Acrobot算法推导与webots仿真探索

前置推导

杆1的质心： $x_{g1}=l_{c1}cos(q_1),y_{g_1}=l_{c1}sin(q_1)$

杆2的质心： $x_{g2}=l_1cos(q_1)+l_{c2}cos(q_1+q_2),$
$y_{g2}=l_1sin(q_1)+l_{c2}sin(q_1+q_2)$

杆1的动能： $T_1=\frac{1}{2}m_1(\dot{x_{g1}}^2+\dot{y_{g1}}^2)=\frac{1}{2}m_1l_{c1}^2\dot{q_1}^2$

杆2的动能：

$$T_2=\frac{1}{2}m_2(\dot{x_{g2}}^2+\dot{y_{g2}}^2)$$

$$\text{直接用}matlab\text{计算}$$

$$=\frac{1}{2}m_2[l_1^2\dot{q}_1^2+l_{c2}^2(\dot{q_1}+\dot{q_2})^2+2l_1l_{c2}\dot{q}_1(\dot{q_1}+\dot{q_2})\cos(q_2)]$$

杆1的势能： $V_1=m_1gl_{c1}sin(q_1)$

杆2的势能： $V_2=m_2g[l_1sin(q_1)+l_{c2}sin(q_1+q_2)]$

拉格朗日方程

参考..\data_analyse\formula.m

$$\boldsymbol{L=T_1+T_2-V_1-V_2}$$

$$=\frac{1}{2}m_1l_{c1}^2\dot{q_1}^2+\frac{1}{2}m_2[l_1^2\dot{q}_1^2+l_{c2}^2(\dot{q_1}+\dot{q_2})^2+2l_1l_{c2}\dot{q}_1(\dot{q_1}+\dot{q_2})\cos(q_2)]$$
$$-m_1gl_{c1}sin(q_1)-m_2g[l_1sin(q_1)+l_{c2}sin(q_1+q_2)]$$

$$\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}}-\frac{\partial L}{\partial q_1}}=[m_1l_{c1}^2+m_2l_1^2+m_2l_{c2}^2+2m_2l_1l_{c2}\cos(q_2)]\ddot{q}_1$$
$$+(m_2l_{c2}^2+m_2l_1l_{c2}\cos(q_2))\ddot{q}_2-2m_2l_1l_{c2}\sin(q_2)\dot{q}_1\dot{q}_2$$
$$-m_2l_1l_{c2}\sin(q_2)\dot{q}_2^2+m_2gl_{c2}\cos(q_1+q_2)+(m_1l_{c1}+m_2l_1)g\cos(q_1)=0$$

$$\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \dot{q_2}}-\frac{\partial L}{\partial q_2}}=[m_2l_{c2}^2+m_2l_1l_{c2}\cos(q_2)]\ddot{q}_1+m_2l_{c2}^2\ddot{q}_2$$
$$+m_2l_1l_{c2}\sin(q_2)\dot{q}_1^2+m_2gl_{c2}\cos(q_1+q_2)=\tau$$

一般动力学方程

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}=\boldsymbol{\tau}_g(\boldsymbol{q})+\boldsymbol{B}(\boldsymbol{q})\boldsymbol{u}$$

其中 $\boldsymbol{q}=[q_1,q_2]^T,\boldsymbol{u}=\tau$ ，且由拉格朗日方程可得：

$$\boldsymbol{M}(\boldsymbol{q})=\begin{bmatrix} m_1l_{c1}^2+m_2l_1^2+m_2l_{c2}^2+2m_2l_1l_{c2}\cos(q_2) & m_2l_{c2}^2+m_2l_1l_{c2}\cos(q_2) \\ m_2l_{c2}^2+m_2l_1l_{c2}\cos(q_2) & m_2l_{c2}^2 \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix} -2m_2l_1l_{c2}\sin(q_2)\dot{q}_2 & -m_2l_1l_{c2}\sin(q_2)\dot{q}_2 \\ m_2l_1l_{c2}\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}$$

$$\boldsymbol{\tau}_g(\boldsymbol{q})=\begin{bmatrix} -m_2gl_{c2}\cos(q_1+q_2)-(m_1l_{c1}+m_2l_1)g\cos(q_1) \\ -m_2gl_{c2}\cos(q_1+q_2) \end{bmatrix}$$

$$\boldsymbol{B}(\boldsymbol{q})=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

