Midterm 2
MAC2302,ODE

Instructor: Tamasan
Date: March 25th, 2016

Student Name:

50 minutes. Write each solution on a separate sheet, order them, and place this on top for stapling. Closed book. NO calculator. Work without justification gets no credit. All problems are equally weighted.

1. Consider the non-homogeneous fourth order linear differential equation

$$y^{(4)} + 8y'' + 16y = 9\sin(t) + 4.$$

a) Find a fundamental set of solutions of the corresponding homogeneous equation. You may use $(a^2+b^2)^2 = a^4 + 2a^2b^2 + b^4$.
b) Find a particular solution y_p as the sum $u + v$, where u solves $u^{(4)} + 8u'' + 16u = 4$ and v solves $v^{(4)} + 8v'' + 16v = 9\sin(t)$.
c) Write down the general solution of the non-homogenous equation (four arbitrary constants should show).
d) How many solutions satisfy $y(0) = y'(0) = 0$? Justify.
e) How many solutions satisfy $y(0) = y'(0) = 0$ and are also bounded? Justify.

2. Find the general solution of the equation

$$(1-x)y''(x) + xy'(x) - y(x) = 4(1-x)^2e^{-x}, \quad x < 1,$$

by answering the following:
a) Check that e^x and x solve the corresponding homogenous equation.
b) Seek a particular solution for the non-homogeneous equation in the form $y_p(x) = c_1(x)e^x + c_2(x)x$, for some unknown functions c_1, c_2 with $c_1'(x)e^x + c_2'(x)x = 0$:
(i) Plug y_p into the equation, to find a second equation for c_1', c_2';
(ii) Solve the linear system for c_1' and c_2';
(iii) Integrate to find c_1 and c_2, and then y_p.
c) Write the general solution of the non-homogenous equation.
d) Can you use the method of the unknown coefficients in this problem? Justify.

3. A large spring is stretched $2ft$ by a hanging object weighing w pounds (this is a force!). The object is attached to a dashpot mechanism with damping constant $\frac{1}{8}lb \cdot sec/ft$ and is acted on by an external force of $\cos(4t)$ lb. Let $x(t)$ be the displacement at time t of the center of mass measured from the dynamic equilibrium and use the gravitation $g = 32ft/sec^2$.
a) Draw two side by side pictures showing the ~~string~~ spring with the weight: at the dynamic equilibrium (on the left), respectively at some displacement x from the dynamic equilibrium (on the right). Show the acting forces.
b) What is the mass (measured in $lb \cdot sec^2/ft$) of the object in terms of its weight w ?
c) What is the elastic constant k (measured in lb/ft) in terms of w?
d) Write down the differential equation for $x(t)$ governing the motion of the system. Your answer should depend on w.
e) Determine the steady state response (which does not die out exponentially in time) of the system .
f) Show that the amplitude of the steady state response is independent of w.

1. $y^{(4)} + 8y'' + 16y = 9\sin(t) + 4$

a) Char. eqn: $r^4 + 8r^2 + 16 = 0 \Rightarrow (r^2+4)^2 = 0 \Rightarrow r^2+4=0 \Rightarrow r = \pm 2i$ (mult. 2)

$\Rightarrow \boxed{\{\cos(2t), \sin(2t), t\cos(2t), t\sin(2t)\}}$ is a fund. set

b) $u = A \Rightarrow 16A = 4$
$u' = 0$
$u'' = 0 \quad \Rightarrow A = \frac{1}{4}$
$u''' = 0$
$u^{(4)} = 0$

$v = A\cos t + B\sin t$
$v' = -A\sin t + B\cos t$
$v'' = -A\cos t - B\sin t$
$v''' = A\sin t - B\cos t$
$v^{(4)} = A\cos t + B\sin t$

$\Rightarrow 16B - 8B + B = 9$
$\Rightarrow B = 1$

Note that A is not involved. or plug in to see A=0.

$\Rightarrow \boxed{y_p = \frac{1}{4} + \sin t}$

c) general solution:

$$\boxed{y = C_1\cos(2t) + C_2\sin(2t) + C_3 t\cos(2t) + C_4 t\sin(2t) + \frac{1}{4} + \sin t}$$

d) There are $\boxed{\text{infinitely many}}$ solutions with $y(0) = y'(0) = 0$. To see why, plug in initial conditions to get $C_1 = -\frac{1}{4}$, $2C_2 + C_3 + 1 = 0$, so there are two degrees of freedom.

e) There is $\boxed{\text{exactly one}}$ bounded solution with $y(0) = y'(0) = 0$. Since $C_3 t\cos(2t) + C_4 t\sin(2t)$ is unbounded if $C_3 \neq 0$ or $C_4 \neq 0$, the boundedness condition forces $C_3 = C_4 = 0$. Thus, the only solution has $C_1 = -\frac{1}{4}$, $C_2 = -\frac{1}{2}$, $C_3 = 0$, $C_4 = 0$.

If you're not fully convinced $C_3 t\cos(2t) + C_4 t\sin(2t)$ is unbounded, put it in the form $R\cos(\omega t - \delta)$. Then, The amplitude is $R = \sqrt{(C_3 t)^2 + (C_4 t)^2} = t\sqrt{C_3^2 + C_4^2}$, which grows to infinity as $t \to \infty$ unless $C_3 = C_4 = 0$.

