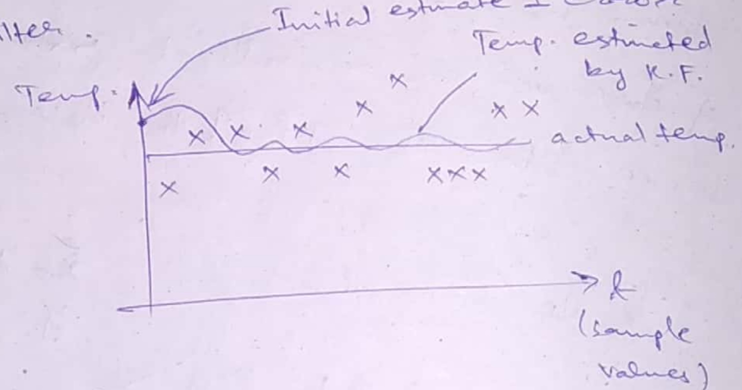
14/03/19

Kalman filter - linear estimator (applicable to stochastic process) (has randomness or noise)

Removes noise - ∴ called filter

A Kalman filter is an iterative mathematical process that contains set of mathematical equations and consequetive data inputs to quickly determine the true value, velocity, position, temp of the object under measurement when the measured value contain unpredicted or random error and uncertainty or variation. The Kalman filter estimates the true value of state from the noisy. Since the Kalman filter estimates or minimizes this noise to estimate the state so it is called filter.



Let the model of the process

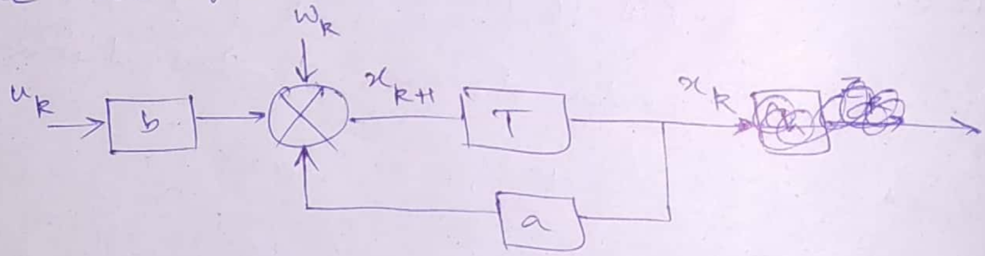


Fig. - Block diagram of process.

The state eq^n is given by

$x_{k+1} = a x_k + b u_k$ where x_k is the state (scaler)

a, b are constants ; $k \to$ time step ; $T \to$ time delay

$u_k \to$ input ; ~~it is const. and $z_k = h x_k$~~

$w_k \to$ noise (white) having

$p(w) \sim N(0, Q)$. 0 → mean

Q → co-variance

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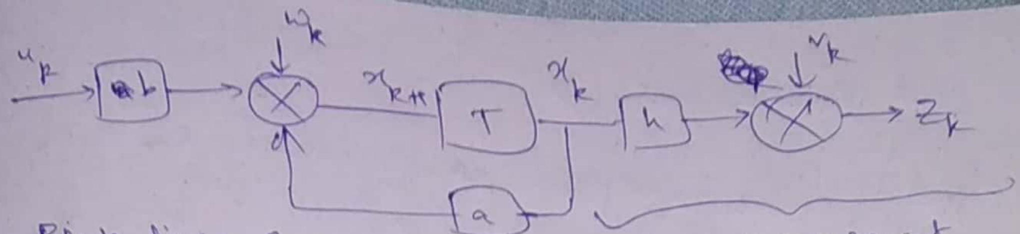

Fig. - Block diagram of process and meas. model

measurement model

$z_k = h x_k + v_k$ $\qquad$ $h \to$ const.

