Short Question

(1) State order & degree of differential equation

$$\sqrt{\frac{\partial^2 y}{\partial x^2}} = \frac{\partial y}{\partial x}$$

squaring both sides we get,

$$\left(\sqrt{\frac{\partial^2 y}{\partial x^2}}\right)^2 = \left(\frac{\partial y}{\partial x}\right)^2$$

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial y}{\partial x}\right)^2$$

order-2, degree-1

(2) State order & degree of differential equation

$$\frac{d^2y}{dx^2} = \left[\left(1+\frac{dy}{dx}\right)^2\right]^{3/2}$$

squaring both sides we get,

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\left[\left(1+\frac{dy}{dx}\right)^2\right]^{3/\cancel{2}}\right)^{\cancel{2}}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(1+\frac{dy}{dx}\right)^6$$

order-2, degree-2

(3) Find differential equation corresponding to the general solution $y = cx^2 - x$

$y = cx^2 - x$ } Given --- (i)

Solution:

differentiating equation (i) w.r.to 'x',

$\frac{dy}{dx} = 2cx - 1$ --- (ii)

again diff ean (ii) w.r. to 'x',

$\frac{d^2y}{dx^2} = 2c$ --- (iii)

again diff ean (iii) w.r. to 'x',

$\frac{d^3y}{dx^3} = 0$ --- (iv)

Now, adding (ii), (iii) & (iv) we get

$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2cx - 1 + 2c$

which is the required diff. equation //

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(4) Find differential equation of all straight lines touching the circle $x^2+y^2=a^2$

Solution

Let differential equation be $\frac{dy}{dx} = m$ --- (i)

Then equation of all straight line touching the given circle may be $y = mx + a\sqrt{1+m^2}$ --- (ii)

Then, eliminating m, we get,

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(y - x\frac{dy}{dx}\right) = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

squaring both sides we get,

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

is required differential equation

(6) Find the differential equation whose general solution is $y = Ae^{2x} + Be^{-x} + x$

Given, $y = Ae^{2x} + Be^{-x} + x$ --- (i)

differentiating eq^n (i) w.r. to 'x',

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x} + 1 \quad \text{--- (ii)}$$

again diff. eq^n (ii) w.r. to 'x', we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + Be^{-x} \quad \text{---- (iii)}$$

Adding (ii) and (iii), we obtain

$$\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = 2Ae^{2x} - \cancel{B}e^{-x} + 1 + 4Ae^{2x} + \cancel{B}e^{-x}$$
$$= 6Ae^{2x} + 1$$

which gives,

$$\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right) = 6Ae^{2x}$$

$$Ae^{2x} = \frac{1}{6}\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right) \quad \text{--- (iv)}$$

Again, adding (i) & (ii), we obtain,

$$y + \frac{dy}{dx} = Ae^{2x} + Be^{-x} + x + 2Ae^{2x} - Be^{-x} + 1$$
$$= 3Ae^{2x} + x + 1 \quad \text{---- (v)}$$

Eliminating Ae^{2x} from (iv) & (v) we get,

$$y + \frac{dy}{dx} = \frac{1}{2}\left(\frac{dy}{dx} + \frac{d^2y}{dx^2} - 1\right) + x + 1$$

or, $$\frac{d^2y}{dx^2} - \frac{dy}{dx} = -2y + 2x + 1 = 0$$

is required equation.

(7) Find the general solution of differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \quad \cdots (1)$$

using operator notation we can write equation (1) as,

$$D^2y - 2Dy + 2y = 0$$

$$y(D^2 - 2D + 2) = 0$$

The auxiliary equation is,

$$m^2 - 2m + 2 = 0$$

$$m = 1+i,\ 1-i$$

Now, the general solution is given by,

$$Y = e^{x}\left[C_1 \cos x - C_2 \sin x\right]$$

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(8) Find general solution of $(D^2 + D)y = 0 \quad \cdots (1)$

Solution

The auxiliary equation is,

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m = 0, -1$$

Now,

General solution is given by,

$$y = C_1 e^{0x} + C_2 e^{-x}$$

$$= C_1 + C_2 e^{-x}$$

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(9) Find general solution of $y'' - 6y' + 9y = 0$

Given $y'' - 6y' + 9y = 0$
using operator notation we can write,
$D^2y - 6Dy + 9y = 0$
$y(D^2 - 6D + 9) = 0$

The auxiliary equation is $m^2 - 6m + 9 = 0$
$m^2 - 3m - 3m + 9 = 0$
$m(m-3) - 3(m-3) = 0$
$(m-3)(m-3) = 0$
$m = 3$ {repeated roots}

