1 Differential equation of the first order

Differential equation of the first order

Differential Equation
Any equation which involves derivative(s) or differential(s) is called a differential equation. an **ordinary differential equation** is defined as an equation that contains a derivative of an unknown function of a variable. In other words, If only one independent variable enters the equation then the derivatives are **ordinary derivatives**, and the equation is called an **ordinary differential equation**. Following are the some examples of differential equations

1. $y' = 2x$
2. $y' + 5y = e^x$
3. $\frac{dy}{dx} = x^2 + 3x$
4. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 0$
5. $(y'')^2 + 3y = 0$

Differential equation of the first order

Order of a differential equation
The order of the highest derivative that occurs in a differential equation is called the **order of the differential equation**.

1. $y' = x^2 + 2x$ is the differential equation of order 1.
2. $y'' - 3y' = 0$ is the differential equation of order 2.

Degree of differential equation
The degree of the highest derivative that occurs in a differential equation is called the **degree of the differential equation**.

1. $(y')^2 + 5y = \sin x$ is the differential equation of degree 2.
2. $y''' + 2(y'')^2 - y' = 0$ is the differential equation of degree 1, but order of the equation is 3.

Differential equation of the first order

Solution of Differential equation
A solution of a differential equation is a relation between the variable involved which contain no derivative(s) or differential(s) and which satisfies the equation identically.

- **General Solution:** If the solution of a differential equation of n^{th} order contains n arbitrary constants then it is called its general solution or complete primitive.

- **Particular Solution:** A solution obtained by giving particular values of arbitrary constants in the general solution is called the particular solution. If we consider a differential equation $y' = \cos x$ Then $y = \sin x + c$ is the general solution, however the equation $y = \sin x + 5$ is particular solution.

Differential equation of the first order

Solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations that have arisen during the investigation of real-life problems, although differential equations are the most important of all the applications of integral calculus. A differential equation is said to be integrable by quadratures if its general solution can be obtained as a result of a finite sequence of elementary actions on the known functions and integrations of those functions. In this chapter we will discuss on the following differential equations which are integrable by quadratures.

1. Variable separable
2. Exact differential equations
3. Homogeneous equations
4. Linear differential equation

1.1 Variables Separable Method

Variables Separable Method

A differential equation which can be expressed in the form

$$f(x)dx = g(y)dy$$

is called variables separable differential equation. to solve this type of equation we simply integrate both sides of this equation.

$$\int f(x)\,dx = \int g(y)\,dy$$

Some Solved Problems by variables separation method

Q. Solve the differential equation: $\left(x^2+1\right)\frac{dy}{dx}=1$
Given differential equation $\left(x^2+1\right)\frac{dy}{dx}=1$ This equation can be written as $dy=\frac{dx}{x^2+1}$ Integrating both sides we get $y=\tan^{-1}x+c$ which is the required solution.

Solve the following differential equation

1. $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$
2. $(\sin x+\cos x)\,dy=(\cos x-\sin x)\,dx$
3. $y\,dx=(e^x+1)\ dy$

Some Solved Problems by variables separation method

Q. Solve the differential equation: $\frac{dy}{dx}=\frac{x}{y}$**.**
Given equation can be written as $x\,dx-y\,dy=0$ Integrating, we get $\frac{x^2}{2}-\frac{y^2}{2}=\frac{c}{2}$ where $\frac{c}{2}$ is arbitrary constant. $\Rightarrow x^2-y^2=c$

Solve the following differential equation

1. $x\,dx+y\,dy=0$
2. $\frac{dy}{dx}=\frac{x^2+1}{y^2+1}$
3. $\frac{dy}{dx}+4x=2e^{2x}$

Some Solved Problems by variables separation method

Q. Solve the differential equation: $(1+x)\,y\,dx+(1+y)\,x\,dy=0$
Dividing both sides by xy, we get $\frac{1+x}{x}\,dx+\frac{1+y}{y}\,dy=0\Rightarrow\frac{1}{x}\,dx+dx+\frac{1}{y}\,dy+dy=0$ Integrating, we get $\log x+x+\log y+y=c\Rightarrow\log(xy)+x+y=c$

