ECE 1675/2570: Robotic Control (Spring 2022)

Module II: Control of Mobile Robots

Lecture 7: Controllability; Observability; Separation Principle; Optimal Control

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Zhi-Hong Mao
Professor of ECE and Bioengineering
University of Pittsburgh, Pittsburgh, PA

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Outline

- Homework 5 (no need to turn in)
- Controllability
- Case study: the Segway robot
- Observer and observability
- Separation principle
- Optimal control

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Controllability

- Motivation
  - *When can we place the eigenvalues however we want using state feedback?*
  - *When is the* **B** *matrix (the actuator configuration) rich enough so that we can make the system do whatever we want it to do?*

The answer revolves around the concept of *controllability.*

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For certain cases, we can't do this

Controllability

- Motivation
- Definition
  - *Controllable* and *uncontrollable:*

    Consider a state equation

    input = u

    $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}.$$

    This state equation or the pair $(\mathbf{A}, \mathbf{B})$ is said to be *controllable* if for any initial state $\mathbf{x}(0) = \mathbf{x}_0$ and any final state $\mathbf{x}_1$, there exists an input that transfer $\mathbf{x}_0$ to $\mathbf{x}_1$ in a finite time. Otherwise the state equation or $(\mathbf{A}, \mathbf{B})$ is said to be *uncontrollable*.

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State instead of output

Controllability

- Motivation
- Definition
  - Controllable and uncontrollable
  - Examples
    - Hand movement control
    - Eye movement control
    - Differential drive mobile robot

Uncontrollable - certain configurations that your hand can't get into

(Horizontally) → 1st order equation: not controllable because they can't move separately

Controllable - can theoretically get to any heading

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Controllability

- Motivation
- Definition
- Insight from an example
  - Consider a discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k, \qquad \mathbf{x}_0 = 0$$

Is it possible to drive this system to a particular target state $\mathbf{x}^*$ in n steps?

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$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k, \qquad \mathbf{x}_0 = 0$$

$$\downarrow$$

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 + \mathbf{B}u_0 = \mathbf{B}u_0$$
$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}u_1 = \mathbf{A}\mathbf{B}u_0 + \mathbf{B}u_1$$
$$\mathbf{x}_3 = \mathbf{A}^2\mathbf{B}u_0 + \mathbf{A}\mathbf{B}u_1 + \mathbf{B}u_2$$
$$\cdots$$
$$\mathbf{x}_n = \mathbf{A}^{n-1}\mathbf{B}u_0 + \cdots + \mathbf{B}u_{n-1}$$

To drive this system to a particular target state $\mathbf{x}^*$ in n steps, we need to solve

$$\mathbf{x}^* = [\mathbf{B}\ \ \mathbf{A}\mathbf{B} \ldots\ \ \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}.$$

This is possible for any target state if and only if

$$\operatorname{rank}([\mathbf{B}\ \ \mathbf{A}\mathbf{B} \ldots\ \ \mathbf{A}^{n-1}\mathbf{B}]) = n.$$

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Discretized version of continuous time system

Controllability

- Motivation
- Definition
- Insight from an example
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are n by n and n by p matrices respectively, is controllable if and only if the n by np *controllability matrix*

$$C = [\mathbf{B}\ \ \mathbf{A}\mathbf{B}\ \ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}]$$

has rank n (full row rank).

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determinant $\neq 0$: invertible, full rank

$C \cdot C^T$ is nonsingular $\rightarrow$ full rank

Controllability

- Motivation
- Definition
- Insight from an example
- Theorems

All eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant matrix $\mathbf{K}$ if and only if $(\mathbf{A},\ \mathbf{B})$ is controllable.

