ECE 1673: Linear Control Systems

Lecture 2: Mathematical Foundation

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Outline of this lecture

- Mathematical foundation
  - Complex variables
  - Differential equations
  - Laplace transform

Complex variables

- Number system
  - Natural number
  - Integer
  - Rational number

    Question: Why $\sqrt{2}$ is not a rational number?
  - Real number
  - Complex number

    Question: How complex numbers can be applied to "the real world"?

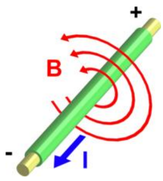
Complex variables

- Number system

Question: How complex numbers can be applied to “the real world”?

Examples of the application of complex numbers:

(1) Electric field and magnetic field.
(2) Complex numbers can be interpreted as being the combination of a phase and a magnitude, e.g., impedance in electric circuits.
(3) Complex numbers sometimes provide a quicker way to solve certain problems (it does appear that some mathematicians have absolutely no intuitive clue concerning the objects they are working with).



$$\int_0^T e^t \cos t \, dt = ?$$

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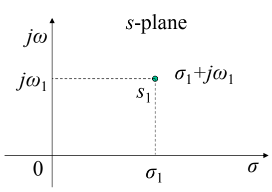
Complex variables

- Number system
- Complex variable
  - A complex variable s has two components: real component σ and imaginary component ω
  - Complex s-plane



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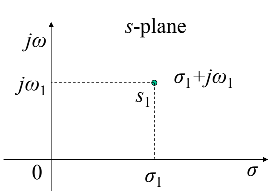
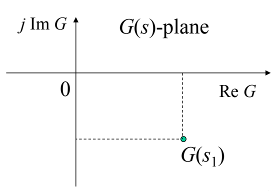
Complex variables

- Number system
- Complex variable
- Functions of a complex variable
  - Function $G(s) = \text{Re}\, G(s) + j\, \text{Im}\, G(s)$





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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
  - A function $G(s)$ of the complex variable s is called an analytic function in a region of the s-plane if the function and all its derivatives exist in the region
  - Example: $G(s)=\dfrac{1}{s(s+1)}$ is analytic at every point in the s-plane except at the points $s=0$ and $s=-1$

Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
- Singularities and poles of a function
  - The singularities of a function are the points in the s-plane at which the function or its derivatives do not exist
  - Definition of a pole: if a function $G(s)$ is analytic in the neighborhood of s_i, it is said to have a pole of order r at $s = s_i$ if the limit

$$\lim_{s\to s_i}(s-s_i)^r G(s)$$

has a finite, nonzero value

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has a finite, nonzero value. In other words, the denominator of $G(s)$ must include the factor $(s-s_i)^r$, so when $s = s_i$, the function becomes infinite. If $r = 1$, the pole at $s = s_i$ is called a simple pole

Complex variables

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  - Examples:

$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^3}$$

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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
- Singularities and poles of a function
- Zeros of a function
  - Definition: If a function $G(s)$ is analytic at $s = s_i$, it is said to have a zero of order r at $s = s_i$ if the limit

$$\lim_{s \to s_i} (s - s_i)^{-r} G(s)$$

has a finite, nonzero value. Or, simply, $G(s)$ has a zero of order r at $s = s_i$ if $1/G(s)$ has an r-th order pole at $s = s_i$

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Differential equations

- Linear ordinary differential equations
  - A wide range of systems in engineering are modeled mathematically by differential equations
  - In general, the differential equation of an n-th order system is written

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

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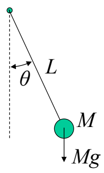
Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
  - Example



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

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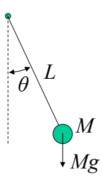
Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
  - Example
  - Linearization of nonlinear differential equations



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

For small value of θ

$$ML\frac{d^2\theta(t)}{dt^2} + Mg\theta(t) = 0$$

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Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example:

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 2x + 1 - 4\cos x + 2e^x$$

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Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example: $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 2x + 1 - 4\cos x + 2e^x$
  - Classical method
    - To find the general homogeneous solution (involving solving the characteristic equation)
    - To find a particular solution of the complete nonhomogeneous equation (involving constructing the family of a function)
    - To solve the initial value problem