Solve the following differential equation

1. $\left(xy^2+x\right)\ dx+(yx^2+y)\,dy=0$
2. $e^{x-y}\,dx+e^{y-x}\,dy=0$
3. $(e^x+1)\,y\,dy=(y+1)\,e^x\,dx$
4. $(1+x)\left(1+y^2\right)\ dx+(1+y)\left(1=x^2\right)\ dy=0$

1.2 Homogeneous Differential Equation

Homogeneous Differential Equation

Definition

If $f(x, y)$ and $g(x, y)$ are homogeneous functions of x and y of the same degree then an equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is called a homogeneous differential equation. Such an equation can always be written in the form $\frac{dy}{dx} = F'\left(\frac{y}{x}\right)$. To solve the equation of this tye we put $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then the equation $\frac{dv}{dx} = F\left(\frac{y}{x}\right)$ is reduced to $v + x\frac{dv}{dx} = F(v)$ this can be written as $\frac{dv}{F(v)-v} = \frac{dx}{x}$ in which the variables are separated.

Homogeneous Differential Equation

Solve: $\frac{dy}{dx} = \frac{3xy+y^2}{3x^2}$

Given Equation $\frac{dy}{dx} = \frac{3xy+y^2}{3x^2}$ Put $y = vx$, then we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$ Now, given differential equatin becomes $v + x\frac{dv}{dx} = \frac{3vx^2+v^2x^2}{3x^2} \Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{3}$ $\Rightarrow x\frac{dv}{dx} = \frac{v^2}{3} \Rightarrow \frac{3dv}{v^2} = \frac{dx}{x} \Rightarrow \frac{-3}{v} = \log x + \log c \Rightarrow \frac{-3}{v} = \log cx \Rightarrow \frac{-3x}{y} = \log cx$ $\Rightarrow -3x = y \log cx \Rightarrow 3x + y \log cx = 0$

Homogeneous Differential Equation

Solve: $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Given differential equation : $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ So given equation can be written as $v + x\frac{dv}{dx} + v = v^2 \Rightarrow \frac{dv}{v(v-2)} = \frac{dx}{x} \Rightarrow$ $\frac{1}{2}\left(\frac{1}{v-2} - \frac{1}{v}\right)dv = \frac{dx}{x}$ Integrating bothsides we get, $\Rightarrow \frac{1}{2}(\log(v-2) - \log v) =$ $\log x + \frac{1}{2}\log c \Rightarrow \frac{1}{2}\left(\log\frac{v-2}{v}\right) = \frac{2\log x + \log c}{2} \Rightarrow \log\frac{(v-2)}{v} = \log x^2 c \Rightarrow \frac{v-2}{v} =$ cx^2 Now,Substituting the value $v = \frac{y}{x}$ and simplifying we get, $y - 2x = cx^2 y$ which is the required solution.

Homogeneous Differential Equation

Solve the following differential equation

1. $x + y\frac{dy}{dx} = 2y$
2. $\frac{dy}{dx} = \frac{x^2 y}{x^3+y^3}$
3. $\left(x^2 + y^2\right) dx + 2xy\, dy = 0$
4. $\left(x^2 + y^2\right) dx = \left(x^2 + xy\right) dy$
5. $\left(x^2 + y^2\right) dy = xy\, dx$
6. $y^2\, dx + \left(xy + x^2\right) dy = 0$

7. $(x+y)\ dx + (y-x)\ dy = 0$

8. $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

9. $x\sin\left(\frac{y}{x}\right) dy = \left(y\sin\left(\frac{y}{x}\right) - x\right)\ dx$

