Five.I Complex Vector Spaces

Linear Algebra
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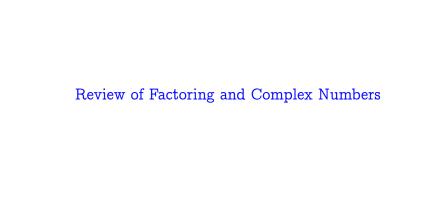
Chapter Five. Similarity

Scalars will now be complex

This chapter requires that we factor polynomials. But many polynomials do not factor over the real numbers; for instance, $x^2 + 1$ does not factor into a product of two linear polynomials with real coefficients; instead it requires complex numbers $x^2 + 1 = (x - i)(x + i)$.

Consequently in this chapter we shall use complex numbers for our scalars, including entries in vectors and matrices. That is, we shift from studying vector spaces over the real numbers to vector spaces over the complex numbers. Any real number is a complex number and in this chapter most of the examples use only real numbers but nonetheless, the critical theorems require that the scalars be complex.

Review of Factoring and Complex Numbers

Division Theorem for Polynomials

Consider a polynomial $p(x) = c_n x^n + \cdots + c_1 x + c_0$ with leading coefficient $c_n \neq 0$. The degree of the polynomial is n. If $n = 0$ then p is a constant polynomial $p(x) = c_0$. Constant polynomials that are not the zero polynomial, $c_0 \neq 0$, have degree zero. We define the zero polynomial to have degree $-\infty$.

Just as integers have a division operation — e.g., '4 goes 5 times into 21 with remainder 1' — so do polynomials.

Example

$$x^2 + x \overline{)\ 3x^3 + 2x^2\ -x + 4x^2 + x)} \quad \overset{3x}{\overline{)\ 3x^3 + 2x^2 - x + 4x^2 + x)}} \quad \overset{3x}{\overline{)\ 3x^3 + 2x^2 - x + 4x^2}} \\ -3x^3 - 3x^2$$

So, $x^2 + x$ goes $3x - 1$ times into $3x^3 + 2x^2 - x + 4$ with remainder 4. In $n = dq + r$ form: $3x^3 + 2x^2 - x + 4 = (x^2 + x) \cdot (3x - 1) + 4$.

1.2 *Theorem*   Let $p(x)$ be a polynomial. If $d(x)$ is a non-zero polynomial then there are *quotient* and *remainder* polynomials $q(x)$ and $r(x)$ such that

$$p(x) = d(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is strictly less than the degree of $d(x)$.

1.4 *Corollary*   The remainder when $p(x)$ is divided by $x - \lambda$ is the constant polynomial $r(x) = p(\lambda)$.

Proof   The remainder must be a constant polynomial because it is of degree less than the divisor $x - \lambda$. To determine the constant, take the theorem's divisor $d(x)$ to be $x - \lambda$ and substitute λ for x.   QED

   If a divisor $d(x)$ goes into a dividend $p(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $d(x)$ is a called a factor of $p(x)$. Any root of the factor, any $\lambda \in \mathbb{R}$ such that $d(\lambda) = 0$, is a root of $p(x)$ since $p(\lambda) = d(\lambda) \cdot q(\lambda) = 0$.

1.5 *Corollary*   If λ is a root of the polynomial $p(x)$ then $x - \lambda$ divides $p(x)$ evenly, that is, $x - \lambda$ is a factor of $p(x)$.

Proof   By the above corollary $p(x) = (x - \lambda) \cdot q(x) + p(\lambda)$. Since λ is a root, $p(\lambda) = 0$ so $x - \lambda$ is a factor.   QED

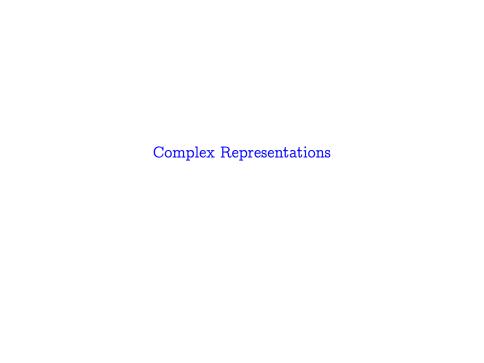## Factoring over the real numbers

1.6 *Theorem*   Any constant or linear polynomial is irreducible over the reals. A quadratic polynomial is irreducible over the reals if and only if its discriminant is negative. No cubic or higher-degree polynomial is irreducible over the reals.

1.7 *Corollary*   Any polynomial with real coefficients factors into a product of linear and irreducible quadratic polynomials with real coefficients. That factorization is unique; any two factorizations have the same factors raised to the same powers.

Factoring over the complex numbers

While $x^2 + 1$ has no real roots and so doesn't factor over the real numbers, if we imagine a root—traditionally denoted i, so that $i^2 + 1 = 0$—then $x^2 + 1$ factors into a product of linears $(x - i)(x + i)$. When we adjoin this root i to the reals and close the new system with respect to addition and multiplication then we have the complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$. (These are often pictured on a plane with a plotted on the horizontal axis and b on the vertical; note that the distance of the point from the origin is $|a + bi| = \sqrt{a^2 + b^2}$.)

In $\mathbb{C}$ all quadratics factor. That is, in contrast with the reals, $\mathbb{C}$ has no irreducible quadratics.

1.11 *Theorem* [Fundamental Theorem of Algebra]   Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

Complex Representations

Recall the definitions of the complex number addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and multiplication.

$$(a + bi)(c + di) = ac + adi + bci + bd(-1)$$
$$= (ac - bd) + (ad + bc)i$$

With those rules for scalars, all of the operations that we've covered for real vector spaces carry over unchanged.

Example

$$\begin{pmatrix} 2 - i & 1 + i \\ i & 4 \end{pmatrix} \begin{pmatrix} 0 & 3 + 3i \\ 1 - i & 2 \end{pmatrix} = \begin{pmatrix} 2 & 9 + 3i \\ 4 - 4i & 5 + 3i \end{pmatrix}$$

We shall carry over unchanged from the previous work everything else that we can. For instance, this

$$\left\langle \begin{pmatrix} 1 + 0i \\ 0 + 0i \\ \vdots \\ 0 + 0i \end{pmatrix}, \ldots, \begin{pmatrix} 0 + 0i \\ 0 + 0i \\ \vdots \\ 1 + 0i \end{pmatrix} \right\rangle$$

is the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and we denote it $\mathcal{E}_n$.