MAC 2311 – Calculus I
Activity Sheet 4 Group Solutions

1. The *Bessel function* of order 0, $y = J(x)$ satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

   (a) To find $J'(0)$, substitute $x = 0$ into the equation $xy'' + y' + xy = 0$ to get

   $$(0)J''(0) + J'(0) + (0)J(0) = 0 \implies J'(0) = 0$$

   (b) To find $J''(0)$, use implicit differentiation rules on the equation:

   $$\frac{d}{dx}\left(xy'' + y' + xy\right) = y'' + xy''' + y'' + y + xy' = xy''' + 2y'' + xy' + y$$

   Now, substitute $x = 0$ to get

   $$(0)J'''(0) + 2J''(0) + (0)J'(0) + J(0) = 0 \implies 2J''(0) + J(0) = 0 \implies 2J''(0) + 1 = 0$$

   so that $J''(0) = -1/2$.

2. Recall that $|x|$ can be written as $\sqrt{x^2}$.

   (a) Using the chain rule, we have

   $$\frac{d}{dx}|x| = \frac{d}{dx}\sqrt{x^2} = \frac{1}{2}\left(x^2\right)^{-1/2}(2x) = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|}$$

   (b) Using part (a) and the chain rule on $f(x) = |\sin(x)|$, we have

   $$\frac{d}{dx}|\sin(x)| = \frac{\sin(x)}{|\sin(x)|}\cos(x)$$

   which is undefined at $x = \pi k$, where k is any integer.

   (c) Using part (a) and the chain rule on $g(x) = \sin|x|$, we have

   $$\frac{d}{dx}\sin|x| = \cos|x|\frac{x}{|x|}$$

   which is undefined at $x = 0$.

3. Let $f_n(x) = \left(1 + \frac{x}{n}\right)^n$. Show that $\lim_{n\to\infty} f_n(x) = e^x$ for any $x > 0$.

   (a) We have

   $$f_n'(x) = n\left(1 + \frac{x}{n}\right)^{n-1}\frac{1}{n} = \left(1 + \frac{x}{n}\right)^{n-1}$$

   (b) We wish to show that $\lim_{n\to\infty} f_n'(x) = e^x$. We do this by showing that the difference between $f_n'(x)$ and $f_n(x)$ approaches zero. That is,

   $$\begin{aligned}\lim_{n\to\infty} f_n'(x) - f_n(x) &= \lim_{n\to\infty}\left[\left(1 + \frac{x}{n}\right)^{n-1} - \left(1 + \frac{x}{n}\right)^n\right] \\ &= \lim_{n\to\infty}\left[\left(1 + \frac{x}{n}\right)^{n-1}\left(1 - \left(1 + \frac{x}{n}\right)\right)\right] \\ &= \lim_{n\to\infty} -\frac{x}{n}\left(1 + \frac{x}{n}\right)^{n-1} \\ &= 0\end{aligned}$$

The limit is zero because the expression $(1+x/n)^{n-1}$ remains bounded, while the expression x/n in front goes to zero as $n \to \infty$. This means that both $f_n(x)$ and $f_n'(x)$ converge to the same function. Using the special property of $f_n(x)$, we have

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\frac{d}{dx} f_n(x) \right) = \lim_{n\to\infty} f_n(x)$$

Thus, $\lim_{n\to\infty} f_n(x)$ is a function that equals its own derivative. This is the definition of e^x, so this function must be e^x, if the limit exists. Now, we're going to cheat a little bit and assume that the limit exists.

(c) In summary, part (b) showed us that $\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} f_n(x)$, which meant that $\lim_{n\to\infty} f_n(x)$ must be the unique function that equals its own derivative, which is e^x.

4. Let f be a one-to-one differentiable function such that its inverse f^{-1} is also differentiable.

(a) To find $(f^{-1})'(x)$, first notice that $f^{-1}(x) = y$ is the same thing as $x = f(y)$. Using implicit differentiation, we have

$$\frac{d}{dx} x = \frac{d}{dx} f(y) \implies 1 = f'(y) \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

(b) If $f(4) = 5$, then we are at the point $(4, 5)$. Notice that this corresponds to $f^{-1}(5) = 4$ for the inverse function. Using part (a) and the fact that $f'(4) = 2/3$, we have

$$\frac{d}{dx} f'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} = \frac{1}{2/3} = \frac{3}{2}$$

5. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2° per minute. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°?

Since the angle is in between the two fixed sides, the length of the third side is given by the law of cosines. Let θ represent the angle, and let $\ell(\theta)$ be the length of the third side. Then, $\ell(\theta) = \sqrt{12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cos\theta}$.

