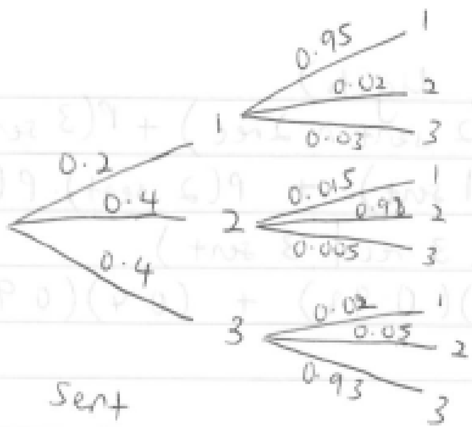
1a)



b) $P(\text{Receiving digit } 2)$

$= P(\text{Rec } 2 \mid \text{sent } 1) \cdot P(1) + P(\text{Rec } 2 \mid \text{sent } 2)\, \text{Prob}(\text{sent } 2)$
$+ \text{Prob}(\text{Rec } 2 \mid \text{sent } 3) \cdot P(\text{sent } 3)$

$= (0.2)(0.02) + (0.4)(0.98) + (0.4)(0.05)$
$= 0.416$

c) $P(2 \text{ sent} \mid 2 \text{ received})$

$= \dfrac{P(2 \text{ sent} \cap 2 \text{ received})}{P(2 \text{ received})} = \dfrac{P(2\text{sent})\, P(2 \text{ rec} \mid 2\text{sent})}{P(2 \text{ received})}$

$= \dfrac{(0.4)(0.98)}{0.416}$

$= 0.9423$

d) $P(1 \text{ sent} \mid 2 \text{ rec})$

$= \dfrac{P(1 \text{ sent} \cap 2\text{rec})}{P(2 \text{ rec})} = \dfrac{P(1 \text{ sent})\, P(2\text{rec} \mid 1 \text{ sent})}{P(2 \text{ received})}$

$= \dfrac{(0.2)(0.02)}{0.416} = 0.0096$

(e) P(Correctly receiving digits)

$= P(1 \text{ sent} \cap 1 \text{ rec}) + P(2 \text{ sent} \cap 2 \text{ rec}) + P(3 \text{ sent} \cap 3 \text{ rec})$

$= P(1 \text{ sent}) \cdot P(1 \text{ rec} \mid 1 \text{ sent}) + P(2 \text{ sent}) \cdot P(2 \text{ rec} \mid 2 \text{ sent})$

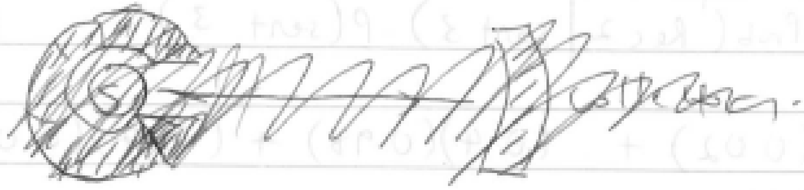
$+ P(3 \text{ sent}) \cdot P(3 \text{ rec} \mid 3 \text{ sent})$

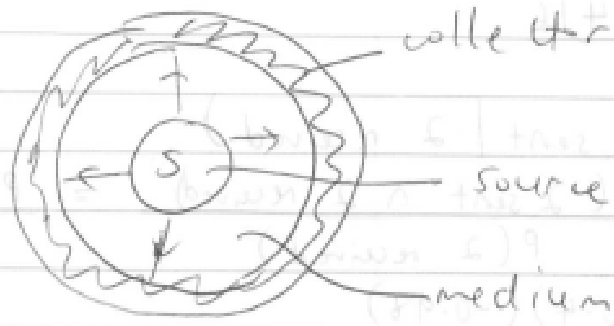
$= (0.2)(0.95) + (0.4)(0.98) + (0.4)(0.93)$

$= 0.954$

The system has over 95% of reliability which is pretty good. The error probability is $(1 - 0.954) = 0.046$, (4% error)

(2) a)





a) Let N be the random variable representing the number of beta particles emitted by the source in the T = 5s measurement time

$$N \sim \text{Poisson}(\alpha = \lambda T = (0.5)(5) = 2.5)$$

$$P_N(n) = \begin{cases} \dfrac{(2.5)^n e^{-2.5}}{n!} & \text{for } n = 0, 1, 2, 3 \ldots \\ 0 & \text{otherwise} \end{cases}$$

