1. $(1-4xy^2)dy - y^3 dx = 0$

Put in form $M dx + N dy = 0 \Rightarrow M = -y^3$
$N = 1-4xy^2$

a) Since M, N are continuously differentiable, $M_y = N_x$ iff the equation is exact. However,

$M_y = \frac{\partial}{\partial y}(-y^3) = -3y^2$ and $\boxed{-3y^2 \neq -4y^2}$

$N_x = \frac{\partial}{\partial x}(1-4xy^2) = -4y^2$ are not equal, so not exact.

b) We want a $\mu(x,y)$, so that $\mu M dx + \mu N dy$ is exact, i.e., such that $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \Rightarrow$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

Assume μ is a function of only y (not x), i.e., $\mu_x = 0$.

Then, $\mu_y M + \mu M_y = \mu N_x \Rightarrow \mu_y M + (M_y - N_x)\mu = 0$

$\Rightarrow \mu_y + \frac{M_y - N_x}{M}\mu = 0$

Since, $M = -y^3$, $M_y = -3y^2$, $N_x = -4y^2$, this is $\mu_y + \frac{-3y^2-(-4y^2)}{-y^3}\mu = 0$

Or $\frac{d\mu}{dy} - \frac{1}{y}\mu = 0$. Now, use integrating factors:
(since we already have μ, I'll call it ν)

$$\nu = e^{\int p\,dy} = e^{-\int \frac{1}{y}dy} = e^{-\ln|y|} = \frac{1}{y}$$

So, $\frac{d}{dy}\left(\frac{1}{y}\mu\right) = 0 \Rightarrow \frac{1}{y}\mu = C \Rightarrow \mu = yC$

μ is unique up to a constant. Any solution will work, so we may take $c=1$ for simplicity

$\therefore \boxed{\mu = y}$

c) Solve: $\overbrace{-y^4}^{M} dx + \overbrace{(y-4xy^3)}^{N} dy = 0$

$\psi_x = M = -y^4 \Rightarrow \psi = \int M dx = -y^4 x + g(y)$

$\psi_y = N = y - 4xy^3 = -4y^3x + g'(y) \Rightarrow g'(y) = y$

$\Rightarrow g(y) = \frac{y^2}{2} + C$

So, $\psi = \frac{y^2}{2} - y^4 x + c$ is constant, and $\boxed{\frac{y^2}{2} - y^4 x = C}$

2. $y'' - 2y' + 2y = e^t, \quad y(0) = 1, \quad y'(0) = 0$

a) $w'' - 2w' + 2w = \delta, \quad w(0) = 1, \quad w'(0) = 0$

Solve, keeping the "δ" part separate as a placeholder:

$$s^2 W - s - 2(sW - 1) + 2W = 1 \Rightarrow$$

Keep separate

$$(s^2 - 2s + 2) W = s - 2 + 1 \Rightarrow$$

Depends on initial conditions

$$W = \frac{(s-1) - 1}{s^2 - 2s + 2} + \frac{1}{s^2 - 2s + 2}$$

transfer function

$s^2 - 2s + 2 \to (s-1)^2 + 1$

$$\boxed{w = \underbrace{e^t \cos t - e^t \sin t}_{\text{initial conditions part}} + \underbrace{e^t \sin t}_{\text{weight function}}}$$

b) Duhamel's principle says that the solution is the sum of the initial conditions part with the convolution of the weight function and the forcing function, i.e.,

convolution operator

$$y = (e^t \cos t - e^t \sin t) + (e^t \sin t) * (e^t)$$

$$= e^t \cos t - e^t \sin t + \int_0^t \underbrace{e^{t-\tau} e^{\tau}}_{= e^t \text{, which is indep. of } \tau} \sin \tau \, d\tau$$

$$= e^t \cos t - e^t \sin t + \underbrace{e^t \int_0^t \sin \tau \, d\tau}_{= e^t \left[-\cos \tau \right]_{\tau=0}^{t} = -e^t \cos t + e^t}$$

$$= \boxed{e^t - e^t \sin t}$$

2. $y'' - 2y' + 2y = e^t$, $y(0) = 1$, $y'(0) = 0$

(Alternate Method)

Since we are not using Duhamel's principle in this class, just solve using the Laplace transform:

