Practice Skills Test 3 – Calculus I
Solutions

1. Find the domain and range of the function $f(x) = \sin(\sin^{-1}(x))$.

   The inverse sine function $\sin^{-1}$ takes a y-coordinate from the interval $[-1, 1]$ and returns the angle θ in the range $[-\pi/2, \pi/2]$ such that $\sin(\theta) = y$.

   The sine function takes any angle θ from the the real line $(-\infty, \infty)$ and returns the y-coordinate on the unit circle. Since $[-\pi/2, \pi/2]$ covers all the possible y-values on the unit circle, the composition $f(x) = \sin(\sin^{-1}(x))$ takes a y-coordinate from the interval $[-1, 1]$ and returns that same y-coordinate, which is still in the interval $[-1, 1]$.

   Thus, $f(x) = x$ with $\boxed{\text{domain } [-1, 1] \text{ and range } [-1, 1]}$.

2. Find the inverse function of $f(x) = \ln(e^x + a)$.

   To find the inverse function, we take $y = \ln(e^x + a)$ and switch the variables x and y to get $x = \ln(e^y + a)$. Then, solve for y. The first step is to undo the log using its inverse function, which is the exponential function with the same base as the logarithm. In this case, we are using the natural logarithm, whose base is the natural number e. Take e to the power of both sides to get

   $$e^x = e^{\ln(e^y + a)} \iff e^x = e^y + a$$

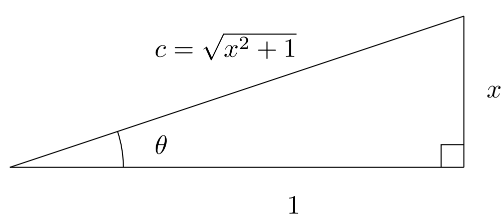
   Next, subtract a from both sides of $e^x = e^y + a$ to get $e^x - a = e^y$. Finally, to remove the e from the y, we take the natural log of both sides to get

   $$y = \ln(e^y) = \boxed{\ln(e^x - a)}$$

3. Simplify the expression $\sin(\tan^{-1}(x))$.

   The inverse tangent function $\tan^{-1}$ takes a ratio of the y-coordinate to the x-coordinate and returns the angle θ in the interval $[-\pi/2, \pi/2]$ such that $\tan(\theta) = y/x$. For our problem, let's call θ the angle that $\tan^{-1}(x)$ gives us, that is, $\theta = \tan^{-1}(x)$. Taking the tangent of both sides to cancel out the $\tan^{-1}$ produces $\tan(\theta) = \tan(\tan^{-1}(x)) = x$.

   Now, we have $\tan(\theta) = x$, and we can think of this as $\tan(\theta) = x/1$, where x is the leg opposite the angle θ in a right triangle, and 1 is the leg adjacent to the angle θ:

   We can then find the length of the hypotenuse using the Pythagorean theorem $a^2 + b^2 = c^2$. We have $c^2 = x^2 + 1$, so that $c = \sqrt{x^2 + 1}$ (remember that distances are always positive).

   Using the fact that $\theta = \tan^{-1}(x)$, we can simplify our expression $\sin(\tan^{-1}(x))$ to $\sin(\theta)$. With the above diagram, we can see how $\sin(\theta)$ relates to x: it is the opposite leg, which is x, over the hypotenuse, which is $\sqrt{x^2 + 1}$. Thus,

   $$\sin(\tan^{-1}(x)) = \sin(\theta) = \boxed{\frac{x}{\sqrt{x^2 + 1}}}$$

4. When $f(x) = \dfrac{1}{x+1}$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

   Our difference quotient is

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1}{(x+h)+1} - \dfrac{1}{x+1}}{h}$$

   First, we get a common denominator to combine the fractions in the numerator. Our common denominator is $(x+h+1)(x+1)$, and our expression becomes

$$\frac{\dfrac{1}{x+h+1}\dfrac{x+1}{x+1} - \dfrac{1}{x+1}\dfrac{x+h+1}{x+h+1}}{h} = \frac{\dfrac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h} = \frac{\dfrac{-h}{(x+h+1)(x+1)}}{h}$$

   Since dividing by h is the same as multiplying by its reciprocal $1/h$, we have

$$\frac{\dfrac{-h}{(x+h+1)(x+1)}}{h} = \frac{-h}{(x+h+1)(x+1)}\left(\frac{1}{h}\right) = \frac{-1}{x^2+xh+x+x+h+1} = \boxed{-\frac{1}{(x+1)^2+h(x+1)}}$$

   Note that there are many ways to write the denominator. We chose to split it into parts with and without an h and factor each part, but other forms are also correct.

5. The point $P(5,-2)$ lies on the curve $y = 2/(4-x)$. The point Q also lies on this curve with x-coordinate equal to 3. Then find slope m_{PQ}

   We have the point $P = (x_p, y_p) = (5,-2)$ and the point $Q = (x_q, y_q)$ with the facts that $x_q = 3$ and $y_q = 2/(4-x_q)$, since Q lies on the curve. Before we can calculate the slope, we must first find the value of y_q. Substituting into the equation for the curve, we have

$$y_q = \frac{2}{4-x_q} = \frac{2}{4-(3)} = \frac{2}{1} = 2$$

   Now that we know $y_q = 2$, we can use the slope formula to calculate m_{PQ}. This slope is the change in y values divided by the change in x values. If we start at P and go to Q, our change in y values is our destination y_q minus our origin y_p. Thus,

$$m_{PQ} = \frac{y_q - y_p}{x_q - x_p} = \frac{(2)-(-2)}{(3)-(5)} = \frac{4}{-2} = \boxed{-2}$$

6. Find the limit, if it exists: $\lim_{x\to a} \dfrac{|x-a|}{x-a}$.

   Since this function has an absolute value, and absolute values are shady when it comes to limits, we suspect it might not exist. One way to show that a limit does not exist is to find the left and right limits and show they are not equal.

