Lecture on Focal Spot

So far in class we have assumed that x rays are from a point source at infinity, which creates a parallel beam of x rays. In radiology, x-ray beams originate from a finite-sized source in relatively close proximity to the object and the detector (at least compare to infinity). What effect will this have on contrast, resolution, noise and SNR?

[draw schematic here]

$$d''_{fs} = \frac{x_2}{x_1} d_{fs} \quad \text{where the magnification factor} = m = \frac{x_1 + x_2}{x_1}$$

If the object is a pinhole then d''_{fs} is the projection of the focal spot on the detector. More formally, if

$f(\vec{r}'')$ is a function that describes the focal spot projected onto the detector; and

$g(\vec{r}'')$ is a function that describes the object projected onto the detector; then

$h(\vec{r}'')$ is a function that describes the image at the detector is given by:

$$h(\vec{r}'') = \left(\frac{a}{b}\right)^2 f\left(-\frac{a}{b}\vec{r}''\right) ** g\left(a\vec{r}''\right) \qquad [1]$$

where

$$a = \frac{x_1}{x_1 + x_2} = \frac{1}{m} \quad , \qquad [2]$$

$$b = \frac{x_2}{x_1 + x_2} = \frac{m-1}{m} \quad \text{and} \qquad [3]$$

$$\frac{a}{b} = \frac{x_1}{x_2} \quad . \qquad [4]$$

Taking the 2D Fourier transform and using the scaling relationship of the Fourier transform:

$$\Im_2\left\{f\left(-\frac{a}{b}\vec{r}''\right)\right\} = \left(\frac{b}{a}\right)^2 F\left(-\frac{b}{a}\vec{\rho}''\right) \quad \text{and} \qquad [5]$$

$$\Im_2\left\{g\left(a\vec{r}''\right)\right\} = \left(\frac{1}{a}\right)^2 G\left(\frac{\vec{\rho}''}{a}\right) \quad ; \qquad [6]$$

then Eq. [1] becomes:

$$H(\vec{\rho}'') = \frac{1}{a^2} F\left(-\frac{b}{a}\vec{\rho}''\right) G\left(\frac{\vec{\rho}''}{a}\right) \qquad [7]$$

Now, if $g(\vec{r}') = \delta(\vec{r}')$, then

$$H(\vec{\rho}'') = \frac{1}{a^2} F\left(-\frac{b}{a}\vec{\rho}''\right) = m^2 F\left(-\frac{x_1}{x_2}\vec{\rho}''\right) = m^2 F\left(-\frac{m-1}{m}\vec{\rho}''\right) \qquad [8]$$

is the optical transfer function (OTF) of the focal spot. This is the focal spot magnified onto the image plane.
AND if $f(\vec{r}') = \delta(\vec{r}')$, then

$$H(\vec{\rho}'') = \frac{1}{a^2} G\left(\frac{\vec{\rho}''}{a}\right) \qquad [9]$$

but m=1/a

$$\therefore H(\vec{\rho}'') = m^2 G(m\vec{\rho}'') \qquad [10]$$

is the effective spatial frequency spectrum of the object projected onto the detector plane.

However, we are interested in what happens in the object plane, as that is where the patient is. Let $D(\vec{\rho}'')$ be the OTF of the detector, then Eq. [7], as measured in the object plane, becomes

$$D(a\vec{\rho}')H(a\vec{\rho}') = \frac{1}{a^2} D(a\vec{\rho}')F(-b\vec{\rho}')G(\vec{\rho}') \,. \qquad [11]$$

Note that $\frac{1}{a^2} D(a\vec{\rho}')F(-b\vec{\rho}')$ is defined as the overall OTF of the system, OTF_{tot} in the object plane. That is,

$$MTF_{tot}(\vec{\rho}') = \frac{|OTF_{tot}(\vec{\rho}')|}{OTF_{tot}(\vec{\rho}')} = \frac{D(a\vec{\rho}')F(-b\vec{\rho}')}{D(0)F(0)} \quad . \qquad [12]$$

Note, $D(a\vec{\rho}') = D\left(\frac{\vec{\rho}'}{m}\right)$ and $F(-b\vec{\rho}') = F\left(-\frac{m-1}{m}\vec{\rho}'\right)$,

$$MTF_{tot}(\vec{\rho}') = \frac{|OTF_{tot}(\vec{\rho}')|}{OTF_{tot}(\vec{\rho}')} = \frac{D\left(\frac{\vec{\rho}'}{m}\right) F\left(-\frac{m-1}{m}\vec{\rho}'\right)}{D(0)F(0)} \quad . \qquad [13]$$

So as m increases (note $m \geq 1$), $D\left(\frac{\vec{\rho}'}{m}\right)$ gets "wider" and $F\left(-\frac{m-1}{m}\vec{\rho}'\right)$ gets "narrower".

Therefore, the overall MTF_{tot} is optimal for a certain magnification value that depends on the focal spot size and the detector MTF. As the focal spot size increases, $F(\vec{\rho}')$ becomes narrower.

Notice that $\because m \geq 1$ then $\frac{1}{m} \leq 1$ and $0 \leq \frac{m-1}{m} \leq 1$.

An Example

Let the detector and focal spot have Gaussian PSFs and therefore Guassian OTFs.

$D(\vec{\rho}'') \propto \exp\left[-\pi\left(\frac{\vec{\rho}''}{\rho''_D}\right)^2\right]$ and $F(\vec{\rho}) \propto \exp\left[-\pi\left(\frac{\vec{\rho}}{\rho_f}\right)^2\right]$

where ρ''_D and ρ_f are characteristic widths of the MTF of the detector and focal spot respectively. These correspond roughly to the full width at half maximum of the Gaussian or more precisely the σ^2 of the Gaussian distribution. Larger values of these parameters give better MTF.

