1. Child goes Trick-or-Treating with bucket that holds 100 pieces more. From i^{th} house, child gets k_i pieces of candy, such that k_i is Poisson distributed with mean = 3. Furthermore, k_i and k_j are independent. Give approximation to probability that the basket is overfilled after visiting exactly 20 houses

Central Limit Theorem

X_i is number of candy pieces from i^{th} house. Total number of pieces of candy is

$$Y = \sum_{i}^{20} X_i \qquad \text{Want } P(Y > 100)$$

$$E[Y] = \sum_{i=1}^{20} E[X_i] = 20 \cdot 3 = 60$$

For independent RV sum, variance of sum is sum of variances

$$Var[Y] = \sum_{i=1}^{20} Var[X_i] = 20 \cdot 3 = 60$$

Var of Poisson RV is same as mean

$$P(Y > 100) = 1 - \Phi\left(\frac{100 - 60}{\sqrt{60}}\right)$$

2. Let $x[n]$ be a WSS random sequence with mean 0 and autocovariance function

$$C_x[m] = \begin{cases} 9 & m = 0 \\ 6 & |m| = 1 \\ 3 & |m| = 2 \\ 0 & |m| = 3 \end{cases}$$

(a) Give the autocorrelation matrix for the vector
$z = [x[n-1] \quad x[n-2]]$

$$[x[n-1] \quad x[n-2]] \begin{bmatrix} x[n-1] \\ x[n-2] \end{bmatrix}$$

WSS

$$\begin{bmatrix} x[n-1]\,x[n-1] & x[n-1]\,x[n-2] \\ x[n-1]\,x[n-2] & x[n-2]\,x[n-2] \end{bmatrix} = C$$

$$\begin{bmatrix} R_{xx}(0) & R_{xx}(1) \\ R_{xx}(1) & R_{xx}(0) \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}$$

(b) Give the mean squared error (MSE) for the linear homogeneous estimate of $x[n]$, given the observations $z = [x[n-1] \quad x[n-2]]$. The estimate, $\hat{x}[n]$, is optimal in terms of the MSE. May leave answer as vectors and matrices, but with all numbers substituted.

$Y = x[n]$

$$MSE = E\{Y^2\} - r_{yz} R_z^{-1} r_{yz}^T$$

$= [E\{x[n]x[n-1] \quad x[n]x[n-2]\}]$

$$r_{yZ} = E\{YZ\} = E[X[n]Z] = E[\{X[n]X[n-1] \quad X[n]X[n-2]\}]$$

$$= [6 \quad 3]$$

$$MSE = 9 - [6 \quad 3] \frac{\begin{bmatrix} 9 & -6 \\ -6 & 9 \end{bmatrix}}{81 - 36} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 4.8$$

(c) Let $A = [a_1 \;\; a_2]$ be vector of optimal estimator coefficients. That is, $\hat{X}[n] = ZA^T$. Don't have to compute values of A. Let $e[n] = \hat{X}[n] - X[n]$ be the error process. Derive a simplified expression for $E\{e[n]\,e[n-1]\}$. Use orthogonality principle in derivation. (error autocorrelation)

3. Let $N(t)$ be Poisson counting process with average rate λ.
Let T be an exponential RV with PDF $f_T(\tau) = \beta e^{-\beta\tau} u(\tau)$,
and let T be independent of $N(t)$. Define a new RP

$$Z(t) = \begin{cases} N(T) & t \geq T \\ N(t) & t < T \end{cases}$$

In other words, $N(t)$ is $Z(t)$ until t is greater than T; after that point $N(t)$ ceases to vary with time. Find the pmf of $Z(t)$

Let $W(t)$ be a Weiner process with diffusion constant α

$$f_{Z(t)} z = \int_{-\infty}^{\infty} f_{Z(t)\,T} (z,\tau)\, d\tau = \int_{-\infty}^{\infty} f_{Z(t)|T} (z|\tau)\, f_T(\tau)\, d\tau$$

$$= \int_0^{\infty} f_{Z(t)|T} (z|\tau)\, \beta e^{-\beta\tau}\, d\tau$$

$$f_{Z(t)|T}(z, \tau) = \begin{cases} \frac{1}{\sqrt{2\pi\alpha t}} e^{z^2/2\alpha t} & t \leq \tau \\ \frac{1}{\sqrt{2\pi\alpha\tau}} e^{z^2/2\alpha t} & t > \tau \end{cases}$$

$$f_{Z(t)}(z) = \int_0^t \frac{1}{\sqrt{2\pi\alpha\tau}} e^{z^2/2\alpha\tau} \beta e^{-\beta\tau}\, d\tau$$

$$+ \int_t^{\infty} \frac{1}{\sqrt{2\pi\alpha t}} e^{z^2/2\alpha t} \beta e^{-\beta\tau}\, d\tau$$