3

a) $P(N=2) = \dfrac{(2.5)^2 e^{-2.5}}{2!}$

$= 0.2565$

b) $P(N \geq 2) = 1 - P(N=0) - P(N=1)$

$= 1 - e^{-2.5}\left[1 + \dfrac{2.5}{1}\right]$

$= 1 - 3.5\, e^{-2.5}$

$= 0.7127$

c) Mean, $E[T] = \alpha = 2.5$

Var, $Var(T) = \alpha = 2.5$

Std-dev $\sigma_T = \sqrt{2.5} = 1.58$

(d) Let Q be the RV representing the number of particles that reach the collector out of the total of n emitted

$Q \sim \text{Binomial}(n, p = 0.8)$

Prob(2 out of 6 reach the collector) $= P(Q=2 \mid N=6)$

$= \binom{6}{2}(0.8)^2(0.2)^4 = \dfrac{6!}{2!\,4!}(0.8)^2(0.2)^4$

$= (15)(0.8)^2(0.2)^4$

$= 0.0154$

(e) $P(Q=q \mid N=n) = \binom{n}{q} p^q (1-p)^{n-q}$

$= \binom{n}{q}(0.8)^q(0.2)^{n-q}$

$= \dfrac{n!}{q!\,(n-q)!}(0.8)^q(0.2)^{n-q}$

where $0 \leq q \leq n$

(f). $\mathrm{Prob}(Q=q)$

$$= \sum_{n=q}^{\infty} \mathrm{Prob}(Q=q \mid N=n)\,\mathrm{Prob}(N=n)$$ using total prob. theorem

$$= \sum_{n=q}^{\infty} \binom{n}{q} p^q (1-p)^{n-q} \cdot \frac{\alpha^n e^{-\alpha}}{n!}$$

$$= \sum_{n=q}^{\infty} \frac{n!}{q!(n-q)!} p^q (1-p)^{n-q} \frac{\alpha^n e^{-\alpha}}{n!}$$

$$= \frac{e^{-\alpha} p^q}{q!} \sum_{n=q}^{\infty} \frac{(1-p)^{n-q}}{(n-q)!} \alpha^n$$

$$= \frac{e^{-\alpha} p^q \alpha^q}{q!} \sum_{n=q}^{\infty} \frac{(1-p)^{n-q}}{(n-q)!} \alpha^{n-q}$$

(let $n-q=w$)

$$= \frac{e^{-\alpha} p^q \alpha^q}{q!} \sum_{w=0}^{\infty} \frac{(1-p)^w}{w!} \alpha^w$$

let $a=(1-p)\alpha$

$$= \frac{(\alpha p)^q e^{-\alpha p}}{q!} \underbrace{\sum_{w=0}^{\infty} \frac{a^w}{w!} e^{-a}}_{=1}$$

$$= \frac{(\alpha p)^q e^{-\alpha p}}{q!}$$

This is the same probability as a Poisson with parameter $\alpha' = \alpha p$, as expected. The absorption ~~does~~ only changes the collection rate, ~~and~~ but the process remains Poisson.

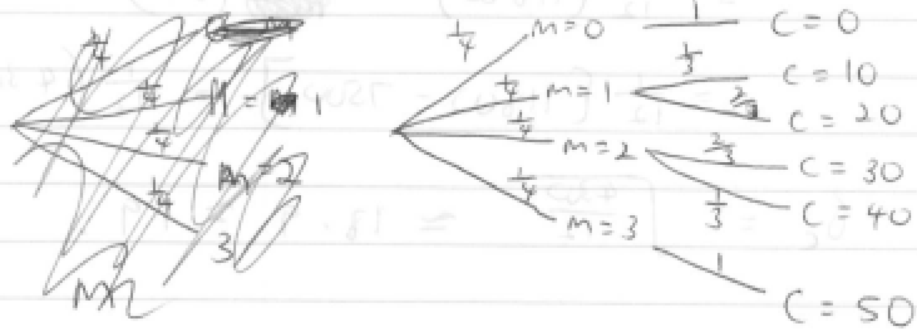
3 a). Number of ways to put the 3 books back on shelf $= 3! = 6 = {}^3P_3$

Prob(books are back in order) $= \frac{1}{6}$.

