Non-ideal Detectors

Assume the imaging system is linear or linearizable. Further assume $s_{in}(u)$ is the input stimulus where u is an independent variable and $s_{out}(u)$ is the output response of the system. If there are two inputs, which produce two outputs, that is: $s'_{out}(u) = s'_{in}(u)$ and $s''_{out}(u) = s''_{in}(u)$, the system is said to be linear if, when both inputs are applied together, $s_{in}(u) = s'_{in}(u) + s''_{in}(u)$, the output is given by: $s_{out}(u) = s'_{out}(u) + s''_{out}(u)$.

For a real (non-ideal) imaging system, the input maybe localized to a location u_0, the response at the output is spread over a range of u centered on u_0. Conversely, any point at the output will depend on input stimuli over a range of positions at the input, that is:

$$s_{out}(u) = \int_{-\infty}^{\infty} p(u,u')s_{in}(u')du' \qquad [1]$$

Let $s_{in}(u) = \delta(u - u_0)$ and recall $\int_a^b f(u)\delta(u - u_0)du = f(u_0)$ if $a < u_0 < b$ and $\int_{-\varepsilon}^{\varepsilon} \delta(u)du = 1$ where $\varepsilon > 0$.

Therefore,

$$s_{out}(u) = p(u,u_0) \qquad [2]$$

where $p(u,u_0)$ is the impulse response function of the system.

For a shift-invariant system, $p(u,u_0) = p(u\text{-}u_0)$. That is, it is only the difference between u and u_0 that is important, not their individual values.

Therefore,

$$s_{out}(u) = \int_{-\infty}^{\infty} p(u-u')s_{in}(u')du' \quad . \qquad [3]$$

This is the definition of a convolution integral. That is, the output response of a detector can be described as a convolution of the input stimulus and the system's impulse response function. We can write

$$s_{out} = p * s_{in} \quad . \qquad [4]$$

SNR for a Real (non-ideal) Detector

The signal can be defined as:

$$signal = s_{out}^{(1)}(u) - s_{out}^{(2)}(u) = p * s_{in}^{(1)}(u) - p * s_{in}^{(2)}(u) \quad , \tag{7}$$

where p is the point spread function (psf).

$$\Im\{signal\} = MTF(f)\Im\{s_{in}^{(1)} - s_{in}^{(2)}\} \quad , \tag{8}$$

where $\Im\{\}$ is the Fourier transform. But the inputs are delta functions,

$$\therefore \Im\{signal\} = MTF(f)\Delta\phi \quad , \tag{9}$$

where $\Delta\phi = \phi_1 - \phi_2$.

Noise can be characterized in terms of square root of the variance:

$$\sqrt{\sigma_{out}^2} = \sqrt{\left\langle s_{out}^2(u)\right\rangle - \left\langle s_{out}(u)\right\rangle^2} = \sqrt{R_{w_{out}}(u) - \left\langle s_{out}(u)\right\rangle^2} \quad , \tag{6}$$

where R(u) is an autocorrelation function.

$$\Im\{noise^2\} = \Im\{R_{w_{out}}(u)\} - \Im\{\left\langle s_{out}(u)\right\rangle^2\} = W'_{out}(f) - \left\langle W'_{out}(f=0)\right\rangle^2 \equiv W_{out}(f) \tag{10}$$

where $W'_{out}(f) = \Im\{R_{w_{out}}(u)\}$; $\left\langle s_{out}(u)\right\rangle^2$ is a constant, so $\Im\{\left\langle s_{out}(u)\right\rangle^2\} = \left\langle W'_{out}(f=0)\right\rangle^2$; and $W_{out}(f)$ is the noise power spectrum.

$$\therefore SNR(f) \equiv \frac{\Im\{signal\}}{\sqrt{\Im\{noise^2\}}} = \frac{\Delta\phi MTF(f)}{\sqrt{W_{out}(f)}} \quad . \tag{11}$$

It can be shown that Eq. 11 is consistent with the Rose model: $SNR = C\sqrt{A\bar{\phi}}$.

