§ 3.6 # 17. $x^2 y'' - 3xy' + 4y = x^2 \ln x$ $\quad x > 0,$

$y_1(x) = x^2$

$y_2(x) = x^2 \ln x$

In order to start with the system,

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

We must have y'' alone because this system was derived using the assumption that $y'' + p(t)y' + q(t)y = g(t)$.

Thus, we solve

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x$$

using Cramer's rule on the system.

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & x^2 \ln x \\ \ln x & 2x\ln x + x \end{vmatrix}}{\begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix}} = \frac{-x^2 (\ln x)^2}{x^3} = -\frac{(\ln x)^2}{x}$$

so that

$$u_1 = \int -\frac{(\ln x)^2}{x} dx = \int -u^2 du = \frac{-u^3}{3} + C_1 = -\frac{(\ln x)^3}{3} + C_1$$

where $u = \ln x$.

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & \ln x \end{vmatrix}}{\begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix}} = \frac{x^2 \ln x}{x^3} = \frac{\ln x}{x}$$

so that

$$u_2 = \int \frac{\ln x}{x} dx = \int u\,du = \frac{u^2}{2} + C_2 = \frac{(\ln x)^2}{2} + C_2$$

where $u = \ln x$ again. Since $y = u_1 y_1 + u_2 y_2$, we have

$$y = x^2\left[-\frac{(\ln x)^3}{3} + C_1\right] + x^2 \ln x\left[\frac{(\ln x)^2}{2} + C_2\right]$$

$$= \boxed{\frac{1}{6} x^2 (\ln x)^3} + \boxed{C_1 x^2 + C_2 x^2 \ln x}$$

homogeneous part (solution)

particular solution