Practice Skills Test 4 Solutions
April 23, 2015

Version A

1. Find the limit, if it exists, of $\lim_{x\to\infty} \dfrac{x^2+3x-5}{5x^2-14x+1}$.

   First, since we are taking an infinite limit, we divide both the numerator and denominator by the highest power of $x$, which is $x^2$:

   $$\lim_{x\to\infty} \frac{x^2+3x-5}{5x^2-14x+1}\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1+3/x-5/x^2}{5-14/x+1/x^2}$$

   Now, since $a/x \to 0$ and $b/x^2 \to 0$, for any constants $a$ and $b$ as $x \to \infty$, we know the limit exists, and

   $$\lim_{x\to\infty} \frac{1+3/x-5/x^2}{5-14/x+1/x^2} = \frac{1+0-0}{5-0+0} = \frac{1}{5}$$

   In this case, we also could have used the horizontal asymptote rule to find the answer of $\boxed{\dfrac{1}{5}}$.

2. Find the derivative of $f(x) = \tan(x-2)/x$.

   First, recall that $\frac{d}{dx}\tan x = \sec^2 x$ (if you forget, use the quotient rule on $\sin(x)/\cos(x)$). There are two ways to proceed; we could use the quotient rule or the product rule.

   – Recall that the quotient rule is $\frac{d}{dx}(f/g) = (f'g - fg')/g^2$. In this case, $f(x) = \tan(x-2)$ and $g(x) = x$, so that $f'(x) = \sec^2(x-2)$ (notice the chain rule is multiplication by one in this case), and $g'(x) = 1$. Thus,

   $$\frac{d}{dx}\left(\frac{\tan(x-2)}{x}\right) = \frac{(\sec^2(x-2))(x) - (\tan(x-2))(1)}{x^2} = \frac{\sec^2(x-2)}{x} - \frac{\tan(x-2)}{x^2}$$

   – Recall that the product rule is $\frac{d}{dx}(fg) = f'g + fg'$. In this case, $f(x) = \tan(x-2)$ and $g(x) = x^{-1}$, so that $f'(x) = \sec^2(x-2)$ and $g'(x) = -x^{-2}$. Thus,

   $$\frac{d}{dx}\tan(x-2)x^{-1} = (\sec^2(x-2))(x^{-1}) + (\tan(x-2))(-x^{-2}) = \frac{\sec^2(x-2)}{x} - \frac{\tan(x-2)}{x^2}$$

   In either case, we find the solution is $\boxed{\dfrac{\sec^2(x-2)}{x} - \dfrac{\tan(x-2)}{x^2}}$

3. Let $f$ be a function that is continuous and differentiable on all real numbers with $f(3) = 14$ and $f'(x) \geq 6$ for all real numbers $x$. What is the minimum value of $f(5)$.

   First, notice that, for any choice of closed interval $[a,b]$, the function $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, so $f$ satisfies the conditions of the Mean Value Theorem. In particular, $f$ satisfies the conditions of the Mean Value Theorem on the interval $[3,5]$. Thus, there is a $c \in (3,5)$ such that $f(5) - f(3) = f'(c)(5-3)$, which we can rewrite as $f(5) = f(3) + 2f'(c)$. However, $f'(x) \geq 6$ for all $x$ and, in particular, $f'(c) \geq 6$. Thus,

   $$f(5) = f(3) + 2f'(c) = 14 + 2f'(c) \geq 14 + 2(6) = 26$$

   Therefore, $f(5)$ can be no smaller than $\boxed{26}$. Notice that this is the same answer as assuming that $f$ is the line containing the point $(3,14)$ with slope 6 and evaluating $f(5)$.

4. Find the intervals on which the function $f(t) = -11t^2 + 22t$ is increasing, if any.

   Recall that a differentiable function is increasing (resp. decreasing) precisely where its first derivative is positive (resp. negative). Since the function $f$ is a polynomial, it is differentiable everywhere, and it suffices to find where the first derivative is positive. We have

$$f'(t) = \frac{d}{dt}(-11t^2 + 22t) = -22t + 22$$

   so that $-22t + 22 > 0$ precise when $t < 1$. Thus, $f$ increases on the interval $\boxed{(-\infty, t)}$.

5. Evaluate $\lim_{x \to \infty} \frac{3}{x} \sin\left(\frac{3}{x}\right)$.

   First, notice that $\lim_{x\to\infty} 3/x = 0$. Now, since $\sin x$ is a continuous function, $\lim_{x\to\infty} \sin(3/x) = \sin(0) = 0$. Thus, the limit of the product exists and equals the product of the limits. That is,

$$\lim_{x \to \infty} \frac{3}{x} \sin\left(\frac{3}{x}\right) = 0 \cdot 0 = 0$$

   So, the limit is $\boxed{0}$.

