Activity Sheet 3 (Individual)
MAC 2311 – Calculus I
Solutions

1. Simplify or expand:

   (a) $(x+2)^2 = x^2 + 2x + 4$

   (b) $\dfrac{\sqrt{t}}{t} - \dfrac{1}{\sqrt{t}} = \dfrac{1}{\sqrt{t}} - \dfrac{1}{\sqrt{t}} = 0$

   Note that $t > 0$, so $\dfrac{\sqrt{t}}{t}$ is positive.

   (c) $\left(x + \dfrac{1}{\sqrt{t}}\right)^2 = x^2 + 2x\dfrac{1}{\sqrt{t}} + \left(\dfrac{1}{\sqrt{t}}\right)^2 = x^2 + 2\dfrac{x}{\sqrt{t}} + \dfrac{1}{t}$

   (d) $\sqrt{x(x+2)+1} = \sqrt{x^2+2x+1} = \sqrt{(x+1)^2} = |x+1|$

   (e) $\dfrac{\sin(2x)}{2} - \sin(x)\cos(x) = \dfrac{2\sin(x)\cos(x)}{2} - \sin(x)\cos(x) = \sin(x)\cos(x) - \sin(x)\cos(x) = 0$

   (f) $\sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1$

2. Find limits:

   (a) $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{x^2 - 3x + 2}$

   Factor the numerator and denominator, cancel, and plug in 2 to find the limit.

$$\begin{aligned}\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 3x + 2} &= \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2)(x-1)} \\ &= \lim_{x\to 2} \frac{x+1}{x-1} \\ &= \frac{(2)+1}{(2)-1} \\ &= 3\end{aligned}$$

   (b) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{9x^6 + 3x^3 + 4} - 7}{x^3 + 3}$

   Because the limit is infinite, divide both the numerator and denominator by $x^3$, the highest order term. Use the substitution $x^3 = -\sqrt{x^6}$ to divide inside the square root, since odd powers of negative numbers are negative.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{9x^6+3x^3+4}-7}{x^3+3} &= \lim_{x\to-\infty} \frac{\sqrt{9x^6+3x^3+4}-7}{x^3+3}\,\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^6+3x^3+4}}{x^3}-\frac{7}{x^3}}{\frac{x^3+3}{x^3}} \\
&= \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^6+3x^3+4}}{-\sqrt{x^6}}-\frac{7}{x^3}}{1+\frac{3}{x^3}} \\
&= \lim_{x\to-\infty} \frac{-\sqrt{9+\frac{3}{x^3}+\frac{4}{x^6}}-\frac{7}{x^3}}{1+\frac{3}{x^3}} \\
&= \frac{-\sqrt{9+0+0}-0}{1+0} \\
&= -3
\end{aligned}$$

(c) $\displaystyle\lim_{x\to-3} \frac{|4x+12|}{x+3}$

Find the left and right limits and show they are not equal to prove the limit does not exist. Recall that the definition of absolute value is

$$|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$

Substituting with this definition, we find that the left limit is

$$\begin{aligned}
\lim_{x\to-3^-} \frac{|4x+12|}{x+3} &= \lim_{x\to-3^-} \frac{4\cdot|x+3|}{x+3} \\
&= \lim_{x\to-3^-} 4\frac{|x+3|}{-|x+3|} \\
&= -4
\end{aligned}$$

and the right limit is

$$\begin{aligned}
\lim_{x\to-3^+} \frac{|4x+12|}{x+3} &= \lim_{x\to-3^+} \frac{4\cdot|x+3|}{x+3} \\
&= \lim_{x\to-3^+} 4\frac{|x+3|}{|x+3|} \\
&= 4
\end{aligned}$$

Since the left and right limits are not equal, the limit does not exist.

3. For $f(x)=x^5$ and $g(x)=5^x$, find derivatives.

   (a) Using the power rule $\frac{d}{dx}x^n = nx^{n-1}$ gives $f'(x)=5x^x$.

(b) Using the chain rule $(f \circ g)'(x) = f'(g(x))g'(x)$, the fact that $\frac{d}{dx}e^x = e^x$, and logarithm / exponent properties, we have

$$g'(x) = \frac{d}{dx}5^x = \frac{d}{dx}e^{\ln(5^x)} = \frac{d}{dx}e^{x\ln(5)} = e^{x\ln(5)}\frac{d}{dx}(x\ln(5)) = \ln(5)e^{x\ln(5)} = \ln(5)\cdot 5^x$$

(c) Using the product rule and substituting our previous values, we have

$$\begin{aligned}(fg)'(x) &= f'(x)g(x) + f(x)g'(x)\\ &= (5x^4)(5^x) + (x^5)(\ln(5)\cdot 5^x)\\ &= x^4\cdot 5^x(5+\ln(5)x)\end{aligned}$$

(d) Using the quotient rule and substituting our previous values, we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}\\ &= \frac{(5x^4)(5^x) + (x^5)(\ln(5)\cdot 5^x)}{(5^x)^2}\\ &= \frac{5^x(5x^4 + \ln(5)x^5)}{(5^x)^2}\\ &= \frac{5x^4 + \ln(5)x^5}{5^x}\end{aligned}$$

4. Tangent line to $f(x) = \dfrac{x + \tan(x)}{x^2 - \cos(x)}$ at $x = 0$.

Using $\frac{d}{dx}\tan(x) = \sec^2(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, and the quotient rule, we have

$$f'(x) = \frac{(1+\sec^2(x))(x^2 - \cos(x)) - (x + \tan(x))(2x + \sin(x))}{(x^2 - \cos(x))^2}$$

Now, without simplifying, substitute $x = 0$.

$$\begin{aligned}f'(x) &= \frac{(1+\sec^2(0))(0^2 - \cos(0)) - (0 + \tan(0))(2\cdot 0 + \sin(0))}{(0^2 - \cos(0))^2}\\ &= \frac{(1+1)(0-1) - (0+0)(0+0)}{(-1)^2} = -2\end{aligned}$$

Now, find the value of the function at $x = 0$.

$$f(0) = \frac{0 + \tan(0)}{0^2 - \cos(0)} = \frac{0+0}{0-1} = 0$$

The equation of the line with slope $-2$ that passes through the point $(0, 0)$ is $y = -2x$.