1. $X[n]$ is iid Gaussian random sequence

$E(X[n]) = 3$

$Var(X[n]) = n$

$Y[n] = X[n] + \beta X[n-1]$ for $0 < \beta < 1$

(a) Complete and simplified expression for $Var(Y[n])$

For independent RVs, the variance of the sum is sum of the variances.

$$Var[Y[n]] = Var[X[n]] + Var[\beta X[n-1]]$$
$$= n + \beta^2 Var\, X[n-1]$$
$$= n + \beta^2 (n-1)$$

(b) Complete and simplified expression for MMSE linear nonhomogeneous estimator of $Y[n+1]$ based on an observation of $X[n]$

Estimator formula: $\hat{Y}[n+1] = aX[n] + b = aX[n] + m_Y[n+1] - am_X[n]$

where $a = \dfrac{Cov(Y[n+1], X[n])}{Var\{X[n]\}}$

$m_Y[n+1] = E\{Y[n+1]\} = E\{X[n+1]\} + \beta E\{X[n]\} = 3(1+\beta)$

$Cov(Y[n+1], X[n]) = \beta n$

$$\hat{Y}[n+1] = \beta X[n] + 3(1+\beta) - 3\beta$$
$$= \boxed{\beta X[n] + 3}$$

$$\hat{Y}[N+1] = E[X[n]] + \mathrm{Cov}(X[n], Y[n+1]) \cdot \left[\frac{y[n+1] - E[y[n+1]]}{\mathrm{Var}[y[n+1]]}\right]$$

$$E[X[n]] = 3$$

$$E[y[n+1]] = E[X[n+1] + \beta X[n]] = E[X[n+1]] + \beta E[X[n]]$$

$$= 3(1+\beta)$$

$$\mathrm{Var}[y[n+1]] = (n+1) + \beta^2 n$$

$$y[n+1] = X[n+1] + \beta X[n]$$

$$\mathrm{Cov}(X[n], y[n+1]) = E[(X[n] - \mu_{X[n]})(y[n+1] - \mu_{y[n+1]})]$$

$$= E[(X[n] - 3)\underline{(Y[n+1] - \mu_Y[n+1])}]$$

$$Y[n+1] - \mu_Y[n+1] = X[n+1] + \beta X[n] - 3(1+\beta)$$

$$= X[n+1] + \beta X[n] - 3 - 3\beta$$

$$= X[n+1] - 3 + \beta(X[n] - 3)$$

$$= E[(X[n] - 3)(X[n+1] - 3 + \beta(X[n] - 3))]$$

$$= E[(X[n]-3)(X[n+1]-3)]$$

$$+ \beta E[(X[n]-3)^2]$$

$$= \underbrace{\mathrm{Cov}(X[n], X[n+1])}_{X \text{ is iid so this is } 0} + \beta \mathrm{Var}(X[n])$$

$$= \beta \mathrm{Var}(X[n]) = \beta n$$

$$a = \frac{\beta n}{n} = \beta$$

$$\hat{Y}[n+1] = \beta X[n] + 3(1+\beta) - 3\beta$$

$$= \boxed{\beta X[n] + 3}$$

2. $X(t)$ is zero mean Gaussian random process with autocorrelation function

$$R_x(r,s) = \frac{\alpha \cos(\pi[r-s])}{1+|r-s|}$$

$t_1, t_2, \ldots$ are consecutive time points from homogeneous Poisson process with average rate λ. Define a random sequence $Y[n] = X[t_n]$, $n = 1, 2, \ldots$. In other words, $Y[n]$ is a sequence of time samples of $X(t)$ taken at random Poisson times.

(a) Is $X(t)$ wide sense stationary? Explain.

Yes - the mean is zero (constant) and the autocorrelation only depends on $r-s$ (not r or s individually)

(b) Using vector-matrix form, give a simplified expression for joint PDF of $X(1)$ and $X(2)$.

$$f_X(x) = \frac{1}{2\pi\sqrt{\det(C)}} \exp\left\{-\frac{1}{2}\left[x-\mu_X\right]^T C^{-1}\left[x-\mu_X\right]\right\}$$

where $X = \begin{bmatrix} X(1) \\ X(2) \end{bmatrix}$

$$C = \begin{bmatrix} R_X(1,1) & R_X(1,2) \\ R_X(2,1) & R_X(2,2) \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha\cos(0\cdot\pi)}{1} & \dfrac{\alpha\cos(-1\cdot\pi)}{1+|-1|} \\ \dfrac{\alpha\cos(1\cdot\pi)}{1+|1|} & \dfrac{\alpha\cos(0\cdot\pi)}{1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & -\dfrac{\alpha}{2} \\ -\dfrac{\alpha}{2} & \alpha \end{bmatrix}$$

$$\det(C) = \alpha^2 - \frac{\alpha^2}{4} = \frac{3\alpha^2}{4}$$

$$C^{-1} = \frac{1}{\frac{3\alpha^2}{4}} \begin{bmatrix} \alpha & \alpha/2 \\ \alpha/2 & \alpha \end{bmatrix}$$

x is zero mean

$$f_X(x) = \frac{1}{2\pi\sqrt{\frac{3\alpha^2}{4}}} \exp\left\{-\frac{1}{2}[x]^T \frac{1}{\frac{3\alpha^2}{4}} \begin{bmatrix} \alpha & \alpha/2 \\ \alpha/2 & \alpha \end{bmatrix} [x]\right\}$$

$$= \frac{1}{\alpha\pi\sqrt{3}} \exp\left\{-\frac{2}{3\alpha^2}[x]^T \begin{bmatrix} \alpha & \alpha/2 \\ \alpha/2 & \alpha \end{bmatrix} [x]\right\}$$

(c) Give a complete and simplified expression for $R_Y[3,4]$, which is the autocorrelation function for $Y[n]$, $R_Y[n,m]$ evaluated at $n=3$ and $m=4$. May leave answer in the form of integrals.

$$R_Y[3,4] = E\{Y(3)Y(4)\} = E\{X(t_3)X(t_4)\}$$

$$= E\left[E\{X(t_3)X(t_4) \mid t_3, t_4\}\right] = E\left[\frac{\alpha\cos(\pi|t_3-t_4|)}{1+|t_3-t_4|}\right]$$

$T = t_3 - t_4$ is an exponentially distributed RV with PDF

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R_Y[3,4] = E\left[\frac{\alpha\cos(\pi T)}{1+T}\right] = \int_0^\infty \frac{\alpha\cos(\pi t)}{1+t}\lambda e^{-\lambda t}\,dt$$