A short tutorial on Spatial Networks and Analysis

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Abstract

This tutorial introduces key fundamentals of Spatial Networks and Analysis of events occurring along with networks. The societal applications include social networks characterization and analysis, predicting pipeline failure risk, analysis of road accidents, analyzing hotspots of disparity in a health care supply chain and spread of disease. All of these scenarios represent complex systems that can be abstracted as spatial networks though in some of the cases (e.g. spread of disease), they could be more appropriately abstracted as spatio-temporal networks that evolve with time. In this tutorial, however, we will limit our focus on characterization and analysis of spatial networks. Societal applications of spatial networks is also presented with two different case studies.

Keywords: spatial networks, mathematical models, SANET.

1 Basic Concepts

- **Space:** Mathematically, space represents a set with some additional structure.

- **Spatial Network:** A Spatial network is a graph whose nodes are geometric objects and edges are spatial elements associated with a notion of distance (e.g. Euclidian distance). Essentially nodes and edges in a spatial network can be viewed as being constrained by geometry (Barthélemy, 2011). Mathematically, spatial networks can be represented as $G = (V, L)$ where, V is a set of N nodes and L is a set of E links that connect these nodes (Barthelemy, 2018). Based on this, an adjacency matrix $A \in \mathcal{R}^{N \times N}$ can be defined that describes the links between different nodes. The ij^{th} element of the matrix A carries value 1 if the nodes are linked and 0 otherwise. It is to note that spatial networks with same adjacency matrix can have different spatial representation as shown in Fig.1.

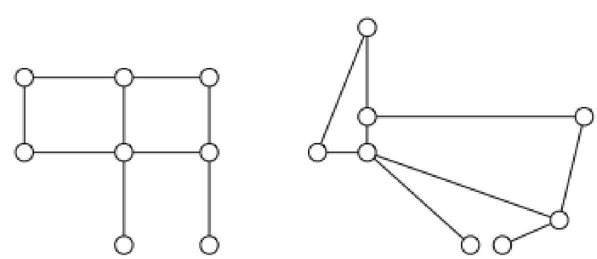


Figure 1: An example of spatial networks with same adjacency matrix but different spatial representation (Barthelemy, 2018).

- **Network Event:** Comprises of events that are constrained by networks (Okabe and Sugihara, 2012). Network events can be classified based on whether they occur directly on a network (e.g. road accidents, pipeline failure, etc.) or occur alongside networks e.g. (disparity persists alongside a health care supply chain network). Based on this classification the network events can be referred to as on-network events or alongside network events.

- **Nearest Neighbor distance:** This is defined as the distance between each node and its nearest neighbor. This distance is often used in network segmentation and partitioning.