So, the general solution is given by,
$Y = e^{3x}(c_1x + c_2)$ #

(10) Solve $p^2 - 3p + 2 = 0$

Solution
$p^2 - (2+1)p + 2 = 0$
$p^2 - 2p - p + 2 = 0$
$p(p-2) - 1(p-2) = 0$
$(p-2)(p-1) = 0$

Either,
$p - 2 = 0$
$\frac{dy}{dx} = 2$
$dy = 2dx$
integrating
$\int dy = \int 2dx$
$y = 2x$
$y - 2x = 0$

Or,
$p - 1 = 0$
$\frac{dy}{dx} = 1$
$dy = 1dx$
integrating
$\int dy = \int dx$
$y = x$
$y - x = 0$

Thus, the solution is
$(y-2x)(y-x) = c$ #

(11) Solve $p^2 - 7p + 12 = 0$ where $p = dy/dx$

Solution

$p^2 - (4+3)p + 12 = 0$

$p^2 - 4p - 3p + 12 = 0$

$p(p-4) - 3(p-4) = 0$

$(p-3)(p-4) = 0$

either,	or,
$p - 3 = 0$	$p - 4 = 0$
$\frac{dy}{dx} = 3$	$\frac{dy}{dx} = 4$
$dy = 3dx$	$dy = 4dx$
integrating	integrating
$\int dy = \int 3dx$	$\int dy = \int 4dx$
$y = 3x$	$y = 4x$
$y - 3x = 0$	$y - 4x = 0$

Thus, the solution is $(y-3x)(y-4x) = C$ //

(12) Solve $p^2 + 5xp + 6x^2 = 0$

$p^2 + 3xp + 2xp + 6x^2 = 0$

$p(p+3x) + 2x(p+3x) = 0$

$(p+3x)(p+2x) = 0$

Either,	Or,
$p + 3x = 0$	$p + 2x = 0$
$dy/dx = -3x$	$dy/dx = -2x$
$dy = -3x\,dx$	$dy = -2x\,dx$
integrating,	integrating,
$\int dy = \int -3x\,dx$	$\int dy = \int -2x\,dx$
$y = -3x^2/2$	$y = -x^2$
$2y + 3x^2 = 0$	$x^2 + y = 0$

Solution is $(x^2+y)(2y+3x^2) = C$ //

(13) Solve: $(x+1)\,dy + (y-1)\,dx = 0$

Solution

$$(x+1)\,dy = -(y-1)\,dx$$

$$\frac{dy}{(y-1)} = -\frac{dx}{(x+1)}$$

integrating both sides we get,

$$\int \frac{dy}{(y-1)} = \int -\frac{dx}{(x+1)}$$

$$\log(y-1) = -\log(x+1) + \log c$$

$$\log(y-1) + \log(x+1) = \log c$$

$$y-1+x+1 = c$$

$x+y = c$ is required solution.

(14) Solve: $\frac{dy}{dx} = 3x^2$

$$dy = 3x^2\,dx$$

integrating both sides,

$$\int dy = \int 3x^2\,dx$$

$$y = \frac{3x^3}{3} + c$$

$y = x^3 + c$ is required solution.

(15) Solve: $(x^2+1)\dfrac{dy}{dx} = 1$

$$dy = \frac{1}{x^2+1}\,dx$$

$$\int dy = \int \frac{1}{x^2+1}\,dx$$

$$y = \tan^{-1}(x) \quad \#$$

(16) Solve: $x\,dy + (x+1)\,dx = 0$

$$x\,dy = -(x+1)\,dx$$

$$dy = -\frac{(x+1)}{x}\,dx$$

$$dy = \left(-1 - \frac{1}{x}\right)dx$$

Integrating both sides,

$$\int dy = \int \left(-1 - \frac{1}{x}\right)dx$$

$$y = -x - \log x + c$$

$x + y + \log x = c$ is required solution

17) Define odd and even function with examples.

⇒ A function is "even" when:

$$f(x) = f(-x) \text{ for all } x$$

eg: $f(x) = \cos x$

$g(x) = x^2$

⇒ A function is "odd" when:

$$-f(x) = f(-x) \text{ for all } x$$

eg: $\sin x$, x^3

18) Define singularity and explain its types.

⇒ A point z_0 is called singular point or singularity of function f if f fails to analytic at some point in every nbd of z_0.

Types:

(a) Isolated singularity: A singular point z_0 is said to be isolated if there is atleast one deleted nbd exist throughout which f is analytic.

(b) Non-isolated singularity: A singular point z_0 is said to be non isolated if every deleted nbd of z_0 contains singular points.

