University of Central Florida Spring Term 2015
Calculus I Exam #1 Version C
9 questions - 90 points

Instructions: You have exactly one hour and twenty minutes to complete this examination. All work must be done in your blue book. You must supply all details to and justifications for your solution to receive full credit. **Simply stating an answer will get scored as zero credit. You must reduce all final answers to their simplest algebraic form to receive full credit. No calculators are permitted.**

#1. Evaluate the limit $\lim_{x\to 5}\frac{5x-x^2}{\sqrt{5}-\sqrt{x}}$ if it exists, if not, explain why.

#2. Evaluate the limit $\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}$ if it exists, if not, explain why.

#3. Evaluate the limit $\lim_{x\to -4}\frac{|x+4|}{x+4}$ if it exists, if not, explain why.

#4. Make a conjecture about the value of the limit $\lim_{x\to 1^+}\sqrt{x-1}\sin(\frac{\pi}{x-1})$ and prove your conjecture using the Squeeze Theorem.

#5. Suppose that $f(x)=\begin{cases} cx-x, x\le -2 \\ 1-cx^2, x>-2 \end{cases}$ Find the constant c such that f(x) is continuous over the entire real number line.

#6. Use the Intermediate Value Theorem to prove there is at least one real solution to the equation $2^x=3-2x$ on the interval [-3,2].

#7. Suppose that $f(x)=\frac{4x^5-x^3}{\sqrt{4x^{10}-3x^7}}$. Find all horizontal asymptotes of $f(x)$, if they exist.

Note: In #8 and #9 you will receive no credit for using derivative rules. You must use the definition of the derivative in order to receive credit.

#8. Using the limit of a difference quotient, find the slope of the tangent line to the curve $f(x)=2x^2+2x-1$ at the point (1,3).

#9. Using the definition of the derivative, calculate the derivative of $f(x)=-\frac{1}{x}$

①

$$\lim_{x \to 5} \frac{5x - x^2}{\sqrt{5} - \sqrt{x}}$$

$$= \lim_{x \to 5} \frac{5x - x^2}{\sqrt{5} - \sqrt{x}} \frac{(\sqrt{5} + \sqrt{x})}{(\sqrt{5} + \sqrt{x})}$$

$$= \lim_{x \to 5} \frac{x\cancel{(5 - x)}(\sqrt{5} + \sqrt{x})}{\cancel{5 - x}}$$

$$= \lim_{x \to 5} x(\sqrt{5} + \sqrt{x})$$

$$= 5(\sqrt{5} + \sqrt{5})$$

$$= 5 \cdot 2\sqrt{5}$$

$$= \boxed{10\sqrt{5}}$$

2 $\lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$

$$= \lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4} \frac{(\sqrt{x^2+9} + 5)}{(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{(x^2+9) - 25}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{x^2 - 16}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9} + 5}$$

$$= \frac{(-4) - 4}{\sqrt{(-4)^2+9} + 5}$$

$$= \frac{-8}{\sqrt{16+9} + 5}$$

$$= \frac{-8}{\sqrt{25} + 5}$$

$$= -8/10$$

$$= \boxed{-4/5}$$

③ $\lim_{x \to -4^-} \frac{|x+4|}{x+4}$

$= \lim_{x \to -4^-} \frac{|x+4|}{-|x+4|}$

$= -1$

$\lim_{x \to -4^+} \frac{|x+4|}{x+4}$

$= \lim_{x \to -4^+} \frac{|x+4|}{|x+4|}$

$= 1$

Since the left and right limits are not equal, the limit $\boxed{\text{does not exist.}}$

(4) Conjecture: $\lim_{x \to 1^+} \sqrt{x-1} \sin\left(\frac{\pi}{x-1}\right) = 0$

Proof:

We know that $-1 \leq \sin\left(\frac{\pi}{x-1}\right) \leq 1$.