"How fast is the length of the third side increasing" means $\frac{d\ell}{dt}$, the derivative of θ with respect to time. Using the chain rule, we have

$$\frac{d\ell}{dt} = \frac{1}{2} \left(12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cos(\theta)\right)^{-1/2} (2 \cdot 12 \cdot 15 \sin(\theta)) \frac{d\theta}{dt}$$

The value $\frac{d\theta}{dt}$ is given in the problem where it says "The angle between them is increasing at a rate of 2° per minute." To avoid problems, convert all degrees to radians (what problems?). Thus, $\frac{d\theta}{dt} = 2\frac{\pi}{180} = \frac{\pi}{90}$, and, at $\frac{\pi}{3}$ (60°), we have

$$\begin{aligned}
\frac{d\ell}{dt}\left(\frac{\pi}{3}\right) &= \frac{1}{2}\left(12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cos\left(\frac{\pi}{3}\right)\right)^{-1/2} \left(2 \cdot 12 \cdot 15 \sin\left(\frac{\pi}{3}\right)\right) \frac{\pi}{90} \\
&= \frac{1}{2}\left(12^2 + 15^2 - 2 \cdot 12 \cdot 15\right)^{-1/2} \left(12 \cdot 15 \cdot \sqrt{3}\right) \frac{\pi}{90} \\
&= \frac{1}{\sqrt{144 + 225 - 90}} \sqrt{3}\pi \\
&= \frac{\pi}{\sqrt{63}} && \approx 0.3958
\end{aligned}$$

Therefore, the third side is increasing at a rate of $\pi/\sqrt{63}$ meters per minute.

6. Suppose \$3000 is invested at 5% interest.

   (a) The formula for compounding interest is $P(1+\frac{r}{n})^{nt}$, where $P = 3000$ is the initial principal, $r = 0.05$ is the interest rate, n is the number of compoundings per time period (usually a year), and $t = 5$ is the number of time periods, since we are concerned with a five year period.

      (i) Annually means $n = 1$, so we have

      $$3000\left(1+\frac{0.05}{1}\right)^{1\cdot 5} \approx 3828.84$$

      (ii) Semiannually means $n = 2$, so we have

      $$3000\left(1+\frac{0.05}{2}\right)^{2\cdot 5} \approx 3840.25$$

      (iii) Monthly means $n = 12$, so we have

      $$3000\left(1+\frac{0.05}{12}\right)^{12\cdot 5} \approx 3850.08$$

      (iv) Weekly means $n = 52$, so we have

      $$3000\left(1+\frac{0.05}{52}\right)^{52\cdot 5} \approx 3851.61$$

      (v) Daily means $n = 365$, so we have

      $$3000\left(1+\frac{0.05}{365}\right)^{365\cdot 5} \approx 3852.01$$

      (vi) Continuously means we take the limit as $n \to \infty$. From exercise 3, we know that $e^x = \lim_{n\to\infty}(1+x/n)^n$. Thus,

      $$\lim_{n\to\infty} 3000\left(1+\frac{0.05}{n}\right)^{5n} = 3000\left[\lim_{n\to\infty}\left(1+\frac{0.05}{n}\right)^{n}\right]^5 = 3000e^{0.05\cdot 5} \approx 3852.08$$

   (b) The initial condition is the initial principal $A(0) = 3000$. To find a differential equation for $A(t)$, we can take the derivative of $A(t) = 3000e^{0.05t}$, which is $A'(t) = 3000e^{0.05t}\cdot 0.05 = A(t)\cdot 0.05$. Thus, our differential equation is the system $A' = 0.05A$ with initial condition $A(0) = 3000$.

7. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas, and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

   Notice that R and n are constants (since n does not change in this particular problem). We seek $\frac{dT}{dt}$, the rate of change of T with respect to time t. Differentiating, we have

   $$\frac{d}{dt}(nRT) = \frac{d}{dt}(PV) \implies nR\frac{dT}{dt} = \frac{dP}{dt}V + P\frac{dV}{dt} \implies \frac{dT}{dt} = \frac{1}{nR}\left(\frac{dP}{dt}V + P\frac{dV}{dt}\right)$$

   Now, the problem tells us that $n = 10$, $R = 0.0821$, $\frac{dP}{dt} = 0.10$ ("increasing at a rate of 0.10 atm/min"), $V = 10$, $P = 8.0$, and $\frac{dV}{dt} = -0.15$ ("decreasing at a rate of 0.15 L/min"). Plugging these values into our equation, at the moment under consideration we have

   $$\frac{dT}{dt} = \frac{1}{(10)(0.0821)}\left((0.10)(10) + (8.0)(-0.15)\right) \approx -0.24$$

Therefore, the rate of change of T with respect to time at the instant under consideration is approximately -0.24 K/min. In particular, our gas is cooling.

8. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room, its temperature has increase to 10°C.

   (a) To model the temperature change, we will use Newton's law of cooling. Newton's law of cooling states, "For a body cooling in a draft, the rate of heat loss is proportional to the difference int temperatures between the body and its surroundings." To write this as an equation, let $T(t)$ be the temperature of the drink at time t, and let $T_s = 20$ be the constant temperature of the surroundings. Then, there is some constant of proportionality k such that

   $$\frac{dT}{dt} = -k(T - T_s)$$

   This is a differential equation with initial condition $T(0) = 5 = T_0$, the initial temperature of the drink. Using Theorem 2 in section 3.8 of the textbook, we can solve this differential equation to find that

   $$T(t) = T_s + (T_0 - T_s)e^{-kt} = 5 + 15e^{-kt}$$

   Now, we can use the facts in the problem to find k. We are given that $T(25) = 10$. Thus,

   $$T(25) = 20 + (5 - 20)e^{-25k} = 10 \implies e^{-25k} = \frac{2}{3} \implies -25k = \ln\left(\frac{2}{3}\right)$$

   Thus, $k = (\ln 1.5)/25 \approx 0.01622$. Knowing the value of k, we can solve for the temperature of the drink after 50 minutes.

   $$T(50) = 20 - 15e^{-50(\ln 1.5)/25} = 20 - 15e^{-2\ln 1.5} \approx 13.33$$

   (b) The temperature is 15°C at time t when $T(t) = 15$ or

   $$\begin{aligned} 20 - 15e^{-t\ln(1.5)/25} = 15 &\implies -15e^{-t\ln(3)/25} = -5 \\ &\implies e^{-t\ln(1.5)/25} = \frac{1}{3} \\ &\implies -t\frac{\ln(1.5)}{25} = \ln\left(\frac{1}{3}\right) \\ &\implies t\frac{\ln(1.5)}{25} = \ln(3) \\ &\implies t = 25\ln(3)/\ln(1.5) \\ &\implies t \approx 67.74 \end{aligned}$$

   Therefore, when $t = 25\ln(3)/\ln(1.5)$ minutes have passed, the temperature of the drink will be 15°C.

9. Two people start from the same point. One walks east at 3 mi/h, and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

   Let $d_1(t)$ be the distance the person headed east has traveled at time t, and let $d_2(t)$ be similarly defined for the other person. The angle between north and northeast is $\pi/4$. The distance between the two people $d(t)$ can be expressed by the law of cosines as

   $$d(t) = \sqrt{d_1(t)^2 + d_2(t)^2 - d_1(t)d_2(t)\cos\left(\frac{\pi}{4}\right)}$$

We seek $\frac{dd}{dt}$. By repeated applications of the chain rule, we have

$$\frac{dd}{dt} = \frac{1}{2}\left(d_1^2 + d_2^2 - 2d_1 d_2 \cos\left(\frac{\pi}{4}\right)\right)^{-1/2}\left(2d_1\frac{dd_1}{dt} + 2d_2\frac{dd_2}{dt} - \frac{dd_1}{dt}d_2\cos\left(\frac{\pi}{4}\right) - d_1\frac{dd_2}{dt}\cos\left(\frac{\pi}{4}\right)\right)$$

The values $\frac{dd_1}{dt} = 3$ and $\frac{dd_2}{dt} = 2$ are the speeds given in the problem statement. Convert 15 minutes into 1/4 hours. Then, $d_1(0.25) = 3/4$, and $d_2(0.25) = 1/2$. Now, plug in these four values to evaluate $\frac{dd}{dt}(0.25)$, which is

$$\begin{aligned}
\frac{dd}{dt} &= \frac{1}{2}\left(\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 - 2\cdot\frac{3}{4}\cdot\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\right)^{-1/2}\left(2\cdot\frac{3}{4}\cdot 3 + 2\cdot\frac{1}{2}\cdot 2 - 2(3\cdot\frac{1}{2} - \frac{3}{4}\cdot 2)\cos\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}\left(\frac{9}{16} + \frac{4}{16} - \frac{6\sqrt{2}}{16}\right)^{-1/2}\left(\frac{18}{4} + \frac{8}{4} - \frac{6\sqrt{2}}{4} - \frac{6\sqrt{2}}{4}\right) \\
&= \frac{\frac{13 - 6\sqrt{2}}{4}}{\frac{\sqrt{13 - 6\sqrt{2}}}{4}} \\
&= \sqrt{13 - 6\sqrt{2}} \\
&\approx 2.1248
\end{aligned}$$

These calculations can be made somewhat simpler by differentiating $d^2 = d_1^2 + d_2^2 - 2d_1 d_2\cos\left(\frac{\pi}{4}\right)$ implicitly. Try it.