1. $\frac{dy}{dt} + \frac{1}{t}y = \frac{\sin t}{t^2 y}$, $t>0$ This is Bernoulli.

$\Downarrow$

Use $v = y^2$, $\frac{dv}{dt} = 2y\frac{dy}{dt} \Rightarrow$

$\frac{1}{2}\frac{dv}{dt} = y\frac{dy}{dt}$

$y\frac{dy}{dt} + \frac{1}{t}y^2 = \frac{\sin t}{t^2} \Rightarrow$

$\frac{1}{2}\frac{dv}{dt} + \frac{1}{t}v = \frac{\sin t}{t^2} \Rightarrow \frac{dv}{dt} + \frac{2}{t}v = 2\frac{\sin t}{t^2}$

Integrating factor $\mu = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$

So, $\frac{d}{dt}(t^2 v) = 2\sin t \Rightarrow$

$t^2 v = 2\int \sin t\, dt = -2\cos t + C \Rightarrow$

$v = \frac{C - 2\cos t}{t^2} \Rightarrow$

$y^2 = \frac{C - 2\cos t}{t^2} \Rightarrow$

$$\boxed{y = \pm\sqrt{\frac{C - 2\cos t}{t^2}}}$$

2. $(x^2+3xy+y^2)\,dx - x^2\,dy = 0$

Note: This is not exact, since $M_y = 3x + 2y \neq -2x = N_x$, and there is no simple integrating factor to make it exact, since $\frac{N_x - M_y}{N}$ is not a function of just $x$, and $\frac{M_y - N_x}{M}$ not of just $y$.

Divide by $x^2 dx$:

Use, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2 - \frac{dy}{dx} = 0 \Rightarrow$$

$$1 + 3v + v^2 - \left(v + x\frac{dv}{dx}\right) = 0 \Rightarrow$$

$$(1+v)^2 = x\frac{dv}{dx} \Rightarrow$$

$$\frac{dx}{x} = \frac{dv}{(1+v)^2} \Rightarrow \ln|x| = -\frac{1}{1+v} + C \Rightarrow$$

$$\ln|x| = -\frac{1}{1+\frac{y}{x}} + C \Rightarrow$$

$$\boxed{\frac{x}{x+y} + \ln|x| = C}$$

Here, we excluded $v = -1$ or $y = -x$, so check:

since $\frac{dy}{dx} = -1$

$$y^2 - 3y^2 + y^2 + y^2 = 0 \Leftrightarrow$$

$$0 = 0 \checkmark$$

So, $\boxed{y = -x}$ is also a solution.

3. a) $V(t) = 50 + (r - q)t$

initial  rate in  rate out

b) $\dfrac{dQ}{dt} = r \cdot S_r - q\,\dfrac{Q}{V}$ , $Q(0) = 50 \cdot s_0$

rate in  concentration in  rate out  concentration out  Initial Volume  initial concentration

c) Full when $V(t) = 100 = 50 + (10-5)t = 50 + 5t \Rightarrow$

$100 = 50 + 5t \Rightarrow \boxed{t = 10}$

d) when tank is full, the rate out is $r = 10$ (since it spills over).

$$\frac{dQ_f}{dt} = r \cdot S_r - r\,\frac{Q}{100} = 10\,S_r - \frac{Q}{10}$$

rate in  conc. in  rate out  conc. out  volume is fixed at 100 gal when full

e) By continuity, the initial value $Q_f(10)$ equals the final value $Q(10)$.

f) The limiting value is $\boxed{S_r.}$ (You can say that, after a long time, it is close to $S_r$)