1.2.1 Equation reducible to homogeneous form

Equation reducible to homogeneous form

An equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ can be reduced to a homogeneous equation by some suitable substitutions. We shall consider two cases here. **Case:I:-** When $\frac{a}{A} \neq \frac{b}{B}$, we put $x = X+h$ and $y = Y+k$ where h and k are constatnts to be chosen so that we get a homogeneous equation. We have $dx = dX$ and $dy = dY$ so that $\frac{dy}{dx} = \frac{dY}{dX}$ $\therefore$ $\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$ Now those h and k such that $ah+bk+c=0$ and $Ah+Bk+C=0$ Then the equation reduces to $\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$, which is homogeneous. It can now be solved by the substitution $Y = VX$. **Case:II:-** When $\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$ say, we cannot solve the equation by the method in case: I. Since the equation is of the form $\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+C'}$ In this case we put $ax+by = v$. Then the variable can be easily separated.

Equation reducible to homogeneous form

Solve: $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Given equation $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ ......(i) Here, $a=1, A=2, b=2, B=1$ and $\frac{a}{A} \neq \frac{b}{B}$ Put $x = X+h$ and $y = Y+k$ then $\frac{dy}{dx} = \frac{dY}{dX}$ and equation (i) becomes $\frac{dY}{dX} = \frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$ ....(ii) Now choose h and k so that, $h+2k-3=0$ and $2h+k-3=0$ On solving we get $h=1, k=1$ So equation (ii) can be written as $\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ .....(iii) Put $Y = VX$ then $\frac{dY}{dX} = V + X\frac{dV}{dX}$ and equation (iii) can be written as $\left(\frac{2}{1-V^2} + \frac{2V}{2(1-V^2)}\right) dV = \frac{dX}{X}$ Integrating both sides we get $2\frac{1}{2.1}\log\left(\frac{1+V}{1-V}\right) - \frac{1}{2}\log\left(1-V^2\right) = \log X + \frac{1}{2}\log C \Rightarrow \frac{1+V}{(1-V)^3} = CX^2$ Now Substituting $X = x-1$ and $Y = y-1$ and solving we get $x+y-2 = C\left(x-y\right)^3$ Which is the required solution.

Equation reducible to homogeneous form

Solve the following differential equation

1. $\frac{dy}{dx} + \frac{2x-y+1}{2y-x-1}$

2. $(2x+3y-5)dy + (3x+2y-5)dx = 0$

3. $(6x-5y+4)dy + (y-2x-1)dx = 0$

4. $(3y-7x+7)dx + (7y-3x+3)dy = 0$

5. $(x-y)dy-(x+y+1)dx=0$

1.3 Exact Differential Equation

Exact Differential Equation

Definition
A differential equation of the form $M\,dx = N\,dy = 0$, where M and N are function of x and y, is said to be exact when there is a function $f(x,y)$ such that $M\,dx + N\,dy = df(x,y)$. i.e. when $M\,dx + N\,dy$ is a perfect differential. **Note:** A necessary and sufficient condition for the differential equation $M\,dx + N\,dy =$ 0to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The rule to solve an exact equation is as follows:

1. First integrate M with respect to x regarding y as constant.
2. Then integrate with respect to y those terms in N which do not contain x
3. Add the above two results and equate the sum to some constant.
4. If N has no term which is free from x, then $\int N\,dy$ is taken as zero

Exact Differential Equation

Integrating Factors
When some differential equations are multiplied by a suitable function. They becomes exact. Such a function is known as integrating factor (written in short as I.F) For example, the equation $x - y\,dx = 0$ is not exact. Multiplying it by $\frac{1}{x^2}$, the equation becomes $\frac{x-y\,dx}{x^2} = 0$ *or* $d\left(\frac{y}{x}\right) = 0$ Which is an exact equation.

Solution by Inspection
In some cases there are certain rules to find integrating factor. But we can often find out the solution easily by suitable grouping of terms to form a perfect differential. In many cases we can also obtain the integrating factors by inspection. we give a few exact differentials which will be useful in making suitable groups or in finding an integrating factor by inspection.

Exact Differential Equation

Solution by Inspection ...