$v_k \to$ noise (white)

$p(v) \sim N(0, R)$ $\qquad$ $R \to$ co-variance

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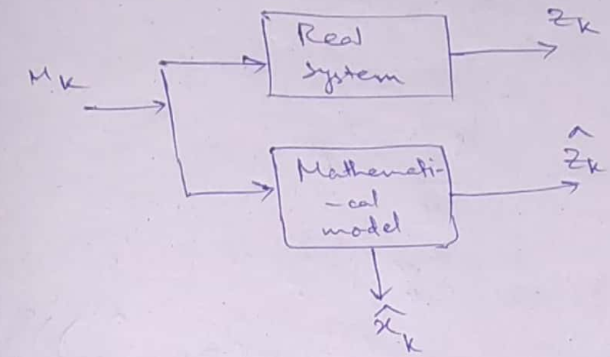

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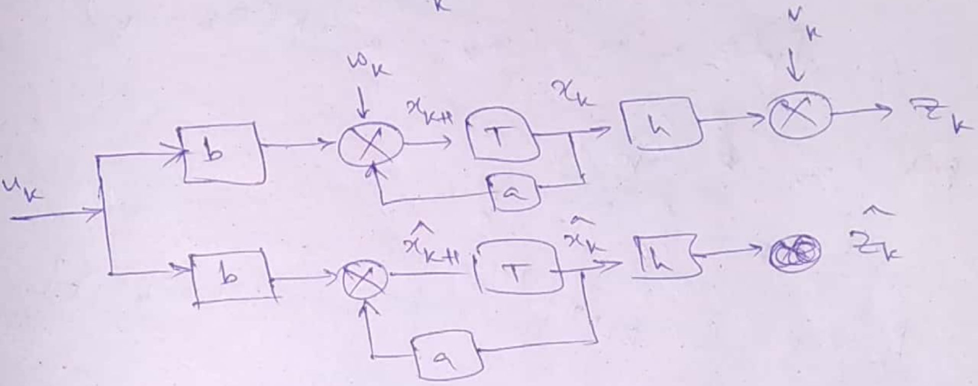

This won't be satisfactory.

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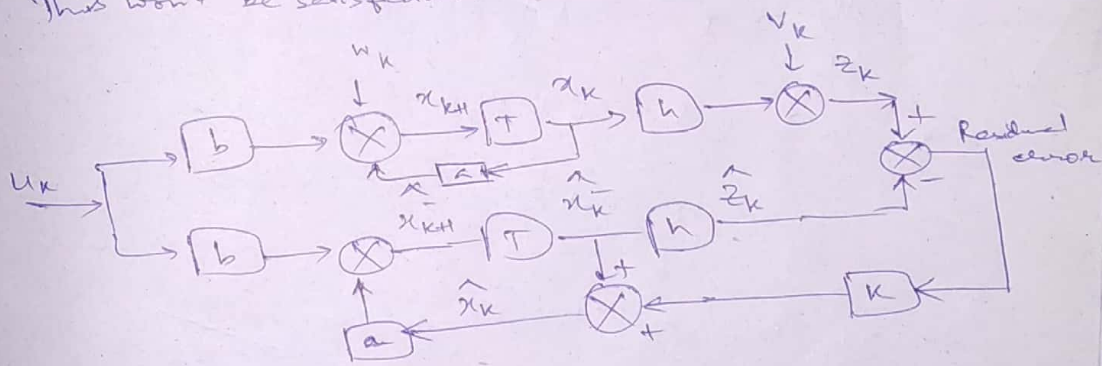

$\hat{x}_k^- \to$ apriori estimate of state x_k

$\hat{z}_k = h \hat{x}_k^-$

$K \to$ kalman filter gain

$\hat{x}_k = \hat{x}_k^- + K(z_k - \hat{z}_k)$

↓

aposteriori estimated state.

If initial assumption, $\hat{x}_k$ is good, error is less and system converges quickly.