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^t\} \Rightarrow$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s-1} \Rightarrow$$

$$s^2 Y - s - 2(sY - 1) + 2Y = \frac{1}{s-1} \Rightarrow$$

$$(s^2 - 2s + 2)Y - s + 2 = \frac{1}{s-1} \Rightarrow$$

Partial Fractions

$$\frac{A}{s-1} + \frac{Bs+C}{s^2-2s+2} = \frac{1}{(s-1)(s^2-2s+2)}$$

$$A(s^2-2s+2) + Bs(s-1) + C(s-1) = 1$$

$s=1 \Rightarrow A = 1$

$s=0 \Rightarrow 2A - C = 1 \Rightarrow C = 1$

$s=2 \Rightarrow 2A + 2B + C = 1 \Rightarrow B = -1$

$$Y = \frac{s - 2 + \frac{1}{s-1}}{s^2 - 2s + 2}$$

$$= \frac{s-2}{s^2-2s+2} + \frac{1}{(s-1)(s^2-2s+2)}$$

$$= \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1} + \frac{1}{s-1} + \frac{1-s}{(s-1)^2+1}$$

These cancel

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2+1}$$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \boxed{e^t - e^t \sin(t)}$$

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3. $y^{(4)} - 2y''' + 2y'' - 2y' + y = t$

a) The characteristic polynomial is

$r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$. Since $\cos(t)$ is a solution, we know that i is a root of the polynomial. Because the coeffients are real, complex roots must appear as conjugate pairs, so $-i$ is a root, and $(x-i)(x+i) = x^2+1$ divides the poly.
(x = r are same here)

Dividing:

$$
\begin{array}{r l}
 & x^2 - 2x + 1 \\
x^2+1 \,\big) & x^4 - 2x^3 + 2x^2 - 2x + 1 \\
 & x^4 + 0 + x^2 \\
\hline
 & 0 - 2x^3 + x^2 \\
 & -2x^3 + 0 - 2x \\
\hline
 & 0 + x^2 + 0 \\
 & x^2 \quad + 1 \\
\hline
 & 0 \qquad 0
\end{array}
$$

Now, $x^2 - 2x + 1 = (x-1)^2$

So...

$r^4 - 2r^3 + 2r^2 - 2r + 1 = 0 \Rightarrow r = \pm i, 1$ multiplicity two, so that

$$\boxed{y_h = c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 t e^t}$$

b) $y_p = At + B$

$y_p' = A$

$y_p'' = y_p''' = y_p^{(4)} = 0$

Plug in: $0 - 2(0) + 2(0) - 2A + At + B = t$

$\Rightarrow A = 1$

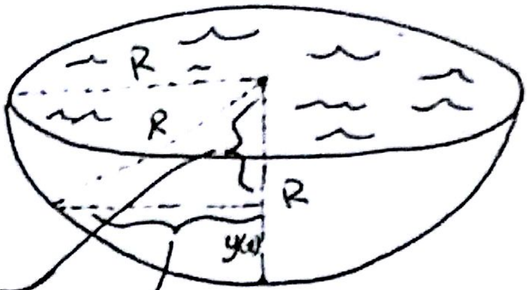
and $B - 2A = 0 \Rightarrow B = 2$

So, $y_p = t + 2$

and the general solution is

$$\boxed{y = c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 t e^t + t + 2}$$

4.

a) 

* Note: The lettering is off on the sample exam.

$R-y$, $\sqrt{R^2-(R-y)}$

b) $A(h) = \pi(\quad)^2 = \pi\left(R^2-(R^2-2Ry+y^2)\right)$

$= \pi(2Ry-y^2)$ $\quad y=h$

$\boxed{= \pi(2Rh-h^2)}$

c) $\dfrac{dV}{dt} = \dfrac{d}{dt}\displaystyle\int_0^{y(t)} A(h)\,dh = A(y)y' = \boxed{\pi(2Ry-y^2)\dfrac{dy}{dt}}$

* d) $-\alpha R^{5/2}\sqrt{y} = V' = \pi(2Ry-y^2)\dfrac{dy}{dt} \Rightarrow$

$\boxed{\left(2\pi R\sqrt{y} - \pi y^{3/2}\right)\dfrac{dy}{dt} = -\alpha R^{5/2}}$ Separate Variables

$\left(\pi y^{3/2} - 2\pi R y^{1/2}\right)dy = \alpha R^{5/2}dt \Rightarrow$ Use $y(0)=R$ to find C.

