Long Questions

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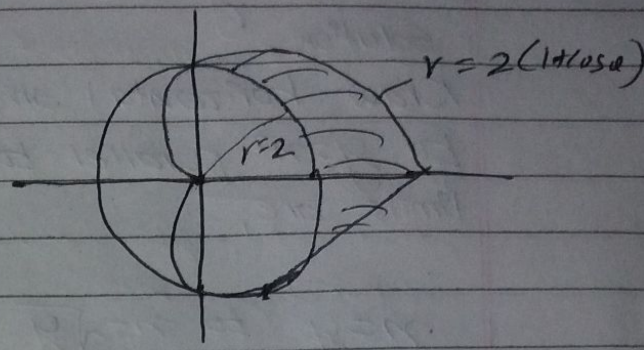
① Find by double integral, calculate area which is inside cardiod $r=2(1+\cos\theta)$ and outside circle $r=2$



$$= 2\cdot\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\cdot\int_0^{\pi/2} \frac{r^2}{2}\Big|_2^{2(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\cdot\int_0^{\pi/2} [2(1+\cos\theta)]^2 - 2^2\,d\theta$$

$$= \int_0^{\pi/2} 4(1+2\cos\theta+\cos^2\theta) - 4\,d\theta$$

$$= \int_0^{\pi/2} 4+8\cos\theta+4\cos^2\theta-4\,d\theta$$

$$= \int_0^{\pi/2} 8\cos\theta+4\cos^2\theta\,d\theta$$

$$= \left[8\sin\theta\right]_0^{\pi/2} + 4\int_0^{\pi/2}\left[\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$\left[\because \cos 2\theta = 2\cos^2\theta - 1,\quad \cos 2\theta + 1 = 2\cos^2\theta,\quad \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= 8 + 4\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8 + 4\left[\frac{\pi}{4} + 0\right]$$

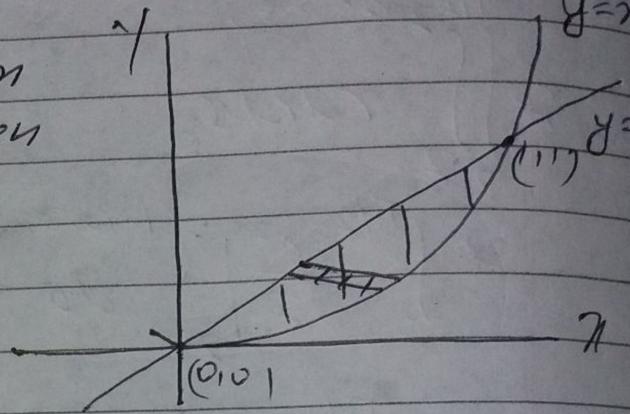
$$= 8+\pi \quad \#$$

$\frac{\pi}{2}$

(2) Evaluate $\iint xy(x+y)\,dx\,dy$ over the diagram bounded by the curves $y=x$ and $y=x^2$

Solution

Draw horizontal strip from $y=x$ to $y=x^2$ parallel to x-axis then limits are,



$x=y$ to $x=\sqrt{y}$

$y=0$ to $y=1$

$$\int_0^1\int_y^{\sqrt{y}} (x^2y+xy^2)\,dx\,dy$$

$$\int_0^1 \left[\frac{x^3y}{3}+\frac{x^2y^2}{2}\right]_y^{\sqrt{y}} dy$$

$$\int_0^1 \left[\frac{y\sqrt{y}\cdot y}{3}+\frac{y\cdot y^2}{2}\right]-\left[\frac{y^4}{3}+\frac{y^4}{2}\right] dy$$

$$\int_0^1 \left[\frac{y^2\cdot y^{1/2}}{3}+\frac{y^3}{2}\right]-\left[\frac{y^4}{3}+\frac{y^4}{2}\right] dy$$

$$\int_0^1 \frac{y^{5/2}}{3}+\frac{y^3}{2}-\left[\frac{2y^4+3y^4}{6}\right] dy$$

$$\int_0^1 \frac{y^{5/2}}{3}+\frac{y^3}{2}-\frac{5y^4}{6}\,dy$$

$$\left[\frac{y^{7/2}}{3\cdot 7/2}+\frac{y^4}{8}-\frac{5y^5}{30}\right]_0^1$$

$$\left[\frac{2y^{7/2}}{21} + \frac{y^4}{8} - \frac{y^5}{6}\right]_0^1$$

$$\left[\frac{2}{21} + \frac{1}{8} - \frac{1}{6}\right]$$

$$\frac{2\times 8 + 1\times 21 - 1\times 28}{168}$$

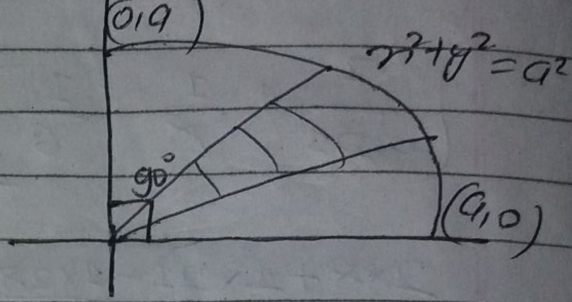
$$\frac{16+21-28}{168}$$

$$\frac{37-28}{168}$$

$$\frac{9}{168}$$

$$\frac{3}{56}$$

③ Evaluate $\iint_R xy\,dxdy$ where R is the positive quadrant of the circle $x^2+y^2=a^2$



