from ... for constant

... s.

We thus have two ways of evaluating double integrals and these correspond to the two orders of integration. To change the order of integration, one should sketch the region R and from it determine the limits of x and y.

The evaluation of double integrals is illustrated by the following examples.

Example 1. Evaluate the double integral

$$\int_0^2 \int_1^2 (x^2+y^2)\, dx\, dy.$$

Here the order of integration is first with respect to x and then with respect to y. We therefore write the double integral as

$$I = \int_0^2 \left[\int_1^2 (x^2+y^2)\, dx \right] dy$$

Now,

$$\int_1^2 (x^2+y^2)\, dx$$

$$= \left[\frac{x^3}{3} + xy^2 \right]_{x=1}^{x=2}, \text{ since } y \text{ is to be treated as a constant.}$$

$$= \frac{1}{3}(8-1) + y^2(2-1) = \frac{7}{3} + y^2.$$

Hence,

$$I = \int_0^2 \left(\frac{7}{3} + y^2 \right) dy = \frac{7}{3}\Big[y \Big]_0^2 + \frac{1}{3}\Big[y^3 \Big]_0^2$$

$$= \frac{7}{3}(2) + \frac{1}{3}(8) = \frac{22}{3}.$$

The reader may verify that the double integral

$$\int_1^2 \int_0^2 (x^2 + y^2)\, dy\, dx$$

also yields the same value as above, since the limits are constants in this case and therefore the order of integration is immaterial.

Example 2. Evaluate

$$\iint_R xy\, dx\, dy$$

where R is the positive quadrant of the circle $x^2 + y^2 = a^2$.

The order of integration is first with respect to x. Hence we consider an elementary strip parallel to the x-axis as in Fig. 6.4. It is seen that such a strip extends from the y-axis, i.e. $x = 0$, to the circle $x^2 + y^2 = a^2$ on which $x = \sqrt{a^2 - y^2}$. Hence the limits of x are $x = 0$ to $x = \sqrt{a^2 - y^2}$. To cover the entire region, such strips should start with the minimum value of y, i.e. the

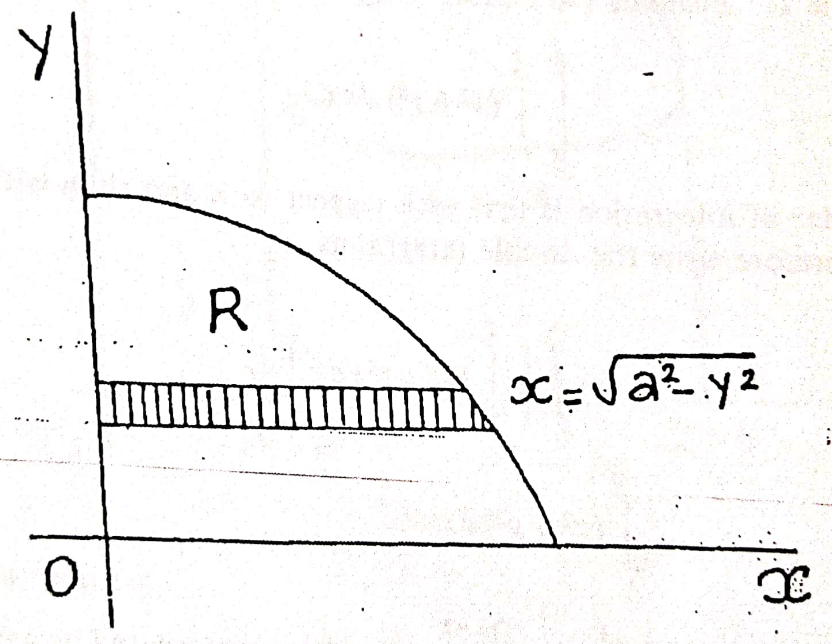

Fig. 6.4.

x-axis to the maximum value of y, i.e. the point where the curve intersects the y-axis. The limits of y are therefore from $y = 0$ to $y = a$. The above double integral is therefore written as

$$I = \int_0^a \int_0^{\sqrt{a^2 - y^2}} xy\, dx\, dy$$

$$= \int_0^a \left[\frac{x^2}{2}\, y\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a (a^2 - y^2)\, y\, dy$$

$$= \frac{1}{2}\left[a^2 \cdot \frac{y^2}{2} - \frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{8}\, a^4.$$

Suppose that we wish to change the order of integration and evaluate the double integral

$$\iint_R xy\, dy\, dx.$$

Now, the order of integration is first with respect to y. Hence, we take an elementary strip parallel to the y-axis as in Fig. 6.5. The limits for y are, obviously, $y=0$ to $y=\sqrt{a^2-x^2}$. The minimum and maximum values of x are