6. Find the absolute maximum of the function $f(x) = 2x^3 - 3x^2 + 1$ on the interval $[-1/2, 4]$.

   To find the absolute maximum, we must first find the local extrema and then compare the values at these points and the values at the end points of the interval to find the largest value. To find the critical numbers, solve

$$f'(x) = 6x^2 - 6x = 0$$

   to find that $x = 0$ or $x = 1$. Now, plug in the x-values: $f(-1/2) = -1/4 - 3/4 + 1 = 0$, $f(0) = 1$, $f(1) = 2 - 3 + 1 = 0$, and $f(4) = 128 - 48 + 1 = 81$. Therefore, the largest value on the interval $[-1/2, 4]$ is $\boxed{81}$.

7. Find an antiderivative of the function $f(x) = 3x - e^x$.

   Recall that, since $\frac{d}{dx}x^n = nx^{n-1}$ and $\frac{d}{dx}e^x = e^x$, the most general antiderivative of $x$ is $x^2/2 + C$, and the most general antiderivative of $e^x$ is $e^x + C$, where $C$ can be any constant. Thus, the most general antiderivative of $f$ is $\frac{3}{2}x^2 - e^x + C$. Since the question asks for *an* antiderivative, not the most general, we may take $C = 0$ and give the answer $\boxed{\frac{3}{2}x^2 - e^x}$.

8. Given the position function of a particle, $s(t) = 6t^4 + 2t^3 - 1$, find the acceleration of the particle at time $t = 3$ seconds.

   Recall that velocity is the first derivative of the position function (with respect to time), and acceleration is the second derivative of the position function (or the first derivative of the velocity function). Let's call $v(t)$ the velocity function and $a(t)$ the acceleration function. Then,

$$v(t) = s'(t) = 24t^3 + 3t^2$$

   and

$$a(t) = v'(t) = 3 \cdot 24t^2 + 6t = 72t^2 + 6t$$

   At $t = 3$, we have $a(3) = 72(3)^2 + 6(3) = 72 \cdot 9 + 18 = 630 + 18 + 18 = \boxed{666}$. Note that we do not know the units because the position function was not completely specified, so just the number is OK.

9. Find the derivative of the function $f(x) = 3x^{3\sin(3x)}$.

   Since there is a complicated function of $x$ in the exponent, we use logarithmic differentiation. Set $y = 3x^{3\sin(3x)}$. Then,

$$\ln y = \ln\left(3x^{3\sin(3x)}\right) = 3\sin(3x)\ln(3x)$$

   Recall that $\frac{d}{dx}\ln x = 1/x$, $\frac{d}{dx}\sin x = \cos x$, and $\frac{d}{dx}(fg) = f'g + fg'$. Now, differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = 3\cos(3x)(3)\ln(3x) + 3\sin(3x)\frac{1}{3x}(3)$$

   (Don't forget the chain rule, which is why the threes in parentheses are there). Simplifying, we have that

$$\frac{dy}{dx} = y\left[9\cos(3x)\ln(3x) + 3\frac{\sin(3x)}{x}\right]$$

   Since $y$ is a function of just $x$ (and not $y$ as well), we can substitute for $y$ to find the answer is $\boxed{3x^{3\sin(3x)}\left[9\cos(3x)\ln(3x) + 3\frac{\sin(3x)}{x}\right]}$.

10. Find the position of a particle after 10 seconds if it has velocity function $v(t) = 5 + 2\sin(4t)$ and satisfies the initial condition $s(0) = 0$, where $s$ is the position function.

    Since velocity is the first derivative of the position function with respect to time, we must find the appropriate antiderivative of $v(t)$. The most general antiderivative is $5t - \cos(4t)/2 + C$, which we can check by differentiating. This doesn't give us enough information, though, so we have to use our initial condition to solve for $C$:

$$0 = 5(0) - \cos(4(0))/2 + C = 0 - 1/2 + C$$

    so that $C = 1/2$. Thus, $s(t) = 5t - \cos(4t)/2 + 1/2$. Now we can solve for the position of the particle when $t = 10$:

$$s(10) = 5(10) - \cos(4(10))/2 + 1/2 = \frac{101}{2} - \cos(40)/2$$

    So, the answer is $\boxed{\frac{101 - \cos(40)}{2}}$.