In polar form

$x=r\cos\theta$, $y=r\sin\theta$, $dxdy=r\,dr\,d\theta$

Changing into polar form

$$I=\iint_R r^2\cos\theta\cdot\sin\theta\; r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^a r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$=\int_0^{\pi/2}\frac{r^4}{4}\cos\theta\cdot\sin\theta\Big|_0^a\,d\theta$$

$$=\int_0^{\pi/2}\frac{a^4}{4}\cos\theta\cdot\sin\theta\,d\theta$$

$$=\frac{a^4}{4}\cdot\frac{1}{2}\int_0^{\pi/2}2\cos\theta\sin\theta\,d\theta$$

$$=\frac{a^4}{8}\int_0^{\pi/2}\sin2\theta\,d\theta$$

$$=\frac{a^4}{8}\;\frac{-\cos2\theta}{2}\Big|_0^{\pi/2}=\frac{a^4}{8}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$=\frac{a^4}{8}$$

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(4) Find the volume of the sphere $x^2+y^2+z^2=a^2$

Using spherical polar coordinate

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$

Changing into polar form,

$x^2+y^2+z^2=a^2$

$r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\phi = a^2$

$r^2\sin^2\phi[\cos^2\theta+\sin^2\theta] + r^2\cos^2\phi = a^2$

$r^2\sin^2\phi + r^2\cos^2\phi = a^2$

$r^2 = a^2$

$r = a$

Now,

$$\text{Volume} = 8\iiint dx\,dy\,dz$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\frac{r^3}{3}\Big|_0^a\sin\phi\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\frac{a^3}{3}\sin\phi\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\frac{\pi}{2}\frac{a^3}{3}\sin\phi\,d\phi$$

$$= \frac{8}{6}\pi a^3[-\cos\phi]_0^{\pi/2}$$

$$= \frac{4}{3}\pi a^3[1]$$

$= 4/3\,\pi a^3$ is required volume

(6) Evaluate $\iiint xyz\, dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$

Solution

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \cancel{xy(a^2-x^2)}\; xy\frac{z^2}{2}\Big|_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\frac{(a^2-x^2-y^2)}{2}\, dx\,dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{a^2xy-x^3y-xy^3}{2}\, dx\,dy$$

$$= \frac{1}{2}\int_0^a \left\{ \frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8}\int_0^a 2a^2x(a^2-x^2) - 2x^3(a^2-x^2) - x(a^2-x^2)^2 - x(a^4-2a^2x^2+x^4)$$

$$= \frac{1}{8}\int_0^a 2a^4x - 2a^2x^3 - 2a^2x^3 + 2x^5 - a^4x + 2a^2x^3 - x^5\, dx$$

$$= \frac{1}{8}\Big[\frac{a^4x^2}{2} - \frac{2a^2x^2}{4} + \frac{x^6}{6} \Big]_0^a$$

$$= \frac{1}{8}\left[\frac{a^6}{2} - \frac{2a^6}{4} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{6a^6 - 6a^6 + 2a^6}{12}\right]$$

$$\frac{2a^6}{96} = a^6/48$$

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(5) Solve: differential equation $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$ ... (I)

Let $y = vx$

diff w.r.to 'x',

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ... (II)

equating (I) & (II)

$v + x\frac{dv}{dx} = \frac{2xy}{x^2-y^2}$

$v + x\frac{dv}{dx} = \frac{2x^2v}{x^2-v^2x^2}$

$v + x\frac{dv}{dx} = \frac{2x^2v}{x^2(1-v^2)}$

$v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$

$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v$

$x\frac{dv}{dx} = \frac{2v-v+v^3}{1-v^2}$

$x\frac{dv}{dx} = \frac{v+v^3}{1-v^2}$

Now, seperating variable & integrating,

$\int \frac{1-v^2}{v+v^3}\,dv = \int \frac{1}{x}\,dx$

$\log x = \int \frac{1-v^2}{v+v^3}\,dv$

$\log x = \int \frac{1-v^2}{v(v^2+1)}\,dv$

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$$\log x = -\int \frac{v^2-1}{v(v^2+1)}\, dv$$

$$\log x = -\int \left(\frac{2v}{v^2+1} - \frac{1}{v} \right) dv \quad \left\{ \text{Partial fraction} \right\}$$

$$\log x = -\int \frac{2v}{v^2+1}\, dv - \int \frac{1}{v}\, dv$$

$$\log x = -\left[\log(v^2+1) - \log v \right]$$

$$\log x = \log v - \log(v^2+1) + \log C$$

$$\log x = \log \frac{v}{1+v^2} \cdot C$$

Hence,

$$x = \frac{Cv}{1+v^2}$$

$$= \frac{C \cdot y/x}{1+y^2/x^2} = \frac{\frac{Cy}{x}}{\frac{x^2+y^2}{x^2}} = \frac{Cxy}{x^2+y^2}$$

or, $x^2+y^2 = Cxy$

Substituting $x=1$ & $y=1$ in above, we obtain C

Hence, particular solution is $x^2+y^2 = 2y$

⑥ Solve: $\frac{dy}{dx} + \frac{2x}{x^2+2} y = \frac{1}{x}$ ....(1)

Here,

$$\frac{dy}{dx} + Py = Q \text{ \{form\}}$$

$$P = \frac{2x}{x^2+2}, \quad Q = \frac{1}{x}$$

$$\int p\,dx = \int \frac{2x}{x^2+2} = \log(x^2+2) + c$$

Now,

$$I.F = e^{\int p\,dx} = e^{\log(x^2+2)} + c$$
$$= (x^2+2) + c$$

Multiplying eqn (1) by I.F we get form,

$$\frac{d}{dx}(y \cdot e^{\int p\,dx}) = Q \cdot e^{\int p\,dx}$$

$$\frac{d}{dx}[y \cdot (x^2+2)] = Q \cdot (x^2+2)$$

integrating we get,

$$x^2y + 2y = \int \frac{1}{x}(x^2+2)\,dx$$

$$x^2y + 2y = \int x + \frac{2}{x}\,dx$$

$$x^2y + 2y = \frac{x^2}{2} + 2\log x + c$$

$$y(x^2+2) = \frac{x^2}{2} + \log x^2 + c$$

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(7) Solve: $\frac{dy}{dx} = \frac{x-y}{x+y}$

