

# Relating Spherical Coordinates In Rotated Frames

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## 1 Introduction

In these notes we will relate typical spherical coordinates (Figure 1a) to obliquely rotated (Figure 1b) and orthogonal (Figure 1c) spherical coordinates.

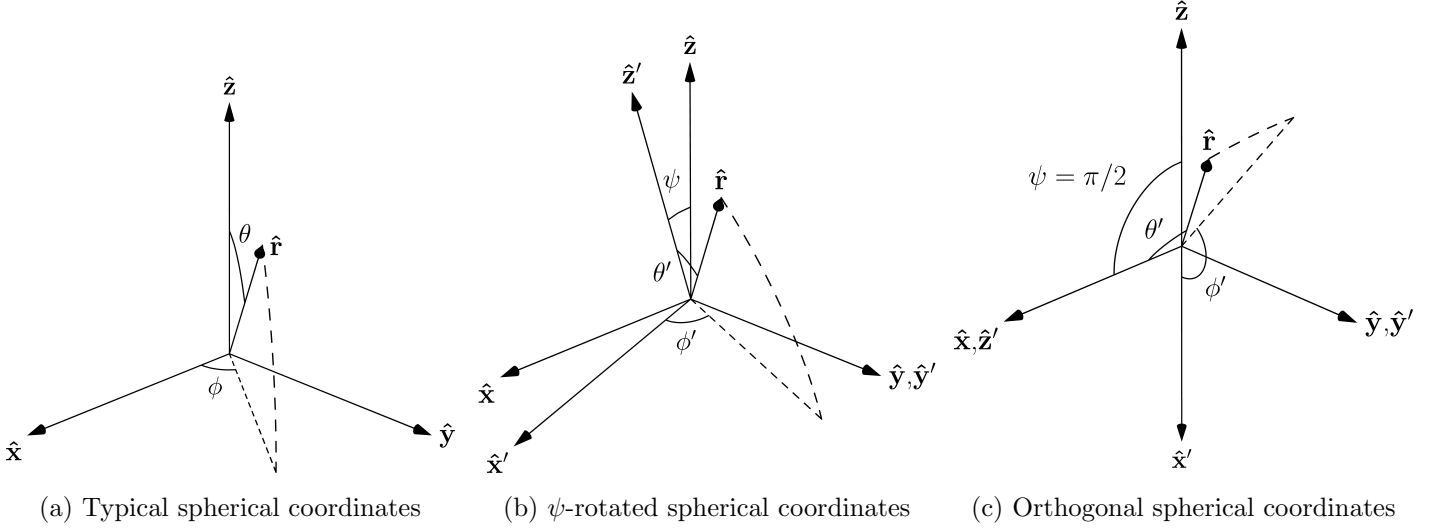


Figure 1: Possible spherical coordinate definitions.

## 2 Relating Spherical Coordinates In Two Oblique Frames

First, we expand a single unit vector  $\hat{\mathbf{r}}$  in two different coordinate systems

$$\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z'\hat{\mathbf{z}}'. \quad (1)$$

Next, we rewrite the vector in spherical coordinates

$$\hat{\mathbf{r}} = \cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} = \cos\phi'\sin\theta'\hat{\mathbf{x}}' + \sin\phi'\sin\theta'\hat{\mathbf{y}}' + \cos\theta'\hat{\mathbf{z}}'. \quad (2)$$

Finally, we relate the coordinate systems. We will only consider coordinate systems that are related by a right handed rotation by angle  $\psi$  about the  $\hat{\mathbf{y}}$  axis, so the coordinate systems are related by

$$\hat{\mathbf{x}} = \cos\psi\hat{\mathbf{x}}' + \sin\psi\hat{\mathbf{z}}' \quad (3a)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}' \quad (3b)$$

$$\hat{\mathbf{z}} = -\sin\psi\hat{\mathbf{x}}' + \cos\psi\hat{\mathbf{z}}'. \quad (3c)$$