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(19) State CR equation for analytic function $f(z)$

⇒ Let $f(z) = u + iv$ be function of z then CR is the equation relating the partial derivatives of real & imaginary parts of an analytic function of complex variable

as, $f(z) = u(x,y) + iv(x,y)$,

by,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ //

(20) Evaluate: $f(z) = \log z$ in form $u(x,y) + iv(x,y)$

$f(z) = \log z$

$z = x + iy$

such that $f(z) = u + iv$

$$\log z = u + iv$$

$$1/z = u + iv$$

$$\frac{1}{x+iy} = u + iv$$

$$\frac{1}{x+iy} \times \frac{x-iy}{x-iy} = u + iv$$

$$\frac{x-iy}{(x)^2 - (iy)^2} = u + iv$$

$$\frac{x-iy}{x^2+y^2} = u + iv$$

$$\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u + iv$$

$$u\left(\frac{x}{x^2+y^2}\right) + iv\left(-\frac{y}{x^2+y^2}\right)$$ is ~~rea form~~ //

(21) Show that derivative of $f(z) = \bar{z}$ doesnot exist anywhere.

$f(z) = \bar{z}$, $z = x+iy$, $\bar{z} = x-iy$

$u+iv = x-iy$

where,

$u = x$, $v = -y$

$\frac{du}{dx} = 1$, $\frac{dv}{dx} = 0$

$\frac{du}{dy} = 0$, $\frac{dv}{dy} = -1$

Hence, $\frac{du}{dx} \neq \frac{dv}{dy}$ it doesnot satisfy C-R eqn

So it doesnot exist anywhere.

(22) Find points at which $f(z) = \frac{\cos z}{z^2+1}$ is not analytic.

Here, the function is not analytic if

$z^2+1=0$

$z^2 = -1$

$z = \pm i$

$\therefore z = \pm i$ is the point where function $f(z) = \frac{\cos z}{z+1}$ is not analytic.

(23) Show $\int_C \frac{z^2}{z-2}\,dz = 0$ where C is unit circle

$|z| = 1$

Solution:

Since $f(z) = \frac{z^2}{z-2}$ has singular point $z=2$

which is outside of C such that $|z| = 1$

$\therefore f(z) = \frac{z^2}{z-2}$ is analytic in closed contour $|z| = 1$

by cauchy integral theorem

$\oint_C f(z)\,dz = 0$

$\int_C \frac{z^2}{z-2}\,dz = 0$ proved

(24) Find real & imaginary part of $\sin z$

$f(z) = \sin z$

$f(z) = u + iv$

$z = x + iy$

Now,

$\sin(x+iy) = u + iv$

$\sin x \cdot \cos iy + \cos x \cdot \sin iy$ $\{\because \sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B\}$

Since, $\cos iy = \cosh y$, $\sin iy = i \sinh y$

Then,

$\sin x \cosh y + \cos x \, i \sinh y = u + iv$

$\sin x \cosh y + i \cos x \sinh y = u + iv$

$\therefore u = \sin x \cosh y$ — real Part

$v = \cos x \sinh y$ — imaginary

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25) Show function, $f(x) = \frac{e^x + e^{-x}}{2}$ is even function

By, defination:

If $f(-x) = f(x)$ then it is even.

So, putting $x = -x$

$$f(-x) = \frac{e^{-x} + e^{-(-x)}}{2}$$
$$= \frac{e^{-x} + e^{x}}{2}$$
$$= \frac{e^{x} + e^{-x}}{2}$$

$\therefore f(-x) = f(x) = \frac{e^x + e^{-x}}{2}$ i.e even function

26) If function $f(z)$ is differentiable at z_0, show $f(z)$ is continuous at $z = z_0$

Solution

$$f'(z_0) = \lim_{z \to z_0} \cancel{f(z_0)} - f(z)$$

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} (z - z_0) \right]$$
$$= \lim_{z \to z_0} \underbrace{\frac{f(z) - f(z_0)}{z - z_0}} \lim_{z \to z_0} (z - z_0)$$

$$= 0$$

Hence, $\lim_{z \to z_0} f(z) = f(z_0)$, which shows $f(z)$ is continuous at $z = z_0$.

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(27) Find derivative of z^2 at $z = z_0$

$f(z) = z^2$

$$f'(z_0) = \underset{h \to 0}{Lt} \frac{(z_0 + h)^2 - z_0^2}{h}$$

$$= \underset{h \to 0}{Lt} \frac{2hz_0 + h^2}{h}$$

$$= 2z_0$$

(28) Calculate residue of $(z) = f(z) = \dfrac{1}{z + \frac{1}{z}}$

$$f(z) = \frac{1}{\frac{z^2+1}{z}}$$

$$= \frac{z}{z^2+1}$$

$$= \frac{z}{(z+i)(z-i)}$$

$z = -i, i$ {simple pole}

at $z = -i$

$$b_1 = \underset{z \to -i}{Lt} (z - z_0) f(z) = \cancel{(z+i)} \cdot \frac{z}{\cancel{(z+i)}(z-i)} = \frac{z}{z-i} = \frac{-i}{-2i} = \frac{1}{2}$$

at $z = i$

$$b_1 = \cancel{z}\underset{z \to i}{Lt} (z - z_0) f(z) = \cancel{(z-i)} \cdot \frac{z}{(z+i)\cancel{(z-i)}} = \frac{z}{z+i} = \frac{i}{2i} = \frac{1}{2}$$