$dy(x+y) = (x-y)\,dx$ { $Mdx + Ndy = 0$ } form

~~dyd~~

$(x-y)\,dx - (x+y)\,dy = 0$

$M = x-y$, $N = -x-y$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -1$ {exact}

Now,

$\frac{du}{dx} = x-y$ ... (i), $\frac{du}{dy} = -x-y$ ... (ii)

integrating (i) w.r.to 'x',

$u(x,y) = \int x-y\,dx$

$= x^2/2 + c(y)$ - (iii)

again diff. (iii) w.r.to 'y',

$\frac{du}{dy} = c'(y)$ ... (iv)

Hence, $c'(y) = -x-y$ { equating (iv) & (ii) }

integrating w.r.to 'y',

$c(y) = -y^2/2$ .. (v)

Substituting value of eqn (v) in (iii) we get

$\frac{x^2}{2} + \frac{y^2}{2} = c$ or, $x^2 - y^2 = c$ is req equation

(8) Solves $x\frac{dy}{dx} + 2y = x^2 \log x$ . . . (1)

$$\frac{dy}{dx} + \frac{2y}{x} = x\log x \left\{ \frac{dy}{dx} + Py = Q \right\} \text{form}$$

$P = \frac{2}{x}$, $Q = x\log x$

Now, computing Integrating Factor,

$I.F = e^{\int P dx}$

$= e^{\int 2/x}$

$= e^{2\log x}$

$= 2x$

Multiplying by I.F in (1) we get,

$\frac{\partial}{\partial x}(y e^{\int P dx}) = Q e^{\int P dx}$

$\frac{\partial}{\partial x}(y \cdot 2x) = x\log x \cdot 2x$

Integrating both sides,

$2xy = \int 2x^2 \log x$

$2xy = 2x^2 \int \log x dx - \int \left[\frac{d}{\partial x}(2x^2) \int \log x dx \right] dx$

$2xy = 2x^2 \cdot \frac{1}{x} - \int \left\{ 4x \cdot \frac{1}{x} \right\} dx$

$2xy = 2x - 4x + c$

$2xy = c - 2x$

$2xy + 2x = c$

$xy + x = c$

$y + 1 = c$ is rea solution

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(0) Solve: $\frac{dy}{dx} = \frac{\tan y - y - 2xy}{\sec^2 y - x\tan^2 y + x^2 + 2}$

$(\sec^2 y - x\tan^2 y + x^2 + 2)dy = (\tan y - y - 2xy)dx$

$(\sec^2 y - x\tan^2 y + x^2 + 2)dy - (\tan y - y - 2xy)dx = 0$ $(Pdx + Qdy = 0)$

Now,

$\frac{\partial P}{\partial y} = -\sec^2 y + 1 + 2x$ $\qquad \frac{\partial Q}{\partial x} = -\tan^2 y + 2x$

$= 2x + 1 - \sec^2 y$ $\qquad = 2x - \tan^2 y$

$= 2x - \tan^2 y$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\{$ exact y

Now,

$\frac{\partial u}{\partial x} = 2xy + y - \tan y$ --- (i)

$\frac{\partial u}{\partial y} = x^2 - x\tan^2 y + \sec^2 y + 2$ .. (ii)

Integrating (1) w.r.to x we get,

$u(x,y) = \int 2xy + y - \tan y \, dx$

$= \frac{2x^2y}{2} + xy - x\tan y + c(y)$

$u = x^2y + xy - x\tan y + c(y)$ -- (iii)

where c is function of y

Now,

diff. partially w.r.to y we get,

$\frac{\partial u}{\partial y} = x^2 + x - x\sec^2 y + c'(y)$ --- (iv) since eqn (II) & (IV) are same we obtain,

$$c'(y) = \sec^2 y - x\tan^2 y + 2 - x + x\sec^2 y$$
$$= \sec^2 y + x(\sec^2 y - \tan^2 y) + 2 - x$$
$$= \sec^2 y + x + 2 - x$$

$$c'(y) = 2 + \sec^2 y$$

integrating w.r. to 'y' we get,

$$c(y) = 2y + \tan y$$

Now,

$$x^2y + xy - x\tan y + 2y + \tan y$$
$$= x^2y + xy + (1-x)\tan y + 2y$$ is required solution

(10) Solve: $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$ ---- (1)

Let $x = X+h$, $y = Y+k$ --- (2)

$dx = dX$, $dy = dY$ --- (3)

subs in eqn (1)

$$\frac{dY}{dX} = \frac{(X+h)+2(Y+k)-3}{2(X+h)+Y+k-3}$$

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \quad \text{--- (4)}$$

on, solving $h+2k-3$ & $2h+k-3$ we get $h=1$, $k=1$

Now eqn ① becomes,

$$\frac{\partial y}{\partial x} = \frac{x+2y}{2x+y} \quad \text{-- ⑤}$$

$$y = vx \quad \text{--- ⑥}$$

$$\frac{\partial y}{\partial x} = v + x\frac{\partial v}{\partial x} \quad \text{--- ⑦}$$

equating ⑤ & ⑦ we get,

$$v + x\frac{\partial v}{\partial x} = \frac{x+2y}{2x+y}$$

$$v + x\frac{\partial v}{\partial x} = \frac{x+2vx}{2x+vx}$$

$$v + x\frac{\partial v}{\partial x} = \frac{x(1+2v)}{x(2+v)}$$

$$v + x\frac{\partial v}{\partial x} = \frac{1+2v}{2+v}$$

$$x\frac{\partial v}{\partial x} = \frac{1+2v}{2+v} - v$$

$$x\frac{\partial v}{\partial x} = \frac{1+2v-2v-v^2}{2+v}$$

$$x\frac{\partial v}{\partial x} = \frac{1-v^2}{2+v}$$

$$\int \frac{2+v}{1-v^2}\,\partial v = \int \frac{1}{x}\,\partial x$$

$$\int \frac{1+1+v}{1-v^2}\,\partial v = \log x + \log c$$

$$\int \frac{1}{1-v^2}\,\partial v + \int \frac{1+v}{1-v^2}\,\partial v = \log x + \log c$$

$$\int \frac{1}{1-v^2}\,\partial v + \int \frac{1}{1-v}\,\partial v = \log x + \log c$$

