Solutions to Tests 1 and 2
MAP 2302 – Ordinary Differential Equations I
December 8, 2015

I.1. Find the solution of the IVP:

$$ty' + (t+1)y = t, \qquad y(\ln 2) = 2, \qquad t > 0$$

First, divide by t to put the equation in the standard form.

$$y' + \frac{t+1}{t}y = 1$$

Now find the integrating factor.

$$\mu(t) = e^{\int (t+1)/t\,dt} = e^{\int 1+1/t\,dt} = e^{t+\ln t} = te^t$$

Multiply the entire equation by $\mu(t)$ and then write the side with y as a single derivative.

$$te^t = te^t y' + (t+1)e^t y = \frac{d}{dt}\left(te^t y\right)$$

Now, integrate.

$$te^t y = \int te^t\,dt = te^t - \int e^t\,dt = (t-1)e^t + C$$

Plug in the initial condition to solve for the constant of integration.

$$(\ln 2)2 \cdot 2 = (\ln 2 - 1)2 + C \implies C = 2\ln 2 + 2$$

Finally, divide by $\mu(t)$ to solve for y.

$$\boxed{y = \frac{t-1}{t} + \frac{2\ln 2 + 2}{te^t}}$$

I.2 Classify and solve both

$$x\frac{dy}{dx} + y = \frac{1}{y^2}, \qquad x > 0$$

and

$$x\frac{dy}{dx} = y + 2xe^{y/x}, \qquad x > 0$$

The first equation is both Bernoulli and separable. It can be solved in two different ways; we will show both. For the first way, subtract y from both sides and combine fractions.

$$x\frac{dy}{dx} = \frac{1}{y^2} - y = \frac{1-y^3}{y^2}$$

Now, move all the x terms and all the y terms to different sides.

$$\frac{y^2}{1-y^3}dy = \frac{1}{x}dx$$

Integrate both sides. Notice that the left side is a u substitution with $u = 1 - y^3$.

$$-\frac{1}{3}\ln|1-y^3| = \ln x + C$$

Exponentiating both sides, we get

$$(1-y^3)^{-1/3} = C_1 x \implies 1 - y^3 = \frac{1}{C_2 x^3} \implies y = \sqrt[3]{1 - \frac{1}{C_2 x^3}}$$

For the second method, let $v = y^3$. Then, $v' = 3y^2 y'$. After multiplying by y^2, we can substitute.

$$xy^2y' + y^3 = 1 \implies \frac{x}{3}v' + v = 1 \implies v' + \frac{3}{x}v = \frac{3}{x}$$

The integrating factor is

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3$$

Multiplying and writing as a single derivative, we get

$$3x^2 = x^3v' + 3x^2v = \frac{d}{dx}(x^3v)$$

Then, integrate and divide by $\mu(x)$.

$$x^3v = \int 3x^2\,dx = x^3 + C \implies v = 1 + Cx^{-3}$$

Since $v = y^3$, take the cubed root to solve for y.

$$\boxed{y = \sqrt[3]{1 + \frac{C}{x^3}}}$$

Notice that this is the same answer as the first method by taking $C = -1/C_2$. The second equation is homogeneous. First, divide by x to get

$$\frac{dy}{dx} = \frac{y}{x} + 2e^{y/x}$$

Now, make the substitution $v = y/x$, using $y' = v + xv'$.

$$v + xv' = v + 2e^v \implies xv' = 2e^v$$

The resulting equation is separable, so move all the v terms and all the x terms to opposite sides.

$$x\frac{dv}{dx} = 2e^v \implies e^{-v}\frac{dv}{2} = \frac{dx}{x}$$

Now integrate.

$$-\frac{1}{2}e^{-v} = \ln x + C \implies e^{-v} = \ln x^{-2} + C_1 \implies -v = \ln\left(\ln x^{-2} + C_1\right)$$

Substitute $v = y/x$ and solve for y.

$$\boxed{y = x\ln(\ln x^{-2} + C_1)^{-1}}$$

3. Show the IVP is exact and solve:

$$2x + y^2 + 2xyy' = 0, \qquad y(2) = 10$$

First, multiply by dx to put the equation in the usual form for exact equations.

$$(2x + y^2)\,dx + 2xy\,dy = 0 \iff M\,dx + N\,dy = 0$$

To verify the equation is exact, differentiate M with respect to y and N with respect to x and show they are equal.

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$$

Either integrate M w.r.t. x or N w.r.t. y.

$$\int 2x + y^2\,dx = x^2 + y^2x + g(y)$$

Now, differentiate the above expression by the other variable – in this case y – and set equal to the other term – in this case N.

$$\frac{\partial}{\partial y}(x^2 + y^2x + g(y)) = 2xy + g'(y) = 2xy = N \implies g'(y) = 0$$

Thus, $g(y)$ is a constant. Therefore,

$$x^2 + y^2x = C$$

for some constant C, which we can find by substituting initial conditions.

$$C = 2^2 + 10^2 \cdot 2 = 4 + 200 = 204$$

The solution is then

$$\boxed{x^2 + y^2x = 204}$$

I.5. Short questions.