2 Introduction

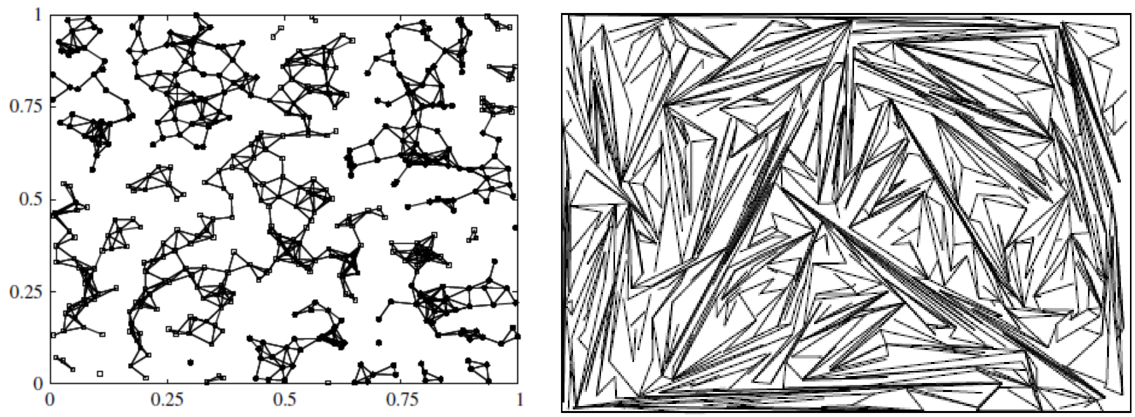
Spatial networks have been long studied in the field of mathematics, quantitative geography and computer science. In simpler words, spatial networks are networks consisting of nodes that are located in space and characterized by a specific metric. In general setting, the space is two-dimensional and the metric defined is the Euclidean distance. Essentially nodes and edges in a spatial network can be viewed as being constrained by geometry (Barthélemy, 2011). Study of river networks (Rodríguez-Iturbe and Rinaldo, 2001), sensor and wireless networks (Akyildiz et al., 2002) and spatial games (Hauert and Szabó, 2005) are some of the subjects developed from the advancement in the science of spatial network. In some literature, a spatial network is also referred to as a social network. This viewpoint was furthered by sociologists like Henri Lefebvre who characterized any kind of space and spatial network be it physical (as in road network) or virtual (as in communication network) as a social space. In his words "All Space is Social" (Lefebvre and Nicholson-Smith, 1991). This notion gave birth to the idea of the spatial network as a relational network (a viewpoint that became prevalent in computer science especially in spatial datasets and databases) as opposed to the abstract network (a mathematical entity). This viewpoint also became a prominent probe of analysis of events happening on or alongside many spatial networks including communication networks, internet and supply chains.

3 Mathematical models of Spatial Networks

The models of spatial networks can be broadly classified into five different classes namely geometric graphs, Erdos-Renyi model, spatial small worlds model, spatial growth models and optimal networks.

3.1 Geometric Graphs

Geometric graphs are the simplest model for spatial networks in which points located in the plane are connected using some given geometric rule such as proximity. In proximity rule, it is assumed that only the nodes located within a specific distance are connected. An example for geometric graph is shown in Fig.2a. For distance measure, usually Euclidean distance is selected to check the proximity condition.



(a) 2D Geometric graph **(b)** Erdos-Renyi graph

Figure 2: An example of two dimensional goemetric graph and Erdos-Renyi graph (Barthélemy, 2011).

3.2 Erdos-Renyi graph

These classes of spatial networks is obtained when the probability of connection between two nodes is dependent on the distance between the respective nodes. It is a type of random graph and can be generated by running through all pairs of nodes and connecting them with a defined probability p. An example of Erdos-Renyi graph is shown in Fig.2b For N number of nodes, the average number of links is given by:

$$E = p\frac{N(N-1)}{2}$$

3.3 Spatial small worlds (Watts-Strogatz model)

This type of model was developed by Watts and Strogratz in 1998 who proposed to incorporate both spatial component and long-range links together to construct a simple yet powerful network. In this type of model, the formulation is started vfrom a regular lattice and then the random links are rewired using a probability value p. An example of Watts-Strogatz model is shown in Fig.3.

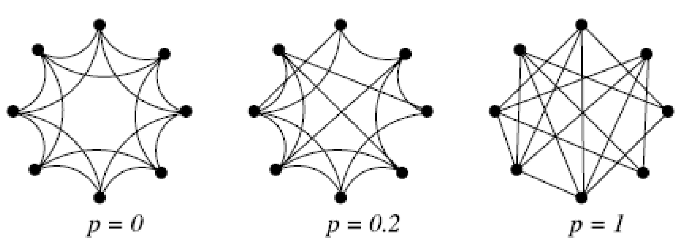


Figure 3: Construction of Watts-Strogatz model for 8 nodes with increasing probability value p (Watts and Strogatz, 1998).

3.4 Spatial growth models

Spatial growth models are spatial extensions of the growth model originally proposed by Albert et al. (1999). In this types of model, there is an inclination for a new node to connect to an already connected one.