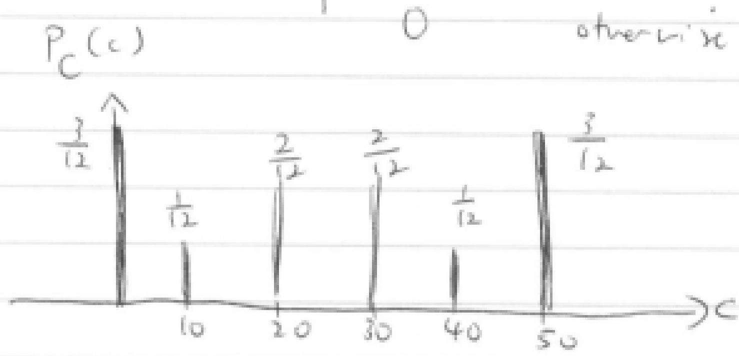
b). Number of ways of choosing 2 books out of 3 $= {}^3C_2 = \frac{3!}{2!(1)!} = \frac{6}{2} = 3$

Prob(both have hardcover) $= \frac{1}{3}$.

c).



$$P_C(c) = \begin{cases} \frac{1}{4} & \text{if } c = 0 \text{ or } 50 \\ \frac{1}{12} & \text{if } c = 10 \text{ or } 40 \\ \frac{2}{12} & \text{if } c = 20 \text{ or } 30 \\ 0 & \text{otherwise} \end{cases}$$



(d) $E(C) = \sum_c c\,P_C(c)$

$= \frac{1}{12}\left[0(3) + 10 + 2(20) + 2(30) + (40) + 3(50)\right]$

$= \frac{1}{12}\left[10 + 40 + 60 + 40 + 150\right]$

$= \frac{300}{12} = \$25$

also from symmetry of PDF.

$$E[C^2] = \sum_c c^2 P_C(c)$$

$$= \frac{1}{12}\left[3(0^2) + (1)(10^2) + (2)(20^2) + 2(30)^2 + 1(40^2) + 3(50^2)\right]$$

$$= \frac{1}{12}\left[100 + 800 + 1800 + 1600 + 7500\right]$$

$$= \frac{1}{12}\left[11800\right]$$

$$\mathrm{Var}(C) = E[C^2] - \left(E[C]\right)^2$$

$$= \frac{1}{12}(11800) - (25)^2$$

$$= \frac{1}{12}\left[11800 - 7500\right] = \frac{1}{12}(4300)$$

$$\sigma_c = \sqrt{\frac{4300}{12}} \simeq 18.9 \simeq 19.$$

Problem 4.

$$f_X(x) = \begin{cases} \frac{2}{7}x + \frac{4}{7}, & -2 \le x \le 0; \\ \frac{4}{7}, & 0 \le x \le 0.5; \\ \frac{8}{7} - \frac{8}{7}x, & 0.5 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

In this problem, it is helpful to obtain the following result:

$$\begin{aligned} \mathsf{E}[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x)\,\mathrm{d}x = \int_{-2}^{0} x^n \left\{\frac{2}{7}x + \frac{4}{7}\right\} \mathrm{d}x + \int_{0}^{0.5} x^n \left(\frac{4}{7}\right) \mathrm{d}x + \int_{0.5}^{1} x^n \left\{\frac{8}{7} - \frac{8}{7}x\right\} \mathrm{d}x \\ &= \frac{2}{7}\left(\frac{4 + 4(-1)^n 2^n - 2^{-n}}{n^2 + 3n + 2}\right); \quad n \ge 0 \qquad \text{(using Maple)} \end{aligned}$$

Hence we have

$$\begin{aligned} \mathsf{E}[X] &= \frac{2}{7}\left(\frac{4 + 4(-1)2 - 2^{-1}}{1 + 3 + 2}\right) = -0.2143 \\ \mathsf{E}[X^2] &= \frac{2}{7}\left(\frac{4 + 4(+1)4 - 2^{-2}}{4 + 6 + 2}\right) = 0.4702 \\ \mathsf{E}[X^3] &= \frac{2}{7}\left(\frac{4 + 4(-1)8 - 2^{-3}}{9 + 9 + 2}\right) = -0.4018 \\ \mathsf{E}[X^4] &= \frac{2}{7}\left(\frac{4 + 4(+1)16 - 2^{-4}}{16 + 12 + 2}\right) = 0.6470 \end{aligned}$$