2. $(1-x)y'' + xy' - y = 4(1-x)^2 e^{-x}$, $x<1$

a) $y = e^x = y' = y'' \Rightarrow$

$(1-x)e^x + xe^x - e^x = 0 \Rightarrow$

$0 = 0$ $\boxed{\checkmark}$

$y = x$, $y' = 1$, $y'' = 0$ $\Rightarrow$ $x \cdot 1 - x = 0 \Rightarrow$ $0 = 0$ $\boxed{\checkmark}$

b) (i)

$$\begin{array}{r|l} -1 & y_p = c_1 e^x + c_2 x \\ x & y_p' = c_1 e^x + c_2 + \overbrace{c_1' e^x + c_2' x}^{\text{set} = 0} \\ 1-x & y_p'' = c_1' e^x + c_1 e^x + c_2' \end{array}$$

$\Rightarrow c_1' e^x + c_2' x = 0$

$c_1' e^x (\cancel{1-x}) + c_2'(\cancel{1-x}) = 4(1-x)^{\cancel{2}} e^{-x}$ cancels one $\Rightarrow$ $\boxed{\begin{array}{l} c_1' e^x + c_2' x = 0 \\ c_1' e^x + c_2' = 4(1-x)e^{-x} \end{array}}$

(ii) Using Cramer's rule: $D = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x = (1-x)e^x$

Then, $c_1' = \dfrac{\begin{vmatrix} 0 & x \\ 4(1-x)e^{-x} & 1 \end{vmatrix}}{D} = \dfrac{-4x(\cancel{1-x})e^{-x}}{(\cancel{1-x})e^x} = \boxed{\dfrac{-4x}{e^{2x}} = c_1'}$

$c_2' = \dfrac{\begin{vmatrix} e^x & 0 \\ e^x & 4(1-x)e^{-x} \end{vmatrix}}{D} = \dfrac{4(\cancel{1-x})}{(\cancel{1-x})e^x} = \boxed{\dfrac{4}{e^x} = c_2'}$

(iii) $c_2 = \int 4e^{-x}dx = \boxed{-4e^{-x} + A}$ ← arbitrary constant

$c_1 = -4\underbrace{\int x e^{-2x} dx}_{} = -4\left(-\frac{x}{2}e^{-2x} + \underbrace{\int \frac{1}{2} e^{-2x} dx}_{-\frac{1}{4}e^{-2x}}\right)$

By parts: $u = x$, $du = dx$, $v = -\frac{1}{2}e^{-2x}$, $dv = e^{-2x}dx$

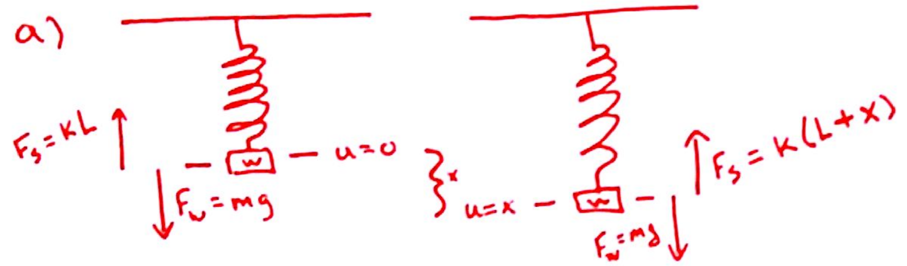
$\Rightarrow c_1 = 2xe^{-2x} + e^{-2x} + B = \boxed{(2x+1)e^{-2x} + B}$

c) general sol'n:

$y = [(2x+1)e^{-2x} + B]e^x + [-4e^{-x} + A]x = \boxed{(1-2x)e^{-x} + Be^x + Ax}$

d) $\boxed{\text{Yes.}}$ Because $g(t) = 4(1-x)e^{-x}$ is a product of an exponential and a polynomial.

3. a)



Dashpot force depends on velocity (not pictured) and acts in the opposite direction.

b) $mg = w \Rightarrow \boxed{m = \frac{w}{32}}$

c) $mg = kL \Rightarrow w = k2 \Rightarrow \boxed{k = \frac{w}{2}}$

d) $mu'' + \gamma u' + ku = F \Rightarrow$

$$\boxed{\frac{w}{32}u'' + \frac{1}{8}u' + \frac{w}{2}u = \cos(4t)}$$

e) Steady state = particular. use $wu'' + 4u' + 16u = 32\cos(4t)$

$$\begin{array}{r|l} 16w & y_p = A\cos(4t) + B\sin(4t) \\ 4 & y_p' = -4A\sin(4t) + 4B\cos(4t) \\ w & y_p'' = -16A\cos(4t) - 16B\sin(4t) \end{array}$$

$\Rightarrow 16wA + 16B - 16Aw = 32 \Rightarrow 16B = 32 \Rightarrow B = 2$

$16wB - 16A - 16Bw = 0 \Rightarrow -16A = 0 \Rightarrow A = 0$

So, $\boxed{y_p = 2\sin(4t)}$

f) Amplitude (which is $R = \sqrt{A^2 + B^2}$) is 2, which is independent of w, $\boxed{\text{since it is a constant.}}$