Practice Skills Test 2 – Calculus I
Solutions

1. Find the domain of the function $f(x) = \dfrac{1}{x^2-1} + 5$.

   Since we are dividing, we must ensure that our denominator is not equal to zero. Thus, our domain is the set $\{x \in \mathbb{R} : x^2 - 1 \neq 0\}$. Our denominator is zero precisely when $x^2 - 1 = 0$. The expression $x^2 - 1$ is a difference of squares, so it factors into $(x-1)(x+1)$, and our restrictions are the solutions $x = 1$ and $x = -1$. In interval notation, $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$.

2. Classify $y = x^e$ as a power function, root function, polynomial, rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. State all that apply.

   - A *logarithmic function* has the base form (before shifting, stretching, or reflecting) $y = \log_b x$, for $b > 0$ a constant.
   - An *exponential function* has the base form b^x for some constant $b > 0$.
   - A *trigonometric function* has the base form $y = \sin x, \cos x, \tan x, \csc x, \sec x, \cot x$.
   - A *power function* has the base form x^a, for some constant a.
   - A *root function* has the base form $x^{1/n}$ for n a positive integer.
   - A *polynomial* is a (finite) sum of power functions, of the form $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, for some constants $a_0, \ldots a_n$.
   - A *rational function* is a quotient of two polynomials, i.e. of the form

   $$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

   - An *algebraic function* is a function that can be defined as some (finite) combination of the algebraic operations of addition or multiplication, which includes subtraction, division, powers, and roots. For example, $y = (\sqrt{x} + 2)/\sqrt[3]{x+2}$ is an algebraic function, but it does not belong in any other of the listed categories.

   Since e is a constant, based on the above descriptions, $y = x^e$ is a $\boxed{\text{power function}}$.

3. Find the zeros/roots of the function $\sin(1/x)$.

   We know from our unit circle that $\sin(\theta)$ (which is the y-coordinate on the unit circle) is zero when $\theta = 0$ or $\theta = \pi$. Since the period of sin is 2π, we can add or subtract any multiple of 2π to θ without changing the value, that is, $\sin(\theta \pm 2\pi n) = \sin(\theta)$ for any integer n. Thus, the solutions of $\sin(\theta) = 0$ are $\theta = \pi n$, for any $n \in \mathbb{Z}$ (any n in the set of integers).

   For this problem, our $\theta = 1/x$. Thus, $1/x = \pi n$, for $n \in \mathbb{Z}$. We take the reciprocal of both sides to solve for x.

   $$x = \left(\frac{1}{x}\right)^{-1} = (\pi n)^{-1} = \boxed{\frac{1}{\pi n} \text{ where } n \text{ is an integer}}$$

4. Obtain the infinite limit: $\lim_{x \to 6^-} \dfrac{4}{x-6}$.

   When $x = 6$, the numerator is not zero but the denominator equals zero. Thus, $4/(x-6)$ has a vertical asymptote at $x = 6$. The minus sign in $x \to 6^-$ means "as x approaches from the left," that is, when $x < 6$. If $x < 6$, then $x - 6 < 0$, so the denominator is negative. The numerator is always positive, and a positive number divided by a negative number is negative. Thus,

$$\lim_{x\to 6^-} \frac{4}{x-6} = \boxed{-\infty}$$

5. Given $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, determine $\lim_{x\to 0} f(x)$ (if it exists).

   If $x \neq 0$, then $x^2 \neq 0$, and $x^2 f(x)/x^2 = f(x)$. We know that $\lim_{x\to 0} x^2 = 0$. By the rules of limits, if $\lim_{x\to a} f(x) = A$ exists, $\lim_{x\to a} g(x) = B$ exists, and the product AB is defined (that is, we're not mulitplying $\pm\infty$ with 0), then $\lim_{x\to a} f(x)g(x)$ exists and equals AB.

   In our case $a = 0$, $f(x) = x^2$, and $g(x) = f(x)/x^2$. Thus,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x^2 \frac{f(x)^2}{x^2}\right] = \left[\lim_{x\to 0} x^2\right]\left[\lim_{x\to 0} \frac{f(x)}{x^2}\right] = 0 \cdot 5 = \boxed{0}$$

6. Find the limit, if it exists. $\lim_{x\to 0^-} (\dfrac{2}{x} - \dfrac{2}{|x|})$

   We are concerned with when x approaches zero from the left (as $x \to 0^-$). When x is to the left of 0, $x < 0$, so x is negative. When x is negative, $|x| = -x$, which is the same as $x = -|x|$. Now, we can substitute $-|x|$ for x to get

$$\lim_{x\to 0^-}\left(\frac{2}{x} - \frac{2}{|x|}\right) = \lim_{x\to 0^-}\left(\frac{2}{-|x|} - \frac{2}{|x|}\right) = \lim_{x\to 0^-} -\frac{4}{|x|} = \boxed{-\infty}$$

   We chose to replace x with $|x|$ instead of the other way around because we know that $|x|$ is positive, so it should be clear that $-4/|x|$ is always negative and grows in magnitude as x gets smaller.