(a) Determine the mean $\mu_x = \mathsf{E}[X]$ of X.
Solution: The mean value from above is

$$\mu_X = \mathsf{E}[X] = -0.2143$$

(b) Determine the variance $\sigma_X^2 = \mathsf{E}\left[(X - \mu_x)^2\right]$ of X.
Solution: The variance is

$$\begin{aligned} \sigma_X^2 &= \mathsf{E}\left[(X - \mu_x)^2\right] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 \\ &= 0.4702 - (-0.2143)^2 = 0.4243 \end{aligned}$$

Hence the standard deviation is

$$\sigma_X = \sqrt{0.4243} = 0.6514.$$

(c) Determine the skewness $\mathcal{S}_X \triangleq \mathsf{E}\left[\left(\frac{X - \mu_x}{\sigma_x}\right)^3\right]$ of X.

Solution: The skewness is

$$\begin{aligned} \mathcal{S}_X &= \mathsf{E}\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^3\right] = \frac{1}{\sigma_X^3}\mathsf{E}\left[X^3 - 3X^2\mu_X + 3X\mu_X^2 - \mu_X^3\right] \\ &= \frac{1}{\sigma_X^3}\left(\mathsf{E}\left[X^3\right] - 3\mu_X \mathsf{E}\left[X^2\right] + 2\mu_X^3\right) = -0.4311 \end{aligned}$$

(d) Determine the kurtosis $\mathcal{K}_X \triangleq \mathsf{E}\left[\left(\frac{X-\mu_x}{\sigma_x}\right)^4\right] - 3$ of X.

Solution: Consider

$$\begin{aligned}\mathsf{E}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^4\right] &= \frac{1}{\sigma_X^4}\mathsf{E}\left[X^4 - 4X^3\mu_X + 6X^2\mu_X^2 - 4X\mu_X^3 + \mu_X^4\right] \\ &= \frac{1}{\sigma_X^4}\left(\mathsf{E}\left[X^4\right] - 4\mu_X\mathsf{E}\left[X^3\right] + 6\mu_X^2\mathsf{E}\left[X^2\right] - 3\mu_X^4\right) = 2.3653\end{aligned}$$

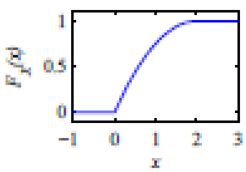
or kurtosis $\mathcal{K}_X = 2.3653 - 3 = -0.6347$.

Problem 5.

(1) Since X is always nonnegative, $F_X(x) = 0$ for $x < 0$. Also, $F_X(x) = 1$ for $x \geq 2$ since its always true that $x \leq 2$. Lastly, for $0 \leq x \leq 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_0^x (1 - y/2)\,dy = x - x^2/4. \tag{1}$$

The complete CDF of X is



$$F_X(x) = \begin{cases} 0 & x < 0, \\ x - x^2/4 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases} \tag{2}$$

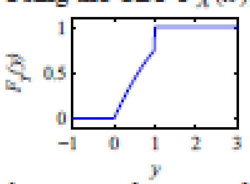
(2) The probability that $Y = 1$ is

$$P[Y = 1] = P[X \geq 1] = 1 - F_X(1) = 1 - 3/4 = 1/4. \tag{3}$$

(3) Since X is nonnegative, Y is also nonnegative. Thus $F_Y(y) = 0$ for $y < 0$. Also, because $Y \leq 1$, $F_Y(y) = 1$ for all $y \geq 1$. Finally, for $0 < y < 1$,

$$F_Y(y) = P[Y \leq y] = P[X \leq y] = F_X(y). \tag{4}$$

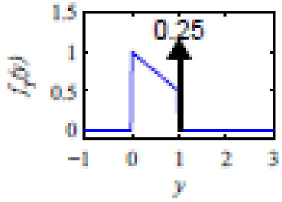
Using the CDF $F_X(x)$, the complete expression for the CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y - y^2/4 & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases} \tag{5}$$

As expected, we see that the jump in $F_Y(y)$ at $y = 1$ is exactly equal to $P[Y = 1]$.

(4) By taking the derivative of $F_Y(y)$, we obtain the PDF $f_Y(y)$. Note that when $y < 0$ or $y > 1$, the PDF is zero.



$$f_Y(y) = \begin{cases} 1 - y/2 + (1/4)\delta(y-1) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$