{0}$ does not have nonzero solutions.

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
    - A real or complex number λ is called an eigenvalue of a square real matrix $\mathbf{A}$ if there exists a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ (a story about eigenvalue and a friend of mine)
    - Any nonzero vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$ is called an eigenvector of $\mathbf{A}$ associated with eigenvalue λ

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of $\mathbf{A}$
    - Defined by $\boxed{\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})}$
    - Every root of the characteristic polynomial of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$ (why?)

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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    - Defined by $\boxed{\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})}$
    - Every root of the characteristic polynomial of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$

Exercise: Prove that $\det \mathbf{A}$ equals product of all eigenvalues of $\mathbf{A}$.

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of $\mathbf{A}$
    - Defined by $\boxed{\Delta(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})}$
    - Every root of the characteristic polynomial of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$

Exercise: What are the eigenvalues of the following matrix (a matrix to rotate a column vector in Cartesian coordinates about the origin by an anti-clockwise angle of θ)?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of $\mathbf{A}$
    - Defined by $\boxed{\Delta(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})}$
    - Every root of the characteristic polynomial of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$

Remark: For an n by n matrix $\mathbf{A}$, the following statements are equivalent to each other:
(1) $\mathbf{A}$ is nonsingular;
(2) $\mathbf{A}$ is invertible;
(3) $\mathbf{A}$ has full rank;
(4) all the columns (rows) of $\mathbf{A}$ are linearly independent;
(5) $\det \mathbf{A}$ is nonzero;
(6) $\mathbf{A}\mathbf{x}=\mathbf{0}$ does not have nonzero solutions;
(7) $\mathbf{A}$ does not have zero eigenvalue.

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of $\mathbf{A}$
  - Companion-form matrices

Exercise: What are the characteristic polynomial of the following matrices?

$$\begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \quad \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of A
  - Companion-form matrices

Exercise: What are the characteristic polynomial of the following matrices?

$$\begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer:
$\Delta(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$

Remark: These matrices (and their transposes) can easily be formed from the coefficients of $\Delta(\lambda)$ and are called companion-form matrices.

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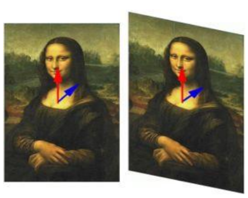
Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of A
  - Companion-form matrices
  - Geometry of eigenvector
    - An eigenvector of a transformation is a non-null vector whose direction is unchanged by that transformation



Question1: Which vector (red or blue) is an eigenvector of the transformation?

Question2: What are the eigenvectors of the rotational transformation?

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of A
  - Companion-form matrices
  - Geometry of eigenvector
  - About “eigen”
    - “Eigen” was introduced from German by Hilbert to denote eigenvalues and eigenvectors, and can be translated as “own”, “peculiar to”, “characteristic”, or “individual”, emphasizing how important eigenvalues are to defining the unique nature of a specific transformation

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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  - Characteristic polynomial of $\mathbf{A}$
  - Companion-form matrices
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  - About “eigen”
    - “Eigen” was introduced from German by Hilbert to denote eigenvalues and eigenvectors, and can be translated as “own”, “peculiar to”, “characteristic”, or “individual”, emphasizing how important eigenvalues are to defining the unique nature of a specific transformation
    - It is common to prefix any natural name for the solution with eigen instead of saying eigenvector. For example, eigenfunction if the eigenvector is a function, eigenmode if the eigenvector is a harmonic mode, eigenstate if the eigenvector is a quantum state, etc.

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
  - Definition
  - Characteristic polynomial of $\mathbf{A}$
  - Companion-form matrices
  - Geometry of eigenvector
  - About “eigen”

$$\boxed{T(\mathbf{v}_\lambda) = \lambda \mathbf{v}_\lambda}$$

Remark: T can be any transformation, including linear transformation (e.g. a matrix or differential operator).

Question: If T is a differential operator d/dt, the eigenvectors are commonly called eigenfunctions. What is the solution to $T(f) = \lambda f$?

