also given that $\frac{dx}{dt} = 0$ *when* $t = \frac{\pi}{2\sqrt{\mu}}$

from (ii)

$0 = (-A\sin\frac{\pi}{2} + B\cos\frac{\pi}{2})\sqrt{\mu}$

$0 = -A\sqrt{\mu}$

$A = 0 (\because \mu > 0)$

$\therefore$ *the required solution is*

$x = a\sin\sqrt{\mu}\, t$

11.5 Particular Integral

From Art 11.3 we know that the general solution of the equation $(D^2 + P_1D + P_2)y = Q.\ Q \neq 0$ is the sum of two parts: (i) complementary function and (ii) particular integral.

The Complementary function is the solution of the above equation when Q = 0 i.e of f(D)y = 0 .Now we will try to find the particular integral of f(D)y = Q.

We define $\frac{1}{f(D)}Q$ as a function of x which is free from any arbitray constant and when it is operated on by f(D), it gives Q.

i.e. $f(D)\left[\frac{1}{f(D)}Q\right] = Q$

Thus the equation f(D) y = Q is satisfied when we put $y = \frac{1}{f(D)}Q$ in it. In other words, $\frac{1}{f(D)}Q$ is solution of the equation and is therefore called the particular integral of the equation.

11.6 To prove: $\frac{1}{(D-\alpha)}Q = e^{\alpha x}\int Q\, e^{-\alpha x}dx$ where α is a constant

Let $\frac{1}{(D-\alpha)}Q = y$

Now we will give some special methods to find the particular integral of $f(D)y = Q$ *when* Q has some special forms

11.7 $\frac{1}{f(D)} e^{ax}$ when $a \neq 0$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \; Provided \; f(a) \neq 0$$

Thus P.I is obtained if we simply put a for D in $\frac{1}{f(D)} e^{ax}$

11.8 The case of $\frac{1}{f(d)} e^{ax}$ when $f(a) = 0$

If $f(a) = 0$. $\frac{1}{f(a)} e^{ax}$ has no meaning and hence the above method fails.

$$\frac{1}{f(D)} e^{ax} = \frac{xe^{ax}}{f'(a)} = x \frac{1}{f'(D)} e^{ax}$$

Case (ii) if $f(D) = (D-a)^2$, *then* $f'(a) = 0$

Using the above result once again, $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(D)} e^{ax}$

Working Rule

To find $\frac{1}{f(D)} e^{ax}$ *when* $f(a) = 0$, differentiate $f(D)$ with respect to D and put $D = a$ and get the denominator. Then multiply the numerator by x.

If the denominator becomes 0 again, repeat the process once again and get the result.

Note: We can also apply Art.11.12 when $f(a) = 0$

11.9 $\frac{1}{f(D)} \sin ax$ and $\frac{1}{f(D)} \cos ax$ when $f(-a^2) \neq 0$

$$\sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$$

$$\therefore \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \; Provided \; f(-a^2) \neq 0$$

Similarly, $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax,$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b),$$

and $\frac{1}{f(D^2)}\cos(ax+b) =$

$\frac{1}{f(-a^2)}\cos(ax,+b)$ Provided $f(-a^2) \neq 0$

Thus we put $-a^2$ for D^2 and get the result in all these cases. If $\frac{1}{f(D)}$ contains both the first and the second powers of D, we proceed as in Ex. 3 below.

11.10 The case of $\frac{1}{F(D^2)}$ sin ax and $\frac{1}{F(D^2)}$ cos ax when $f(-a^2) = 0$.

If $f(-a^2) = 0, \frac{1}{f(-a^2)}$ sin ax and $\frac{1}{f(-a^2)}$ cos ax have no meaning and hence the above method fails. Such a linear equation of the second order is of the type $(D^2 + a^2)\, y = \sin ax$ or $(D^2 + a^2)\, y = \sin ax$. We may find their P.I., by the method given below.

Equating real parts, $u = x\frac{1}{2D}\cos ax$ and equating imaginary part.

$V = x\frac{1}{2D}\sin ax.$

Working Rule

To find $\frac{1}{D^2+a^2}$ sin ax and $\frac{1}{D^2+a^2}$ cos ax, when $D^2 = -a^2$, differentiate $D^2 + a^2$ with respect to D so that it is $2D$ and multiply the result of $\frac{1}{2D}$ sin ax or $\frac{1}{2D}$ cos ax by x.

Note: We can also apply art. 11.12 when $f(-a^2) = 0$.

11.11 $\frac{1}{f(D)}x^m$, where m is a positive integer

Use Binomial theorem and expand $[f(D)]^{-1}$ in ascending power of D and then operate on x^m with each term of the expansion. The power of D beyond m need not be retained because the $(m+1)^{th}$ and higher derivatives of x^m *are zero.*

The following expansions should also be remembered:

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots\dots\dots\dots$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots\dots\dots\dots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots\dots\dots\dots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots\dots\dots\dots$$

11.12 $\frac{1}{f(D)}$ $e^{ax}V$, where V is a function of x or a constant

$$\therefore \quad \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.$$

Thus we take out e^{ax}, write $D + a$ for D so that $\frac{1}{f(D+a)}$ operates on V.

We can also use this method to calculate $\frac{1}{f(D)}$ e^{ax} when $f(a) = 0$.

The rule is:

Put $D = a$ in that factor of $f(D)$ which does not become 0 when $x = a$. Then find the particular integral of e^{ax}. 1 by the above method.

The method can also be used to calculate $\frac{1}{D^2+a^2}e^{iax}$ then deduce the value of $\frac{1}{D^2+a^2}\cos ax$ and $\frac{1}{D^2+a^2}\sin ax$ easily by equating the real and imaginary parts respectively.

The methods will be illustrated below.

11.13 $\frac{1}{f(D)}x^m \cos(ax+b)$ and $\frac{1}{f(D)}x^m \sin(ax+b)$

$$\frac{1}{f(D)}x^m \cos(ax+b) = \frac{1}{f(D)}[\text{real part of } x^m e^{i(ax+b)}]$$

$$\frac{1}{f(D)}x^m \sin(ax+b) = \frac{1}{f(D)}\left[\text{coefficient of } i \text{ in } x^m e^{i(ax+b)}\right]$$

11.14 $\frac{1}{f(D)}xV$ where V is a function of x

Here we use the formula

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{(f(D))^2}V \text{ without proof.}$$

11.15 Method of Partial Fractions

It is sometimes possible to express $\frac{1}{f(D)}$ into partial fractions so that $\frac{1}{f(D)}Q = \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots\right)Q.$

Using the result

$\frac{1}{D-\alpha}Q = e^{\alpha x}\int Q\,e^{-\alpha x}\,dx$ of art 11.5 we get

$$\frac{1}{f(D)}Q = A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x}Q\,dx + A_2 e^{\alpha_2 x}\int e^{-\alpha_2 x}Q\,dx + \cdots$$