Homework 2.6 (Selected Problem)
MAP 2302 – Ordinary Differential Equations I
February 1, 2016
§2.6 # 26

2.6.26. Solve the DE:

$$y' = e^{2x} + y - 1$$

We suspect that the equation is exact, so we put it in the form $Mdx + Ndy = 0$ by subtracting y' and multiplying both sides of the equation by dx.

$$dx\left[0 = e^{2x} + y - 1 - y'\right] \implies 0 = (e^{2x} + y - 1)dx - dy$$

Here, we have $M = e^{2x} + y - 1$ and $N = -1$. If $M_y = N_x$, then the equation is exact. (The subscript here denotes differentiation with respect to the given variable).

$$M_y = 1 \neq 0 = N_x$$

Unfortunately, the equation is not exact. However, we notice that $(M_y - N_x)/N = (1-0)/(-1) = -1$ is a function that does not depend on y. Thus, we try to find an integrating factor $\mu(x)$ by solving the equation

$$\begin{aligned} \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = -\mu &\implies \frac{d\mu}{\mu} = -dx \\ &\implies \ln|\mu| = -x + C \\ &\implies \mu = Ce^{-x} \end{aligned}$$

Since we only need one such μ, we may pick $C = 1$ for convenience, so that $\mu(x) = e^{-x}$. Multiplying by our original equation, we get

$$e^{-x}\left[0 = (e^{2x} + y - 1)dx - dy\right] \implies 0 = (e^{x} + ye^{-x} - e^{-x})dx - e^{-x}dy$$

Our new functions are $\tilde{M} = e^x + ye^{-x} - e^{-x}$ and $\tilde{N} = -e^{-x}$. Now, if we check the partials, we find

$$\tilde{M}_y = e^{-x} = \tilde{N}_x$$

so that our integrating factor worked, and our new equation is exact. We can now proceed with the solution as we would with any other exact equation. Integrate $\tilde{N}$ with respect to y. (Note that we could have integrated $\tilde{M}$ with respect to x, but $\tilde{N}$ seemed simpler).

$$\int \tilde{N}\, dy = \int -e^{-x}\, dy = -e^{-x}y + g(x) \tag{1}$$

Since we're dealing with multivariate functions, our "constant" of integration is actually an arbitrary function of x. Remember, the partial derivative treats other variables like constants. Now, take the derivative with respect to the other variable (in this case, x).

$$\frac{\partial}{\partial x}(-e^{-x}y + g(x)) = e^{-x}y + g'(x)$$

This should be equal to $\tilde{M}$, so, by setting them equal, we can solve for $g(x)$ (up to a constant).

$$\begin{aligned} e^{-x}y + g'(x) = e^x + ye^{-x} - e^{-x} &\implies g'(x) = e^x - e^{-x} \\ &\implies g(x) = \int e^x - e^{-x}\, dx \\ &\implies g(x) = e^x + e^{-x} + C \end{aligned}$$

Now, we can plug $g(x)$ into equation (1) and set this all equal to a constant.

$$-e^{-x}y + e^x + e^{-x} = C \implies ye^{-x} = e^x + e^{-x} + C \implies y = e^{2x} + 1 + Ce^x$$

Now, let's check to ensure we're right. Compute the derivative.

$$y' = 2e^{2x} + Ce^x$$

Now, substitute into the very first equation.

$$\begin{aligned} y' \stackrel{?}{=} e^{2x} + y - 1 &\implies 2e^{2x} + Ce^x \stackrel{?}{=} e^{2x} + (e^{2x} + 1 + Ce^x) - 1 \\ &\implies 2e^{2x} + Ce^x \stackrel{\checkmark}{=} 2e^{2x} + Ce^x \end{aligned}$$

And now we're fairly certain we have the right answer.

$$\boxed{y = e^{2x} + 1 + Ce^x}$$