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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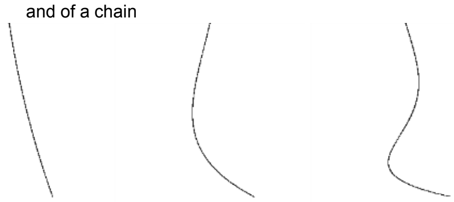
  - Characteristic polynomial of $\mathbf{A}$
  - Companion-form matrices
  - Geometry of eigenvector
  - About “eigen”
    - A simple experiment: eigenmodes of a compound pendulum and of a chain



Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
  - A set of linearly independent vectors in $\mathbf{R}^n$ is called a basis if every vectors in $\mathbf{R}^n$ can be expressed as a unique linear combination of the set
  - Any set of n linearly independent vectors can be used as a basis of $\mathbf{R}^n$

Review of linear algebra

- Determinant
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  - Any set of n linearly independent vectors can be used as a basis of $\mathbf{R}^n$
  - Let $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ be a basis, and then every vector $\mathbf{x}$ can be expressed uniquely as

$$\mathbf{x} = \alpha_1\mathbf{q}_1 + \alpha_2\mathbf{q}_2 + \cdots + \alpha_n\mathbf{q}_n$$
$$= [\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix} \equiv \mathbf{Q}\bar{\mathbf{x}}$$

We call $[\alpha_1, \alpha_2, \ldots, \alpha_n]'$ the representation of the vector $\mathbf{x}$ with respect to the basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
  - Every matrix with distinct eigenvalues can be diagonalized
    - Let λ_i, $i = 1, \ldots, n$, be the eigenvalues of $\mathbf{A}$ and be all distinct
    - Let $\mathbf{q}_i$ be an eigenvector of $\mathbf{A}$ associated with λ_i: $\mathbf{A}\mathbf{q}_i = \lambda_i\mathbf{q}_i$
    - Then the set of eigenvectors $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is linearly independent

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$$\mathbf{A}\underline{[\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n]} = [\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n]\underline{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}$$

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\hat{\mathbf{A}}$$

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \hat{\mathbf{A}}$$

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
  - Every matrix with distinct eigenvalues can be diagonalized

$$\mathbf{A}[\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n] = [\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\hat{\mathbf{A}}$$

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \hat{\mathbf{A}}$$

Question: Is the matrix $\mathbf{Q}$ unique?

Exercise: Prove that $\det \mathbf{A}$ equals product of all eigenvalues of $\mathbf{A}$ (this also holds for the case where eigenvalues of $\mathbf{A}$ are not all distinct).

MATLAB command to find the eigenvalues and eigenvectors of a matrix:

```
[Q, D] = eig(A)
```


Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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- Diagonal form
  - Every matrix with distinct eigenvalues can be diagonalized

$$\mathbf{A}[\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n] = [\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\hat{\mathbf{A}}$$

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \hat{\mathbf{A}}$$

Example: Consider the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$ and linear transformation $\mathbf{Q}$ such that $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$ and $\mathbf{y} = \mathbf{Q}\bar{\mathbf{y}}$, then

$$\mathbf{A}\mathbf{x} = \mathbf{y} \Rightarrow$$

$$\mathbf{A}\mathbf{Q}\bar{\mathbf{x}} = \mathbf{Q}\bar{\mathbf{y}} \Rightarrow$$

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\bar{\mathbf{x}} = \bar{\mathbf{y}} \Rightarrow$$

$$\hat{\mathbf{A}}\bar{\mathbf{x}} = \bar{\mathbf{y}}$$

Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
  - Every matrix with distinct eigenvalues can be diagonalized

$$\mathbf{A}[\mathbf{q}_1\ \mathbf{q}_2\ \cdots \mathbf{q}_n] = [\mathbf{q}_1\ \mathbf{q}_2\ \cdots \mathbf{q}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\mathbf{AQ} = \mathbf{Q}\hat{\mathbf{A}}$$

$$\mathbf{Q}^{-1}\mathbf{AQ} = \hat{\mathbf{A}}$$

Exercise: Considering a linear transformation $\mathbf{Q}$ such that

$$\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}},$$

what will the following state equations change to?