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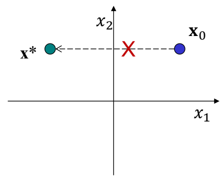
Controllability

- Motivation
- Definition
- Insight from an example
- Theorems

- A remark
  - Being controllable does not mean that we can follow any arbitrary trajectories. For example:

Newton's Second Law

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



In this example, we cannot follow a line directly linking $\mathbf{x}_0$ to $\mathbf{x}^*$, but have to take a route penetrating the region of negative velocity.

x_1 is position along line

x_2 is velocity

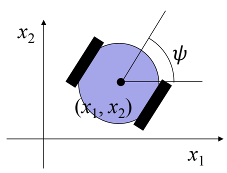
Because velocity is always forward

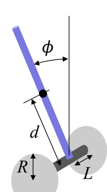
IE: Car is moving forward, can't somehow move backwards

Case study: the Segway robot

Case study: the Segway robot

- Model
  - Unicycle + inverted pendulum





The base:

$$\boxed{\begin{aligned} \dot{x}_1 &= v\cos\psi \\ \dot{x}_2 &= v\sin\psi \\ \dot{\psi} &= \omega \end{aligned}}$$

Extra states: $\boxed{v, \omega}$

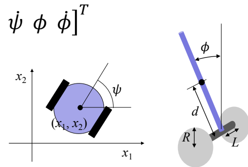
The "pendulum": $\boxed{\phi, \dot{\phi}}$

Inputs: $\boxed{\tau_l, \tau_r}$

Highly Nonlinear

Case study: the Segway robot

- Model
  - Unicycle + inverted pendulum
  - State vector: $\mathbf{x} = [x_1 \; x_2 \; v \; \psi \; \dot{\psi} \; \phi \; \dot{\phi}]^T$
  - Input vector: $\mathbf{u} = [\tau_l, \tau_r]^T$
  - Dynamics:



$$3(m_w + m_b)\,\dot{v} - m_b d(\cos\phi)\ddot{\phi} + m_b d(\sin\phi)(\dot{\phi}^2 + \dot{\psi}^2) = -\frac{1}{R}(\tau_l + \tau_r)$$

$$\left(\left(3L^2 + \frac{1}{2R^2}\right) m_w + m_b d^2 \sin^2\phi + I_2\right)\ddot{\psi} + m_b d^2(\sin\phi)(\cos\phi)\dot{\phi}\dot{\psi} = \frac{L}{R}(\tau_l - \tau_r)$$

$$m_b d(\cos\phi)\dot{v} - (m_b d^2 + I_3)\ddot{\phi} + m_b d^2(\sin\phi)(\cos\phi)\dot{\phi}^2 + m_b g d \sin\phi = \tau_l + \tau_r$$

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Case study: the Segway robot

- Model
  - Unicycle + inverted pendulum
  - State vector: $\mathbf{x} = [x_1 \; x_2 \; v \; \psi \; \dot{\psi} \; \phi \; \dot{\phi}]^T$
  - Input vector: $\mathbf{u} = [\tau_l, \tau_r]^T$
  - Dynamics
  - Linearization (linearizing the equations around $\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0}$ and plugging in the physical parameters):

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.16 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 72.5 & 0 \end{bmatrix}}_{\substack{A \\ 7\times 7}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.67 & -1.67 \\ 0 & 0 \\ 0.029 & -0.029 \\ 0 & 0 \\ -24.2 & -24.2 \end{bmatrix}}_{\substack{B \\ 7\times 2}} \mathbf{u}$$

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Case study: the Segway robot

- Model
- Controllability

In MATLAB:

```
>> rank(ctrb(A,B))
ans =
      6
```

Need rank 7, so some rows are linearly dependent and linearized system is not controllable

Case study: the Segway robot

- Model
- Controllability
  - Rank of the controllability matrix = $6 < n$ → uncontrollable
  - The unicycle dynamics gets messed up when linearized (around $\mathbf{0}$)

$$\dot{x}_1 = v\cos\psi \approx v(1 - \frac{\psi^2}{2}) \approx v$$
$$\dot{x}_2 = v\sin\psi \approx v\psi \approx 0$$
$$\dot{\psi} = \omega$$

↓

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \psi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix}$$

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Case study: the Segway robot

- Model
- Controllability
  - Rank of the controllability matrix = $6 < n$ → uncontrollable
  - The unicycle dynamics gets messed up when linearized (around $\mathbf{0}$)
  - A smaller system: if we can control v and ω, that should be "enough," so let us simply remove the unicycle part

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Case study: the Segway robot