7. If $f(x)$ is odd, then is $x^2 f(x)$ even or odd or neither?

   We test $g(x) = x^2 f(x)$ by plugging in $-x$ and seeing if $g(-x) = g(x)$ (even), $g(-x) = -g(x)$ (odd), or if neither condition holds. Since $f(x)$ is odd, we know that $f(-x) = -f(x)$. Thus,

$$g(-x) = (-x)^2 f(-x) = x^2\left[-f(x)\right] = -\left[x^2 f(x)\right] = -g(x)$$

   Since $g(-x) = -g(x)$, we conclude that the given function is $\boxed{\text{odd}}$.

8. Obtain the limit, if it exists: $\lim_{t\to -6} \dfrac{t^2 - 36}{2t^2 + 13t + 6}$.

   Since we have a rational function, we first factor and simplify. The numerator is a difference of squares, so it factors into $t^2 - 36 = (t-6)(t+6)$.

   To factor the denominator, we use the AC method:

   (1) From $2t^2 + 13t + 6$, we get $a = 2$, $b = 13$, $c = 6$.

   (2) We find two integers p and q such that the product $pq = ac = (2)(6) = 12$ and the sum $p + q = b = 13$. To make sure we cover all our options, factor $pq = 12$ into its prime factorization $12 = 2^2 \cdot 3$. Then, all the possibilities are the different ways to split the factors of 12 between p and q. That is, either $p = 1$ and $q = 2^2 \cdot 3$, $p = 2$ and $q = 2 \cdot 3$, $p = 2^2$ and $q = 3$, or $p = 3$ and $q = 2^2$. Through trial and error, we find that $p = 1$ and $q = 2^2 \cdot 3 = 12$.

   (3) Rewrite $2t^2 + 13t + 6$ by replacing b with $p + q$ and then factor by grouping:

$$2t^2 + (1+12)t + 6 = (2t^2 + t) + (12t + 6) = t(2t+1) + 6(2t+1) = (t+6)(2t+1)$$

   Now, we can simplify our limit. Since $x + 6$ appears both in the numerator and the denominator, there is a hole at $x = -6$, and the $x + 6$ terms cancel out everywhere else, so that the limit is the same if we cancel those terms out. Thus,

$$\lim_{t\to -6}\frac{t^2-36}{2t^2+13t+6}=\lim_{t\to -6}\frac{(t-6)(t+6)}{(t+6)(2t+1)}=\lim_{t\to -6}\frac{t-6}{2t+1}=\frac{(-6)-6}{2(-6)+1}=\frac{-12}{-11}=\boxed{\frac{12}{11}}$$

9. If a ball is thrown in the air with a velocity 34 ft/s, its height in feet t seconds later is given by $y = 34t - 16t^2$. Find the average velocity for the time period beginning when $t = 2$ and lasting until 0.5 second (Be careful about the unit).

   The average velocity is the total change in position divided by the total change in time. Here, the time interval starts at time $t = 2$ and lasts for 0.5 seconds until the end time $t = 2.5$. The start position is $y(2)$, and the end position is $y(2.5)$. The average is the difference quotient

$$\begin{aligned}\frac{y(t+h)-y(t)}{(t+h)-t}&=\frac{[34(t+h)-16(t+h)^2]-[34t-16t^2]}{h}\\&=\frac{34t+34h-16(t^2+2ht+h^2)-34t+16t^2}{h}\\&=\frac{34h-32ht-16h^2}{h}=34-32t-16h\end{aligned}$$

   In our specific case, $t = 2$ and $h = 0.5$, so that the average velocity is

$$34-32t-16h=34-32(2)-16(0.5)=34-64-8=\boxed{-38\text{ ft/s}}$$

   Note that a negative velocity is OK, as the negative just means "downward."

10. Solve the equation: $\log_5(nx) - 5 = b$

   We want to solve for x in terms of n and b. Right now, the x is trapped inside the log, and we need to remove each layer encasing it, so we work from the outside inward. First, at 5 to both sides of the equation to get $\log_5(nx) = b + 5$. Now, to undo a log, we use its inverse function: exponentiation to the base of the log. We take five to the power of each side to get

$$5^{\log_5(nx)}=5^{b+5}\iff nx=5^{b+5}$$

   Finally, the last step is to divide both side by n to get $x = \boxed{5^{b+5}/n}$.