3.5 Optimal Networks

Optimal networks are very popular in the field of mathematics, transportation science and computer science. This type of networks are obtained by performing minimization of a pre-defined cost function. The nodes of the network are embedded within a $d-$dimensional Euclidean space, degree is almost always limited and connections are restricted to the neighbors only. Depending on how the cost functions are defined, the optimization of the same networks lead to different results. For example, Fig.4 shows two different transportation networks that optimizes the user travel time and distance, and the one which minimizes the construction costs.

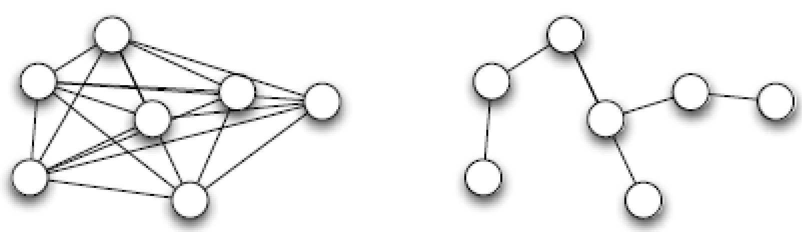


Figure 4: Optimal transportation networks that optimizes the user travel time and distance (left) and the one that optimizes the construction costs (right)

4 Case Studies

4.1 Spatial Distribution Networks for optimal facility location

In the study by Gastner and Newman (2006), the goal is to design the optimal location of facilities (also referred to as $p-$median problem) such as hospitals, airports and malls in the country such that the average distance between a person's home and the nearest facility is minimized. The study is conducted over United States with inputs of then recent population data. Two different methods, one consisting of regression analysis and another method based on density-dependent map projections is utilized. Their study verified the analytic argument that optimal density of facilities is proportional to the the population density to the two-thirds power.

