Relating Spherical Coordinates In Rotated Frames

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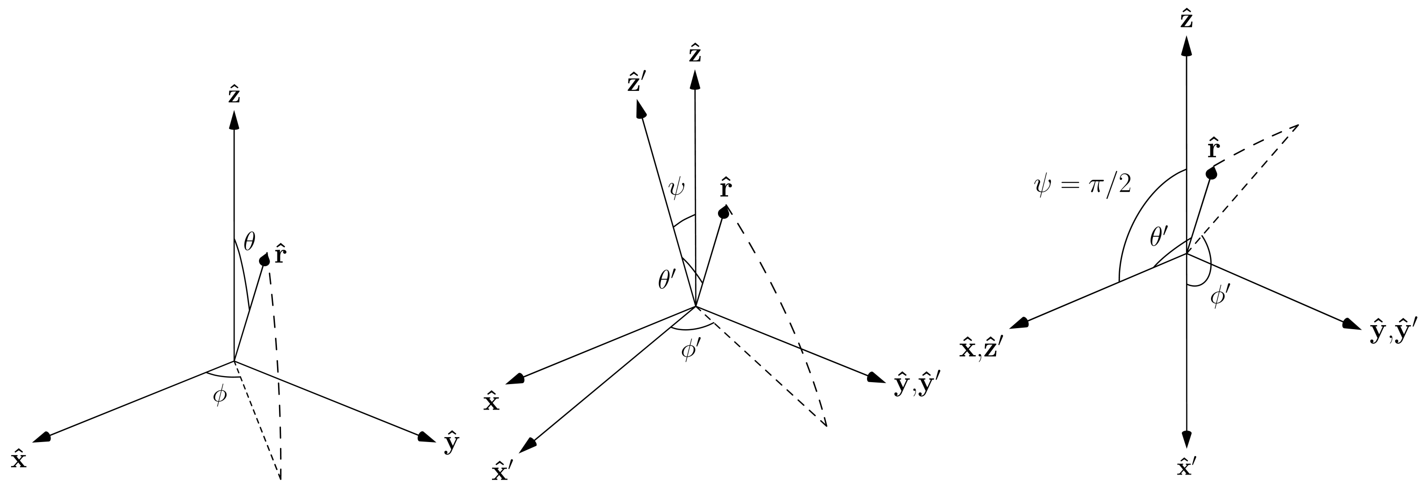
1 Introduction

In these notes we will relate typical spherical coordinates (Figure 1a) to obliquely rotated (Figure 1b) and orthogonal (Figure 1c) spherical coordinates.



(a) Typical spherical coordinates (b) ψ-rotated spherical coordinates (c) Orthogonal spherical coordinates

Figure 1: Possible spherical coordinate definitions.

2 Relating Spherical Coordinates In Two Oblique Frames

First, we expand a single unit vector $\hat{\mathbf{r}}$ in two different coordinate systems

$$\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z'\hat{\mathbf{z}}'. \tag{1}$$

Next, we rewrite the vector in spherical coordinates

$$\hat{\mathbf{r}} = \cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} = \cos\phi'\sin\theta'\hat{\mathbf{x}}' + \sin\phi'\sin\theta'\hat{\mathbf{y}}' + \cos\theta'\hat{\mathbf{z}}'. \tag{2}$$

Finally, we relate the coordinate systems. We will only consider coordinate systems that are related by a right handed rotation by angle ψ about the $\hat{\mathbf{y}}$ axis, so the coordinate systems are related by

$$\hat{\mathbf{x}} = \cos\psi\hat{\mathbf{x}}' + \sin\psi\hat{\mathbf{z}}' \tag{3a}$$
$$\hat{\mathbf{y}} = \hat{\mathbf{y}}' \tag{3b}$$
$$\hat{\mathbf{z}} = -\sin\psi\hat{\mathbf{x}}' + \cos\psi\hat{\mathbf{z}}'. \tag{3c}$$

Plugging equation 3 into equation 2 and equating each component gives

$$\cos\psi\cos\phi\sin\phi - \sin\psi\cos\theta = \cos\phi'\sin\theta' \tag{4a}$$
$$\sin\phi\sin\theta = \sin\phi'\sin\theta' \tag{4b}$$
$$\sin\psi\cos\phi\sin\theta + \cos\psi\cos\theta = \cos\theta'. \tag{4c}$$

Solving for θ' and ϕ' in terms of θ, ϕ, and ψ gives

$$\theta' = \arccos\left(\sin\psi\cos\phi\sin\theta + \cos\psi\cos\theta\right) \tag{5a}$$

$$\phi' = \arccos\left(\frac{\cos\psi\cos\phi\sin\phi - \sin\psi\cos\theta}{\sqrt{1-(\sin\psi\cos\phi\sin\theta+\cos\psi\cos\theta)^2}}\right). \tag{5b}$$

Equation 5a follows directly from equation 4c. Equation 5b follows from plugging equation 5a into equation 4a and using the identity $\sin(\arccos(x)) = \sqrt{1-x^2}$.

The inverse equations can be found by the substituting $\psi \to -\psi$:

$$\theta = \arccos\left(-\sin\psi\cos\phi'\sin\theta' + \cos\psi\cos\theta'\right) \tag{6a}$$

$$\phi = \arccos\left(\frac{\cos\psi\cos\phi'\sin\phi' + \sin\psi\cos\theta'}{\sqrt{1-(\sin\psi\cos\phi'\sin\theta'+\cos\psi\cos\theta')^2}}\right). \tag{6b}$$

3 Relating Spherical Coordinates In Two Orthogonal Frames

Setting $\psi = \pi/2$ in equation 5 gives

$$\theta' = \arccos\left(\cos\phi\sin\theta\right) \tag{7a}$$

$$\phi' = \arccos\left(\frac{-\cos\theta}{\sqrt{1-\cos^2\phi\sin^2\theta}}\right). \tag{7b}$$

The inverse equations are

$$\theta = \arccos\left(-\cos\phi'\sin\theta'\right) \tag{8a}$$

$$\phi = \arccos\left(\frac{\cos\theta'}{\sqrt{1-\cos^2\phi'\sin^2\theta'}}\right). \tag{8b}$$

We'll test these equations with $\theta = \pi/4$ and $\phi = \pi/3$ as illustrated in Figure 1a. Plugging into equation 7 gives

$$\theta' = \arccos\left(\cos(\pi/3)\sin(\pi/4)\right) = 1.21 = 69.0^\circ \tag{9a}$$

$$\phi' = \arccos\left(\frac{-\cos(\pi/4)}{\sqrt{1-\cos^2(\pi/3)\sin^2(\pi/4)}}\right) = 2.43 = 139.1^\circ \tag{9b}$$

which is consistent with Figure 1c. We'll test the inverse equation as well. Plugging equation 10 into equation 8 gives

$$\theta = \arccos\left(-\cos(2.43)\sin(1.21)\right) = \pi/4 \tag{10a}$$

$$\phi = \arccos\left(\frac{\cos(1.21)}{\sqrt{1-\cos^2(2.43)\sin^2(1.21)}}\right) = \pi/3 \tag{10b}$$

as expected.

4 Discussion

Equations 7 and 8 will be useful for analyzing data collected with the polarization diSPIM. If we measure θ and ϕ in both views with the optical axis along the $\hat{\mathbf{z}}$ axis, then we can use the equations 7 and 8 to convert the measurements to the same coordinate system. Note that these equations will only work if the $\hat{\mathbf{y}}$ axis is defined as the normal to the plane containing the optical axes.