- Model
- Controllability
  - Rank of the controllability matrix = $6 < n$ → uncontrollable
  - The unicycle dynamics gets messed up when linearized (around $\mathbf{0}$)
  - A smaller system
    - State vector: $\mathbf{x} = [v \;\; \omega \;\; \phi \;\; \dot{\phi}]^T$
    - Input vector: $\mathbf{u} = [\tau_l, \tau_r]^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 2.16 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 72.5 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -1.67 & -1.67 \\ 0.029 & -0.029 \\ 0 & 0 \\ -24.2 & -24.2 \end{bmatrix}\mathbf{u}$$

In MATLAB:

```
>> rank(ctrb(A,B))
ans =
     4
```

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System is controllable now

Case study: the Segway robot

- Model
- Controllability
- A tracking problem
  - Tracking instead of stabilizing to $\mathbf{0}$
    - Desirable state: $\mathbf{x}_d = [v_d \;\; \omega_d \;\; 0 \;\; 0]^T$

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Case study: the Segway robot

- Model
- Controllability
- A tracking problem
  - Tracking instead of stabilizing to $\mathbf{0}$
  - Solution: turning the tracking problem to a stabilization problem
    - Desirable state: $\mathbf{x}_d = [v_d \;\; \omega_d \;\; 0 \;\; 0]^T$
    - Let $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d$

$$\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_d = \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = \mathbf{A}(\tilde{\mathbf{x}} + \mathbf{x}_d) + \mathbf{Bu}$$

Note that (luckily) $\mathbf{Ax}_d = \mathbf{0}$ so

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{Bu}$$

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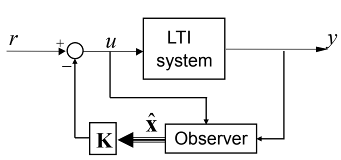
Observer and observability

- Motivation
  - We now know how to design effective controllers using state feedback, but what about output feedback?



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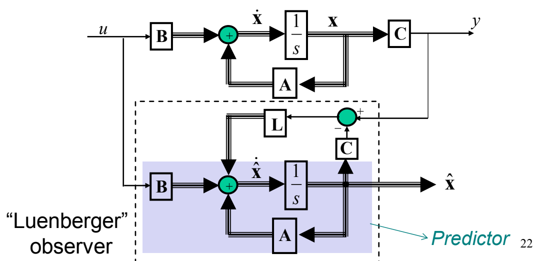
Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
  - *Predictor:* making a copy of the system: $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u$



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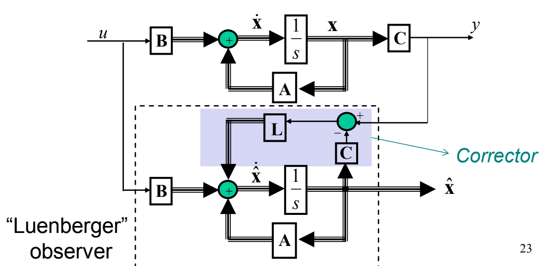
Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
  - *Predictor*
  - *Corrector:* adding a notation of how wrong the estimate is to the model: $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})$



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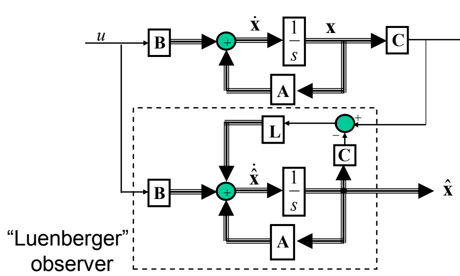
Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
  - *Predictor*
  - *Corrector*
  - Picking the observer gain



Question:
What are the equations for $\hat{\mathbf{x}}$ and $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$?

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Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
  - *Predictor*
  - *Corrector*
  - Picking the observer gain
    - We want to stabilize (drive to zero) the estimation error $\mathbf{e}$
    - We can just pick $\mathbf{L}$ such that the eigenvalues $\mathbf{A}-\mathbf{LC}$ have negative real parts—it is pole placement again!

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A}-\mathbf{LC})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$
$$\dot{\mathbf{e}} = (\mathbf{A}-\mathbf{LC})\mathbf{e}$$

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Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
  - *Predictor*
  - *Corrector*
  - Picking the observer gain
    - We want to stabilize (drive to zero) the estimation error $\mathbf{e}$
    - We can just pick $\mathbf{L}$ such that the eigenvalues $\mathbf{A}-\mathbf{LC}$ have negative real parts—it is pole placement again!
    - *Does this always work?*

No. The answer revolves around the concept of *observability*.

