Name: Key

MAP 2302 – Ordinary Differential Equations I

September 18, 2015

Quiz 2

1. A 100 gallon tank initially contains 50 gallons of salt water at a concentration of s_0 pounds of salt per gallon. The water is emptied from the tank at a rate of q gallons per minute. Salt water at a concentration of s_r pounds of salt per gallon enters the tank at a rate of r gallons per minute. The top of the tank is open, and, if filled beyond capacity, the excess will spill out.

   (a) Find an equation $v(t)$ for the volume of the salt water in the tank at time t that is valid before the tank becomes either full or empty.

   $$v(t) = (\underset{\text{rate in}}{r} - \underset{\text{rate out}}{q})\underset{\text{time}}{t} + \underbrace{50}_{\text{initial volume}}$$

   (b) Let $Q(t)$ denote the pounds of salt in the tank at time t. Write an initial value problem for $Q(t)$ that is valid before the tank becomes full or empty.

   $$\boxed{\frac{dQ}{dt} = \underset{\frac{\text{pounds}}{\text{gallon}}}{S_r} \cdot \underset{\frac{\text{gallons}}{\text{minute}}}{r} - \underbrace{\frac{\overset{\text{pounds}}{Q(t)}}{\underset{\text{gallons}}{v(t)}}}_{\text{concentration in tank}} \overset{\frac{\text{gallons}}{\text{minute}}}{q}}$$

   (c) If $r = 10$ and $q = 5$, at what time t will the tank become full?

   $$100 = v(t) = (10-5)t + 50 \Rightarrow 50 = 5t \Rightarrow \boxed{t = 10} \text{ (minutes)}$$

   (d) If $r = 10$ and $q = 5$, write a differential equation that $Q_f(t)$ satisfies *after* the tank becomes full. ~~(The subscript f denotes that the tank is full).~~

   $$\boxed{\frac{dQ_f}{dt} = S_r \cdot r - \frac{Q_f(t)}{V(t)} r}$$

   When tank spills over, rate in = rate out.

   (e) Without solving for $Q(t)$, what is the initial value for $Q_f(t)$?

   $Q(10)$ ← by continuity of solution

   (f) If $r = 10$ and $q = 5$, what is the concentration in the tank after a long amount of time?

   $$0 = \lim_{t\to\infty} \frac{dQ_f}{dt} = S_r \cdot r - \frac{\lim_{t\to\infty} Q_f(t)}{\lim_{t\to\infty} V(t)} r$$

   By linearity of limit

   $$\Rightarrow \frac{\lim_{t\to\infty} Q_f(t)}{\lim_{t\to\infty} V(t)} \cancel{r} = S_r \cdot \cancel{r}$$

   $$\Rightarrow \frac{\overset{\text{pounds}}{\lim_{t\to\infty} Q_f(t)}}{\underset{\text{Volume}}{100}} = \boxed{S_r}$$

   Note: $\lim_{t\to\infty} V(t) = 100$ but we don't need to split $\lim_{t\to\infty} \frac{Q_f(t)}{V(t)}$ to find concentration.