(a) Find all value r for which x^r is a solution of $x^2y'' - 2y = 0$.
Differentiation, we have

$$y' = rx^{r-1} \quad \text{and} \quad y'' = r(r-1)x^{r-2}$$

Substituting into the differential equation gives

$$(r(r-1) - 2)x^r = 0 \implies r^2 - r - 2 = 0 \implies (r+1)(r-2) = 0$$

Thus, $\boxed{r = -1, 2}$.

(b) Construct a first order differential equation such that all solutions are asymptotic to $y = t^2 - 1$.
Starting with a curve that's asymptotic to $t^2 - 1$, such as $y = t^2 - 1 + e^{-t}$, take its derivative $y' = 2t - e^{-t}$. Combine these to form a differential equation with the desired property.

$$\boxed{y + y' = t^2 + 2t - 1}$$

II.1. Find the solution of the IVP:

$$y'' + 2y' + 10y = 50t, \qquad y(0) = y'(0) = 0$$

First, solve the homogeneous problem with characteristic equation

$$r^2 + 2r + 10 = 0 \implies r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 10}}{2} = -1 \pm 3i$$

Thus, the complementary solution is

$$y = e^{-t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Now, the derived family is $\{t, 1\}$. Since this does not conflict with any homogeneous solution, there is no need for revision. Assume the particular solution has the form $y = At + B$. Then, $y' = A$ and $y'' = 0$. Substituting into the original equation, we have

$$2A + 10(At + B) = 50t \implies (2A + B) + 10At = 50t$$

Equating coefficients, $10A = 50 \implies A = 5$ and $2A + 10B = 0 \implies B = -1$. The general solution is therefore

$$y = y_c + y_p = e^{-t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right) + 5t - 1$$

Substitute the initial condition $y(0) = 0$ to solve for c_1.

$$0 = e^{-0}(c_1 \cos(3 \cdot 0) + c_2 \sin(3 \cdot 0)) + 5 \cdot 0 - 10 \implies 0 = c_1 - 1 \implies c_1 = 1$$

Differentiate and use $y'(0) = 0$ to solve for c_2.

$$0 = -e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + e^{-t}(-3c_1 \sin(3t) + 3c_2 \cos(3t)) + 5 \implies$$
$$0 = -1 + 3c_2 + 5 \implies c_2 = -4/3$$

So, the final solution is

$$\boxed{y = e^{-t}(\cos(3t) - \frac{4}{3}\sin(3t)) + 5t - 1}$$

II.2. Consider $y''' + 2y'' + y' = g(x)$.

(a) Find a fundamental set of solutions of the associated homogeneous equation.
The characteristic equation is

$$0 = r^3 + 2r^2 + r = r(r^2 + 2r + 1) = r(r+1)^2$$

so a fundamental set of solutions is

$$\boxed{\{1, e^{-x}, xe^{-x}\}}$$

(b) Determine the form of a particular solution, without solving, for each choice of $g(x)$.

(i) $g(x) = x^2 + 2$
The derived family is $\{x^2, x, 1\}$. Since 1 is part of the fundamental set of solutions, we must revise the derived family by multiplying each member by x to become $\{x^3, x^2, x\}$. Thus, the particular solution has the form

$$\boxed{y_p = Ax^3 + Bx^2 + C}$$

(ii) $g(x) = xe^{-x}$

The derived family is $\{xe^{-x}e^{-x}\}$. We must multiply by x^2 to avoid overlap with the fundamental set of solutions, so particular solution has the form

$$\boxed{y_p = Ax^3e^{-x} + x^2e^{-x}}$$

(iii) $g(x) = e^{2x} + \sin x$

Split this up at the plus sign into two problems. The derived family of e^{2x} is $\{e^{2x}\}$, and the derived family of $\sin x$ is $\{\sin x, \cos x\}$. Thus, the particular solution has the form

$$\boxed{y_p = Ae^{2x} + B\sin x + C\cos x}$$

II.3 Given $y_1 = e^{-t}$ and $y_2 = te^{-t}$, use variation of parameters to solve

$$y'' + 2y' + y = \frac{e^{-t}}{t^2}$$

(a) Write the system of equations for $u_1' and u_2'$.

The equations are

$$\begin{aligned} y_1u_1' + y_2u_2' = 0 &\iff e^{-t}u_1' + te^{-t}u_2' = 0 \\ y_1'u_1' + y_2'u_2' = g(t) &\iff -e^{-t}u_1' + (1-t)e^{-t}u_2' = t^{-2}e^{-t} \end{aligned}$$

(b) Solve the equations for u_1' and u_2'.