Since the general solution is a decaying exponential (possibly negative) plus some constant, $\dfrac{dQ_f}{dt} \to 0$,

and $$0 = \lim_{t\to\infty} \frac{dQ_f}{dt} = 10\,S_r - \left(\lim_{t\to\infty} \frac{Q}{100}\right)10 \Rightarrow$$

$$\lim_{t\to\infty} \frac{Q}{100} = S_r$$

4. $K = 3, \quad m = 2, \quad \gamma = 1, \quad f(t) = 3\cos(3t) - 2\sin(3t)$

Steady state means particular

$mu'' + \gamma u' + Ku = f(t) \Rightarrow$

$2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t)$

$$\begin{array}{c|l} 3 & y_p = A\cos(3t) + B\sin(3t) \\ 1 & y_p' = -3A\sin(3t) + 3B\cos(3t) \\ 2 & y_p'' = -9A\cos(3t) - 9B\sin(3t) \end{array}$$

$(3A + 3B - 18A)\cos 3t + (3B - 3A - 18B)\sin(3t) = f(t)$

$\Rightarrow \quad -15A + 3B = 3 \quad \Rightarrow \quad -75A + 15B = 15 \Rightarrow$

$\quad -3A - 15B = -2 \qquad -78A = 13 \Rightarrow$

$A = -\frac{1}{6}$

$-3(-\frac{1}{6}) - 15B = -2 \Rightarrow$

$-15B = -\frac{5}{2} \Rightarrow -3B = -\frac{1}{2} \Rightarrow B = \frac{1}{6}$

$y_p = -\frac{1}{6}\cos(3t) + \frac{1}{6}\sin(3t)$

$R = \sqrt{A^2 + B^2} = \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6}$

In quad IV, but should be in quad II

$\omega = 3$

Quadrant adjustment

$\delta = \operatorname{atan}\left(\frac{B}{A}\right) + \underline{\qquad} = \operatorname{atan}(-1) + \pi = \frac{3\pi}{4}$

So, $\boxed{\frac{\sqrt{2}}{6}\cos\left(3t - \frac{3\pi}{4}\right) = y_p}$

5. $1 + \left(\frac{x}{y} - \sin y\right) y' = 0 \Rightarrow$ multiply by $y\,dx$

$$\underbrace{y}_{M}\,dx + \underbrace{(x - y\sin y)}_{N}\,dy = 0$$

$M_y = 1 = N_x$, so it's exact.

$\Psi_x = M \Rightarrow \Psi = \int M\,dx = \int y\,dx = xy + g(y) \Rightarrow$

$N = \Psi_y = x + g'(y) \Rightarrow x - y\sin y = x + g'(y)$

$\Rightarrow g'(y) = -y\sin y \Rightarrow$

$g(y) = -\int y \sin y\,dy = -\left[-y\cos y + \int \cos y\,dy\right] = y\cos y - \sin y + C$

$u = y \qquad v = -\cos y$

$du = dy \qquad dv = \sin y\,dy$

$\Psi = xy + y\cos y - \sin y + c \Rightarrow$

$$\boxed{xy + y\cos y - \sin y = C}$$

---

Finding integrating factor '$y$' from

$$\underbrace{dx}_{M=1} + \underbrace{\left(\frac{x}{y} - \sin y\right)}_{N} dy = 0$$

$M_y = 0 \neq \frac{1}{y} = N_x \Rightarrow$ <u>not exact</u>

However, $\frac{M_y - N_x}{M} = \frac{-\frac{1}{y}}{1} = -\frac{1}{y}$ is a function of <u>Just</u> $y$!

So, $(\mu M)dx + (\mu N)dy = 0$ is exact when

$\mu_y M + \mu M_y = \mu_x N + \mu N_x = \mu N_x$ (since ) $\Rightarrow$

$\frac{d\mu}{dy} + \mu\left(\frac{M_y - N_x}{M}\right) = 0 \Rightarrow \frac{d\mu}{dy} - \frac{1}{y}\mu = 0$

$\Rightarrow \frac{d}{dy}\left(\frac{1}{y}\mu\right) = 0$

$\Rightarrow \boxed{\mu = C \cdot y}$ And we may take $C=1$ for simplicity.

