25. $\underbrace{(3x^2y + 2xy + y^3)}_{M} + \underbrace{(x^2 + y^2)}_{N} y' = 0$

§2.6
p.102

$M\,dx + N\,dy = 0$

$M_y = 3x^2 + 2x + 3y^2 \neq 2x = N_x \Rightarrow$ Not exact

Find μ s.t. $(\mu M)_y = (\mu N)_x \Leftrightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$

Assume $\mu_y = 0$, i.e. μ is a function of x and not y.

$\mu(M_y - N_x) = \mu_x N \Rightarrow 0 = \mu_x + \mu\left(\frac{N_x - M_y}{N}\right)$

$\Rightarrow 0 = \frac{d\mu}{dx} + \mu\left(\frac{2x - (3x^2 + 2x + 3y^2)}{x^2 + y^2}\right)$

$\Rightarrow 0 = \frac{d\mu}{dx} - 3\mu$

$3\mu = \frac{d\mu}{dx} \Rightarrow$

$3\,dx = \frac{d\mu}{\mu} \Rightarrow$

$3x + C = \ln|\mu| \Rightarrow$

$\mu = e^{3x}$

take $C=0$ for simplicity

Now, solve $\mu M\,dx + \mu N\,dy = 0$

or $(3x^2y + 2xy + y^3)e^{3x}dx + (x^2 + y^2)e^{3x}dy = 0$

Find Ψ s.t. $\Psi_x = M\mu$ and $\Psi_y = N\mu$.

$\Psi = \int \mu N\,dy = \int (x^2 + y^2)e^{3x}dy = (x^2y + \frac{y^3}{3})e^{3x} + g(x)$

$\Psi_x = M\mu = (\cancel{3x^2y} + \cancel{2xy} + \cancel{y^3})e^{3x} = \cancel{2xy}e^{3x} + \cancel{3x^2y}e^{3x} + \cancel{y^3}e^{3x} + g'(x)$

$\Rightarrow g'(x) = 0$

$\Rightarrow g(x) = C$

So, $\Psi = (x^2y + \frac{y^3}{3})e^{3x} + C$, and we have

$$\boxed{(x^2y + \frac{y^3}{3})e^{3x} = C}$$

Note: There is a typo in the solution in the book

28. $y + (2xy - e^{-2y})y' = 0, \quad M\,dx + N\,dy, \quad M = y$
§2.6 $\quad N = 2xy - e^{-2y}$
p.102 $\quad M_y = 1 \neq 2y = N_x \quad$ (Assume) Take $\mu_x = 0$.

$$\mu_y M + \mu(M_y - N_x) = 0 \Rightarrow$$

$$y\frac{d\mu}{dy} + \frac{1-2y}{1}\mu = 0 \Rightarrow \frac{d\mu}{dy} + \underbrace{\left(\frac{1}{y} - 2\right)}_{p(y)}\mu = 0$$

$$\nu = e^{\int p(t)\,dt} = e^{\int \frac{1}{y} - 2\,dy} = y e^{-2y}$$

$$\frac{d}{dy}\left(ye^{-2y}\mu\right) = 0 \Rightarrow$$

$$ye^{-2y}\mu = C \Rightarrow \mu = \frac{C}{y}e^{2y}$$

take $C = 1$

$$\mu = \frac{e^{2y}}{y}$$

So, $\underbrace{e^{2y}}_{M}dx + \underbrace{\left(2xe^{2y} - \frac{1}{y}\right)}_{N}dy = 0$ should be exact

$M_y = 2e^{2y}, \quad N_x = 2e^{2y} \Rightarrow$ exact!

$\Psi = \int M\,dx = xe^{2y} + g(y), \quad \Psi_y = \cancel{2xe^{2y}} + g'(y) = N = \cancel{2xe^{2y}} - \frac{1}{y} \Rightarrow g'(y) = -\frac{1}{y}$

$\Rightarrow g(y) = -\ln|y| + C$

So, $\Psi = xe^{2y} + C - \ln|y|$, and

$$\boxed{xe^{2y} - \ln|y| = C}$$

Note: $y=0$ is also a solution.