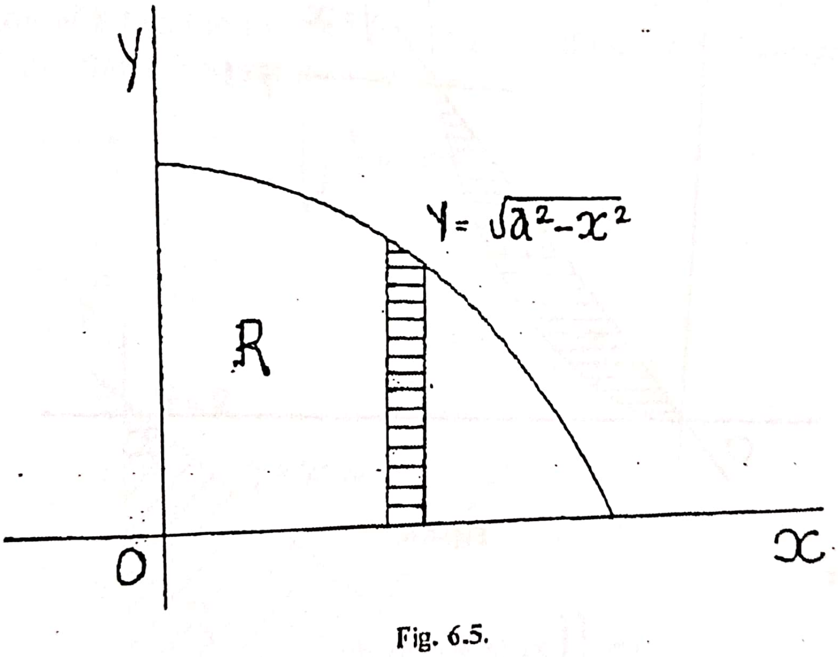

Fig. 6.5.

$x=0$ to $x=a$ which are therefore the limits for x. Hence the double integral can be written as

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_0^a \left[x\,\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)\,dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

$$= \frac{1}{8}a^4, \quad \text{as before.}$$

Example 3. Evaluate the double integral

$$\iint xy\,(x+y)\,dx\,dy$$

over the region bounded by the curves $y=x$ and $y=x^2$.

The region of integration is shown in Fig. 6.6.

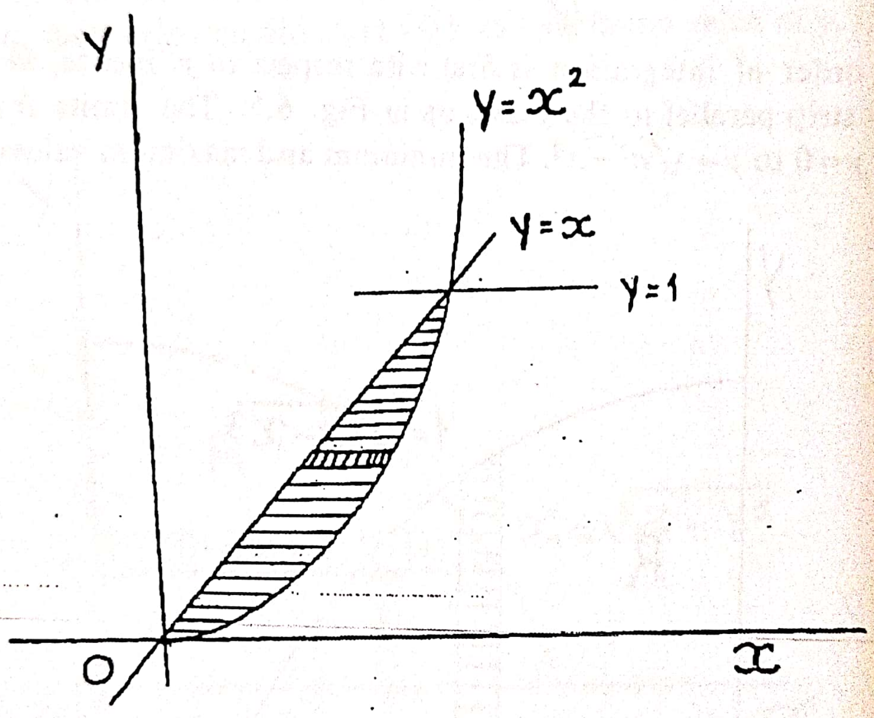

Fig. 6.6.

We have

$$I = \iint xy\,(x+y)\,dx\,dy.$$

Now the order of integration is first with respect to x and then with respect to y. Hence, we take an elementary strip parallel to the x-axis. The limits for x are easily seen to be from $x=y$ to $x=\sqrt{y}$. The limits for y are then from $y=0$ to $y=1$. Hence, we write

$$I = \int_0^1\int_y^{\sqrt{y}} xy\,(x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} y + \frac{x^2}{2} y^2 \right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{y^{3/2} y}{3} + \frac{y^3}{2} - \frac{y^4}{3} - \frac{y^4}{2} \right] dy$$

$$= \int_0^1 \left[\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{5}{6} y^4 \right] dy$$

$$= \left[\frac{2}{7} \frac{y^{7/2}}{3} + \frac{y^4}{8} - \frac{5}{6} \frac{y^5}{5} \right]_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{6} = \frac{3}{56}.$$

