Faster Techniques For 3D Orientation Reconstruction

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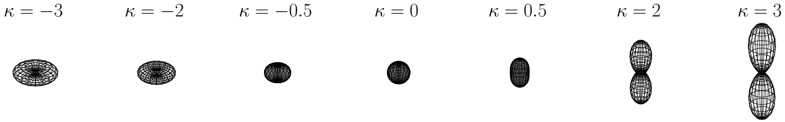
Background: Real Projective Plane $\mathbb{RP}^2$

- Set of all infinite lines through the origin
- Two dimensional space
- Not a vector space!
- Single dipole orientations live in the real projective plane
- The sphere $\mathbb{S}^2$ is a *double cover* of $\mathbb{RP}^2$
- The sphere $\mathbb{S}^2$ can be *embedded* in 3D Euclidean space $\mathbb{R}^3$

Watson Distribution



$$f(r; \hat{\mu}, \kappa) = \frac{1}{4\pi {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, \kappa\right)} \exp\{\kappa(\hat{\mu}^T r)^2\}$$

$$r \in \mathbb{RP}^2, \hat{\mu} \in \mathbb{RP}^2, \kappa \in \mathbb{R}$$

- Expensive special function ${}_1F_1$
- Can't take integrals or derivatives wrt κ
- Can't take intensity integrals wrt $\hat{\mu}$

Central Angular Gaussian Distribution

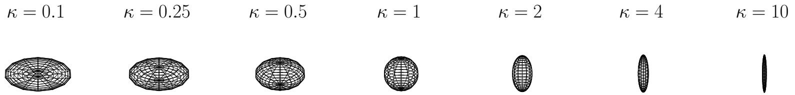
$$f(r; \mathbf{A}) = \frac{r^T \mathbf{A} r}{|\mathbf{A}|}$$

$r \in \mathbb{RP}^2$

$\mathbf{A}$ is a 3×3 positive definite matrix

- Projection of a Gaussian in $\mathbb{R}^3$ into $\mathbb{RP}^2$
- $\mathbf{A}$ defines a ellipsoid.
- Too general for us. We want rotational symmetry.

Spheroid Distribution



$$f(r; \mathbf{A}) = \frac{r^T \mathbf{A} r}{|\mathbf{A}|}$$

- **A** is a 3×3 positive definite matrix with 3 orthonormal evecs.
- The first evec is pointed along the symmetry axis $\hat{\mu} \in \mathbb{RP}^2$
- The other two evecs are have the same eigenvalue.
- The ratio of the symmetry evalue to orthogonal evalues is κ^2.
- Cheap to compute. Can take derivatives and integrals.
- κ has an easy interpretation—ratio of symmetry axis radius to orthogonal axis radius
- Matches Rudolf's previous work with birefringent materials. We're comfortable reasoning about spheroids.

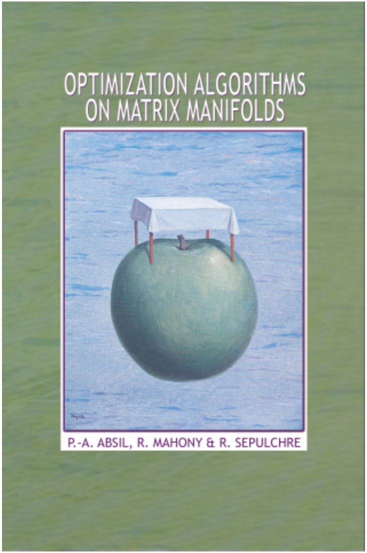
Next Topic: Optimization Problem

$$f : \mathbb{S}^2 \to \mathbb{R}$$

$$x^* = \underset{x \in \mathbb{S}^2}{\operatorname{argmin}} \; f(x)$$

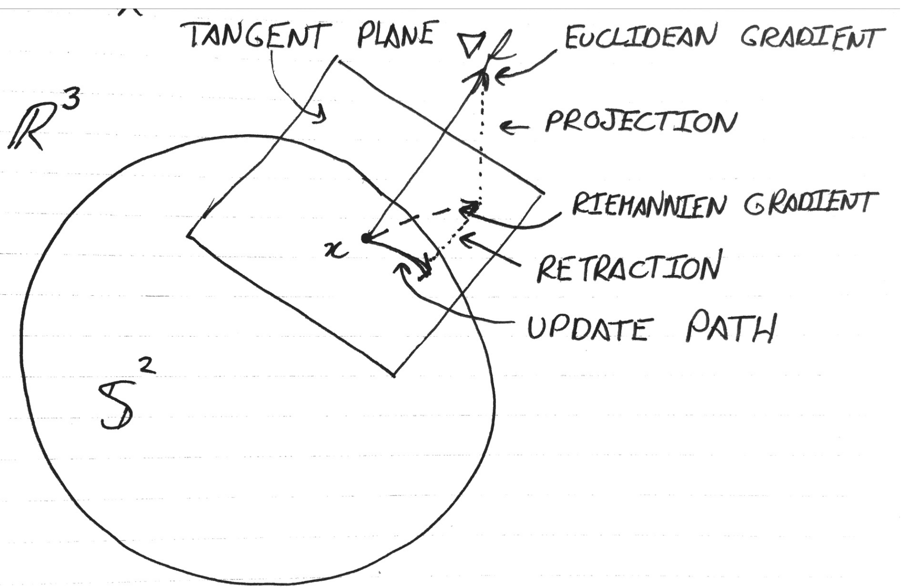
- What is the gradient of f?
- Use spherical coordinates (θ, ϕ). $\nabla f \stackrel{?}{=} \left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}\right)$
- Step size is different at different points on the sphere.
- Gradient depends on the coordinate choice.
- Trouble at the poles!

Instead optimize on a manifold



Rough Recipe For Optimizing On A Manifold

1. Let $f : \mathbb{S}^2 \to \mathbb{R}$ be the function we want to optimize.
2. Embed the manifold in Euclidean space and define a new smooth function $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$. If we constrain $\bar{f}$ to $\mathbb{S}^2$ then we recover f.
3. Choose an initial guess on $\mathbb{S}^2$.
4. Take the usual Euclidean gradient of $\bar{f}$. $\nabla \bar{f} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$
5. Project the Euclidean gradient onto the tangent space of $\mathbb{S}^2$. This projected gradient is called the *Riemannian gradient*.
6. Update your guess by moving along the Riemannian gradient.
7. *Retract* your new guess from the tangent space back onto $\mathbb{S}^2$.
8. Repeat from step 4.


TANGENT PLANE
∇f
EUCLIDEAN GRADIENT
$\mathbb{R}^3$
PROJECTION
RIEMANNIEN GRADIENT
x
RETRACTION
UPDATE PATH
S^2

Summary + Work In Progress

- A central angular Gaussian distribution with rotational symmetry gives the spheroid distribution.
- Spheroid distributions are easier to handle than Watson distributions.
- I'm working on integrating the spheroid distribution analytically which will yield a much faster forward model.
- Optimizing on an embedded manifold allows us to calculate gradients correctly.
- Multiple seed gradient methods will be much faster than the gradient-free particle swarm methods I've been using.
- Initial seed could be generated with Rudolf's proposed change of coordinates.