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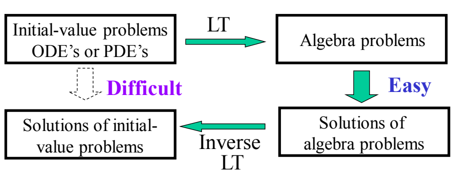
Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example
  - Classical method
  - Laplace transform



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Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example
  - Classical method
  - Laplace transform

Examples (about "usefulness" of mathematical transforms)
(1) log and exp pairs.
(2) Multiplication of polynomials.
(3) Compact representation of data.

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Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

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Laplace transform

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$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- The inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

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Laplace transform

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  – We seldom use the above equation to calculate an inverse Laplace transform; instead we use the equation of Laplace transform to construct a table of transforms for useful time functions. Then we use the table to find the inverse transform

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Laplace transform table

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

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Laplace transform table (continued)

$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\frac{C-aB}{\omega}\right)\sin\omega t\right]$	$\frac{Bs+C}{(s+a)^2+\omega^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

  – Example: $$\frac{c}{(s+a)(s+b)}=\frac{k_1}{s+a}+\frac{k_2}{s+b}$$

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Laplace transform

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  – Case 1: $D(s)$ does not have repeated roots. Then $F(s)$ can be expressed as

$$F(s)=\frac{N(s)}{\prod_{i=1}^{n}(s-p_i)}=\frac{k_1}{s-p_1}+\cdots+\frac{k_n}{s-p_n},$$

where $k_j=(s-p_j)F(s)\big|_{s=p_j}$.

k_j is also called the residue of $F(s)$ in the pole at $s=p_j$

Laplace transform

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$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

  – Case 1: $D(s)$ does not have repeated roots

  – Case 2: $D(s)$ has repeated roots. Then $F(s)$ can be expanded as in the example

$$F(s)=\frac{N(s)}{(s-p_1)(s-p_2)^r}=\frac{k_1}{s-p_1}+\frac{k_{21}}{s-p_2}\cdots+\frac{k_{2r}}{(s-p_2)^r},$$

where $k_{2j}=\frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}[(s-p_2)^r F(s)]\big|_{s=p_2}$

Laplace transform

- The Laplace transform
- The inverse Laplace transform

- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

  – Case 1: $D(s)$ does not have repeated roots

  – Case 2: $D(s)$ has repeated roots

  – Examples: Find inverse Laplace transforms of the following functions

$$F_1(s)=\frac{5}{s^2+3s+2},\quad F_2(s)=\frac{2s+3}{s^3+2s^2+s}$$

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem is valid only if $sF(s)$ does not have any poles on the $j\omega$ axis and in the right half of the s-plane.

Examples:

$$F_1(s) = \frac{5}{s(s^2+s+2)}, \quad F_2(s) = \frac{\omega}{s^2+\omega^2}$$

Laplace transform

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Examples:

$$F_1(s) = \frac{5}{s(s^2+s+2)}, \quad F_2(s) = \frac{\omega}{s^2+\omega^2}$$

$$\lim_{t\to\infty} f_1(t) = 5/2 \qquad f_2(t) = \sin\omega t$$

Final value theorem does not apply in the second example.

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0^-),$$

$$L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^-) - \cdots - f^{n-1}(0^-),$$

$$\text{where } f(0^-) = \lim_{t\to 0} f(t), \quad t<0$$

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- **Theorems of the Laplace transform**
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  - Differential theorem

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- **Theorems of the Laplace transform**
  - Final value theorem
  - Differential theorem
  - Integral theorem

$$L\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

  - Shifting theorem

$$L[f(t - t_0)u(t - t_0)] = e^{-t_0 s} F(s)$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- **Theorems of the Laplace transform**
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem

$$L[e^{-at} f(t)] = F(s + a)$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem

$$L[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$$

Same signs

$$L[e^{-at}f(t)] = F(s+a)$$

Different signs

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem
  - Theorem of convolution integral

$$L^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)d\tau = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

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