Example 4. Change the order of integration and hence evaluate the double integral

$$\int_0^1 \int_x^{2-x} \frac{x}{y} \, dy \, dx.$$

The region of integration is shown in Fig. 6.7. If the order of integration is changed, the above integral may be written as

$$\iint_R \frac{x}{y} \, dx \, dy,$$

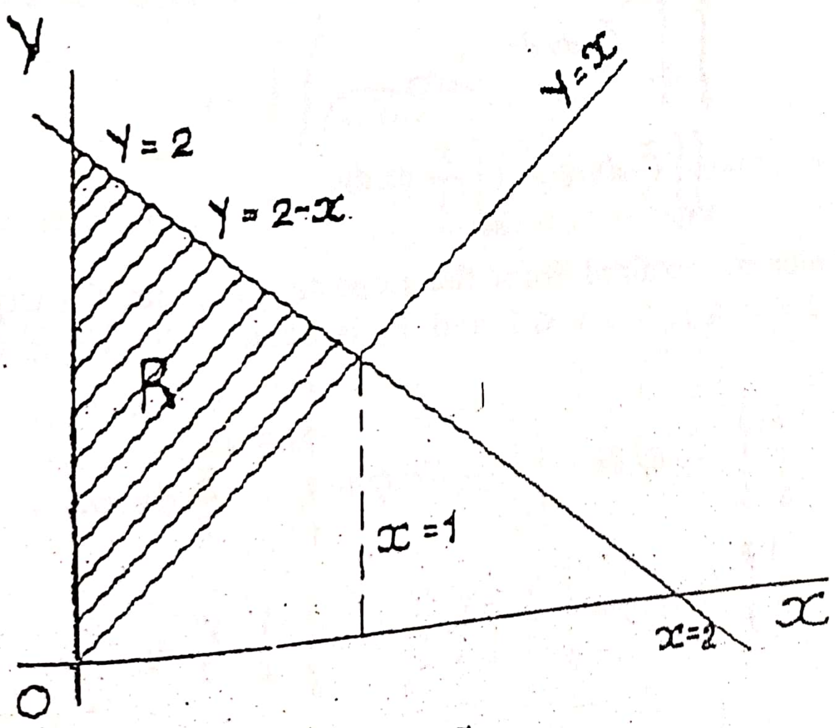

Fig. 6.7.

where the limits are to be determined from the region R. Since the integration is first with respect to x, we take elementary strips parallel to the x-axis. Two such strips are shown in Fig. 6.8, and it is seen that they touch two different lines on the right, viz.

$$y = 2 - x \quad \text{and} \quad y = x.$$

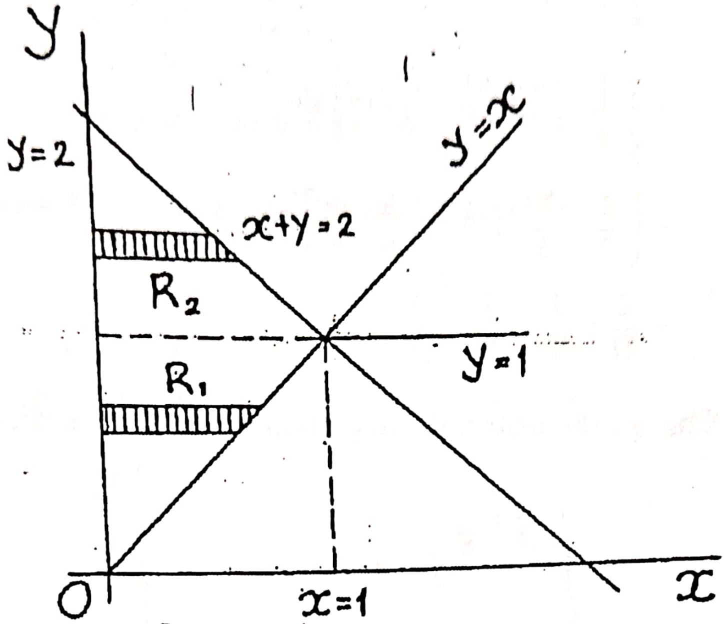

Fig. 6.8.

To change the order of integration, we therefore divide the integral into two distinct parts corresponding to the two regions R_1 and R_2. Hence we have

$$\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$$

$$= \iint_{R_1} \frac{x}{y}\, dx\, dy + \iint_{R_2} \frac{x}{y}\, dx\, dy,$$

where the limits are obtained from the respective regions. Clearly, R_1 is bounded by $0 \leqslant x \leqslant y$, $0 \leqslant y \leqslant 1$ and R_2 is bounded by $0 \leqslant x \leqslant 2-y$, $1 \leqslant y \leqslant 2$.