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$$\frac{1}{2\cdot 1}\log\left(\frac{1+v}{1-v}\right)+\frac{\log(1-v)}{-1}=\log x+\log c$$

$$\frac{1}{2}\log\left(\frac{1+v}{1-v}\right)-2\log(1-v)=2(\log x+\log c)$$

$$\log\left(\frac{1+y/x}{1-y/x}\right)-2\log\left(1-\frac{y}{x}\right)=2(\log x+\log c)$$

$$\log\left(\frac{x+y}{x-y}\right)-2\log\left(\frac{x-y}{x}\right)=2\log x+2\log c$$

$$\log\left(\frac{x+y}{x-y}\right)-2\log(x-y)+2\log x=2\log x+2\log c$$

$$\log\left(\frac{x+y}{x-y}\right)-\log[(x-y)^2]=2\log c$$

$$\log\left\{\frac{x+y}{(x-y)^2}\right\}=\log c^2$$

$$\frac{x+y}{(x-y)^2}=c^2$$

$$\frac{x-1+y-1}{(x-1-y+1)^2}=k$$

$$\frac{x+y-2}{(x-y)^2}=k$$

$$x+y-2=k(x-y)^2 \text{ Ans}$$

(11) Solve the second order differential equation

$$\frac{\partial^2 y}{\partial x^2} - 2\frac{\partial y}{\partial x} + 4y = e^x \sin x$$

Using operator notation,

$$D^2y - 2Dy + 4y = e^x \sin x$$

$$y(D^2 - 2D + 4) = e^x \sin x$$

Auxiliary Equation: $m^2 - 2m + 4 = 0$

$$m = 1 \pm \sqrt{3}\, i$$

$$Y_c = e^x(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)$$

Particular integral given by,

$$Y_P = \frac{1}{D^2 - 2D + 4}(e^x \sin x)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} \sin x$$

$$= e^x \frac{1}{D^2 + 3} \sin x$$

$$= e^x \frac{1}{-1+3} \sin x$$

$$= \frac{1}{2} e^x \sin x$$

$$Y = Y_c + Y_p$$

$$= e^x(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x) + \frac{1}{2} e^x \sin x$$

(12) Find complete solution of differential equation

$$\frac{\partial^2 y}{\partial x^2} + 4\frac{\partial y}{\partial x} + 4y = e^{3x} + \cos 5x$$

using operator notation,

$D^2y + 4Dy + 4y = 0$

$(D^2 + 4D + 4)y = 0$

Auxiliary Equation $= m^2 + 4m + 4 = 0$

$m = -2$ (twice)

$y_c = e^{-2x}(c_1x + c_2)$

for, y_p

$$y_p = \frac{1}{f(D)}(e^{3x} + \cos 5x)$$

$$= \frac{1}{D^2+4D+4}e^{3x} + \frac{1}{D^2+4D+4}\cos 5x$$

$$= \frac{e^{3x}}{25} + \frac{1}{-25+4D+4}\cos 5x$$

$$= \frac{e^{3x}}{25} + \frac{\cos 5x}{4D-21}$$

$$= \frac{e^{3x}}{25} + \frac{4D+21}{(4D)^2-(21)^2}\cos 5x$$

$$= \frac{e^{3x}}{25} + \frac{4D+21}{16D^2-441}\cos 5x$$

$$= \frac{e^{3x}}{25} + \frac{4D+21}{16(-25)-441}\cos 5x$$

$$= \frac{e^{3x}}{25} - \frac{(4D+21)\cos 5x}{841}$$

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$$\frac{e^{3x}}{25}-\frac{1}{841}\left[4(-5\sin 5x)+21\cos 5x)\right]$$

$$\frac{e^{3x}}{25}-\frac{1}{841}(-20\sin 5x+21\cos 5x)$$

$$\frac{e^{3x}}{25}+\frac{20}{841}\sin 5x-\frac{21}{841}\cos 5x$$

$$y=y_c+y_p$$
$$=e^{-2x}(c_1x+c_2)+\frac{e^{3x}}{25}+\frac{20}{841}\sin 5x-\frac{21}{841}\cos 5x$$

(Qno.13) $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=7-6x-3x^2+x^3$

using operator notation,

$D^3y-D^2y-Dy+y=0$

$(D^3-D^2-D+1)y=0$

Auxiliary Equation $= m^3-m^2-m+1=0$

$m=-1,1,1$

$y_c=c_1e^{-x}+(c_2+c_3x)e^{x}$

$$y_p=\frac{1}{D^3-D^2-D+1}(7-6x-3x^2+x^3)$$

$$=\frac{1}{(1+D)(1-D)^2}(7-6x-3x^2+x^3)$$

$$=\frac{1}{4}\left(\frac{1}{1+D}+\frac{1}{1-D}+\frac{2}{(1-D)^2}\right),$$ Partial fraction

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$$= \frac{1}{4}\left(\frac{1}{1+D} + \frac{1}{1-D} + \frac{2}{(1-D)^2}\right)(7-6x-3x^2+x^3)$$

$$= \frac{1}{4}\left[(1-D+D^2-D^3+\cdots)+(1+D+D^2+D^3+\cdots)+2(1+2D+3D^2+\cdots)\right]\cdot(7-6x-3x^2+x^3)$$

$$= \frac{1}{4}\left[4+4D+8D^2+8D^3+\cdots\right](7-6x-3x^2+x^3)$$

$$= (1+D+2D^2+2D^3+\cdots)(7-6x-3x^2+x^3)$$

$$= 7-6x-3x^2+x^3+(0-6-6x+3x^2)+2(-6+6x)+2(6)]$$

$$= 1+x^3$$

$$y = y_c + y_p$$

$$= c_1e^{-x}+(c_2+c_3x)e^{x}+1+x^3$$

(14) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x \cos 2x$

$D^2y - 4Dy + 3y = \frac{1}{2}(\sin 5x + \sin x)$ .....(i)

