Test 1
MAC2302, Ordinary Differential Equations

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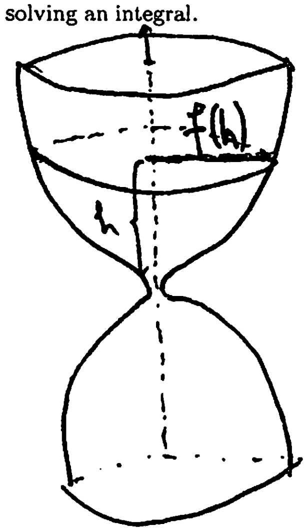
Student Name: Key

50 minutes. Closed book. NO calculator. Work without justification gets no credit. All problems are equally weighted. Start each problem on a new sheet, place them in order, put this cover page on top, and staple it.

1. Consider the Bernoulli equation $2y'(t) - 2y\cos(t) = -\cos(t)y^3$.
a) Make an appropriate substitution (v in terms of y) and reduce the nonlinear equation for y to a linear equation for v.
b) Find an integrating factor for the linear equation in v.
c) Solve the linear equation for v. Here you may want to use the method of substitution for solving an integral.
d) Find the general solution $y > 0$ for the original equation.

2. An hourglass (clepsydra) is a leaking tank filled with fine sand (which leaks into a bottom tank, which plays no role here). The tank is obtained by rotating a specific curve about the vertical axis, in such a way that the height h is **decreasing at a constant rate.** The (upper) tank is $1ft$ **tall and it empties in exactly one hour** ($3600sec$). Let $V(t)$ denote the volume of sand in the tank and by $h(t)$ the height at time t. Toricelli's law for fine sand leaking through a circular hole of radius r tells that the rate of change of the volume



$$\frac{dV(t)}{dt} = -\frac{1}{8}\pi r^2\sqrt{2gh(t)},$$

where $g = 32ft/sec^2$.
a) Given that it is constant, what should the rate dh/dt be (in ft/sec)?
b) Let $f(h)$ denote the radius of the tank for each height h. Given that the volume of sand at time t is $V(t) = \int_0^{h(t)} \pi f^2(s)ds$ find the differential equation governing the rate of change in height.
c) Given that the hole has a radius of $1/60ft$, find the formula for the radius $f(h)$ at each height h.
d) If at time $t = 0$ the upper tank is full, what is the volume of sand needed for your clepsydra?

3. Find all the solutions (including singular solutions, if any) of the differential equation

$$y'(x) = -\frac{4x + 3y}{2x + y}, \quad \text{for } x > 0,$$

by noting that the right hand side is a function of y/x.

4. Consider the differential equation

$$(-e^{-x}\sin y + 2e^{2y}\sin x)dy - (e^{-x}\cos y - e^{2y}\cos x)dx = 0.$$

a) Check that it is exact.
b) Find the general solution of the equation in the implicit form (a constant C should show).
c) Find the particular solution which also satisfies $y(0) = \frac{\pi}{2}$.

1. $2y' - 2y\cos t = -(\cos t)y^3$

a) $v = y^{1-n} = y^{-2}$, $v' = -2y^{-3}y'$, so multiply by $-y^{-3}$

$$-2y^{-3}y' + 2y^{-2}\cos t = \cos t \Rightarrow$$

and substitute

$$\boxed{v' + (2\cos t)v = \cos t}$$

b) $p = 2\cos t$, so $\mu = e^{\int p\,dt} = e^{\int 2\cos t\,dt} = e^{2\sin t}$

$$\boxed{\mu(t) = e^{+2\sin t}}$$

c) Quick check: $\mu'(t) = 2(\cos t)e^{2\sin t} \overset{\checkmark}{=} p(t)\mu(t)$

So, $\frac{d}{dt}(e^{2\sin t}v) = (\cos t)e^{2\sin t} \Rightarrow$

$$e^{2\sin t}v = \frac{1}{2}\int 2\cos t\, e^{2\sin t}\,dt = \frac{1}{2}e^{2\sin t} + C \Rightarrow$$

$$\boxed{v = \frac{1}{2} + Ce^{-2\sin t}}$$

d) Plug in $v = y^{-2}$

$$y^{-2} = \frac{1 + \underset{\text{new constant}}{C}e^{-2\sin t}}{2} \Rightarrow$$

$$\boxed{y = \pm\sqrt{\frac{2}{1 + Ce^{-2\sin t}}}}$$

$\boxed{\text{Also, } y \equiv 0}$

2. $$\frac{dV}{dt} = -\frac{1}{8}\pi r^2\sqrt{2gh(t)} = -\frac{1}{8}\pi r^2\sqrt{64h} = -\pi r^2\sqrt{h}$$

a) Since 1 foot drains in 3600 seconds at a constant rate,

$$\boxed{\frac{dh}{dt} = \frac{-1}{3600}}$$ (ft./sec.) [It's negative b/c decreasing].