Therefore, $$\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx = \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\, dx\, dy. \qquad \ldots \text{(i)}$$

Now, $$\int_0^1 \int_0^y \frac{x}{y}\, dx\, dy = \int_0^1 \frac{1}{y} \left[\frac{x^2}{2}\right]_0^y dy = \int_0^1 \frac{1}{y} \cdot \frac{y^2}{2}\, dy$$

$$= \left[\frac{y^2}{4}\right]_0^1 = \frac{1}{4},$$

and
$$\int_1^2 \int_0^{2-y} \frac{x}{y}\, dx\, dy = \int_1^2 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_1^2 \frac{1}{2y}(2-y)^2\, dy = \frac{1}{2}\int_1^2 \frac{1}{y}(4+y^2-4y)\, dy$$

$$= \frac{1}{2}\left[4\log_e y + \frac{y^2}{2} - 4y\right]_1^2$$

$$= \frac{1}{2}\left[4\log_e 2 + 2 - 8 - 0 - \frac{1}{2} + 4\right]$$

$$= \frac{1}{2}\left[4\log_e 2 - \frac{5}{2}\right]$$

$$= 2\log_e 2 - \frac{5}{4}.$$

Hence,
$$\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx = \frac{1}{4} + 2\log_e 2 - \frac{5}{4} = 2\log_e 2 - 1.$$

This example demonstrates the fact that it is sometimes easier to evaluate a double integral by changing the order of integration. The reader may verify this by attempting to evaluate the double integral on the left-hand side of (i).

Example 5. Evaluate the double integral

$$I = \int_0^a \int_0^x \frac{f'(y)}{\sqrt{(a-x)(x-y)}}\, dy\, dx.$$

Here the order of integration is first with respect to y and then with respect to x. Clearly, the integration with respect to y is rather complicated. We, therefore, change the order of integration. The region of integration is shown in Fig. 6.9 (page 530).

To change the order of integration, we first consider an elementary strip parallel to Ox as in the figure. Its limits are seen to be from $x=y$ to $x=a$. The limits for y are then from $y=0$ to $y=a$. Hence, we write

$$I = \int_0^a \int_y^a \frac{f'(y)}{\sqrt{(a-x)(x-y)}}\, dx\, dy$$

$$= \int_0^a f'(y)\, dy \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}}$$

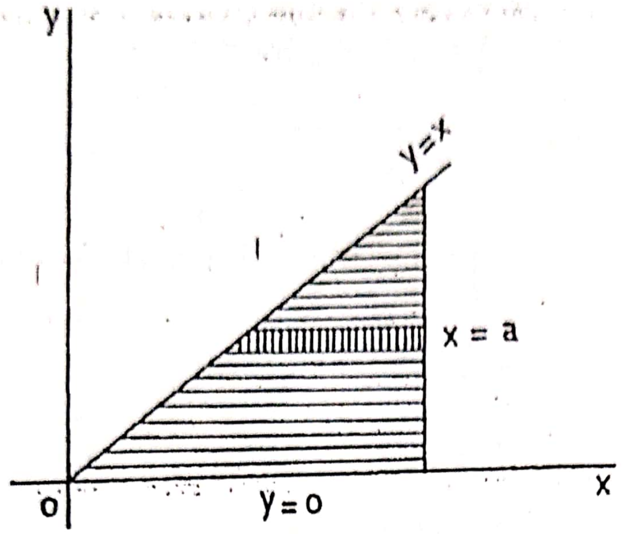

Fig. 6.9.

We first evaluate the integral

$$I_1 = \int_y^a \frac{1}{\sqrt{(a-x)(x-y)}}\,dx = \int_y^a \frac{1}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x-\frac{a+y}{2}\right)^2}}\,dx.$$

Using the substitution $x-\frac{a+y}{2}=\frac{a-y}{2}\sin\theta$, it will be found that the value of this integral is $-\pi$. Hence, we obtain

$$I=\int_0^a f'(y)\,dy\,(-\pi) = -\pi\left[f(a)-f(0)\right].$$

6.1.2 Double Integrals in Polar Coordinates

Certain types of double integrals may be evaluated more conveniently by changing to polar coordinates. In this case, we subdivide the region R in the following way. Let the region R be bounded by $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=a$ and $r=b$ as in Fig. 6.10.

Let the radial interval (a, b) be divided into m parts by concentric circular arcs at intervals of Δr, and, similarly, let the angular interval (θ_1, θ_2) be divided into n parts by drawing radial lines at intervals of $\Delta\theta$, so that

$$\Delta r = \frac{b-a}{m} \quad \text{and} \quad \Delta\theta = \frac{\beta-\alpha}{n} \qquad \ldots (6.9)$$