Optimization of X-ray Imaging: X-ray Spectrum

Recall, the Rose model of SNR:

$$SNR = C\sqrt{A\bar{\phi}} \quad , \qquad [1]$$

where A is the cross-sectional area of the object, $\bar{\phi}$ is the mean x-ray fluence incident on the detector and C is the radiation contrast defined as:

$$C = \frac{\Delta\phi}{\bar{\phi}} = \frac{2(\phi_1 - \phi_2)}{(\phi_1 + \phi_2)} \quad . \qquad [2]$$

For a poly-energetic x-ray beam:

$$\phi_1 = \sum_E \phi_0(E)\exp[-\mu_1(E)t_1] \quad \text{and} \qquad [3]$$

$$\begin{aligned}\phi_2 &= \sum_E \phi_0(E)\exp[-\mu_2(E)t_2]\exp[-\mu_1(E)(t_1 - t_2)] \\ &= \sum_E \phi_0(E)\exp[-\Delta\mu(E)t_2]\exp[-\mu_1(E)t_1]\end{aligned} \quad . \qquad [4]$$

Therefore, substituting Eq. 3 and 4 into 2, gives:

$$C = \frac{\sum_E \phi_0(E)\exp[-\mu_1(E)t_1]\{1 - \exp[-\Delta\mu(E)t_2]\}}{\frac{1}{2}\sum_E \phi_0(E)\exp[-\mu_1(E)t_1]\{1 + \exp[-\Delta\mu(E)t_2]\}} \quad . \qquad [5]$$

Note as t_1 or μ_1 increases, low energy x-rays are preferentially attenuated from the beam. This is known as **beam hardening**. One consequence of beam hardening is that the radiation contrast decreases.

For a mono-energetic beam

$$C = \frac{1 - \exp[-\Delta\mu t_2]}{\frac{1}{2}\{1 + \exp[-\Delta\mu t_2]\}} \quad . \qquad [6]$$

Note, the radiation contrast for a mono-energetic beam is independent of t_1, therefore there is no beam hardening.

For small $\Delta\mu t_2$ (i.e. low contrast object) [using Taylor's expansion exp(x)~1+x]

$$C \approx \Delta\mu t_2 \quad . \qquad [7]$$

We will not use this approximation here, but it is a useful approximation for low contrast objects.

For poly-energetic x-ray beams, Eq. 1 becomes

$$SNR = \frac{\Delta\phi\sqrt{A}}{\sqrt{\phi}} = \frac{\sqrt{A}\sum_E \phi_0(E)\exp[-\mu_1(E)t_1]\{1-\exp[-\Delta\mu(E)t_2]\}}{\sqrt{\frac{1}{2}\sum_E \phi_0(E)\exp[-\mu_1(E)t_1]\{1+\exp[-\Delta\mu(E)t_2]\}}} \qquad [8]$$

For mono-energetic x-ray beams, SNR becomes:

$$SNR = \frac{\sqrt{2A}[1-\exp(-\Delta\mu t_2)]\sqrt{\phi_0\exp(-\mu_1 t_1)}}{\sqrt{1+\exp(-\Delta\mu t_2)}} \quad . \qquad [9]$$

Therefore, one can write

$$\frac{SNR^2}{X_{ent}} = \frac{2A\exp(-\mu_1 t_1)}{K}\frac{[1-\exp(-\Delta\mu t_2)]^2}{1+\exp(-\Delta\mu t_2)} \quad , \qquad [10]$$

where $X_{ent} = K\phi_0$ is the entrance exposure to the patient and K is the conversion factor from fluence to exposure.

Rose found that the minimum SNR for detection of an object is approximately 5. Therefore the minimum entrance exposure for an object to be detectable is

$$(X_{ent})_{\min} = \frac{25K\exp(+\mu_1 t_1)}{2A}\frac{[1+\exp(-\Delta\mu t_2)]}{[1-\exp(-\Delta\mu t_2)]^2} \quad . \qquad [11]$$

Note that $\sqrt{A\bar{\phi}}$ means that the noise is measured assuming an aperture with area equal to the object.

IMPORTANT: In these calculations it is assumed that the only source of noise is x-ray quantum noise (which is assumed to be Poisson). Further it is assumed that the detector is perfect (all x-ray quanta are completely absorbed, the detector adds no noise and there is no image blurring).