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Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
- Observability
  - Definition: Consider an n-dimensional p-input q-output state equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}.$$

This state equation or the pair $(\mathbf{A}, \mathbf{C})$ is said to be *observable* if for any unknown initial state $\mathbf{x}(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over $[0, t_1]$ suffices to determine uniquely the initial state $\mathbf{x}(0)$. Otherwise, equation is said to be *unobservable*.

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Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
- Observability
  - Definition
  - A modest example

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \qquad y_k = \mathbf{C}\mathbf{x}_k$$

$$y_0 = \mathbf{C}\mathbf{x}_0$$
$$y_1 = \mathbf{C}\mathbf{x}_1 = \mathbf{CA}\mathbf{x}_0$$
$$\vdots$$
$$y_{n-1} = \mathbf{CA}^{n-1}\mathbf{x}_0$$

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \overbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}}^{\text{Observability matrix}} \mathbf{x}_0$$

The initial condition can be estimated from the outputs when the observability matrix has full rank.

Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
- Observability
  - Definition
  - A modest example
  - Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{C})$, where $\mathbf{A}$ and $\mathbf{C}$ are n by n and q by n matrices respectively, is observable if and only if the nq by n *observability matrix*

$$O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

has rank n (full column rank).

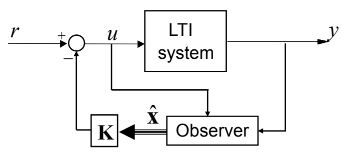
Observer and observability

- Motivation
- Predictor-corrector architecture for observer design
- Observability
  - Definition
  - A modest example
  - Theorems

All eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant vector $\mathbf{L}$ if and only if $(\mathbf{A},\ \mathbf{C})$ is observable.

The separation principle

- Design of feedback from estimated states
  - *Step 1:* Design the state feedback as if we had $\mathbf{x}$ (which we don't)
  - *Step 2:* Estimate $\mathbf{x}$ using an observer



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$
$$u = r - \mathbf{K}\hat{\mathbf{x}}$$

The separation principle

- Design of feedback from estimated states
  - *Step 1:* Design the state feedback as if we had $\mathbf{x}$ (which we don't)
  - *Step 2:* Estimate $\mathbf{x}$ using an observer

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$
$$u = r - \mathbf{K}\hat{\mathbf{x}}$$

Questions:
(1) The eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ are obtained from $u = r - \mathbf{K}\mathbf{x}$. Do we still have the same set of eigenvalues in using estimated state variables?
(2) Will the eigenvalues of the observer be affected by the connection?
(3) What is the effect of the observer on the transfer function from r to y?

Now two systems are intertwined (L and K)

Luckily, these processes can be decoupled

The separation principle

- Design of feedback from estimated states
- The separation principle

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$
$$u = r - \mathbf{K}\hat{\mathbf{x}}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} r$$
$$y = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} r$$
$$y = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

The separation principle

- Design of feedback from estimated states
- The separation principle

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Remarks:
1) The eigenvalues are the union of those of $\mathbf{A}-\mathbf{BK}$ and $\mathbf{A}-\mathbf{LC}$.
2) Inserting the observer does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the observer affected by the connection.
3) The design of state feedback and the design of observer can be carried out independently. This is called the *separation principle*.

Optimal control

- Formulation of optimal control problems
  - Objective function (or performance index)

Example: Linear quadratic regulator (LQR) problem

$$J = \mathbf{x}^T(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\, dt$$

Optimal control

- Formulation of optimal control problems
  - Objective function (or performance index)
  - Decision variables: $\mathbf{x}$, $\mathbf{u}$
  - Constraints: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, etc.