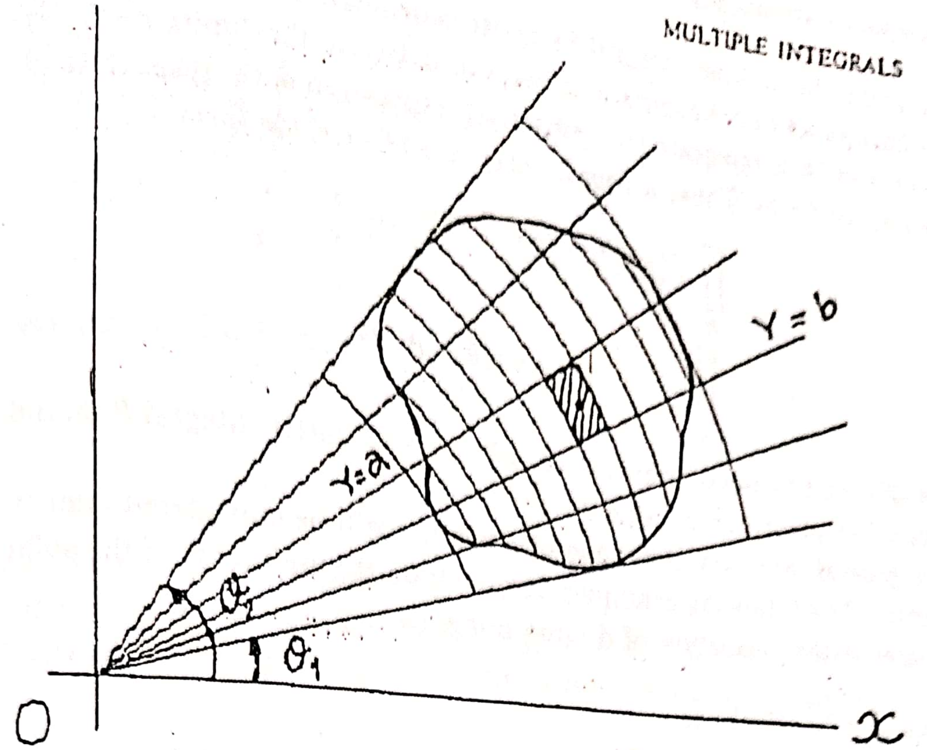

Fig. 6.10.

We have thus divided the range into infinitesimal curvilinear rectangles and partial rectangles. But the total area of the partial rectangles will be small provided that Δr and $\Delta\theta$ are sufficiently small.

Let (r_i, θ_i) be the coordinates of the centre of a typical element so that area ΔA_i is given by

$$\Delta A_i = \frac{1}{2}\left(r_i + \frac{1}{2}\Delta r\right)^2 \Delta\theta - \frac{1}{2}\left(r_i - \frac{1}{2}\Delta r\right)^2 \Delta\theta.$$

$$= \frac{1}{2}(2r_i \cdot \Delta r\,\Delta\theta) = r_i\,\Delta r\,\Delta\theta. \qquad \ldots (6.10)$$

We evaluate the function, $f(r, \theta)$ say, at the centre of each of these elements and from the sum of the products of these functional values and the corresponding values of ΔA_i. We then obtain

$$S = \sum_{i=1}^{N} f(r_i, \theta_i)\,\Delta A_i = \sum_{i=1}^{N} f(r_i, \theta_i)\, r_i\,\Delta r\,\Delta\theta, \qquad \ldots (6.11)$$

where N is the total number of elements.

Then, $\lim\limits_{N\to\infty} \sum\limits_{i=1}^{N} f(r_i, \theta_i)\, r_i\,\Delta r\,\Delta\theta$ is defined as the *double integral* of f over R and we write

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(r_i, \theta_i)\, r_i\,\Delta r\,\Delta\theta = \iint_R f(r, \theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r, \theta)\, r\,dr\,d\theta \qquad \ldots (6.12)$$

To evaluate the double integral in polar coordinates, we first integrate $f(r, \theta) \cdot r$ with respect to r keeping θ constant between the limits $r=f_1(\theta)$ and $r=f_2(\theta)$, and then integrate the remaining expression with respect to θ between $\theta=\theta_1$, to $\theta=\theta_2$. Thus, a *double integral in Cartesian form*

i.e.
$$\iint_R f(x, y)\, dx\, dy$$

becomes
$$\iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

when transformed to polar coordinates, where in the latter integral R should be expressed in polar coordinates.

The general problem of change of variables will be considered later in this chapter. The following examples demonstrate the usefulness of the polar coordinates in the evaluation of double integrals.

Example 1. Evaluate
$$\iint_R xy\, dx\, dy$$

where R is the positive quadrant of the circle $x^2+y^2=a^2$.

r^2

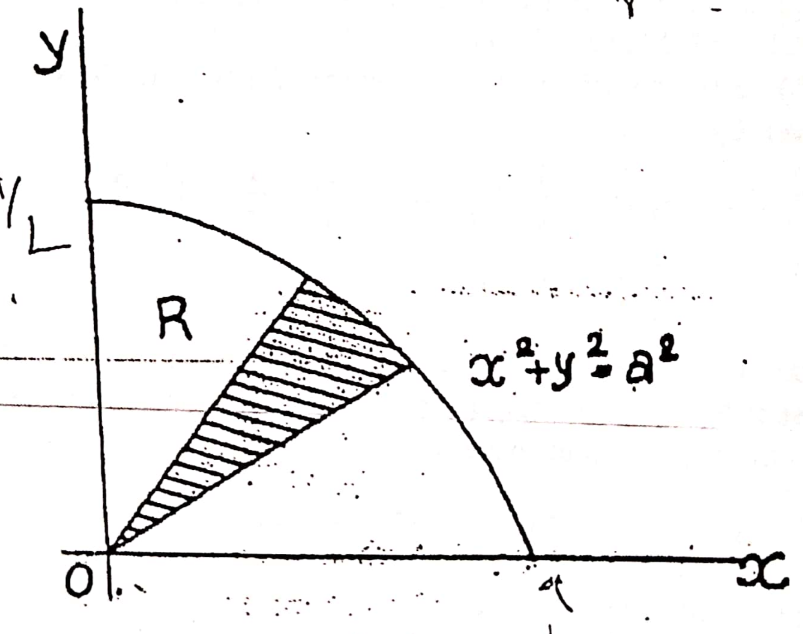

Fig. 6.11.