论文中的相关公式推导

——based on 《Angular momentum based balance controller for an under-actuated planar robot》

复习一下**角动量**的计算公式：**角动量是位矢和线动量的叉积**

$$L=r\times mv=I\omega$$

理论上来说，论文中的形式是通过 $I\omega$ 计算得到的，但是在计算杆2相对于原点的惯量比较复杂，所以采取从**线动量**开始推导的方式：

杆1质心的线动量为：

$$\begin{aligned} p_{x1}&=m_1\dot{x_{g1}}=-m_1l_{c1}\sin(q_1)\dot{q}_1 \\ p_{y1}&=m_1\dot{y_{g1}}=m_1l_{c1}\cos(q_1)\dot{q}_1 \end{aligned}$$

$$\mathbf{p}_1=\begin{pmatrix} -m_1l_{c1}\sin(q_1)\dot{q}_1 \\ m_1l_{c1}\cos(q_1)\dot{q}_1 \end{pmatrix}$$

杆2质心的线动量为：

$$\begin{aligned} p_{x2}&=-m_2\dot{x_{g2}}=-m_2\left[l_1\sin(q_1)\dot{q}_1+l_{c2}\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)\right] \\ p_{y2}&=m_2\dot{y_{g2}}=m_2\left[l_1\cos(q_1)\dot{q}_1+l_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)\right] \end{aligned}$$

$$\mathbf{p}_2=\begin{pmatrix} -m_2\left(l_1\sin(q_1)\dot{q}_1+l_{c2}\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)\right) \\ m_2\left(l_1\cos(q_1)\dot{q}_1+l_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)\right) \end{pmatrix}$$

杆1质心相对于原点的矢经：

$$\mathbf{r}_1 = \begin{pmatrix} -l_{c1}\sin(q_1) \\ l_{c1}\cos(q_1) \end{pmatrix}$$

杆2质心相对于原点的矢径：

$$\mathbf{r}_2 = \begin{pmatrix} -l_1\sin(q_1) - l_{c2}\sin(q_1+q_2) \\ l_1\cos(q_1) + l_{c2}\cos(q_1+q_2) \end{pmatrix}$$

经过叉乘计算之后：

杆1的角动量 $\boldsymbol{L}_1 = \boldsymbol{r}_1 \times \boldsymbol{p}_1 = m_1 l_1^2 \dot{q}_1 \boldsymbol{k}$

杆2的角动量

$\boldsymbol{L}_2 = \boldsymbol{r_2} \times \mathbf{p2} = [[m_2 l_1^2 + m_2 l_{c2}^2 + 2m_2 l_1 l_{c2} cos(q_2)]\dot{q}1 + [m_2 l_{c2}^2 + m_2 l_1 l_{c2} cos(q2)]\dot{q}2]\boldsymbol{k}$

$L = L_1 + L_2 = [m_1 l_1^2 + m_2 l_1^2 + m_2 l_{c2}^2 + 2m_2 l_1 l_{c2} cos(q_2)]\dot{q}_1 + [m_2 l_{c2}^2 + m_2 l_1 l_{c2}]\dot{q}_2$

（和代码里的H11和H12对上了）

计算一下**机器人整体质心在x轴方向的位置**：

$$X_g = \frac{1}{m_1 + m_2}(x_{g1} + x_{g2}) = (l_{c1} + l_1)cos(q_1) + l_{c2}cos(q_1 + q_2)$$

$$\dot{L} = -(c_4 g cos(q_1) + c_5 g(cos(q_1 + q_2))) = -cgX$$

$$\dot{L} = -((m_1 l_{c1}) g cos(q_1) + m_2 l_{c2} g(cos(q_1 + q_2)))$$

根据角动量定理可知，质点在某一过程中角动量的变化量等于质点在这个过程中所受的冲量矩：

$\frac{dL}{dt} = \dot{L} = M = -(m1 + m2)gX_g$ (顺负逆正)