b) $\frac{dV}{dt}$ is given one way by Toricelli's law. We can find another expression for $\frac{dV}{dt}$ by differentiating $V = \int_0^h \pi f^2(s)\,ds$. By the Fundamental Theorem of Calculus and the chain rule, $$\frac{dV}{dt} = \pi f^2(h)\frac{dh}{dt}.$$ Setting this equal to Toricelli's law gives

$$\pi f^2(h)\frac{dh}{dt} = -\pi r^2\sqrt{h} \Rightarrow$$

$$\boxed{f^2(h)\frac{dh}{dt} = -r^2\sqrt{h}}$$

c) We know $\frac{dh}{dt} = -\frac{1}{3600}$ and $r = \frac{1}{60}$, so, plugging in, we have

$$f^2(h)\left(-\frac{1}{3600}\right) = -\left(\frac{1}{3600}\right)\sqrt{h} \Rightarrow$$

$$f^2(h) = \sqrt{h} \Rightarrow \boxed{f(h) = \sqrt[4]{h}}$$

plug in

d) If it's full, then $h(0) = 1$, so that

$$V(t) = \int_0^1 \pi\left(\sqrt[4]{s}\right)^2 ds = \int_0^1 \pi\sqrt{s}\,ds$$

$$= \pi\left[\frac{2}{3}s^{3/2}\right]_0^1$$

$$= \boxed{\frac{2\pi}{3}}$$

3. $v = \frac{y}{x}$, $y' = v + xv'$ $\Rightarrow$

$$v + xv' = -\frac{4+3v}{2+v} \Rightarrow xv' = -\frac{4+3v}{2+v} - \frac{2v+v^2}{2+v}$$

$$= \frac{-4-5v-v^2}{2+v} \Rightarrow$$

$$-\frac{2+v}{4+5v+v^2}\,dv = \frac{1}{x}\,dx \Rightarrow$$

$$-\int \frac{2+v}{(v+4)(v+1)}\,dv = \ln|x| + C$$

$$\frac{2+v}{(v+4)(v+1)} = \frac{A}{v+4} + \frac{B}{v+1} \Rightarrow 2+v = A(v+1) + B(v+4)$$

$v=-1 \Rightarrow B = 1/3$

$v=-4 \Rightarrow A = 2/3$

So,

$$\ln|x| + C = -\int \frac{\frac{2}{3}}{v+4} + \frac{\frac{1}{3}}{v+1}\,dv$$

$$= -\frac{2}{3}\ln|v+4| - \frac{1}{3}\ln|v+1| \Rightarrow$$

new constant

$$\ln|x|^{-3} + C = \ln|v+4|^2 + \ln|v+1| \Rightarrow$$

new constant

$$C \cdot \frac{1}{|x|^3} = (v+4)^2 \cdot |v+1| \Rightarrow$$

$$\frac{C}{|x|^3} = \left(\frac{y}{x}+4\right)^2 \cdot \left|\frac{y}{x}+1\right| = \frac{(y+4x)^2\,|y+x|}{|x|^3} \Rightarrow$$

$$\boxed{(y+4x)^2 \cdot |y+x| = C}$$

Singular Solutions:
(From $v+4=0$, $v+1=0$)

$$\boxed{\begin{array}{l} y = -4x \\ y = -x \end{array}}$$

$$-\frac{4x+3(-4x)}{2x+(-4x)} = -\frac{-8x}{-2x} = -4 \checkmark \qquad -\frac{4x+3(-x)}{2x+(-x)} = -\frac{x}{x} = -1 \checkmark$$

4. $\underbrace{\left(-e^{-x}\sin y + 2e^{2y}\sin x\right)}_{N} dy - \underbrace{\left(e^{-x}\cos y - e^{2y}\cos x\right)}_{M} dx = 0$

a) $M_y \stackrel{?}{=} N_x$ $\quad M_y = 2e^{2y}\cos x + e^{-x}\sin y$

$N_x = e^{-x}\sin y + 2e^{2y}\cos x$

$M_y \stackrel{\checkmark}{=} N_x$

arbitrary fnct. of x

b) $\int 2e^{2y}\sin x - e^{-x}\sin y \, dy = e^{2y}\sin x + e^{-x}\cos y + g(x)$

take derivative

$\cancel{e^{2y}\cos x} - \cancel{e^{-x}\cos y} + g'(x) = \cancel{e^{2y}\cos x} - \cancel{e^{-x}\cos y} \Rightarrow$

$g'(x) = 0 \Rightarrow g(x) = C$, so

new constant

$$\boxed{e^{2y}\sin x + e^{-x}\cos y = C}$$

c) Plug in $y = \frac{\pi}{2}$, $x = 0$

$e^{2(\frac{\pi}{2})}\overset{0}{\cancel{\sin(0)}} + e^{-0}\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} = C \Rightarrow c = 0$

$$\boxed{e^{2y}\sin x + e^{-x}\cos y = 0}$$