§ 6.5 #7. $y'' + y = \delta(t-2\pi)\cos t$, $y(0)=0$, $y'(0)=1$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{\delta(t-2\pi)\cos t\}$$

$$= \int_0^\infty e^{-st}\,\delta(t-2\pi)\cos t\,dt$$

$$(*) \quad = \int_{-\infty}^\infty e^{-st}\,\delta(t-2\pi)\cos t\,dt$$

$$(**) \quad = e^{-s\cdot 2\pi}\cos(2\pi)$$

$$= e^{-2\pi s} \quad \Rightarrow$$

$$s^2Y - 1 + Y = e^{-2\pi s} \quad \Rightarrow$$

$$(s^2+1)Y = 1 + e^{-2\pi s} \quad \Rightarrow$$

$$Y = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} \quad \Rightarrow$$

$$\boxed{y = \sin t + u_{2\pi}(t)\sin(t-2\pi)}$$

$(*)$ is true because the integral only depends on the values near the point 2π, since the δ function is defined by a function zero outside $[2\pi-\tau, 2\pi+\tau]$, and we take the limit as $\tau \to 0^+$.

$(**)$ is true by the property of the delta function in equation (16) of section 6.5.