Multiplying everything by $\sqrt{x-1}$ gives

$-\sqrt{x-1} \leq \sqrt{x-1} \sin\left(\frac{\pi}{x-1}\right) \leq \sqrt{x-1}$. Since

$\lim_{x \to 1^+} -\sqrt{x-1} = 0 = \lim_{x \to 1^+} \sqrt{x-1}$, by the

squeeze theorem, $\lim_{x \to 1^+} \sqrt{x-1} \sin\left(\frac{\pi}{x-1}\right) = 0$

⑤ $f(-2) = -2c + 2$

$\lim_{x \to -2^-} f(x)$

$= \lim_{x \to -2^-} (cx - x)$

$= c(-2) - (-2)$

$= -2c + 2$

$\lim_{x \to -2^+} f(x)$

$= \lim_{x \to -2^+} (1 - cx^2)$

$= 1 - c(-2)^2$

$= 1 - 4c$

$-2c + 2 = 1 - 4c$

$\Rightarrow \quad 2c + 2 = 1$

$\Rightarrow \quad 2c = -1$

$\Rightarrow \quad c = -1/2$

If $c = -1/2$, then the left and right limits are equal, so $\lim_{x \to -2} f(x) = 1 - 4(-\frac{1}{2})$
$= 3$

If $c = -1/2$, then $f(-2) = -2(-1/2) + 2 = 3$.

Since $f(-2) = \lim_{x \to -2} f(x)$, f is continuous at $x = -2$, if $c = -1/2$.

Since $cx - x$ and $1 - cx^2$ are continuous functions, $f(x)$ is alway continuous for $x \neq -2$.

Thus, if $\boxed{c = -1/2}$ f is continuous on the whole real number line.

⑥ Let $f(x) = 2^x + 2x - 3$. Then, f is continuous.

$$f(-3) = 2^{(-3)} + 2(-3) - 3$$
$$= \frac{1}{2^3} - 6 - 3$$
$$= \frac{1}{8} - 9 < 0$$
$$f(2) = 2^{(2)} + 2(2) - 3$$
$$= 4 + 4 - 3$$
$$= 5 > 0$$

Since f is continuous, $f(-3) < 0$, and $f(2) > 0$, by the IVT, there is a c in $(-3, 2)$ such that $f(c) = 0$. Thus,

$$2^c + 2c - 3 = 0$$
$$\Rightarrow 2^c = 3 - 2c$$

and there is at least one real solution.

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$$\lim_{x\to-\infty} \frac{4x^5-x^3}{\sqrt{4x^{10}-3x^7}}$$

$$= \lim_{x\to-\infty} \frac{4x^5-x^3}{\sqrt{4x^{10}-3x^7}} \frac{\frac{1}{x^5}}{\frac{1}{x^5}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4x^5-x^3}{x^5}}{\frac{\sqrt{4x^{10}-3x^7}}{x^5}}$$

$$= \lim_{x\to-\infty} \frac{4-\frac{1}{x^2}}{\frac{\sqrt{4x^{10}-3x^7}}{x^5}}$$

$$= \lim_{x\to-\infty} \frac{4-\frac{1}{x^2}}{\frac{\sqrt{4x^{10}-3x^7}}{-\sqrt{x^{10}}}}$$

$$= \lim_{x\to-\infty} \frac{4-\frac{1}{x^2}}{-\sqrt{\frac{4x^{10}-3x^7}{x^{10}}}}$$

$$= \lim_{x\to-\infty} -\frac{4-\cancel{\frac{1}{x^2}}^{\to 0}}{\sqrt{4-\cancel{\frac{3}{x^3}}^{\to 0}}}$$

$$= -\frac{4}{\sqrt{4}} = -2$$

$$\lim_{x\to\infty} \frac{4x^5-x^3}{\sqrt{4x^{10}-3x^7}} \frac{\frac{1}{x^5}}{\frac{1}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\frac{4x^5-x^3}{x^5}}{\frac{\sqrt{4x^{10}-3x^7}}{\sqrt{x^{10}}}}$$

$$= \lim_{x\to\infty} \frac{4-\cancel{\frac{1}{x^2}}^{\to 0}}{\sqrt{4-\cancel{\frac{3}{x^3}}^{\to 0}}}$$

$$= \frac{4}{\sqrt{4}} = 2$$

There are two horizontal asymptotes:

$y=-2$ and $y=2$

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$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(1+h)^2 + 2(1+h) - 1\right] - \left[2(1)^2 + 2(1) - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{2(1+2h+h^2) + 2 + 2h - 1 - 2 - 2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{6h + 2h^2}{h}$$

$$= \lim_{h \to 0} (6 + 2h)$$

$$= \boxed{6}$$

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[-\frac{1}{x+h}\right] - \left[-\frac{1}{x}\right]}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x} - \frac{1}{x+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x(x+h)} - \frac{x}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{x}x(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x(x+(0))}$$

$$= \boxed{\frac{1}{x^2}}$$