Now, by Eq. [12]

$$MTF_{tot}(\vec{\rho}') = \frac{\left|OTF_{tot}(\vec{\rho}')\right|}{OTF_{tot}(\vec{\rho}')} = \frac{D(a\vec{\rho}')F(-b\vec{\rho}')}{D(0)F(0)} = \exp\left[-\pi\rho'^2\left(\frac{1}{(m\rho''_D)^2} + \frac{(m-1)^2}{m^2\rho_f^2}\right)\right]$$

MTF_{tot} will be an extremum when

$$\frac{d}{dm}\left[\frac{1}{(m\rho''_D)} + \frac{(m-1)^2}{m^2\rho_f^2}\right] = 0$$

Therefore, the optimum magnification is when $m = 1 + \left(\frac{\rho_f}{\rho''_D}\right)^2$. As ρ_f tends to zero (large focal spot), the optimum magnification tends towards 1.

Noise

The Wiener (or noise power spectrum) measured in the detector plane is given by:

$$W(\vec{\rho}'') = \lim_{\substack{x\to\infty \\ y\to\infty}} \left\langle \frac{1}{4XY} \left| \int_{-X}^{X}\int_{-Y}^{Y} \Delta I(\vec{r}'') \exp\left[(-2\pi i)(\vec{r}'' \bullet \vec{\rho}'')\right] d\vec{r}'' \right|^2 \right\rangle \qquad [14]$$

where $\Delta I(\vec{\rho}'') = I(\vec{\rho}'') - \bar{I}$, $I(\vec{\rho}'')$ is the intensity distribution in the detector plane and $\bar{I}$ is the mean. Similarly, in the object plane, the noise power spectrum is given by:

$$W(\bar{\rho}') = \lim_{\substack{x\to\infty \\ y\to\infty}} \left\langle \frac{1}{4XY} \left| \int_{-X}^{X}\int_{-Y}^{Y} \Delta I(\vec{r}') \exp\left[(-2\pi i)(\vec{r}' \bullet \bar{\rho}')\right] d\vec{r}' \right|^2 \right\rangle \qquad [15]$$

Also, $\vec{\rho}''$ is the spatial frequency conjugate to $\vec{r}''$ in the detector plane. Then,

$$\vec{r}' = a\vec{r}'' \qquad [16]$$

Then Eq. [15] becomes:

$$W(\bar{\rho}') = \lim_{\substack{x\to\infty \\ y\to\infty}} \left\langle \frac{1}{4XY} \left| \int_{-X}^{X}\int_{-Y}^{Y} \Delta I(a\vec{r}'') \exp\left[-(2\pi i)(a\vec{r}'' \bullet \vec{\rho}')\right] a d\vec{r}'' \right|^2 \right\rangle$$

$$W(\bar{\rho}') = a^2 \lim_{\substack{x\to\infty \\ y\to\infty}} \left\langle \frac{1}{4XY} \left| \int_{-X}^{X}\int_{-Y}^{Y} \Delta I(a\vec{r}'') \exp\left[(-2\pi i)(a\vec{r}'' \bullet \vec{\rho}')\right] d\vec{r}'' \right|^2 \right\rangle \qquad [17]$$

Then by Eq. [14],

$$W(\vec{\rho}') = a^2 W(a\vec{\rho}'')$$

$$W(\vec{\rho}') = \frac{1}{m^2} W\left(\frac{\vec{\rho}''}{m}\right) \qquad [18]$$

That is, compared to the noise power measured in the detector plane, the noise power in the object plane is reduced in magnitude by the magnification factor squared, but the spectrum is wider (just as the MTF is wider in the object plane). The noise is decreased in magnitude by m^2 because the object aperture is projected onto the detector plane and is enlarged. Alternatively, the noise pattern is minified in going from the detector to the object plane. Therefore, the number of photons per unit area increases, so that the noise measured by the object aperture is reduced.

Effect of Geometric Magnification on SNR

Previously we showed that in the absence of geometric magnification, the SNR as a function of spatial frequency can be written as:

$$SNR(\rho') = \frac{\Im\{signal\}}{\sqrt{\Im\{noise^2\}}} = \frac{\Delta\phi MTF(\rho')}{\sqrt{W_{out}(\rho')}} \quad . \qquad [1]$$

This was assuming that the input was composed of two delta functions separated in space. More generally

When geometric magnification is used, then using Eqs. 13 and 18

$$SNR(\rho') = \frac{\Delta\phi S(\rho') K D\left(\frac{\vec{\rho}'}{m}\right) F\left(\frac{m-1}{m}\vec{\rho}'\right)}{\sqrt{\frac{1}{m^2} W\left(\frac{\vec{\rho}''}{m}\right)}} \quad . \qquad [2]$$

where we have not assumed that the input are two delta functions and so have include the spatial frequency spectrum of the object (i.e., the Fourier transform of the object), $S(\rho')$; and

$$K = \frac{1}{D(0)F(0)} \quad .$$

$$SNR^2(\rho') = \frac{m^2 \Delta\phi^2 K^2 \left|S(\rho')\right|^2 D^2\left(\frac{\vec{\rho}'}{m}\right) F^2\left(\frac{m-1}{m}\vec{\rho}'\right)}{W\left(\frac{\vec{\rho}''}{m}\right)} \quad . \qquad [3]$$