6. $y''' + y' = \sec t$

First, solve homogeneous:

$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r = 0,\ r = \pm i$

$y_h = c_1 + c_2 \cos t + c_3 \sin t$

Now, let $c_1, c_2, c_3$ be functions of $t$:

set $= 0$

$$\begin{array}{l|l}
0 & y_p = c_1(t) + c_2(t)\cos t + c_3(t)\sin t \\
1 & y_p' = -c_2 \sin t + c_3 \cos t + \underline{c_1' + c_2' \cos t + c_3' \sin t} = 0 \\
0 & y_p'' = -c_2 \cos t - c_3 \sin t \qquad - c_2' \sin t + c_3' \cos t = 0 \\
1 & y_p''' = c_2 \sin t - c_3 \cos t \qquad - c_2' \cos t - c_3' \sin t = \sec t
\end{array}$$

Plug In to get

$$D = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} = \sin^2 t + \cos^2 t = 1$$

$$c_1' = \frac{\begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \sec t & -\cos t & -\sin t \end{vmatrix}}{D} = \sec t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \sec t \Rightarrow c_1 = \ln|\sec t + \tan t|$$

$$c_2' = \frac{\begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \sec t & -\sin t \end{vmatrix}}{D} = \begin{vmatrix} 0 & \cos t \\ \sec t & -\sin t \end{vmatrix} = -1 \Rightarrow c_2 = -t$$

$$c_3' = \frac{\begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \sec t \end{vmatrix}}{D} = \begin{vmatrix} -\sin t & 0 \\ -\cos t & \sec t \end{vmatrix} = -\sin t \sec t = -\tan t \Rightarrow c_3 = \ln(\cos t)$$

So, $y = y_h + c_1 + c_2 \cos t + c_3 \sin t$

Right here, switch back to $c_1, c_2, c_3$ as constants.

$$= \boxed{c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec t + \tan t| - t\cos t + (\sin t)\ln(\cos t)}$$

$c_1, c_2, c_3$ are now arbitrary constants for the homogeneous problem.

This notation was used because $c_1, c_2$, etc. are often used as functions on the test, but your book has used $c_1, c_2$, etc. as arbitrary constants.

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7. $y'' + y = \delta(t-\pi)\cos t + g(t)$, $y(0)=1$, $y'(0)=1$, $g(t) = \begin{cases} 0, & t<7 \\ 1, & 7\le t<8 \\ t, & 8\le t \end{cases}$

Write $g(t)$ in terms of heaviside function:

$$g(t) = u_7(t) + (t-1)u_8(t)$$

Take Laplace transform of both sides:

These must match!

$$s^2Y - s - 1 + Y = \mathcal{L}\{\delta(t-\pi)\cos t + u_7(t) + (t-8)u_8(t) + 7u_8(t)\}$$

$$= e^{-s\pi}\cos\pi + \frac{e^{-7s}}{s} + \frac{e^{-8s}}{s^2} + 7\frac{e^{-8s}}{s}$$

$$\Rightarrow Y = \frac{s+1}{s^2+1} - \frac{e^{-s\pi}}{s^2+1} + \frac{e^{-7s} + 7e^{-8s}}{s(s^2+1)} + \frac{e^{-8s}}{s^2(s^2+1)}$$

Partial Fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow A(s^2+1) + (Bs+C)s = 1$$

$s=0 \Rightarrow A = 1$

$s=1 \Rightarrow 2A+B+C=1 \Rightarrow B+C=-1$

$s=-1 \Rightarrow 2A+B-C=1 \Rightarrow B-C=-1$

$\Rightarrow C=0,\ B=-1$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow As(s^2+1) + B(s^2+1) + (Cs+D)s^2 = 1$$