因此继续推导： $\ddot{L} = -(m_1 + m_2)g\dot{X}_g$

满足平衡的条件为： $L = \dot{L} = \ddot{L} = 0$

将角动量视为输出函数，并定义一个反馈控制器： $\tau = k_{dd}\ddot{L} + k_d\dot{L} + k_p L$

加入重力项（类似前馈？）使其能够在任何不稳定平衡配置以及垂直配置中都能稳定机器人：

$\tau_d = m_2 g l_{c2} cos(q_1 + q_2)$

$$\tau = k_{dd}\ddot{L} + k_d\dot{L} + k_p L + \tau_d$$

如何根据模型求出 k_{dd}, k_d, k_p 就是此论文最难的地方

令 $\boldsymbol{x}=[q_1-q_1^d,q_2-q_2^d,\dot{q}_1,\dot{q}_2]^T$ ，则

$$\dot{\boldsymbol{x}}=\begin{bmatrix}\dot{\boldsymbol{q}}\\ \ddot{\boldsymbol{q}}\end{bmatrix}=\begin{bmatrix}\dot{\boldsymbol{q}}\\ \boldsymbol{M^{-1}(q)[\tau_g(q)+B(q)u-C(q,\dot{q})\dot{q}]}\end{bmatrix}=f(\boldsymbol{x},u)$$

线性化之后得到：

$$\dot{\boldsymbol{x}}=\boldsymbol{A}_{lin}\boldsymbol{x}+\boldsymbol{B}_{lin}u$$

求解 $\boldsymbol{A_{lin}},\boldsymbol{B_{lin}}$ 矩阵

我们可以看到，系统的非线性函数可以分为两个部分：状态方程 $\dot{\boldsymbol{q}}$ 和动力学方程 $\ddot{\boldsymbol{q}}$ 。因此，雅可比矩阵 $\boldsymbol{A}_{lin}$ 将具有如下形式：

$$\boldsymbol{A}_{lin}=\left.\frac{\partial f(\boldsymbol{x},\boldsymbol{u})}{\partial\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0,\boldsymbol{u}=\boldsymbol{u}_0}=\begin{bmatrix}\frac{\partial\dot{\boldsymbol{q}}}{\partial\boldsymbol{q}}&\frac{\partial\dot{\boldsymbol{q}}}{\partial\dot{\boldsymbol{q}}}\\ \frac{\partial\ddot{\boldsymbol{q}}}{\partial\boldsymbol{q}}&\frac{\partial\ddot{\boldsymbol{q}}}{\partial\dot{\boldsymbol{q}}}\end{bmatrix}$$

易知： $\dfrac{\partial\dot{\boldsymbol{q}}}{\partial\boldsymbol{q}}=0,\dfrac{\partial\dot{\boldsymbol{q}}}{\partial\dot{\boldsymbol{q}}}=\boldsymbol{I}$

因为 $\ddot{\boldsymbol{q}}=\boldsymbol{M}^{-1}(\boldsymbol{q})[\tau_g(\boldsymbol{q})+\boldsymbol{B}(\boldsymbol{q})u-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}]$

所以 $\dfrac{\partial\ddot{\boldsymbol{q}}}{\partial\boldsymbol{q}}=\boldsymbol{M}^{-1}\left(\dfrac{\partial\tau_g}{\partial\boldsymbol{q}}+\dfrac{\partial\boldsymbol{B}}{\partial\boldsymbol{q}}u-\dfrac{\partial\boldsymbol{C}}{\partial\boldsymbol{q}}\dot{\boldsymbol{q}}\right)$ ，第三项由易知可得为0，

则 $\boldsymbol{A}_{lin}=\begin{bmatrix}\boldsymbol{0}&\boldsymbol{I}\\ \boldsymbol{M}^{-1}\left(\frac{\partial\tau_g}{\partial\boldsymbol{q}}+\frac{\partial\boldsymbol{B}}{\partial\boldsymbol{q}}u\right)&\boldsymbol{0}\end{bmatrix}_{\boldsymbol{x}=\boldsymbol{0},\boldsymbol{u}=\boldsymbol{0}}$

$$\boldsymbol{B}_{lin}=\left.\frac{\partial f(\boldsymbol{x},\boldsymbol{u})}{\partial\boldsymbol{u}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0,\boldsymbol{u}=\boldsymbol{u}_0}=\begin{bmatrix}\frac{\partial\dot{\boldsymbol{q}}}{\partial u}\\ \frac{\partial\ddot{\boldsymbol{q}}}{\partial u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{M^{-1}(q)B(q)}\end{bmatrix}$$

