1. $(x^2+1)y' + 2xy = xe^{x^2}$, $y(0) = \frac{1}{2}$

Since $\frac{d}{dx}(x^2+1) = 2x$, this equation is already in the right form:

$$(x^2+1)y' + 2xy = \frac{d}{dx}\left[(x^2+1)y\right] = xe^{x^2} \Rightarrow$$

$$(x^2+1)y = \int xe^{x^2}dx = \frac{e^{x^2}}{2} + C \Rightarrow$$

$$y = \frac{e^{x^2}}{2(x^2+1)} + \frac{C}{x^2+1}$$ Solve for C:

$$\frac{1}{2} = \frac{e^{(0)^2}}{2((0)^2+1)} + \frac{C}{x^2+1} = \frac{1}{2} + C \Rightarrow C = 0,$$

so $$\boxed{y = \frac{e^{x^2}}{2(x^2+1)}}$$

Find integrating factor mechanically:

Put in standard form:

$$y' + \underbrace{\frac{2x}{x^2+1}}_{p(x)} y = \underbrace{\frac{xe^{x^2}}{x^2+1}}_{g(x)}$$

Integrating factor

$$\mu = e^{\int p(x)dx} = e^{\int \frac{2x}{x^2+1}dx} = e^{\ln(x^2+1)} = x^2+1$$

Remember, this is derived from $\frac{2x}{x^2+1}\mu = \mu'$,

Since this condition means that

$$\mu y' + \mu\frac{2x}{x^2+1}y = \frac{xe^{x^2}}{x^2+1}\mu$$

is in the right form.

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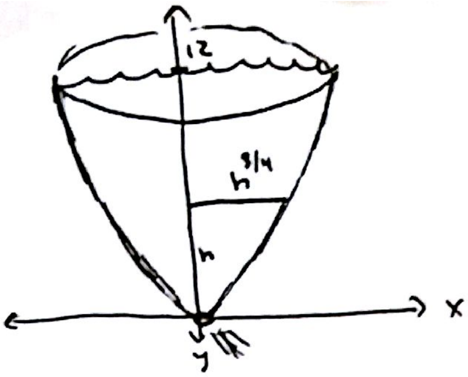
2. (a)



(b) $V(t) = \int_0^{y(t)} A(s)\,ds$ By chain rule and fundamental thm of calculus,

$\frac{dV}{dt} = A(y(t))\,y'(t)$, where $A(h) = \pi\left(h^{3/4}\right)^2 = \pi h^{3/2}$

So, $\boxed{\frac{dV}{dt} = \pi y^{3/2} \frac{dy}{dt}}$

(c) $V' = -a\sqrt{2gy} = -a\sqrt{2(32)y} = -8a\sqrt{y}$

But also

$V' = \pi y^{3/2} y'$, so, setting them equal, or $-a\sqrt{2g} = \pi y y'$

$-8a\sqrt{y} = \pi y^{3/2} y' \Rightarrow \boxed{-8a = \pi y y'}$

Now, separate variables:

$-8a\,dt = \pi y\,dy \Rightarrow \frac{\pi y^2}{2} = -8at + C$

(d) $y(0) = 12$ from initial condition $\Rightarrow \frac{\pi(12)^2}{2} = -8a(0) + C \Rightarrow C = 72\pi$

$y(1) = 6$ given $\Rightarrow \frac{\pi(6)^2}{2} = -8a(1) + 72\pi \Rightarrow a = \frac{18\pi - 72\pi}{-8}$ or

whoops! It looks like we weren't supposed to use $g = 32\ ft/s^2$, but we can fix that

$\rightarrow a = \frac{18\pi - 72\pi}{-\sqrt{2g}} \Rightarrow$

$\boxed{a = \frac{27\pi\sqrt{2}}{2g}}$

(e) The tank is empty when $y(t) = 0$ or

$\frac{\pi(0)^2}{2} = -8at + 72\pi \Rightarrow 8at = 72\pi \Rightarrow \boxed{t = \frac{9\pi}{a}}$

or $t = \frac{72\pi}{a\sqrt{2g}}$ if g is unspecified

3. $m=1,\ \gamma=0,\ k=?$

Resonance occurs when a homogeneous solution and a forcing function are linearly dependent.

$$mu'' + \gamma u' + ku = f(t) \Rightarrow$$

$$u'' + ku = \cos\left(\frac{2\pi}{3}t\right) + \sin(\pi t)$$

So, resonance occurs if $y_h = c_1\cos\left(\frac{2\pi}{3}t\right) + c_2\sin\left(\frac{2\pi}{3}t\right)$

or if $y_h = c_1\cos(\pi t) + c_2\sin(\pi t)$

(Note that we are assuming k is a real number for physical reasons, so complex roots appear in conjugate pairs).

The characteristic equation $r^2 + k = 0$ must therefore have either $r = \pm\frac{2\pi}{3}i$ or $r = \pm\pi i$ as solutions, and

$$k = -r^2 = -\left(\frac{2\pi}{3}i\right)^2 = \frac{4\pi^2}{9} \quad \text{or}$$

$$k = -r^2 = -(\pi i)^2 = \pi^2$$

So, $\boxed{k = \pi^2,\ \frac{4\pi^2}{9}}$

4. $y'' + \lambda y = 0$, $y(0) = y'(2) = 0$

The characteristic equation is

$$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$$

- If $\lambda < 0$, then $-\lambda > 0$, so r is real, and the solution is $y = c_1 e^{rt} + c_2 e^{-rt}$ which cannot be zero because exponentials are always positive.

  or

  $y(0) = 0 \Rightarrow 0 = c_1 + c_2$

  and $y'(2) = 0 \Rightarrow 0 = c_1 r e^{rt} - c_2 r e^{rt}$ $\rightarrow$ Solving this system gives $c_1 = c_2 = 0$