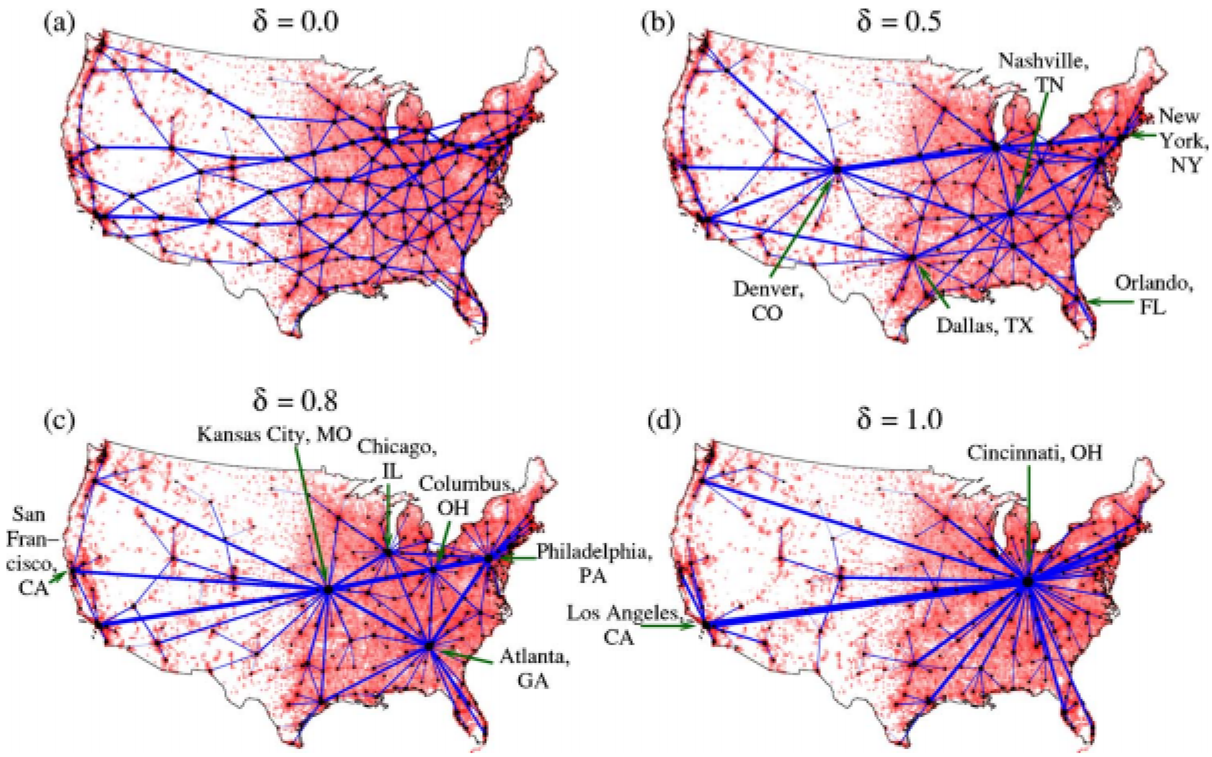


Figure 5: Optimal networks for the population distribution of the United States (Gastner and Newman, 2006).

4.2 Spatial Networks for understanding universal relations in food webs

Food webs are usually used to represent the structure of ecological communities in which species are represented by vertices connected through links that represents the predation. In the study by Garlaschelli et al. (2003), authors propose to describe these food webs as analogous to transportation networks. This analogy helps to reduce the complexity of the whole structure of the food web and help to study how the resources are delivered in the ecosystem. Furthermore, it helps to analyze the observation of universality in food webs that highlight organizing principles across different ecosystem.

5 SANET

SANET is an acronym for Spatial Analysis along Networks (Okabe et al., 2006). Network events that exist in real world can be broadly classified into two types namely on-network events in which the events are occurring exactly on networks and alongside-networks in which events occur alongside networks. Conventionally, both of these types of network events are analyzed using methods that utilize Euclidean distance defined over a plane (also referred to as the planar spatial analysis). However, there exist difference between Euclidean distance

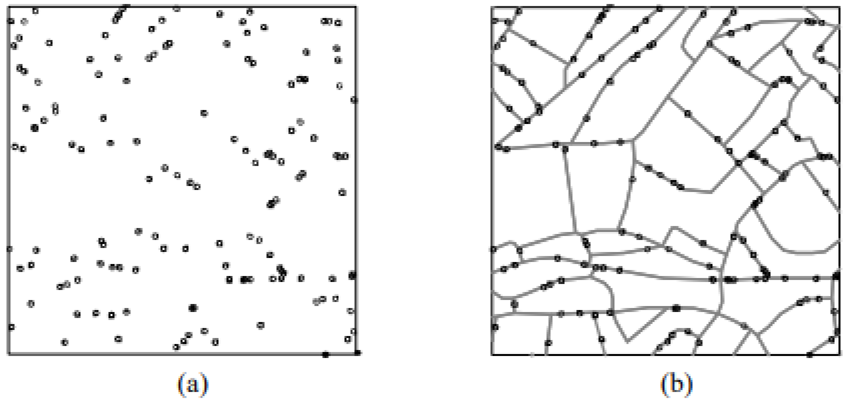


Figure 6: (a) Non-randomly distributed points on a bounded plane and (b) randomly distributed points on a network (Okabe and Sugihara, 2012).

and shortest-path distance especially over urbanized areas (as much as 20% difference can be seen for Euclidean distance < 400 m).

Unlike planar spatial analysis, network spatial analysis utilize the shortest-path distance on networks. The advantage of more practical investigation of network events comes with a downside of heavy geometrical and topological computation. To overcome the burden of programming skill necessary for such network analysis, SANET provides an easier platform for network spatial analysis to assist in programming. SANET helps to avoid misleading statistical inference. For example, in Fig.6a, a seemingly non-random distribution of points on a bounded plane is actually a uniformly distributed points over the network (Fig.6b).

6 Learning Objectives

- Understand key ideas behind spatial networks.
- Mathematical Models of Spatial Networks.
- Application of spatial networks to solve real-world problems.
- SANET: An open-source Tool for Spatial Network Analysis.

7 Instructional Assessment Questions

- What are the key characteristics of a spatial network?
- Select the properties of a typical adjacency matrix A
  (a) square
  (b) symmetric
  (c) square and symmetric

- In transportation, is the optimal transportation network unique?

8 Additional Resources

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