1. $x\,dy + y\,dx = d(xy)$
2. $x\,dx + y\,dy = \frac{1}{2}d\left(x^2+y^2\right)$
3. $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

4. $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
5. $\frac{y\,dx - x\,dy}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\log\left(\frac{y}{x}\right)$
6. $\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$
7. $\frac{2xy\,dy - y^2\,dx}{x^2} = d\left(\frac{y^2}{x}\right)$
8. $\frac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
9. $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

Exact Differential Equation

Solve: $(3x - 2y + 1)\,dx + (3y - 2x - 1)\,dy = 0$
Given differential equation is $(3x - 2y + 1)\,dx + (3y - 2x - 1)\,dy = 0$ Here, $M = 3x - 2y + 1,\ \ N = 3y - 2x - 1$ $\frac{\partial M}{\partial y} = -2$ and $\frac{\partial N}{\partial x} = -2 \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the equation is exact. Now, $\int M\,dx$ taking y as constant $= \int(3x - 2y + 1)\,dx = \frac{3x^2}{2} - 2xy + x$ Again the term free from x in N is $3y - 1$ So, $\int(3y-1)\,dy = \frac{3y^2}{2} - y$ Thus the required equation is $\frac{3x^2}{2} - 2xy + x + \frac{3y^2}{2} - y = k \Rightarrow 3x^2 - 4xy + 2x + 3y^2 - 2y = c$

Exact differential equation

Solve the following differential equation

1. $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$
2. $\left(x^2 + 2xy^2\right)\,dx + \left(2x^2y + y^2\right)\,dy = 0$
3. $\left(x^2 - ay\right)\,dx - \left(ax - y^2\right)\,dy = 0$
4. $x\,dy + (x+1)y\,dx = 0$
5. $x\frac{dy}{dx} + y = y^2\log x$ **Solution:** The given equation can be written as $x\,dy + y\,dx = y^2\log x\,dx \Rightarrow \frac{x\,dy + y\,dx}{x^2y^2} = \frac{\log x}{x^2}\,dx \Rightarrow \frac{d(xy))}{x^2y^2} = \frac{\log x}{x^2}\,dx$ Integrating we get $\Rightarrow \frac{-1}{xy} = \frac{-1}{x}(\log x + 1) + c \Rightarrow 1 + cxy = y(\log x + 1)$

Exact differential equation

Solve: $(x + y)(dx - dy) = dx + dy$
Given equation can be written as $dx - dy = \frac{dx+dy}{x+y} \Rightarrow dx - dy = \frac{d(x+y)}{x+y}$ Integrating we get $x - y = \log(x + y) + c$

Exact differential equation

Solve: $\left(x^2+y^2+2x\right)dx+2y\,dy=0$
Given differential equation can be written as $\left(x^2+y^2\right)dx+2x\,dx+2y\,dy=0$ $\Rightarrow \left(x^2+y^2\right)dx+d\left(x^2+y^2\right)=0$ Dividing by x^2+y^2 we get $dx+\frac{d\left(x^2+y^2\right)}{x^2+y^2}=0$ Integrating we get, $x+\log\left(x^2+y^2\right)=c$

Exact differential equation

Solve: $\left(x^2+y^2+2x\right)dx+xy\,dy=0$
Multiplying the given differential equation by IF x we get $\left(x^3+xy^2+2x^2\right)dx+x^2y\,dy=0 \Rightarrow \left(x^3+2x^2\right)dx+\frac{1}{2}\left(2xy^2\,dx+2x^2y\,dy\right)=0 \Rightarrow \left(x^3+2x^2\right)dx+\frac{1}{2}d\left(x^2y^2\right)=0$ Integrating we get, $\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2y^2}{2}=\frac{c}{12} \Rightarrow 3x^4 8x^3+6x^2y^2=c$

1.4 Linear Differential Equation

Linear Differential Equation

Definition
An equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x alone or constant is called a linear differential equation of the first order.