由于 u 也可以视为

TODO这里后面的推导暂时还没弄懂，论文里是在附录直接给出的

这里直接给个论文里推系数的结论：

系统特征多项式为：

$$\alpha\lambda^4+(bk_{dd})\lambda^3+(bk_d-\beta)\lambda^2+(bk_p)\lambda+\gamma=0$$

$$\alpha = c_1 c_2 - c_3^2 \cos^2(q_2^d)$$
$$b = g\left(c_4 \sin(q_1^d)(c_2 + c_3\cos(q_2^d)) - c_5 \sin(q_1^d + q_2^d)(c_1 + c_3 \cos(q_2^d))\right)$$
$$\beta = g\left(c_1 c_5 \sin(q_1^d + q_2^d) + c_2 c_4 \sin(q_1^d)\right)$$
$$\gamma = c_4 c_5 g^2 \sin(q_1^d + q_2^d)\sin(q_1^d)$$

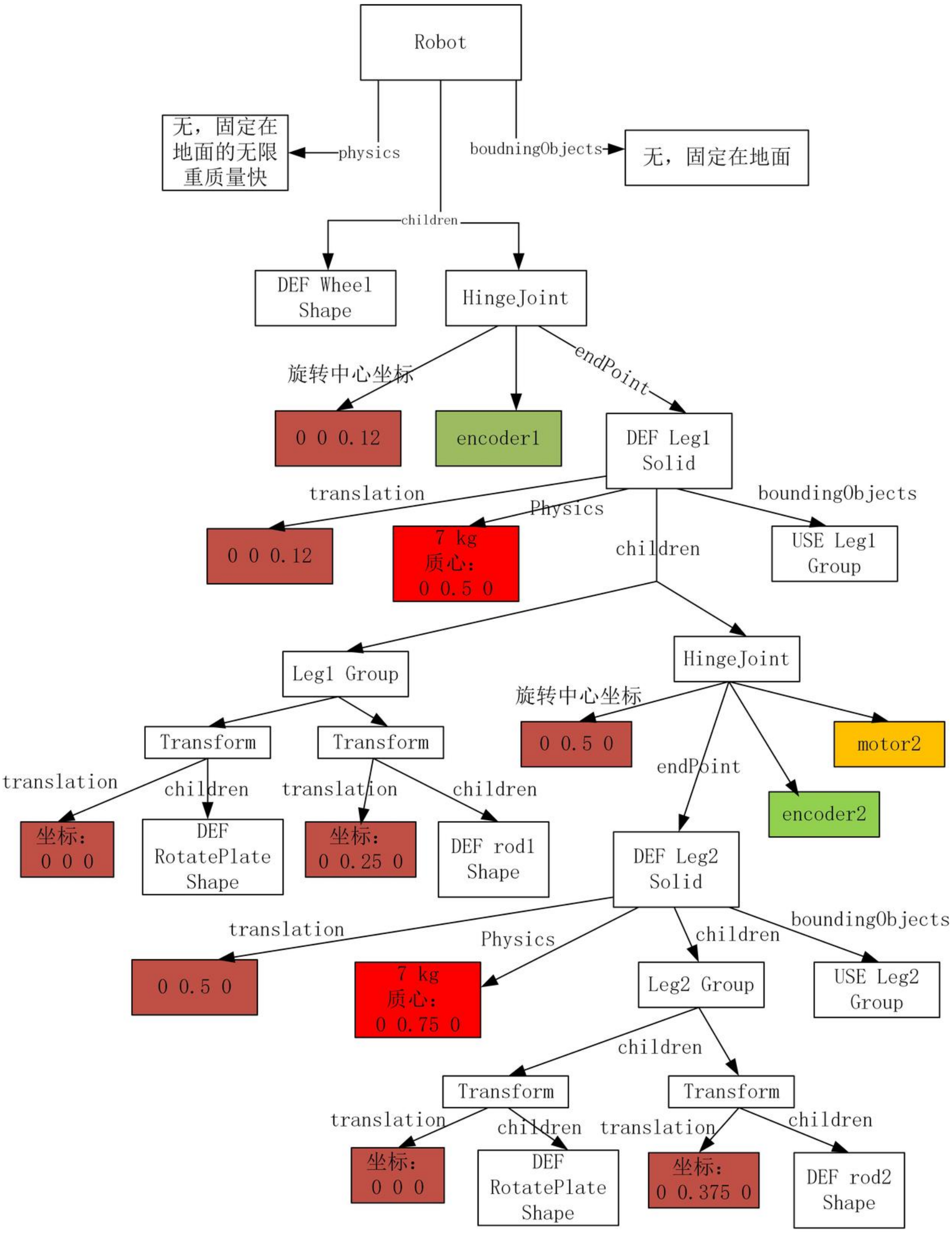
$$c_1 = m_1 l_{c1}^2 + m_2 l_1^2$$
$$c_2 = m_1 l_{c2}^2$$
$$c_3 = m_2 l_1 l_{c2}$$
$$c_4 = m_1 l_{c1} + m_2 l_1$$
$$c_5 = m_2 l_{c2}$$