$A \cdot E = m^2 - 4m + 3 = 0$

$m = 1, 3$

complementary function given by,

$Y_c = C_1 e^{x} + C_2 e^{3x}$

for particular integral,

$Y_P = \frac{1}{D^2 - 4D + 3} \cdot \frac{1}{2}(\sin 5x + \sin x)$

$= \frac{1}{2} \frac{1}{-25 - 4D + 3} \sin 5x + \frac{1}{2} \frac{1}{-1 - 4D + 3} \sin x$

$= \frac{1}{2} \frac{1}{-22 - 4D} \sin 5x + \frac{1}{2} \frac{1}{2 - 4D} \sin x$

$= \frac{-1}{2} \frac{22 - 4D}{44D - 16D^2} \sin 5x + \frac{1}{2} \frac{2 + 4D}{4 + 6D^2} \sin x$

$= \frac{-1}{2} \frac{1}{844}(22 - 4D)\sin 5x + \frac{1}{40}(2 + 4D)\sin x$

$= \frac{1}{844}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$

Hence solution (i) given by

$y = C_1 e^{x} + C_2 e^{3x} + \frac{1}{844}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$

(15) Verify C-R equation for $e^x(\cos y + i\sin y)$

where, $u = e^x \cos y$

$v = e^x \sin y$

Now, differentiating real & imaginary part we gets,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

Now, by C-R equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cos y\, e^x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = e^x \sin y$$

Hence, C-R equation verified.

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(16) Find analytic function of $f(z) = u + iv$, if $u = e^x \sin y$.

solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^x \sin y) \quad \cdots (i) \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^x \sin y).$$

$$= e^x \sin y \qquad\qquad = e^x \cos y \quad \cdots (ii)$$

using C-R equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \sin y$$

or, $$\frac{\partial v}{\partial y} = e^x \sin y \quad \cdots (iii)$$

integrating (iii) w.r.to 'y'

$$v(x,y) = \int e^x \sin y \, dy$$

$$= +e^x \cos y + c(x) \quad \cdots (iv)$$

Also,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(+e^x \cos y + c(x)) \quad \cdots (v)$$

$$= +e^x \cos y + c'(x)$$

from (ii) & (v)

$$e^x \cos y = +e^x \cos y + c'(x)$$

$$\therefore c'(x) = 0$$

integrating w.r. to 'x', we get,

$$c(x) = C$$

Then, equation (iv) becomes

$$v(x,y) = +e^x \cos y + C$$

$$f(z) = u + iv$$

$$= e^x \sin y + i(+e^x \cos y) + C$$

(17) Find analytic function whose real part is $u(x,y) = y^3 - 3x^2y$

$\frac{du}{dx} = \frac{d}{dx}(y^3 - 3x^2y)$ $\qquad$ $\frac{du}{dy} = \frac{d}{dy}(y^3 - 3x^2y)$

$= -6xy$ $\qquad$ $= 3y^2 - 3x^2$

by, C.R equation,

$\frac{du}{dx} = \frac{dv}{dy} = -6xy$ --- (i)

$\frac{du}{dy} = \frac{-dv}{dx} = 3y^2 - 3x^2$ --- (ii)

as, $\frac{dv}{dy} = -6xy$

$\int \frac{dv}{dy} = \int -6xy \, dy$

$v(x,y) = -\frac{6xy^2}{2} + c(x)$

$v(x,y) = -3xy^2 + c(x)$ --- (iii)

Now,

$\frac{dv}{dx} = -3y^2 + c'(x)$ - (iv)

from (ii) & (iv)

$c'(x) = 3x^2$

$\int c'(x)dx = \int 3x^2 dx$

$c(x) = x^3$ --- (v)

Substituting value of $c(x)$ in (iii) we get

$v(x,y) = -3xy^2 + x^3$

Now,

$u(y^3 - 3x^2y) + iv(-3xy^2 + x^3)$ is analytic function

Find Taylor's series for $f(z) = \frac{1}{z}$ at $z = 1$

We have Taylor's series

$$f(z) = f(z) + f'(z)(z-a) + f''(z)\frac{(z-a)^2}{2!} + \cdots$$

Now,

$$f(z) = \frac{1}{z} = \frac{1}{1} = 1$$

$$f'(1) = \frac{-1}{z^2} = \frac{-1}{1} = -1$$

$$f''(1) = \frac{2}{z^3} = \frac{2}{1} = 2$$

$$f'''(1) = \frac{-6}{z^4} = \frac{-6}{1} = -6$$

Then,

$$f(z) = 1 - (z-a) + \frac{2}{2!}(z-a)^2 - \frac{6}{3!}(z-a)^3 + \cdots$$

$$= 1 - (z-a) + (z-a)^2 - (z-a)^3 + \cdots$$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

is required series //

Enpand the function $f(z) = \log(1+z)$ about point $z = 1$

$$f(z) = f(1) + f'(1)\frac{(z-1)}{1!} + f''(1)\frac{(z-1)^2}{2!} + f'''(1)\frac{(z-1)^3}{3!} + \cdots$$

$$f(1) = \log(1+z) = \log 2$$

$$f'(1) = \frac{1}{1+z} = \frac{1}{2}$$

$$f''(1) = \frac{-1}{(1+z)^2} = \frac{-1}{4}$$

$$f'''(1) = \frac{2}{(1+z)^3} = \frac{1}{4}$$

$$f(z) = \log 2 + \frac{1}{2}(z-1) - \frac{1}{4}\frac{(z-1)^2}{2!} + \frac{1}{4}\frac{(z-1)^3}{3!} + \cdots$$