The Wronskian is

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = (1-t)e^{-2t} + te^{-2t} = e^{-2t}$$

Using Cramer's rule, we have

$$u_1' = \frac{\begin{vmatrix} 0 & te^{-t} \\ t^{-2}e^{-t} & (1-t)e^{-t} \end{vmatrix}}{W} = \frac{-t^{-1}e^{-2t}}{e^{-2t}} = -\frac{1}{t}$$

and

$$u_2' = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & t^{-2}e^{-t} \end{vmatrix}}{W} = \frac{t^{-2}e^{-2t}}{e^{-2t}} = \frac{1}{t^2}$$

(c) Find $y_p(t)$.

First, integrate to find u_1 and u_2. The constants of integration don't matter here because they are part of the homogeneous solution.

$$u_1 = \int u_1'\,dt = \int -\frac{1}{t}\,dt = -\ln|t|$$

and

$$u_2 = \int u_2'\,dt = \int \frac{1}{t^2}\,dt = -\frac{1}{t}$$

Therefore, a particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 = -\ln|t|e^{-t} - t^{-1}te^{-t} = -\ln|t|e^{-t} - e^{-t}$$

We can make this a little cleaner by removing any linear combination of homogeneous solutions.

$$\boxed{y_p(t) = -\ln|t|e^{-t}}$$

II.4 Complex roots.

(a) Find all complex roots of the algebraic equation $z^4 + 16 = 0$.
To solve $z^4 = -16$, first put -16 into polar form $-16 = 16e^{\pi i}$. Now, use the formula for $z^{1/n}$, where $z = re^{i\theta}$.

$$z^{1/n} = \sqrt[n]{r}e^{\theta i/n + 2\pi ki/n} \implies (-16)^{1/4} = \sqrt[4]{16}e^{\pi i/4 + \pi ki/2} = 2e^{\pi i/4 + \pi ki/2}$$

where $k = 0, 1, 2, 3$ (in general, $k = 0, 1, \ldots, n-1$). Using Euler's formula $e^{\theta i} = \cos\theta + i\sin\theta$, we find that

$$z = 2\left(\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right) = \boxed{\pm\sqrt{2} \pm i\sqrt{2}}$$

(b) Find a fundamental set of solutions to $y^{(4)}(t) + 16y(t) = 0$.
A fundamental set of solutions is

$$\boxed{\{e^{\sqrt{2}t}\cos\sqrt{2}t, e^{\sqrt{2}t}\sin\sqrt{2}t, e^{-\sqrt{2}t}\cos\sqrt{2}t, e^{-\sqrt{2}t}\sin\sqrt{2}t\}}$$

(c) Find by inspection a particular solution of $y^{(4)}(t) + 16y(t) = 10$.
The derived family is $\{1\}$, so we see a constant solution. If y is constant, then $y^{(4)} \equiv 0$, so we must have $16y = 10$. Thus, $y = 10/16$ or

$$\boxed{y = 5/8}$$

II.5. Other questions.

(a) Show that the set of function $f(t) = 2t - 3$, $g(t) = t^2 + 1$, $h(t) = 2t^2 - t$ is linearly independent on $(-\infty, \infty)$.
To show a collection of functions is linearly independent, show the Wronskian does not equal zero. That is,

$$\begin{aligned}
\begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix} &= (2t-3)\begin{vmatrix} 2t & 4t-1 \\ 2 & 4 \end{vmatrix} - (t^2+1)\begin{vmatrix} 2 & 4t-1 \\ 0 & 4 \end{vmatrix} + (2t^2-t)\begin{vmatrix} 2 & 2t \\ 0 & 2 \end{vmatrix} \\
&= (2t-3)(2) - (t^2+1)(8) + (2t^2-t)(4) \\
&= 4t - 6 - 8t^2 - 8 + 8t^2 - 4t \\
&= -14
\end{aligned}$$

$$\boxed{\text{Since the Wronskian is never zero, the functions are linearly independent.}}$$

Note that, if the question asks for linear *dependence*, **computing the Wronskian does not suffice**. You *must* write one function as a linear combination of the others.

(b) If $a(t) > 0$ is differentiable, use Abel's formula to determine the Wronskian of two (independent) solutions of

$$(a(t)y'(t))' + q(t)y(t) = 0$$

Expand the first term using the product rule to put the equation in standard form.

$$a'(t)y'(t) + a(t)y''(t) + q(t)y(t) = 0 \implies y''(t) + \frac{a'(t)}{a(t)} + \frac{q(t)}{a(t)}y(t) = 0$$

By Abel's formula, the Wronskian is

$$W = Ce^{-\int p(t)\,dt} = Ce^{-\int a'(t)/a(t)\,dt} = Ce^{-\ln a(t)}$$

so that the Wronskian is

$$\boxed{W = \frac{C}{a(t)}}$$