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Finding out Kalman gain K

Two types of error : apriori estimate error $= e_k^- = x_k - \hat{x}_k^-$

: a posteriori " " $= e_k = x_k - \hat{x}_k$

Mean Square Error (MSE) : $E\{(e_k^-)^2\} = P_k^-$ → apriori error covariance

and $E\{(e_k)^2\} = P_k$ → aposteriori "

The Kalman filter minimizes P_k by suitable choosing the value of K .

For optimization $\frac{dP_k}{dK} = 0$.

$$P_k = E\{(x_k - \hat{x}_k)^2\}$$

$$= E\{(x_k - \hat{x}_k^- - K(z_k - \hat{z}_k))^2\}$$

put $(x_k - \hat{x}_k^-) = a$

$(z_k - \hat{z}_k) = b$

$$P_k = E\{(a - Kb)^2\} = E\{(a^2 - 2aKb + K^2b^2)\}$$

$$\frac{dP_k}{dK} = E\{-2ab + 2Kb^2\}$$

$$\Rightarrow 0 = E\{-2ab + 2Kb^2\}$$

$$= -2E\{(ab)\} + 2KE\{(b^2)\}$$

$$\Rightarrow 2E\{(ab)\} = 2KE\{b^2\}$$

$$\Rightarrow K = \frac{E\{ab\}}{E\{b^2\}}$$

Numerator $= E\{ab\}$

$$= E\{(x_k - \hat{x}_k^-)(z_k - \hat{z}_k)\}$$

$$= E\{x_k z_k - h\hat{x}_k^- x_k - \hat{x}_k^-(hx_k + v_k) + h(\hat{x}_k^-)^2\}$$

$$= E\{x_k(hx_k + v_k) - h\hat{x}_k^- x_k - \hat{x}_k^-(hx_k + v_k) + h(\hat{x}_k^-)^2\}$$

$= E\{hx_k^2 + v_k x_k - 2hx_k \hat{x}_k^- - v_k \hat{x}_k^- + h(\hat{x}_k^-)^2\}$

$= E\{h(x_k^2 - 2x_k \hat{x}_k^- + (\hat{x}_k^-)^2) + v_k(x_k - \hat{x}_k^-)\}$

$= hE\{(x_k - \hat{x}_k^-)^2\} + E\{v_k(x_k - \hat{x}_k^-)\}$

Now, $E\{v_k(x_k - \hat{x}_k^-)\} = E\{v_k e_k^-\} = 0$

$\therefore$ Numerator $= hE\{(x_k - \hat{x}_k^-)^2\} = hP_k^-$

Denominator $= E\{b^2\}$

$= E\{(z_k - \hat{z}_k)^2\}$

$= E\{z_k^2 - 2z_k \hat{z}_k + \hat{z}_k^2\}$

$= E\{(hx_k + v_k)^2 - 2h\hat{x}_k^-(hx_k + v_k) + h^2(\hat{x}_k^-)^2\}$

$= E\{h^2 x_k^2 + 2hx_k v_k + v_k^2 - 2h^2 x_k \hat{x}_k^- - 2h\hat{x}_k^- v_k + h^2(\hat{x}_k^-)^2\}$

$= E\{h^2 x_k^2 - 2h^2 x_k \hat{x}_k^- + h^2(\hat{x}_k^-)^2 + 2h v_k(x_k - \hat{x}_k^-) + v_k^2\}$

$= h^2 E\{x_k^2 - 2x_k \hat{x}_k^- + (\hat{x}_k^-)^2\} + 2hE\{v_k(x_k - \hat{x}_k^-)\}$
$+ E\{(v_k^2)\}$

$= h^2 E\{(x_k - \hat{x}_k^-)^2\} + 2hE\{\cancel{(v_k e_k^-)}^{0}\} + E\{(v_k^2)\}$

$= h^2 P_k^- + R$

$$\therefore K = \frac{hP_k^-}{h^2 P_k^- + R}$$

$R \rightarrow$ Measurement noise co-variance

(1) P_k^- is very large (as compared to R in the denominator)

$$K = \frac{1}{h}$$

and $\hat{x}_k = \hat{x}_k^- + K(z_k - \hat{z}_k)$

$= \hat{x}_k^- + \frac{1}{h}(z_k - h\hat{x}_k^-)$

$= \frac{z_k}{h}$

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② P_k^- is very small

K will then be very small.