Changing to polars, the region R becomes $r=a$ from $\theta=0$ to $\theta=\pi/2$. Hence, the given integral becomes:

$$\int_0^{\pi/2}\int_0^a r\cos\theta \cdot r\sin\theta \cdot r\, dr\, d\theta.$$

$$= \int_0^{\pi/2}\int_0^{a} r^3\,dr \cos\theta \sin\theta\, d\theta$$

$$= \left[\frac{r^4}{4}\right]_0^a \cdot \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{a^4}{4} = \frac{a^4}{8}.$$

Example 2. Evaluate

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy,$$

by changing into polar coordinates.

The region of integration is the upper half of the circle $x^2+y^2=2ax$, (Fig. 6.12). If we change to polars, this region becomes $r=2a\cos\theta$ from

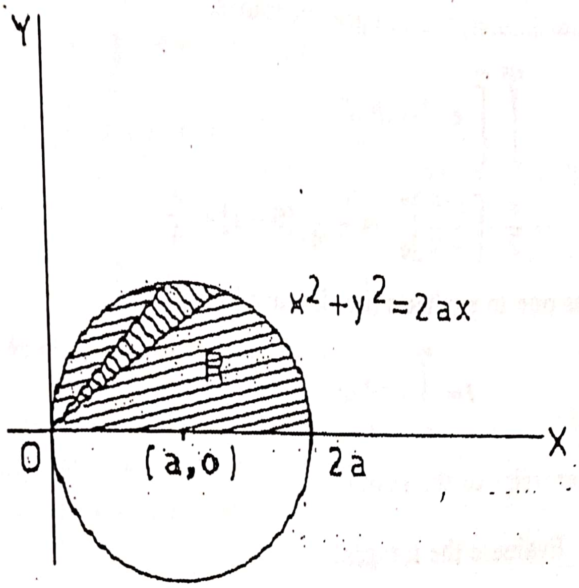

Fig. 6.12.

$\theta=0$ to $\theta=\pi/2$. Hence the given integral becomes:

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 . r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} dx$$

$$=\frac{1}{4}\int_0^{\pi/2} 16a^4\cos^4\theta\, d\theta$$

$$=4a^4\int_0^{\pi/2}\cos^4\theta\, d\theta=4a^4\cdot\frac{3}{16}\pi$$

$$=\frac{3}{4}a^4\pi.$$

Example 3. Evaluate

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy,$$

by changing into polar coordinates.

The region of integration here is clearly the first quadrant of the xy-plane. If we use polar coordinates, the integral becomes:

$$\int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta.$$

$$=\frac{\pi}{2}\cdot\left[\frac{e^{-R}}{-2}\right]_0^\infty=-\frac{\pi}{4}[0-1]=\frac{\pi}{4}\cdot$$

This result enables one to evaluate the definite integral

$$I=\int_0^\infty e^{-x^2}\,dx.$$

This is left as an exercise to the reader.

Example 4. Evaluate the integral

$$\iint xy\,(x^2+y^2)^{n/2}\,dx\,dy$$

over the positive quadrant of the circle $x^2+y^2=a^2$, supposing $n+3>0$.

Changing into polars, the region becomes $r=a$ from $\theta=0$ to $\theta=\pi/2$. Hence the given integral becomes:

$$\int_0^{\pi/2}\int_0^a r\cos\theta\cdot r\sin\theta\cdot r^n\cdot r\,dr\,d\theta.$$

$$=\int_0^{\pi/2}\int_0^a r^{n+3}\cos\theta\sin\theta\,d\theta.$$

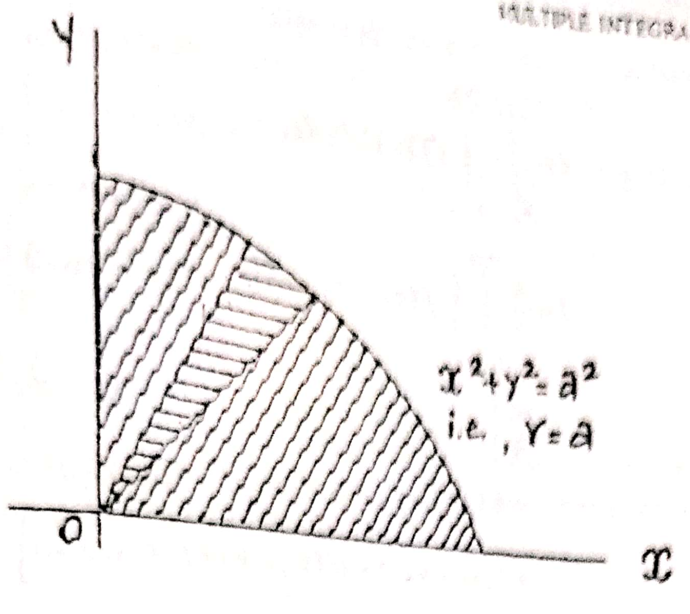

Fig. 6.13.