Optimal control

- Formulation of optimal control problems
- Approaches to optimal control
  - Calculus of variations (Pontryagin's maximum principle)
  - Dynamic programming

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Optimal control

- Formulation of optimal control problems
- Approaches to optimal control
  - Calculus of variations
  - Dynamic programming
  - Closed-form solutions to LQR problems
    - For an LTI system described by $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \mathbf{x}(0) = \mathbf{x}_0$ with a quadratic cost function defined as

$$J = \int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\, dt$$

the feedback control law that minimizes the value of the cost is $\mathbf{u} = -\mathbf{Kx}$ where $\mathbf{K}$ is given by $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ and $\mathbf{P}$ is found by solving the *algebraic Riccati equation* (ARE):

$$\boxed{\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} = \mathbf{0}}$$

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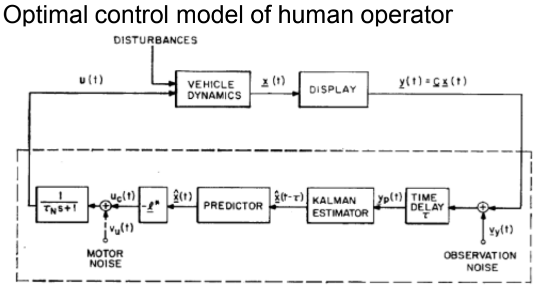
Optimal control

- Formulation of optimal control problems
- Approaches to optimal control
- Optimal control model of human operator



(Kleinman, Baron, and Levison, 1970)

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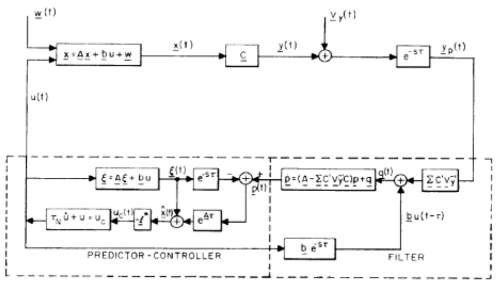
Optimal control

- Formulation of optimal control problems
- Approaches to optimal control
- Optimal control model of human operator



(Kleinman, Baron, and Levison, 1970)

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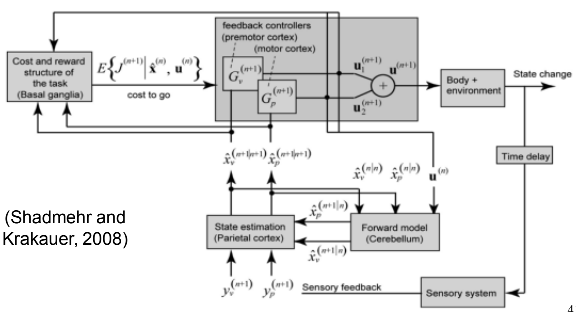
Optimal control

- Formulation of optimal control problems
- Approaches to optimal control
- Optimal control model of human operator



(Shadmehr and Krakauer, 2008)

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References

- M. Egerstedt, Lecture Notes for Control of Mobile Robots, Georgia Institute of Technology, 2016.
- C.-T. Chen. Linear System Theory and Design, 4th Edition, Oxford University Press, 2013.
- K. J. Astrom and R. M. Murray, Feedback Systems: An Introduction for Scientists and Engineers, 2010.
- L. Kleinman, S. Baron, and W. H. Levison, “An optimal control model of human response—part I: Theory and validation,” Automatica 5, 357-369, 1970.
- R. Shadmehr and J. W. Krakauer, “A computational neuroanatomy for motor control,” Experimental Brain Research 185, 359-381, 2008.
- https://uncrate.com/segway-robot/

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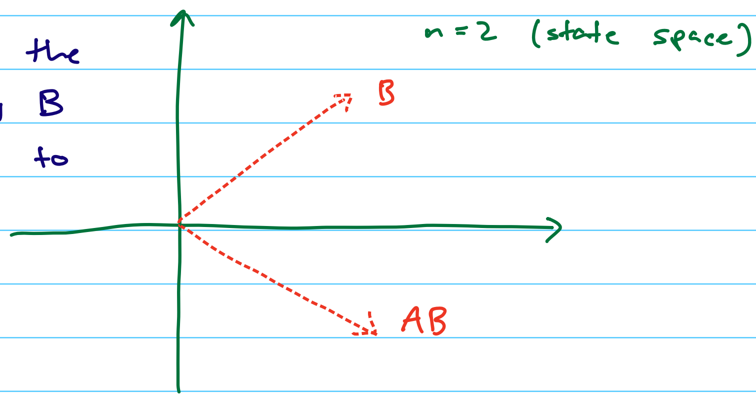
Controllability Example (Slide 6)

$$X_{k+1} = AX_k + Bu_k, \quad X_0 = 0$$

$x_k \in \mathbb{R}^n$
$u_k \in \mathbb{R}$ (scalar)