Measurement is reliable — Kalman gain is high — Estimation is unstable

$\updownarrow$ $\updownarrow$ $\updownarrow$

Measurement is unreliable — Kalman gain is low — Estimation stable

*. Kalman Filter is a linear optimal estimator.

*. Finding out a posteriori Error covariance P_k

$$P_k = E\{(x_k - \hat{x}_k)^2\}$$

$$= E\{(x_k - \hat{x}_k^- - K(z_k - \hat{z}_k))^2\}$$

$$\cancel{= E\{(x_k - \hat{x}_k^- - Kz_k + hK\hat{x}_k^-)^2\}}$$

$$\cancel{= E\{(x_k - \hat{x}_k^- - K(hx_k + v_k) + hK\hat{x}_k^-)^2\}}$$

$$= E[(x_k - \hat{x}_k^-)^2] + K^2 E[\underbrace{(z_k - \hat{z}_k)^2}_{b}] - 2KE[ab]$$

$$= P_k^- + K^2(h^2 P_k^-) - 2KhP_k^-$$

$$= (1 - Kh)P_k^-$$

and $P_k^- = E[(x_k - \hat{x}_k^-)^2]$

$$P_k^- = E[(a(x_{k-1} - \hat{x}_{k-1}) + w_{k-1})^2]$$

$$= a^2 E[(x_{k-1} - \hat{x}_{k-1})^2] + E[w_{k-1}^2] + 2aE[(\,)w_{k-1}] \quad 0$$

$$= a^2 P_{k-1} + Q$$

$$\boxed{P_k = (1 - Kh)P_k^-} \quad \text{and} \quad \boxed{P_k^- = a^2 P_{k-1} + Q}$$

03-19

The Discrete Kalman Filter Algorithm :-

Kalman filter is a linear optimal estimator, can be applied to both time variant and invariant system. (Extended KF → for non-linear systems)

It estimates the state of a dynamic process perturbed by noise (white) by considering

(i) Knowledge of process and measurement dynamics

(ii) Statistical description of noise sources.

(iii) Any Information regarding initial condition of the variables of interest.

Equations can be considered into two types :-

① Time Update Equations (Predictor Step)

② Measurement Update Equations (~~Corrector~~ Corrector Step)

→ These equations are responsible for projecting forward (ahead in time) an estimate of state and error co-variance to obtain a priori estimate of state and a priori co-variance for the next time step i.e. (k-1) to k or k to (k+1) where

k → time-step

→ These equations are responsible for providing feedback, i.e., incorporating a new measurement based on a priori state estimate to obtain new improved a posteriori estimate of state.

From state equations, in general,

$$x_{k+1} = a x_k + b u_k + w_k$$

$$z_k = h x_k + v_k$$

Here x_k is the state measured and in here we assume it is one state, so scalar. It can be vector too.

Here, u_k is the i/p (scalar)

$x_k \to$ state (scalar) $\quad a, b, h \to$ const.

$w_k, v_k \to$ process and measurement noises.