$$= \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \cdot \int_0^a r^{n+3} \, dr$$

$$= \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \cdot \left[\frac{r^{n+4}}{n+4}\right]_0^a, \quad \text{since } n+3>0.$$

$$= \frac{1}{2} \cdot \frac{a^{n+4}}{(n+4)} = \frac{a^{n+4}}{2(n+4)}.$$

6.1.3 Numerical Double Integration

Formulae for the evaluation of a double integral can be obtained by repeatedly applying the trapezoidal and Simpson's rules derived in § 4.1.3. We consider, as an example, the double integral defined by

$$I = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y) \, dx \, dy \qquad \ldots (6.13)$$

where $x_{i+1} = x_i + h$ and $y_{j+1} = y_j + k$.
By the repeated application of the trapezoidal rule to (6.13), we get

$$I = \frac{h}{2} \int_{y_j}^{y_{j+1}} \left[f(x_i, y) + f(x_{i+1}, y) \right] dy$$

$$= \frac{hk}{4} \left[f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1}) \right] \qquad \ldots (6.14)$$

$$= \frac{hk}{4} \left[f_{i,j} + f_{i+1,j} + f_{i,j+1} + f_{i+1,j+1} \right]$$

where $f_{i,j} = f(x_i, y_j)$, etc.

Similarly, applying Simpson's rule to the integral

$$I=\int_{y_{j-1}}^{y_{j+1}}\int_{x_{i-1}}^{x_{i+1}} f(x, y)\,dx\,dy, \qquad \ldots(6.15)$$

we obtain,

$$I=\frac{h}{3}\int_{y_{j-1}}^{y_{j+1}}\Big[f(x_{i-1}, y)+4f(x_i, y)+f(x_{i+1}, y)\Big]dy$$

$$=\frac{hk}{9}\Big[f(x_{i-1}, y_{j-1})+4f(x_{i-1}, y_j)+f(x_{i-1}, y_{j+1})$$

$$+4\{f(x_i, y_{j-1})+4f(x_i, y_j)+f(x_i, y_{j+1})\}$$

$$+f(x_{i+1}, y_{j-1})+4f(x_{i+1}, y_j)+f(x_{i+1}, y_{j+1})\Big]$$

$$=\frac{hk}{9}\Big[f_{i-1,j-1}+f_{i-1,j+1}+f_{i+1,j-1}+f_{i+1,j+1}$$

$$+4(f_{i-1,j}+f_{i,j-1}+f_{i,j+1}+f_{i+1,j})+16f_{i,j}\Big]. \qquad \ldots(6.16)$$

The following numerical example demonstrates the use of trapezoidal and Simpson's rules in evaluating a double integral.

Example. Evaluate

$$I=\int_0^1\int_0^1 e^{x+y}\,dx\,dy,$$

using the trapezoidal and Simpson's rules.

With $h=k=0.5$, the following table of values of e^{x+y} is prepared.

y \ x	0	0.5	1.0
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1.0	2.7183	4.4817	7.3891

(i) Using the trapezoidal rule (6.14) repeatedly, we obtain:

$$I=\frac{0.25}{4}\Big[1.0+4(1.6487)+6(2.7183)+4(4.4817)+7.3891\Big]$$

$$=\frac{12.3050}{4}$$

$$=3.0762.$$

(ii) If we use Simpson's rule (6.16) repeatedly, we have:

$$J = \frac{0.25}{9}\Big[1.0 + 2.7183 + 7.3891 + 2.7183 + 4\,(1.6487 + 4.4817 + 4.4817 + 1.6487) + 16\,(2.7183)\Big]$$

$$= \frac{26.59042}{9}$$

$$= 2.9545.$$

The exact value of the double integral is 2.9525 and therefore the result obtained by using Simpson's rule is about sixty times more accurate than that obtained by the trapezoidal rule.

EXERCISES 6.1

Evaluate the following double integrals (Exs. 1–15):

1. $\displaystyle\int_0^1\int_1^2 (x^2+y^2)\,dx\,dy.$

2. $\displaystyle\int_0^3\int_1^2 xy\,(x+y)\,dx\,dy.$

3. $\displaystyle\int_0^a\int_0^b (x^2+y^2)\,dx\,dy.$

4. $\displaystyle\int_1^2\int_3^4 \frac{1}{(x+y)^2}\,dx\,dy.$

5. $\displaystyle\int_0^1\int_x^1 \frac{y^2\,dy}{\sqrt{x^2+y^2}}.$

6. $\displaystyle\int_1^4\int_0^{\sqrt{4-x}} xy\,dy\,dx.$

7. $\displaystyle\int_1^2\int_x^{x\sqrt{3}} xy\,dx\,dy.$

8. $\displaystyle\int_1^2\int_1^x xy^2\,dy\,dx.$

9. $\displaystyle\int_0^{\pi/4}\int_0^{\pi/2} \sin(x+y)\,dx\,dy.$

10. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} y^3\,dy\,dx.$

11. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-y} x^2\,dx\,dy.$

12. $\displaystyle\int_0^2\int_{x^2}^{2x} (2x+3y)\,dy\,dx.$

13. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx.$

14. $\displaystyle\int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx.$

15. $\displaystyle\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx.$

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41. $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$