$$\Rightarrow As^3 + As + Bs^2 + B + Cs^3 + Ds^2 = 1$$

$$\Rightarrow (A+C)s^3 + (B+D)s^2 + As + B = 1 + 0s + 0s^2 + 0s^3$$

$$\Rightarrow A+C=0,\quad B+D=0,\quad A=0,\quad B=1$$

$C=0$, $D=-1$

So,

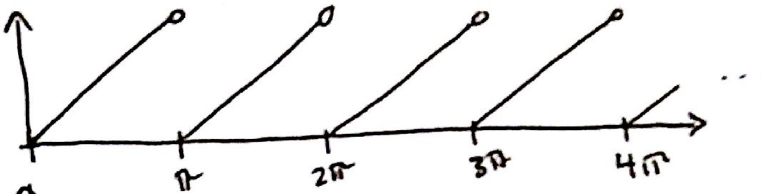
$$Y = \frac{s}{s^2+1} + \frac{1}{s^2+1} - e^{-s\pi}\left(\frac{1}{s^2+1}\right) + \left(e^{-7s} + 7e^{-8s}\right)\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + e^{-8s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

and

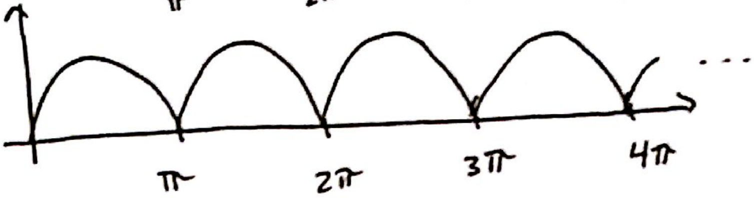
$$\boxed{\begin{aligned} y = \cos t + \cancel{\sin t} - \sin(t-\pi)u_\pi(t) + u_7(t) + 7u_8(t) \\ - \cos(t-7)u_7(t) - 7\cos(t-8)u_8(t) \\ + (t-8)u_8(t) - \sin(t-8)u_8(t) \end{aligned}}$$

8. $\mathcal{L}\{f * g\}$

$f(t) =$




$g(t) =$




From section 6.6, $\mathcal{L}\{f * g\} = F(s)\,G(s)$. Now,

$$\int_0^{\pi} e^{-st}\, t\, dt = \mathcal{L}\{f(t)\} - \mathcal{L}\{f(t-\pi)u_{\pi}(t)\}$$

$$= F(s) - e^{-\pi s}F(s) \Rightarrow$$

$$F(s) = \frac{\int_0^{\pi} e^{-st}\, t\, dt}{1-e^{-\pi s}} = \frac{\left[t\frac{e^{-st}}{-s}\right]_0^{\pi} + \int_0^{\pi}\frac{e^{-st}}{s}\,dt}{1-e^{-\pi s}}$$

$$= \frac{-\frac{\pi e^{-s\pi}}{s} + \left[-\frac{e^{-s\pi}}{s^2} + \frac{1}{s^2}\right]}{1-e^{-s\pi}}$$

and

$$G(s) = \frac{\int_0^{\pi} e^{-st}\sin t\, dt}{1-e^{-\pi s}}$$

$$\int_0^{\pi} e^{-st}\sin t\, dt = \left[-\frac{\sin t}{s}e^{-st}\right]_0^{\pi} + \frac{1}{s}\underbrace{\int_0^{\pi} e^{-st}\cos t\, dt}$$

$u = \sin t \quad v = -\frac{1}{s}e^{-st}$

$du = \cos t\, dt \quad dv = e^{-st}dt$

$u = \cos t$

$du = -\sin t\, dt$

$dv = e^{-st}dt$

$v = -\frac{1}{s}e^{-st}$

$$= \left[-\frac{\cos t}{s}e^{-st}\right]_0^{\pi} - \frac{1}{s}\int \sin t\, e^{-st}dt$$

$$\Rightarrow \left(1+\frac{1}{s^2}\right)\int_0^{\pi} e^{-st}\sin t\, dt = \frac{1}{s}\left[\frac{1}{s} + \frac{e^{-\pi s}}{s}\right] \Rightarrow \int_0^{\pi} e^{-st}\sin t\, dt = \frac{2e^{-st}}{s^2+1}$$

So,

$$\boxed{\mathcal{L}\{f * g\} = F(s)\,G(s) = \left[\frac{1-e^{-s\pi}(\pi s+1)}{s^2(1-e^{-s\pi})}\right]\left[\frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}\right]}$$