$z_k \to$ o/p (scalar)

Time Update Equations :-

Form a priori estimate of state and error co-variance based on the previous estimate of state and current value of the i/p

$$\hat{x}_k^- = a\hat{x}_{k-1} + bu_k$$

and a priori error co-variance.

$$P_k^- = a^2 P_{k-1} + Q$$

Measurement Update Equations :-

Compute Kalman Filter Gain

$$K_k = \frac{hP_k^-}{h^2 P_k^- + R}$$

This gain is then used together a priori estimate of state to obtain new improved a posteriori estimate of state

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h\hat{x}_k^-)$$

Compute error in updated estimate or a posteriori error co variance

$$P_k = (1 - hK_k) P_{k-1}^-$$

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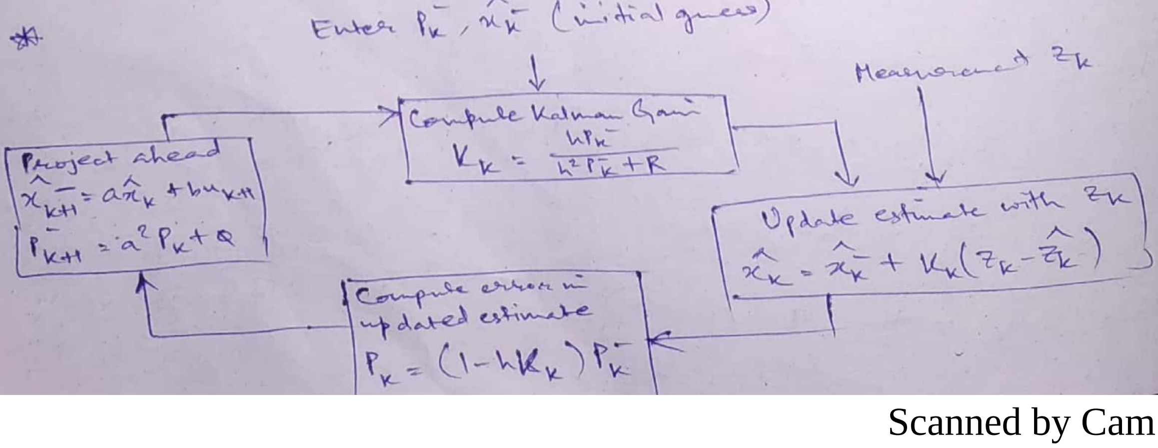

(Fig) - Flowchart of Kalman Filter Algorithm

eg: True temp. 72°c

Kalman gain = KG ; Error in estimate = E_{est} ;

Error in Measurement = E_{meas} ; Initial ~~gain~~ estimate = 68c

Initial error = 2°c ; Initial Measurement = 75°c ;

Measurement Noise Co-variance = 4°c ; The o/p gain = 1 .

→ ~~$\hat{x}_k = \hat{x}_k^- + KG(z_k - \hat{z}_k)$~~

~~$= 68 + KG(-$~~

$x_k = 72$

$P_k^- = 2$

$z_k = 75$

$R = 4$

$h = 1$

time	Measurement z_k	Initial estimate x_k^-	Error in initial estimate P_k	Kalman Gain	Error in updated estimate
k-1		68	2		
k	75			0.33	1.33
k+1					
k+2					
k+3					
k+4					
k+5					

$$KG = \frac{hP_k^-}{h^2P_k^- + R} = \frac{1.2}{1^2.2+4} = 0.33$$

Updated state estimate

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h\hat{x}_k^-)$$

$$= 68 + 0.33(75 - 1 \times 68) = 70.33$$

Error in updated estimate, $P_k = (1 - hK_k)P_{k-1}$

$$= (1 - 1 \times 0.33) \times 2 = 1.33$$

Say, (k+1), $z_k = 71$ (will be given), $(z_k)_{k+2} = 70$

$(z_k)_{k+3} = 74$

k+4 = 74

k+5 = 74

$$KG = \frac{1 \times 1.33}{1.33 + 4} = 0.25$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h\hat{x}_k^-)$$

$$= 68 + 0.25(71 - 1 \times 68)$$

$$= 67.25$$

$$P_k = (1 - hK_k)P_{k-1}$$

$$= (1 - 1 \times 0.25)\, 1.33 = 0.9975$$