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
  - Every matrix with distinct eigenvalues can be diagonalized
  - If a square matrix has repeated eigenvalues, then it may not be diagonalized; however, it has a block-diagonal and triangular-form representation (Jordan form)

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$

$$\boxed{\begin{aligned} &\mathbf{A}^k \equiv \mathbf{AA}\cdots\mathbf{A}\ \ (k \text{ terms}) \\ &\mathbf{A}^0 \equiv \mathbf{I}\ \ (\text{identity matrix}) \end{aligned}}$$

$$\boxed{f(x) = x^3 + 4x^2 + 3x + 1 \Rightarrow f(\mathbf{A}) = \mathbf{A}^3 + 4\mathbf{A}^2 + 3\mathbf{A} + \mathbf{I}}$$

$$\boxed{\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \Rightarrow \mathbf{A}^k = \begin{bmatrix} \mathbf{A}_1{}^k & 0 \\ 0 & \mathbf{A}_2{}^k \end{bmatrix} \Rightarrow f(\mathbf{A}) = \begin{bmatrix} f(\mathbf{A}_1) & 0 \\ 0 & f(\mathbf{A}_2) \end{bmatrix}}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \ \text{ or } \ \mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1} \Rightarrow$$
$$f(\hat{\mathbf{A}}) = \mathbf{Q}^{-1} f(\mathbf{A})\mathbf{Q} \ \text{ or } \ f(\mathbf{A}) = \mathbf{Q} f(\hat{\mathbf{A}})\mathbf{Q}^{-1}$$

Why?

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$

Exercise: Given

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$

find A^{20}. Hint:

$$\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}, \ \mathbf{Q} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \ \hat{\mathbf{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$
    - Cayley-Hamilton theorem: A matrix satisfies its own characteristic polynomial

Let $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$

be the characteristic polynomial of $\mathbf{A}$. Then

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbf{I} = \mathbf{0}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$
    - Cayley-Hamilton theorem: A matrix satisfies its own characteristic polynomial
    - Cayley-Hamilton theorem implies that, for any polynomial $f(\lambda)$, no matter how large its degree is, $f(\mathbf{A})$ can always be expressed as

$$f(\mathbf{A}) = \beta_{n-1}\mathbf{A}^{n-1} + \cdots + \beta_1\mathbf{A} + \beta_0\mathbf{I}$$

In other words, every polynomial of $\mathbf{A}$ can be expressed as a linear combination of $\{\mathbf{A}^{n-1}, \cdots, \mathbf{A}, \mathbf{I}\}$

Question: Can you prove this?

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$
  - Exponential function $e^{\mathbf{A}t}$

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots + \frac{\lambda^n t^n}{n!} + \cdots$$

$$e^{\mathbf{A}t} = I + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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  - Polynomials of a square matrix $\mathbf{A}$
  - Exponential function $e^{\mathbf{A}t}$

$$e^{\mathbf{A}t} = 1 + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

$$e^{\mathbf{0}} = \mathbf{I}$$

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}$$

$$[e^{\mathbf{A}t}]^{-1} = e^{-\mathbf{A}t}$$

$$e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t} e^{\mathbf{B}t} \text{ in general}$$

$$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t} e^{\mathbf{B}t} \text{ only if } \mathbf{AB} = \mathbf{BA}$$

$$L[e^{\mathbf{A}t}] = (s\mathbf{I} - \mathbf{A})^{-1}$$

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Review of linear algebra

- Determinant
- Eigenvalue and eigenvector
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- Diagonal form
- Functions of a square matrix
  - Polynomials of a square matrix $\mathbf{A}$
  - Exponential function $e^{\mathbf{A}t}$

Exercise: Calculate $e^{\mathbf{A}t}$ for the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

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