- If $\lambda = 0$, then $r = 0$ with multiplicty 2, so that

  $y = c_1 + t c_2$ and $y(0) = 0 \Rightarrow c_1 = 0$
  $y'(2) = 0 \Rightarrow c_2 = 0$

- If $\lambda > 0$, then $-\lambda < 0$, so r is pure imaginary. Say, $r = \pm \omega i$, where $\omega = \sqrt{\lambda}$. Then,

  $y = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ and

  $y(0) = 0 \Rightarrow c_1 = 0$

  $y'(2) = 0 \Rightarrow 0 = c_2 \omega \cos(2\omega)$
  so either $c_2 = 0$ or $\cos(2\omega) = 0$
  (remember, $\omega = \sqrt{\lambda} > 0$)

$\cos(2\omega) = 0$ iff $2\omega = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$ or $\sqrt{\lambda} = \frac{(2k+1)\pi}{4} \Rightarrow \boxed{\lambda = \frac{(2k+1)^2\pi^2}{16}, k \in \mathbb{Z}}$

Note that, under this condition on λ, we have the freedom to choose any value of c_2, so there are certainly nonzero solutions.

5. $(1-x^2)y'' - 2xy' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$ centered about $x_0 = 0$

(a) since $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y(0) = c_0$ and $y'(0) = c_1$, s

$$\boxed{c_0 = 1 \quad \text{and} \quad c_1 = 0}$$

(b) $y(x) = \sum_{n=0}^{\infty} c_n x^n$ plug in: $(1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$y'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$ equate powers $\Rightarrow$ $\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} c_n n(n-1)x - 2\sum_{n=1}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$y''(x) = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

So, $\sum_{n=0}^{\infty}\left[c_{n+2}(n+2)(n+1) - c_n n(n-1) - 2c_n n + 2c_n\right]x^n = 0 \Rightarrow$

$$c_{n+2}(n+2)(n+1) - n^2 c_n + c_n \cdot n - 2c_n \cdot n + 2c_n = 0 \Rightarrow$$

$$c_{n+2} = \frac{c_n \cdot n^2 + c_n \cdot 1n - c_n \cdot 2}{(n+2)(n+1)} = \frac{c_n\cancel{(n+2)}(n-1)}{\cancel{(n+2)}(n+1)}$$

or $\boxed{c_{n+2} = \dfrac{c_n(n-1)}{n+1}}$

(c) Since $c_1 = 0$, the odd terms all equal $\boxed{\text{zero.}}$

(d) $c_2 = -c_0$ $\quad c_8 = -\frac{\cancel{5}}{7}\frac{\cancel{3}}{\cancel{5}}\frac{1}{\cancel{3}}c_0$

$c_4 = -\frac{1}{3}c_0$ $\quad \vdots$ $\quad \boxed{c_{2n} = -\dfrac{1}{2n-1}c_0}$ or $\boxed{c_{2n} = -\dfrac{1}{2n-1}}$ since $c_0 = 1$

$c_6 = -\frac{\cancel{3}}{5}\cdot\frac{1}{\cancel{3}}c_0$

(e) This depends on the values of the domain under consideration.

6. $w'' + 4w' + 5w = \delta_0(t)$, $w(0) = 1$, $w'(0) = -3$

a)

Since we will be using Duhamel's principle, we will keep the δ term separate.

Take Laplace transform

$$s^2 W - s + 3 + 4(sW - 1) + 5W = 1$$

keep separate

$$(s^2 + 4s + 5)W = s + 1 + 1$$

transfer function

$$W = \frac{s+1}{(s+2)^2+1} + \frac{1}{(s+2)^2+1} = \frac{(s+2) - 1}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}$$

$$\boxed{w = \underbrace{e^{-2t}\cos(t) - e^{-2t}\sin(t)}_{\text{Depends on initial conditions}} + \underbrace{e^{-2t}\sin(t)}_{\text{weight function}}}$$

b) $x'' + 4x' + 5x = e^{-2t}$, $x(0) = 1$, $x'(0) = -3$

By Duhamel's principle:

$$x = e^{-2t}\cos(t) - e^{-2t}\sin(t) + \underbrace{\int_0^t e^{-2(t-\tau)} e^{-2\tau} \sin(\tau)\, d\tau}$$

$$= e^{-2t} \int_0^t \sin(\tau)\, d\tau$$

$$= e^{-2t} \Big[-\cos(\tau)\Big]_{\tau=0}^{t}$$

$$= e^{-2t}\left[1 - \cos(t)\right]$$

So,

$$\boxed{x = e^{-2t} - e^{-2t}\sin(t)}$$

7. $\underbrace{(2xye^{x^2y} + \cos x)}_{M}dx + \underbrace{x^2e^{x^2y}}_{N}dy = 0$

$M_y = 2xe^{x^2y} + 2x^3ye^{x^2y}$ Since $\boxed{M_y = N_x}$, the equation is exact!

$N_x = 2xe^{x^2y} + 2x^3e^{x^2y}$

Let Ψ be ~~the~~ an equation s.t. $\Psi_x = M$ and $\Psi_y = N$.

Then, $\Psi = \int N\,dy = \int x^2e^{x^2y}\,dy = e^{x^2y} + g(x)$

and $2xye^{x^2y} + \cos x = M = \Psi_x = 2xe^{x^2y} + g'(x) \Rightarrow$

$\cos x = g'(x) \Rightarrow g(x) = +\sin(x) + C$

So, $\Psi = e^{x^2y} + \sin(x) + C$

and $\boxed{e^{x^2y} + \sin(x) = C}$