Plugging equation 3 into equation 2 and equating each component gives

$$\cos\psi\cos\phi\sin\theta - \sin\psi\cos\theta = \cos\phi'\sin\theta' \quad (4a)$$

$$\sin\phi\sin\theta = \sin\phi'\sin\theta' \quad (4b)$$

$$\sin\psi\cos\phi\sin\theta + \cos\psi\cos\theta = \cos\theta'. \quad (4c)$$

Solving for  $\theta'$  and  $\phi'$  in terms of  $\theta$ ,  $\phi$ , and  $\psi$  gives

$$\theta' = \arccos(\sin \psi \cos \phi \sin \theta + \cos \psi \cos \theta) \quad (5a)$$

$$\phi' = \arccos\left(\frac{\cos \psi \cos \phi \sin \theta - \sin \psi \cos \theta}{\sqrt{1 - (\sin \psi \cos \phi \sin \theta + \cos \psi \cos \theta)^2}}\right). \quad (5b)$$

Equation 5a follows directly from equation 4c. Equation 5b follows from plugging equation 5a into equation 4a and using the identity  $\sin(\arccos(x)) = \sqrt{1 - x^2}$ .

The inverse equations can be found by the substituting  $\psi \rightarrow -\psi$ :

$$\theta = \arccos(-\sin \psi \cos \phi' \sin \theta' + \cos \psi \cos \theta') \quad (6a)$$

$$\phi = \arccos\left(\frac{\cos \psi \cos \phi' \sin \theta' + \sin \psi \cos \theta'}{\sqrt{1 - (\sin \psi \cos \phi' \sin \theta' + \cos \psi \cos \theta')^2}}\right). \quad (6b)$$

**Note:** The arccos in equations 5 and 6 depends the range of  $\phi$ . If  $\phi \in [-\pi, \pi]$  then use

$$\phi' = \begin{cases} \arccos\left(\frac{\cos \psi \cos \phi \sin \theta - \sin \psi \cos \theta}{\sqrt{1 - (\sin \psi \cos \phi \sin \theta + \cos \psi \cos \theta)^2}}\right) & 0 \leq \phi < \pi \\ -\arccos\left(\frac{\cos \psi \cos \phi \sin \theta - \sin \psi \cos \theta}{\sqrt{1 - (\sin \psi \cos \phi \sin \theta + \cos \psi \cos \theta)^2}}\right) & -\pi \leq \phi < 0 \end{cases} \quad (7)$$

### 3 Relating Spherical Coordinates In Two Orthogonal Frames

Setting  $\psi = \pi/2$  in equation 5 gives

$$\theta' = \arccos(\cos \phi \sin \theta) \quad (8a)$$

$$\phi' = \arccos\left(\frac{-\cos \theta}{\sqrt{1 - \cos^2 \phi \sin^2 \theta}}\right). \quad (8b)$$

The inverse equations are

$$\theta = \arccos(-\cos \phi' \sin \theta') \quad (9a)$$

$$\phi = \arccos\left(\frac{\cos \theta'}{\sqrt{1 - \cos^2 \phi' \sin^2 \theta'}}\right). \quad (9b)$$

We'll test these equations with  $\theta = \pi/4$  and  $\phi = \pi/3$  as illustrated in Figure 1a. Plugging into equation 8 gives

$$\theta' = \arccos(\cos(\pi/3) \sin(\pi/4)) = 1.21 = 69.0^\circ \quad (10a)$$

$$\phi' = \arccos\left(\frac{-\cos(\pi/4)}{\sqrt{1 - \cos^2(\pi/3) \sin^2(\pi/4)}}\right) = 2.43 = 139.1^\circ \quad (10b)$$

which is consistent with Figure 1c. We'll test the inverse equation as well. Plugging equation 11 into equation 9 gives

$$\theta = \arccos(-\cos(2.43) \sin(1.21)) = \pi/4 \quad (11a)$$

$$\phi = \arccos\left(\frac{\cos(1.21)}{\sqrt{1 - \cos^2(2.43) \sin^2(1.21)}}\right) = \pi/3 \quad (11b)$$

as expected.

## 4 Discussion

Equations 8 and 9 will be useful for analyzing data collected with the polarization diSPIM. If we measure  $\theta$  and  $\phi$  in both views with the optical axis along the  $\hat{\mathbf{z}}$  axis, then we can use the equations 8 and 9 to convert the measurements to the same coordinate system. Note that these equations will only work if the  $\hat{\mathbf{y}}$  axis is defined as the normal to the plane containing the optical axes.