Find Maclaurin's enparsion of $f(z) = \sin z$ at $z = 0$

$f(0) = \sin 0 = 0$

$f'(0) = \cos z = 1$

$f''(z) = -\sin z = 0$

$f'''(z) = -\cos z = -1$

$$f(z) = 0 + \frac{z}{1!} - \frac{z^3}{3!} + 0 - \frac{z^5}{5!} + \cdots$$

$$= \frac{z}{1!} - \frac{z^3}{2!} - \frac{z^5}{6!} + \cdots$$

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Find Laurent's series for $f(z) = \frac{1}{(1-z)(z+2)}$ valid for the domain $1 < |z| < 2$

Solution

Given, $f(z) = \frac{1}{(1-z)(z+2)}$

$$= \frac{1}{3(1-z)} + \frac{1}{3(z+2)} \quad \cdots (\mathrm{i})$$

for, $|z| > 1$, $\frac{1}{1-z} = \frac{-1}{z-1} = \frac{-1}{z(1-\frac{1}{z})} = \frac{-1}{z}\left(1 - \frac{1}{z}\right)^{-1}$

$$= \frac{-1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$= -\sum_{1}^{\infty} \frac{1}{z^n} \quad \cdots (\mathrm{ii})$$

for, $|z| < 2$, $\frac{1}{z+2} = \frac{1}{2(1+z/2)} = \frac{1}{2}\left(1 + \frac{z}{2}\right)^{-1}$

$$= \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}} \quad \cdots (\mathrm{iii})$$

using (ii) & (iii) in (i)

$$f(z) = -\sum_{1}^{\infty} \frac{1}{3z^n} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3(2^{n+1})}$$

which is valid for $1 < |z| < 2$

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Find Laurent series for $f(z) = \frac{1}{(z+1)(z+3)}$

valid for domain $1 < |z| < 3$

Given, $f(z) = \frac{1}{(z+1)(z+3)}$

$$= \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}\right)$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \frac{1}{3}\sum_{0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$$

/-

Find Residue of $f(z) = \frac{\cos z}{z^3}$ at its poles

Given $f(z) = \frac{\cos z}{z^3}$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$= \frac{1}{z^3} - \frac{z}{2!z^3} + \frac{z^4}{4!z^3} - \frac{z^6}{6!z^3} + \cdots$$

$$= \frac{1}{z^3} - \frac{1}{2!z^2} + \frac{z}{4!} - \frac{z^3}{6!}$$

The principal part of $f(z)$ terminate with term $\frac{1}{z^3}$

i.e, Pole of order 3 at 0

Now,

$$b_1 = \frac{1}{(m-1)!} \underset{z \to z_0}{Lt} \frac{d^{m-1}}{dz^{m-1}}\left\{(z-z_0)^m \cdot f(z)\right\}$$

$$= \frac{1}{2!} \underset{z \to 0}{Lt} \frac{d^2}{dz^2}\left[(z-0)^3 \cdot \frac{\cos z}{z^3}\right]$$

$$= \frac{1}{2} \underset{z \to 0}{Lt} -\cos z$$

$$= \frac{-1}{2} \text{ is req residue}$$

Find Residue of $f(z) = \dfrac{3z-4}{z(z-1)(z-2)}$ at each of its poles

We get,

$z = 0, 1, 2$ all simple pole

at $z=0$

$$\text{res } f(0) = \lim_{z \to 0} \cancel{(z-0)} \frac{3z-4}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{-4}{-1 \cdot -2}$$

$$= -2$$

at $z=1$

$$\text{res } f(1) = \lim_{z \to 1} \cancel{(z-1)} \frac{3z-4}{z\cancel{(z-1)}(z-2)}$$

$$= \frac{3 \cdot 1 - 4}{1 \cdot (1-2)}$$

$$= -1/-1$$

$$= 1$$

at $z=2$

$$\text{res } f(2) = \lim_{z \to 2} \cancel{(z-2)} \frac{(3z-4)}{z(z-1)\cancel{(z-2)}}$$

$$= \frac{6-4}{2(2-1)}$$

$$= 2/2 = 1$$

$\therefore$ residue are $-2, 1, 1$ respectively.

A curve is such that the abscissa of the point of contact of a tangent and perpendicular from the origin to the tangent have equal lengths.
Find equation of the curve

Let (x,y) be a point on the curve Then the equation of the tangent to the curve at (x,y) is given by

$$Y-y=\frac{dy}{dx}(X-x) \quad \text{--} \quad (i)$$

then the perpendicular from origin to (i) is equal to

$$\frac{-y+x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Since this is equal to the abscissa x, we have

$$\frac{-y+x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=x$$

Squaring both sides & simplifying, we obtain

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy} \quad \text{--} \quad (ii)$$

Equation (ii) is a homogeneous differential equation Using the method outlined in § 1.1.3, its solution may be written as $y^2+x^2=cx$, which is therefore required equation of the curve

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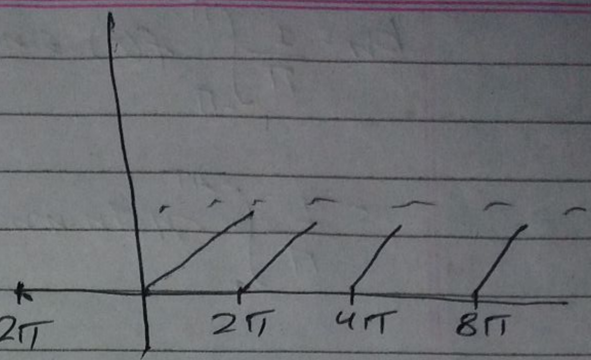
Find fourier series for function
$f(x) = x, \quad 0 < x < 2\pi$



Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx$$

$$= \frac{1}{\pi} \left.\frac{x^2}{2}\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2}{2} - 0\right]$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} - \int \left[\frac{\sin nx}{n}\right] dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\right] - \left[0 + \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} - \int\left(-\frac{\cos nx}{n}\right) dx \right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{-2\pi \cos n(2\pi)}{n} + \frac{\sin n(2\pi)}{n^2} \right\} - 0$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n} + 0 \right]$$

$$= \frac{-2}{n}$$

Now, fourier series is given by,

$$f(x) = \frac{1}{2} a_0 + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2} 2\pi + 0 - \frac{2}{n}\sum_{1}^{\infty} \sin nx$$

$$= \pi - \frac{2}{n}\sum_{1}^{\infty} \sin nx$$

//

Find fourier series for the function

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & \pi \le x \le 2\pi \end{cases}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} (-1)\,dx\right]$$

$$= \frac{1}{\pi}\left[x\Big|_0^{\pi} + -x\Big|_{\pi}^{2\pi}\right.$$

$$= \frac{1}{\pi}\left[(\pi - 0) + (-2\pi + \pi)\right]$$

$$= \frac{1}{\pi}\{\pi - \pi\}$$

$$\therefore a_0 = 0$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\,dx - \int_{\pi}^{2\pi} \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{\sin nx}{n}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left\{0 - 0\right]$$

$$\therefore a_n = 0$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx - \int_{\pi}^{2\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{ \left.\frac{-\cos nx}{n}\right|_0^{\pi} - \left.\frac{-\cos nx}{n}\right|_{\pi}^{2\pi} \right.$$

$$= \frac{1}{\pi}\left\{ \frac{-\cos n\pi}{n} + \frac{1}{n}\right] + \left\{\frac{\cos 2n\pi}{n} - \frac{\cos n\pi}{n}\right]$$

if n is odd

$$= \frac{1}{\pi}\left[\left\{\frac{1}{n} + \frac{1}{n}\right\} + \left\{\frac{1}{n} + \frac{1}{n}\right\}\right]$$

$$= \frac{4}{n}\pi$$

if n is even

$$= \frac{1}{\pi}\left\{\left(\frac{-1}{n} + \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n}\right)\right\}$$

$$= 0$$

$$\therefore b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 4/n\pi & \text{if } n \text{ is odd} \end{cases}$$

We have fourier series,

$$f(x) = a_0/2 + \sum_1^{\infty} a_n\cos nx + \sum_1^{\infty} b_n \sin nx$$

$$= \sum_1^{\infty} b_n \sin nx$$

$$= \frac{4}{\pi}\sin nx + \frac{4}{3\pi}\sin 3x + \cdots$$

#

Q. Find fourier series for function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 0\,dx \right]$$

$$= \frac{1}{\pi} \left[\pi + k_1 \right]$$

$$a_0 = 1 + \frac{k_1}{\pi}$$

And,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \cos nx\,dx + \int_{\pi}^{2\pi} 0 \cdot \cos nx\,dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{\sin nx}{n} \right)_0^{\pi} + k_2 \right]$$

$$= \frac{1}{\pi} \left[k_2 \right]$$

$$a_n = k_2/\pi$$

again, $b_n = \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx + \int_{\pi}^{2\pi} 0\cdot\sin nx\,dx\right]$

$= \frac{1}{\pi}\left[\left.\frac{-\cos nx}{n}\right|_0^{\pi} + k_3\right]$

is n is odd

$= \frac{1}{\pi}\left[\left(\frac{1}{n}+\frac{1}{n}\right)+k_3\right]$

$= \frac{1}{\pi}\left[\frac{2}{n}+k_3\right]$

$= \frac{2}{n\pi} + \frac{k_3}{\pi}$

if n is even

$= \frac{1}{\pi}\left[\left(\frac{-1}{n}+\frac{1}{n}\right)+k_3\right]$

$= \frac{k_3}{\pi}$

Since, $\begin{cases} \frac{2}{n\pi}+\frac{k_3}{\pi} & \text{if } n \text{ is odd} \\ \frac{k_3}{\pi} & \text{if } n \text{ is even} \end{cases}$

fourier series is given by,

$f(x) = \frac{1}{2}\left(1+\frac{k_1}{\pi}\right) + \frac{k_2}{\pi} \sum_{1}^{\infty} \cos nx + \left(\frac{2}{n\pi}+\frac{k_3}{\pi}\right)\sum_{n=2n+1}^{\infty} \sin nx$

$+ \frac{k_3}{\pi}\sum_{n=2n}^{\infty} \sin nx$

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Expand $f(x) = x$ as cosine series $0 \le x \le \pi$

Also prove $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Now,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} - \int\left(\frac{\sin nx}{n}\right)dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

When n is even

$$= \frac{2}{\pi}\left[0 + \left(\frac{-1}{n^2} + \frac{1}{n^2}\right)\right] = 0$$

When n is odd

$$= \frac{2}{\pi}\left[0 + \left(\frac{-1}{n^2} - \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-2}{n^2}\right]$$

$$= \frac{-4}{n^2\pi}$$

Now, the cosine series is given by,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Put $x = 0$

$$0 = \pi/2 - 4/\pi\left[1 + 1/3^2 + 1/5^2 + \cdots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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Obtain half-range cosine series for the $f(x) = \sin x$ in $0 \le x \le \pi$

Hence show, $\sum_{1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} \sin x dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[1+1\right]$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(n+1)x - \sin(n-1)x dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\cos(n-1)\pi}{n-1} - 1\right\} - \left\{\frac{\cos(n+1)\pi}{n+1} - 1\right\}\right]$$

$$\begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

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Cosine series given by,

$$\therefore \sin x = \frac{2}{\pi} + \sum_{1}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{2n}^{\infty} \frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots \right]$$

put $x=0$ in (1)