$\frac{2}{5}\pi y^{5/2} - \frac{4}{3}\pi R y^{3/2} = \alpha R^{5/2}t + C$

$\frac{2}{5}\pi R^{5/2} - \frac{4}{3}\pi R^{5/2} = \alpha R^{5/2}\cdot 0 + C \Rightarrow C = \left(\frac{2}{5}-\frac{4}{3}\right)\pi R^{5/2}$

$= -\frac{14}{15}\pi R^{5/2}$

* e) $y(1) = \frac{1}{4}R \Rightarrow \frac{2}{5}(\pi)\left(\frac{R}{4}\right)^{5/2} - \frac{4}{3}\pi\left(\frac{R}{4}\right)^{3/2}\cdot R = \alpha R^{5/2} - \frac{14}{15}\pi R^{5/2}$

$\Rightarrow \boxed{\alpha = \frac{2}{5}\pi\left(\frac{1}{4}\right)^{5/2} - \frac{4}{3}\pi\left(\frac{1}{4}\right)^{3/2} + \frac{14}{15}\pi}$

* f) The tank is empty when $y(t)=0$, or

$0 = \alpha R^{5/2}t - \frac{14}{15}\pi R^{5/2} \Rightarrow \boxed{t = \dfrac{14\pi}{15\alpha}}$

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5. $(1-x^2)y'' - 2xy' + 2y = x, \quad y(0) = y'(0) = 0$

a) $y = \sum_{n=0}^{\infty} c_n x^n$

First, differentiate term-by-term. Then, plug into equation.

$y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$

$y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$

Plugging in:

$$(1-x^2)\sum_{n=0}^{\infty} c_n (n-1) n x^{n-2} - 2x\sum_{n=0}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = x \Rightarrow$$

$$\sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} c_n n(n-1) x^n - 2\sum_{n=0}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n x^n = x$$

note that first 2 terms are zero

Now, match powers of 'x' and then match starting indices.

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n - \sum_{n=0}^{\infty} c_n n(n-1) x^n - 2\sum_{n=0}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n x^n = x$$

If $n=0$, we have $c_2 \cdot 2 \cdot 1 - 0 - 2 \cdot 0 + 2 \cdot c_0 = 2c_2 + 2c_0 = 0$

If $n=1$, we have $6 \cdot c_3 - 0 - 2 \cdot c_1 + 2 \cdot c_1 = 6c_3 = 1$ coeff. of 'x' on RHS

For $n \geq 2$, we have the recurrence relation

$$c_{n+2}(n+2)(n+1) - c_n(n)(n-1) - 2c_n(n) + 2c_n = 0 \Rightarrow$$

$$\boxed{c_{n+2} = \frac{c_n \cdot n(n-1) + 2n \cdot c_n - 2c_n}{(n+2)(n+1)} = \frac{c_n(n^2+n-2)}{(n+2)(n+1)} = \frac{c_n(n-1)}{(n+1)}}$$

b) Since $y(0)=0$, $\boxed{c_0 = 0}$, and, since $y'(0)=0$, $\boxed{c_1 = 0}$. Thus, $2(c_2 + c_0) = 0$ implies $\boxed{c_2 = 0}$, and, finally, $6c_3 = 1 \Rightarrow \boxed{c_3 = 1/6}$.

c) $c_0 = 0$ and $c_{n+2} = c_n \frac{(n-1)}{(n+1)}$ means all even indices are zero, i.e., $\boxed{c_{2n} = 0}$

d) $c_3 = \frac{1}{6}, c_5 = \frac{1}{12}, c_7 = \frac{1}{18}, \ldots$ the pattern is $\boxed{c_{2n+1} = \frac{1}{3} \cdot \frac{1}{2n} = \frac{1}{6n}}$

e) $\frac{1}{12}x^5 + \frac{1}{18}x^7 + \frac{1}{24}x^9 + \cdots = \sum_{n=2}^{\infty} \frac{1}{6n} x^{2n+1}$

take $x = 1/4$

$$\leq \sum_{n=2}^{\infty} \frac{1}{12} x^{2n+1} = \frac{1}{12} x^5 \frac{1}{1-x^2} = \frac{1}{12}\left(\frac{1}{4}\right)^5 \frac{16}{15}$$

$$= \frac{1}{12}\left(\frac{1}{4}\right)^3 \frac{1}{15}$$

$$= \frac{1}{11520}$$

With $\boxed{\text{one term}}$,

$\frac{1}{6}x^3 = \frac{1}{6}\left(\frac{1}{4}\right)^3 = \frac{1}{384} \approx 0.0026$ has error less than $\frac{1}{11520}$

So it is correct to 3 decimal places.

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6. Suppose $f(t)$ grows at most exponentially.
That means there are constants A, B s.t. $|f(t)| \leq Ae^{Bt}$.

Then,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \leq \int_0^\infty |e^{-st} f(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} \cdot Ae^{Bt}\, dt$$

$$= \mathcal{L}\{Ae^{Bt}\}$$

$$= \frac{A}{s-B}$$

However,

$$\frac{s^2}{s^2-1} \leq \frac{A}{s-B}$$ is false for s sufficiently large

since
$$\lim_{s \to \infty} \frac{s^2}{s^2-1} = 1 > 0 = \lim_{s \to \infty} \frac{A}{s-B}$$

Therefore, there can be no such function.

$\boxed{\text{No.}}$