$x_1 = Bu_0$

x_1 will be on the line defined by B
→ can't move to any point in state space



$$x_2 = Ax_1 + Bu_1 = A(Bu_0) + Bu_1$$

AB is 2×1 vector (same dimension as B)
→ Now can move to any point in state space by expressing any point as linear combination of previous two vectors because these vectors are linearly dependent

∴ This is controllable

→ If B and AB are linearly dependent, it is not controllable because you can't get to points that are off the line

→ Need n steps because n is the minimum number of steps needed to form a basis of $\mathbb{R}^n$

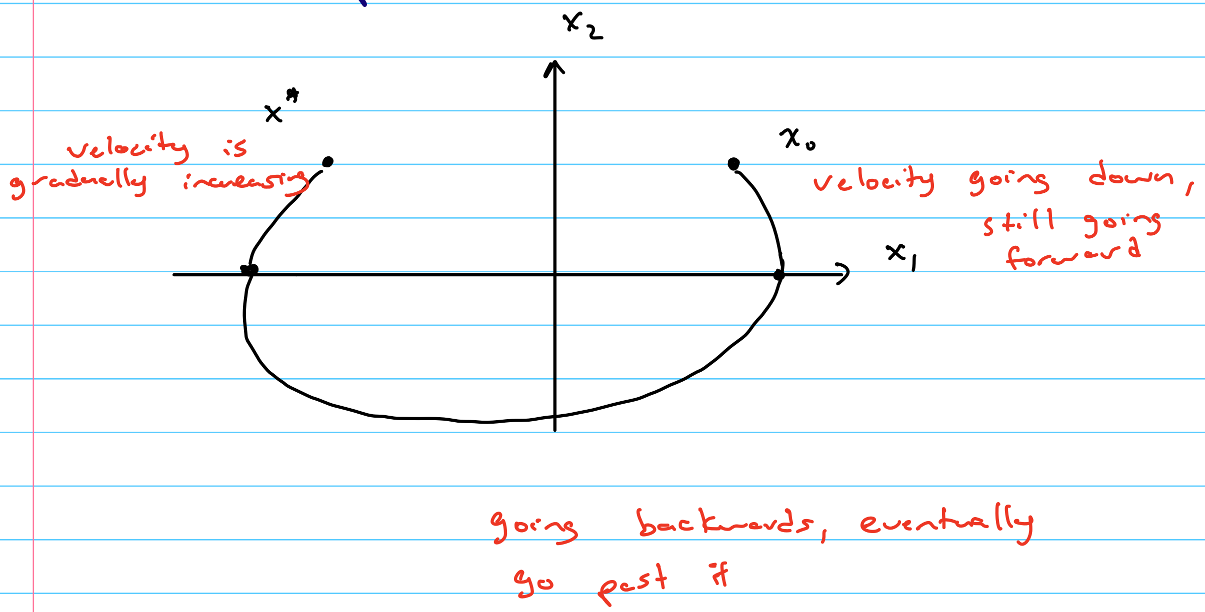
→ Why do we stop at n basis vectors? Adding more doesn't really do more because additional vectors are linearly dependent on previous vectors

→ Characteristic equation of matrix A is

$$|\lambda I - A| = 0$$

→ Any matrix A should solve its own characteristic equation, which implies that $A^n B$ can be written as linear combination of previous columns

Slide 10 example



Notes on system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Each point in state space gives you a position and velocity

Slide 14 Example - Linearized Segway robot

$A: 7\times7$ $B: 7\times2$

$x \in \mathbb{R}^7$ $u \in \mathbb{R}^2$

$u_1 : \tau_L$

$u_2 : \tau_R$

Controllability Matrix:

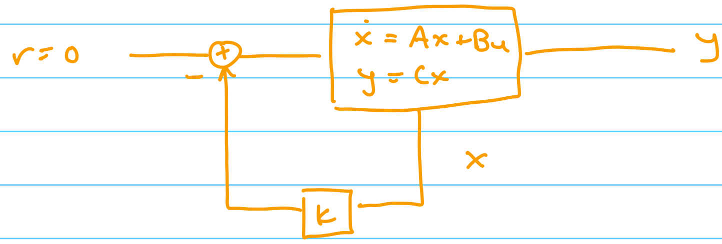
$$C = [\underbrace{B}_{7\times2} \ \underbrace{AB}_{7\times2} \ \cdots \ \underbrace{A^{n-1}B}_{7\times2}] = 7\times14$$