   By the definition of absolute value, if $x - a < 0$, then $|x-a| = -(x-a)$. Thus, if we approach a from the left ($x \to a^-$), then

$$\lim_{x\to a^-} \frac{|x-a|}{x-a} = \lim_{x\to a^-} \frac{-(x-a)}{x-a} = \lim_{x\to a^-} -1 = -1$$

   Similarly, if $x - a > 0$, then $|x-a| = x-a$, so that, if we approach a from the right ($x \to a^+$), then

$$\lim_{x\to a^+} \frac{|x-a|}{x-a} = \lim_{x\to a^+} \frac{x-a}{x-a} = \lim_{x\to a^+} 1 = 1$$

   Since the left limit (-1) does not equal the right limit (1), the limit $\boxed{\text{does not exist}}$.

7. Obtain the point/s of discontinuity, if any, for

$$f(x) = \begin{cases} \pi^x, & \text{if } x < 0 \\ x^\pi, & \text{if } x \geq 0 \end{cases}$$

The functions π^x and x^π, as power and exponential functions, are continuous, so we only need to check for discontinuity at the point $x = 0$. Since π^x is continuous, $\lim_{x\to 0} \pi^x = \pi^0 = 1$, and, since x^π is continuous, $\lim_{x\to 0} x^\pi = 0^\pi = 0$.

Thus, the left limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \pi^x = 1$$

but the right limit

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0} x^\pi = 0$$

Since the left and right limits are not equal, the limit $\lim_{x\to 0} f(x)$ does not exist, so $f(x)$ has a point of discontinuity at $\boxed{x = 0}$.

8. Obtain the limit, if it exists: $\lim_{h\to 0} \dfrac{\sqrt{64+h} - 8}{h}$.

At $h = 0$, the numerator is $\sqrt{64 + (0)} - 8 = \sqrt{64} - 8 = 8 - 8 = 0$, and the denominator is also zero, so we must manipulate the limit to make the denominator nonzero. Notice that we have something of the form $\sqrt{x} + y$ (in this case, $x = 64 + h$ and $y = -8$). In situations like this, multiplying by the conjugate $\sqrt{x} - y$ can help, so we try that and see what happens:

$$\frac{\sqrt{64+h} - 8}{h} \frac{\sqrt{64+h} + 8}{\sqrt{64+h} + 8} = \frac{\left(\sqrt{64+h}\right)^2 - 8^2}{h\left(\sqrt{64+h} + 8\right)} = \frac{64 + h - 64}{h\left(\sqrt{64+h} + 8\right)} = \frac{h}{h\left(\sqrt{64+h} + 8\right)} = \frac{1}{\sqrt{64+h} + 8}$$

It worked! Now we have an function that is continuous when the denominator is not equal to zero, and, if we plug in $h = 0$, we get

$$\lim_{h\to 0} \frac{\sqrt{64+h} - 8}{h} = \lim_{h\to 0} \frac{1}{\sqrt{64+h} + 8} = \frac{1}{\sqrt{64 + (0)} + 8} = \boxed{\frac{1}{16}}$$

9. Find the number a such that the limit exists. $\lim_{x\to -2} \dfrac{3x^2 + ax + a + 6}{x^2 + x - 2}$

First, take a look at the denominator. It factors into $x^2 + x - 2 = (x + 2)(x - 1)$. Since the powers of the factors are odd numbers, we know that, if we have a vertical asymptote at -2, our function will approach positive infinity from one side and negative infinity from the other, so the limit wouldn't exist. Therefore, $x = -2$ must be a zero of our numerator. That means, plugging in $x = -2$, we must have

$$0 = 3(-2)^2 + a(-2) + a + 6 = 12 - 2a + a + 6 = 18 - a$$

Therefore, since $0 = 18 - a$, we must have $\boxed{a = 18}$. For completeness, we find

$$\lim_{x\to -2} \frac{3x^2 + 6x + 24}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3(x+2)(x+4)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3(x+4)}{(x-1)} = \frac{3(-2+4)}{(-2-1)} = -2$$

10. Suppose a continuous function $f(x)$ obeys the relation $2x - 1 \leq f(x) \leq 3x^2 - 2$ for all x, then find $\lim_{x\to 1} f(x)$.

    Since $f(x)$ is squeezed in between the functions $2x - 1$ and $3x^2 - 2$, we will use the squeeze theorem. The squeeze theorem says that, if the limits on the left and right exist and are equal, then the limit in the middle exists and equals that same thing. Thus,

$$1 = \lim_{x\to 1}(2x - 1) \leq \lim_{x\to 1} f(x) \leq \lim_{x\to 1}(3x^2 - 2) = 1$$

    means that $\lim_{x\to 1} f(x) = \boxed{1}$.