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{2n}^{\infty} \frac{1}{n^2-1}$$

$$\frac{4}{\pi} \sum_{2n}^{\infty} \frac{1}{n^2-1} = \frac{2}{\pi}$$

$$\sum_{2n}^{\infty} \frac{1}{n^2-1} = \frac{1}{2}$$

$$\sum_{1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \text{ proved}$$

Find Fourier sine series for $f(x) = x^2$ in interval $0 < x < 3$

Here, $a_0 = a_n = 0$,

and, $b_n = \frac{2}{3}\int_0^3 x^2 \sin\frac{n\pi x}{3}\,dx$

$$= \frac{2}{3}\left[\frac{-x^2\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right]_0^3 + \frac{2}{3}\int_0^3 \frac{6x}{n\pi}\cos\frac{n\pi x}{3}\,dx$$

$$= \frac{-18}{n\pi}\cos n\pi + \frac{4}{n\pi}\left[x\frac{\sin\frac{n\pi x}{3}}{n\pi/3} - \int_0^3 \frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}\,dx\right]$$

$$= \frac{-18}{n\pi}\cos n\pi + \frac{4}{n\pi}\left[\frac{9}{n\pi}\sin n\pi + \frac{9}{n^2\pi^2} + \frac{9}{n^2\pi^2}\cos n\pi - \frac{9}{n^2\pi^2}\right]$$

$$= \frac{-18}{n\pi}\cos n\pi + \frac{36}{n^3\pi^3}\cos n\pi - \frac{36}{n^3\pi^3}$$

$$= \begin{cases} -\frac{18}{n\pi} & \text{if } n \text{ is even} \\ \frac{18}{n\pi} - \frac{72}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}$$

half-series given by,

$$x^2 = \sum_1^\infty b_n \sin\frac{n\pi x}{3}$$

$$= \left(\frac{18}{\pi} - \frac{72}{\pi^3}\right)\sin\frac{\pi x}{3} - \frac{18}{2\pi}\sin\frac{2\pi x}{3} + \left(\frac{18}{3\pi} - \frac{72}{27\pi^3}\right)\sin\frac{3\pi x}{3} + \cdots$$

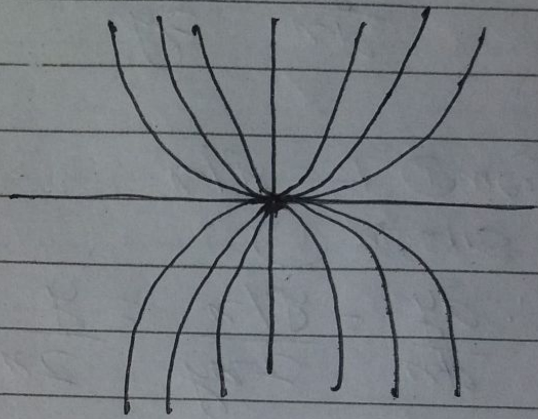
Find orthogonal trajectories of curve given by $y=kx^2$.

A trajectory of a family of curves which cuts the members of family according to given law;

Orthogonal trajectory: Orthogonal trajectory is a curve which intercepts every member of family of right angles

Solution: Given, $y=kx^2$ -- (i)

represents family of parabola whose vertex at O as shown in figure.



Then, $\frac{dy}{dx} = 2kx$ --- (ii)

from (i) $k = y/x^2$ --- (iii)

substitute in (ii)

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{-- (iv)}$$

Now, for orthogonal trajectories

$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$2y\,dy = -x\,dx$$

$$y^2 = \frac{-x^2}{2} + c$$

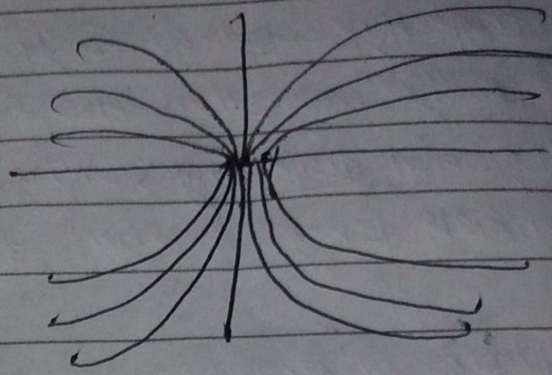
$$x^2 + 2y^2 = c$$

$$\frac{x^2}{c} + \frac{y^2}{c/2} = 1 \text{ ellipse form}$$

Find orthogonal trajectories of family of curves $y^2 = kx$

Solution

Given $y^2 = kx$ --- ① represent family of parabola at vertex 0



Then, $2y\frac{dy}{dx} = k$

$$\frac{dy}{dx} = \frac{k}{2y} \quad \text{--- (II)}$$

from ① $k = y^2/x$

subs in (II)

$\therefore \frac{dy}{dx} = \frac{y^2/x}{2y} = \frac{y}{2x}$ represent slope of tangent of family at any point

Now, for orthogonal trajectories

$$\frac{dy}{dx} = \frac{-2x}{y}$$

$$y\,dy = -2x\,dx$$

$$\frac{y^2}{2} = -x^2 + c$$

$$y^2 + 2x^2 = c$$

$$\frac{x^2}{c/2} + \frac{y^2}{c} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

lz

Find the equation of orthogonal trajectories of circle $r = a\cos\theta$
solution:

Let ψ be angle between tangent to the curve at point (r, θ) and radius vector, then we have,

$$\tan\psi = \frac{r\,d\theta}{dr} \quad \text{--- (i)}$$

If ψ_g and ψ_0 be value of ψ for given curves and orthogonal trajectories respectively. Then,

$$\psi_0 = \psi_g + \pi/2$$

$$\tan\psi_0 = -\cot\psi_0 = -\frac{1}{\tan\psi_g} \quad \text{--- (ii)}$$

Now, for circles, $r = a\cos\theta$.

$$\tan\psi_g = \frac{r\,d\theta}{dr}$$

$$= \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta \quad \text{--- (iii)}$$

& here, $\tan\psi_0 = \frac{-1}{\tan\psi_g} = \tan\theta$

From (i) & (ii) diff. eqn of orthogonal trajectories be,

$$\frac{r\,d\theta}{dr} = \tan\theta$$

Solving we get,

$r = b\sin\theta$

Thus,