Rank must be 7 to be controllable

Review of Control, Observability Intro

learned output feedback and state feedback

Regulation problem:



State feedback diagram

Don't actually know x $\longrightarrow$ $\boxed{\text{need to estimate it}}$

Design the observer/state estimator for x

Known: A, B, C, D $\qquad\qquad \dot{x} = Ax + Bu$

u, y $\qquad\qquad y = Cx$

Method 1: Use a simulator

$\dot{\hat{x}} = A\hat{x} + Bu$

How to solve for $x(t)$?

$$X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\hat{x}(t) = e^{At}\hat{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

But $x(0)$ is not known, so this doesn't work

In reality, A B C might not be entirely accurate

If A is unstable, $\|e^{At}(x(0) - \hat{x}(0))\| \to \infty$ (discrepancy)

Method 2: $x = C^{-1}y$

C might not be invertible (square)!

Too expensive or infeasible to make C square

Method 3: $y = Cx \longrightarrow \dot{y} = C\dot{x} = CAx$

$\ddot{y} = C\ddot{x} = CA^2x$

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} x$$

$y \in \mathbb{R}^1$ $\quad x \in \mathbb{R}^n$ $\quad C = 1\times n$

$CA = 1\times n$

$CA^2 = 1\times n$

$\vdots$

Estimate up to y^{n-1} so observability matrix is invertible

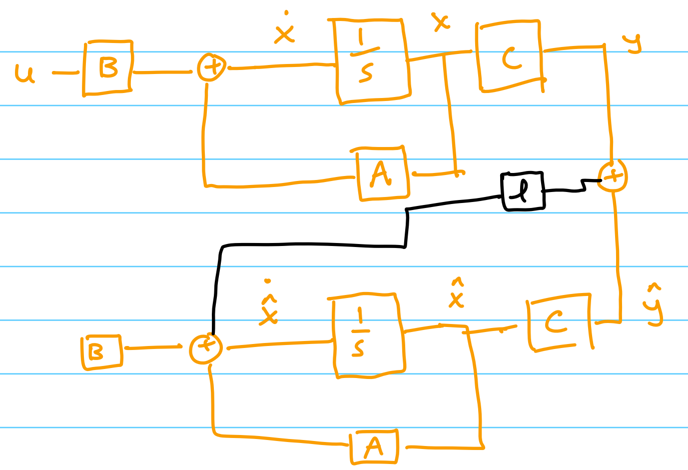
$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \\ CA^{n-1} \end{bmatrix} x$$

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{n-1} \end{bmatrix} = O x \longrightarrow x = O^{-1} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{n-1} \end{bmatrix}$$

Main issue is noise with higher-order derivatives

(Final exam) Method 4:

Describing the system in more detail



l is $n \times 1$
(column vector)

Estimate the output and compare it to y, use error output to correct as you go

Goal: Derive equation for $x - \hat{x} = e$ $\quad e \in \mathbb{R}^n$

$\dot{e} = \dot{x} - \dot{\hat{x}}$

$\dot{x} = Ax + Bu$

$\dot{\hat{x}} = A\hat{x} + Bu + l(\underbrace{y - \hat{y}}_{lcx - lc\hat{x}})$

$\dot{e} = Ax + Bu - (A\hat{x} + Bu + lcx - lc\hat{x})$

$= A(x - \hat{x}) - lc(x - \hat{x})$

$= (A - lc)(x - \hat{x})$

$\boxed{\dot{e} = (A - lc)e}$

Design goal: Find l such that $\hat{x}$ will converge to x for any $e(0)$

Equivalently, find l such that $e(t) \to 0$ for any $e(0)$

$\Rightarrow$ $A - lc$ is asymptotically stable

Last lecture: $\dot{x} = (A - BK)x \longrightarrow (A - BK)$ is asymptotically stable

Very similar problem

$A' - l'c'$ is the same problem