Homework 2.4 (Selected Problem)
MAP 2302 – Ordinary Differential Equations I
January 31, 2016
§2.4 # 28

2.4.28. Solve the following DE:

$$t^2y' + 2ty - y^3 = 0, \qquad t > 0$$

First, notice that this equation is Bernoulli, since it can be written in the form $y' + p(t)y = g(t)y^n$. Add y^3 to both sides and divide by t^2 to get it in that form.

$$\frac{1}{t^2}\left[t^2y' + 2ty = y^3\right] \implies y' + \frac{2}{t}y = \frac{1}{t^2}y^3$$

Here, $p(t) = 2/t$, $g(t) = 1/t^2$, and $n = 3$. We make the substitution $v = y^{1-n} = y^{1-3} = y^{-2}$ and (using implicit differentiation and the chain rule) $v' = -2y^{-3}y'$, which we can write as $-\frac{1}{2}v' = y^{-3}y'$. Before we make this substitution, though, we divide both sides by y^3.

$$\begin{aligned}\frac{1}{y^3}\left[y' + \frac{2}{t}y = \frac{1}{t^2}y^3\right] &\implies y^{-3}y' + \frac{2}{t}y^{-2} = \frac{1}{t^2}\\ &\implies -\frac{1}{2}v' + \frac{2}{t}v = \frac{1}{t^2}\\ &\implies v' - \frac{4}{t}v = -\frac{2}{t^2}\end{aligned}$$

Now we have a first order linear differential equation, and we can use the method of integrating factors to solve for v. The integrating factor is

$$\mu(t) = e^{\int -4/t\,dt} = e^{-4\ln|t|} = |t|^{-4} = \frac{1}{t^4}$$

We quickly check to make sure we did it right.

$$\frac{d}{dt}\mu(t) = \frac{d}{dt}\frac{1}{t^4} = -4t^{-5} = -\frac{4}{t}\frac{1}{t^4} \stackrel{\checkmark}{=} p(t)\mu(t)$$

Keep in mind that the $p(t)$ here is with respect to v. That is, it's from the new, linear equation we found, not from the original problem. Now, multiply by $\mu(t)$.

$$\begin{aligned}\frac{1}{t^4}\left[v' - \frac{4}{t}v = -\frac{2}{t^2}\right] &\implies \frac{1}{t^4}v' - \frac{4}{t^5}v = -\frac{2}{t^6}\\ &\implies \frac{d}{dt}\left(\frac{1}{t^4}v\right) = -\frac{2}{t^6}\end{aligned}$$

And then integrate.

$$\frac{1}{t^4}v = \int -\frac{2}{t^6}\,dt = \frac{2}{5}t^{-5} + C \implies v = \frac{2}{5}t^{-1} + Ct^4$$

Since we have no initial condition, we must leave the constant C unspecified. Solve for y using $v = y^{-2}$ and simplify.

$$\begin{aligned}y^{-2} = \frac{2}{5t} + Ct^4 = \frac{2 + Ct^5}{5t} &\implies y^2 = \frac{5t}{2 + Ct^5}\\ &\implies y = \pm\sqrt{\frac{5t}{2 + Ct^5}}\end{aligned}$$

Just for fun, let's check our answer. It's not as difficult as it looks. First, we'll use implicit differentiation and the quotient rule on the solution without the root.

$$\frac{d}{dt}\left(y^2 = \frac{5t}{2+Ct^5}\right) \implies 2yy' = \frac{(2+Ct^5)5 - 5t(5Ct^4)}{(2+Ct^5)^2} = \frac{10-20Ct^5}{(2+Ct^5)^2}$$

Since we know $y \equiv 0$ is a singular solution and not the one in which we're interested, we can multiply the original equation by y.

$$y\left[t^2y' + 2ty - y^3 = 0\right] \implies t^2yy' + 2ty^2 - y^4 = 0$$

Now, we can substitute for yy' and y^2. (Remember, $y^4 = (y^2)^2$).

$$t^2\frac{5-10Ct^5}{(2+Ct^5)^2} + 2t\frac{5t}{2+Ct^5} - \left(\frac{5t}{2+Ct^5}\right)^2 \stackrel{?}{=} 0$$

Multiply both sides by $(2+Ct^5)^2$ and simplify.

$$\begin{aligned} t^2(5-10Ct^5) + 10t^2(2+Ct^5) - 25t^2 \stackrel{?}{=} 0 &\implies 5t^2 - 10Ct^7 + 20t^2 + 10Ct^7 - 25t^2 \stackrel{?}{=} 0 \\ &\implies 0 \stackrel{\checkmark}{=} 0 \end{aligned}$$

Now, we have confidence in our answer.

$$\boxed{y = \pm\sqrt{\frac{5t}{2+Ct^5}}}$$