Solving methods of Linear differential equation

- Compute: Integrating factor$(I.F.)=e^{\int Pdx}$
- Multiply both sides of equation by $I.F.$
- equation will be of the form $\frac{d}{dx}\left(ye^{\int Pdx}\right)=Qe^{\int Pdx}$
- Integrating both sides we get

$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$$

which is the solution of the given differential equation

Linear differential equation

Solve: $\frac{dy}{dx}+2y=4x$
Given equation, $\frac{dy}{dx}+2y=4x$........(i) Equation (i) is linear differential equation with $P=2$ and $Q=4x$ Now, $I.F.=e^{\int Pdx}=e^{\int 2dx}=e^{2x}$ Multiplying equation (i) by $I.F.$ we get $e^{2x}\frac{dy}{dx}+2e^{2x}y=4xe^{2x}$ $\frac{d}{dx}\left(ye^{2x}\right)=4xe^{2x}$ Integrating both sides, we get $ye^{2x}=4\int xe^{2x}dx$ $ye^{2x}=4\left[\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}\right]+C \Rightarrow y=(2x-1)+Ce^{-2x}$ which is the required solution.

Linear differential equation

Solve the following differential equation:

1. $\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^2}$
2. $\left(1+x^2\right)\frac{dy}{dx}+2xy=4x^2$
3. $\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{(1+x^2)^2}$
4. $x\left(x-1\right)\frac{dy}{dx}-y=x^2\left(x-1\right)^2$
5. $(1+x)\frac{dy}{dx}-xy=1-x$
6. $\frac{dy}{dx}+y\cot x=2\cos x$
7. $\frac{dy}{dx}+y\tan x=\sec x$
8. $\cos x\frac{dy}{dx}+y\sin x=\sec^2 x$
9. $\frac{dy}{dx}+2y\tan x=\sin x$

Linear differential equation

Note:
If a linear differential equation is of the form $\frac{dx}{dy}+Px=Q$, where P and Q are functions of y alone or constants, then the $I.F.=e^{\int Pdy}$.

Solve:$(x+y+1)\frac{dy}{dx}=1$
Given equation can be written as $\frac{dx}{dy}-x=y+1$ .....(i) Here $P=-1$ and $I.F.=e^{\int Pdy}=e^{-\int dy}=e^{-y}$ Now, multiplying equation (i) by $I.F.$ we get $\frac{d}{dy}\left(xe^{-y}\right)=e^{-y}\left(y+1\right)$ Integrating we get $xe^{-y}=\int e^{-y}\left(y+1\right)dy$ On solving we get, $(x+y+2)=Ce^{y}$ which is the required solution.

Linear differential equation

Equation Reducible to linear form
An equation of the form $\frac{dy}{dx}+Py=Qy^n$, where P and Q are function of x alone is called Bernoulli's equation. We can easily reduce it to the linear form using following steps

- Dividing by y^n, the equation becomes

$$\frac{1}{y^n}\frac{dy}{dx}+Py^{1-n}=Q$$

- Put $y^{1-n}=v$, then $(1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$
- The equation is reduced to $\frac{dv}{dx}+(1-n)Pv=(1-n)Q$, which is a linear equation.

Linear differential equation

Solve: $\frac{dy}{dx} + \frac{y}{x} = y^2$
Dividing both sides of given equation by y^2, it becomes $\frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{y}\frac{1}{x} = 1 \quad ....(i)$ Put, $\frac{1}{y} = v$ then we have, $-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$ Now equation (i) becomes $-\frac{dv}{dx} + \frac{1}{x}.v = 1 \Rightarrow \frac{dv}{dx} - \frac{1}{x}.v = -1 \;...(ii)$, Which is linear in v Now, $I.F. = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$ Now multiplying equation (i) by $I.F.$ and integrating we get, $v.\frac{1}{x} = -\int \frac{1}{x} dx \quad \Rightarrow \frac{v}{x} = -\log x + c$ Substituting $v = \frac{1}{y}$ and solving we get $xy\,(C - \log x) = 1$ which is the required solution.

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THANK YOU

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