调参系数为：

$$p = \sqrt[4]{\frac{\gamma}{\alpha}}$$
$$k_p = \frac{4\alpha p^3}{b}$$
$$k_d = \frac{6\alpha p^2 + \beta}{b}$$
$$k_{dd} = \frac{4\alpha p}{b}$$

仿真搭建

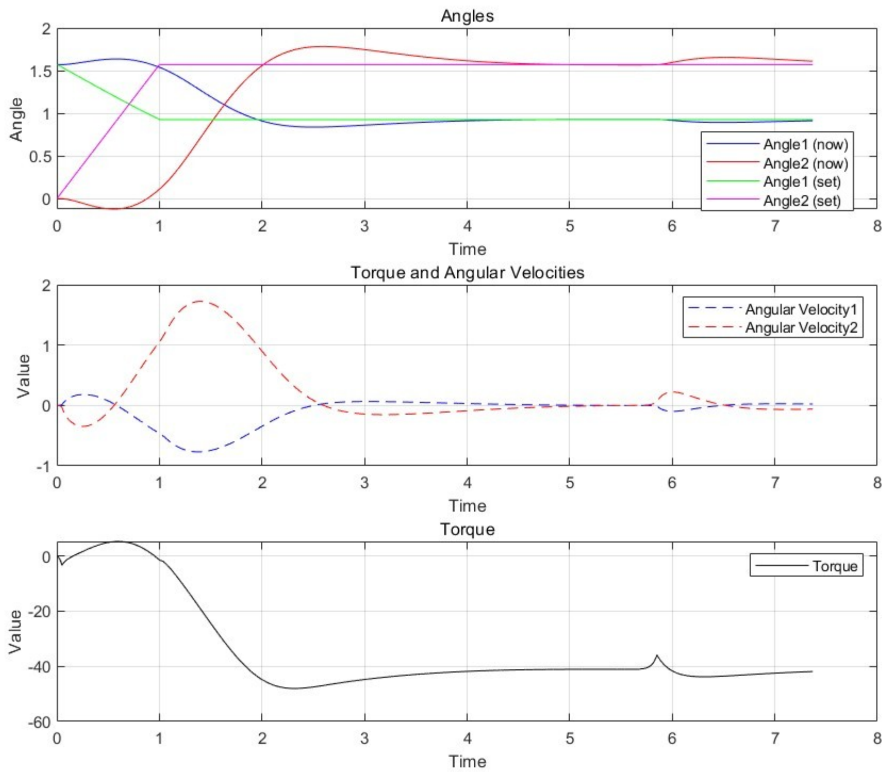
(图搞得有点复杂。。sry)



需要注意的是质心的坐标设置是“相对坐标”，即相对于该solid的原点的坐标（之前设置成“绝对坐标”即相对于世界系原点的坐标，导致一直仿真不对）；以及Transform节点下的坐标也是“相对坐标”，是相对于Transform的根solid节点的原点的坐标，巧用Transform可以让坐标系建立更加清晰。

最终效果：

acrobot_success.mp4
4.7MB


Angles
Angle
Time
Angle1 (now)
Angle2 (now)
Angle1 (set)
Angle2 (set)
Torque and Angular Velocities
Value
Time
Angular Velocity1
Angular Velocity2
Torque
Value
Time
Torque