$\therefore$ 1/2, 1/2 are residue

29. Find residue of $\frac{z}{z^2-5z+6}$ at $z=3$

$$f(z) = \frac{z}{z^2-5z+6}$$

$$= \frac{z}{(z-3)(z-2)}$$

$$b_1 = \lim_{z \to 3} \cancel{(z-3)} \cdot \frac{z}{\cancel{(z-3)}(z-2)} \quad \left\{ \lim_{z \to z_0} (z-z_0) \cdot f(z) \right\}$$

$$\frac{3}{3-2}$$

$$b_1 = 3$$

$\therefore$ 3 is the required residue of $f(z)$

30. Find residue of $f(z) = \frac{\cos z}{z^2}$

pole of order 2, at $z=0$

Now,

$$\text{Res}\, f(0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$= \lim_{z \to 0} \frac{d}{dz} \left\{ \cancel{(z-0)^2} \cdot \frac{\cos z}{\cancel{z^2}} \right\}$$

$$= \lim_{z \to 0} -\sin z$$

$$= 0$$

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(31) Find zeros of $f(z) = \left(\frac{2z-1}{z^2-1}\right)^3$

Solution

$f(z_0)=0$

i.e $(2z-1)^3 = 0$

or, $8z^3 - 3(2z)^2 + 3\cdot 2z\cdot 1^2 - 1 = 0$

or, $8z^3 - 12z^2 + 6z - 1 = 0$

$z_0 = 1/2$

$z_0 = 1/2$ is the zeros of given function.

(32) Find zeros of $f(z) = \left(\frac{z+1}{z^2+1}\right)^3$

Solution

$f(z_0)=0$

i.e $(z+1)^3 = 0$

$z^3 + 3z^2 + 3z + 1 = 0$

$z_0 = -1$

$\left\{ \begin{array}{c} -1^3 + 3\cdot(-1)^2 + 3\cdot(-1) + 1 \\ -1 + 3 - 3 + 1 \\ = 0 \end{array} \right\}$

$z_0 = -1$ is the zeros of given function

(33) Express Taylors series of $\cos z$

Solution

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$

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(34) Test whether $(2x-3y)dx - 3xdy = 0$ is exact or not

$M = 2x - 3y$, $N = -3x$

$\frac{\partial M}{\partial y} = -3$, $\frac{\partial N}{\partial x} = -3$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -3$ {exact}

(35) Define fourier series.

⇒ A fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series makes use of the orthogonality relationships of the sine and cosine functions.

The series of $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$

where $a_0, a_1, a_2, \ldots b_1, b_2$ are constants

and $k_0 + k_1 \sin(x + \alpha_1) + k_2 \sin(2x + \alpha_2) + \cdots$

is called fourier series.

36 Evaluate: $\int_0^1\int_1^2 (x^2+y^2)\,dx\,dy$

Solution:

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_1^2 dy$$

$$= \int_0^1 \left[\frac{8}{3} + 2y^2\right] - \left[\frac{1}{3} + y^2\right] dy$$

$$= \int_0^1 \left[\frac{7}{3} + y^2\right] dy$$

$$= \left[\frac{7}{3}y + \frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{7}{3} + \frac{1}{3}\right] - 0$$

$$= 8/3$$

//

(37) Evaluate: $\int_0^3 \int_1^1 xy(x+y)\, dx dy$

$$\int_0^3 \int_1^2 (x^2y + xy^2)\, dx dy$$

$$\int_0^3 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_1^2 dy$$

$$\int_0^3 \left[\frac{8y}{3} + \frac{4y^2}{2}\right] - \left[\frac{y}{3} + \frac{y^2}{2}\right] dy$$

$$\int_0^3 \left[\frac{7y}{3} + \frac{3y^2}{2}\right] dy$$

$$\left[\frac{7y^2}{6} + \frac{3y^3}{6}\right]_0^3$$

$$\frac{7\times 9}{6} + \frac{3\times 27}{6} - 0$$

$$\frac{63}{6} + \frac{81}{6}$$

$$\frac{21}{2} + \frac{27}{2}$$

$$= \frac{48}{2}$$

$$= 24$$

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