42. $\int_0^a\int_0^x \frac{f'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}}.$

43. $\int_0^{2a}\int_{y^2/4a}^{3a-y} (x^2+y^2)\,dx\,dy.$

44. $\int_0^c\int_x^c \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx.$

45. $\int_0^a\int_0^x \frac{1}{\sqrt{(a-x)(x-y)(a-y)(a+y)}}\,dy\,dx.$

46. $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_1^9\int_{(y-3)/2}^{\sqrt{y}} dx\,dy.$

47. $\int_0^{\pi}\int_0^{\sin x} x^2y\,dy\,dx.$

48. $\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\,dx.$

49. $\int_0^2\int_0^{x^2} (\sin xy)\,dy\,dx.$

50. Evaluate the double integral

$$I=\int_4^{4.2}\int_2^{2.3} \frac{1}{xy}\,dx\,dy.$$

using the trapezoidal and Simpson's rules and compare the results with the exact value.

6.2 APPLICATIONS OF DOUBLE INTEGRALS

In this section, we shall discuss some important applications of double integrals which occur quite often in science and engineering. These include problems involving area, volume, mass, centroid, centre of mass, moment of inertia, etc. The simplest application is that of finding the area of a region of the xy-plane and this is considered first.

6.2.1 Area by Double Integration

We wish to find the area enclosed between $x=a$, $x=b$, $y=f(x)$ and $y=F(x)$.

Let the area be divided into rectangular elements of the type $PQRS$ (Fig. 6.14) of area $\Delta x\,\Delta y$ where $P\equiv(x, y)$ and $R\equiv(x+\Delta x,\ y+\Delta y)$.

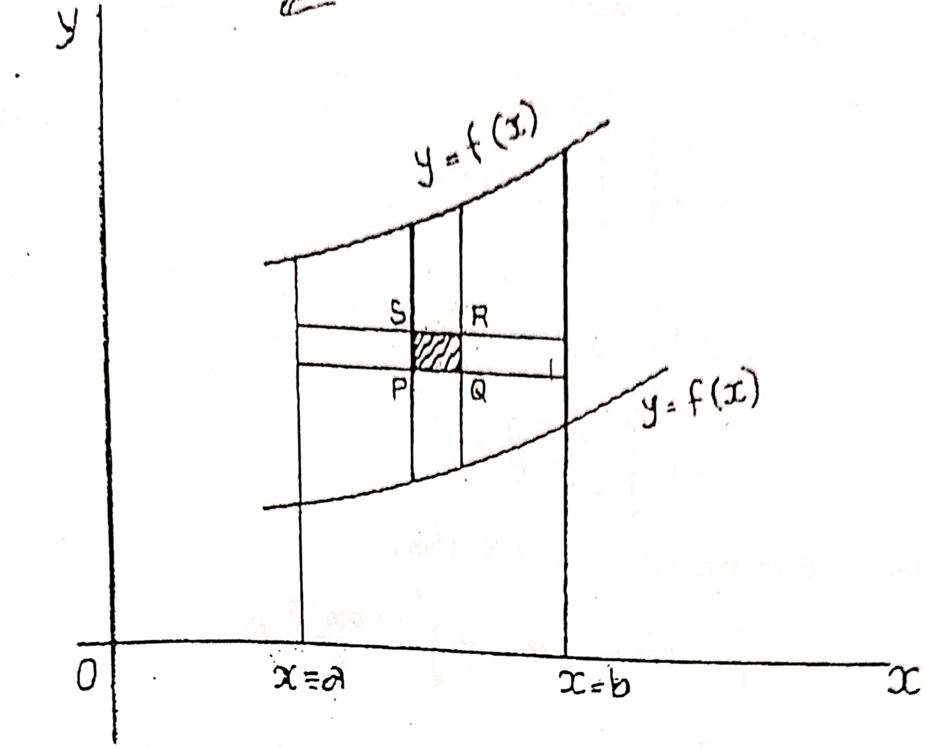

Fig. 6.14.

If this element is moved along the vertical strip from $y=f(x)$ to $y=F(x)$, we obtain

$$\Delta x \cdot \lim_{\Delta y \to 0} \sum_{y=f(x)}^{y=F(x)} \Delta y, \text{ since } \Delta x \text{ is a constant.}$$

This can be written as

$$\Delta x \cdot \int_{f(x)}^{F(x)} dy,$$

which represents the area of the vertical strip. Adding up all such strips between $x=a$ and $x=b$, we get the required area as

$$\lim_{\Delta x \to 0} \sum_{a}^{b} \Delta x \int_{f(x)}^{F(x)} dy = \int_a^b dx \int_{f(x)}^{F(x)} dy = \int_a^b \int_{f(x)}^{F(x)} dx\, dy. \qquad \ldots (6.17)$$

In a similar way, the area A, in polar coordinates, becomes:

$$A = \iint r\, dr\, d\theta, \qquad \ldots (6.18)$$

over the given region R.

Example 1. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

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Because of symmetry, we consider the area in the first quadrant and write

$$A=4\int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} dx\,dy.$$

$$=4\int_0^b a\sqrt{1-\frac{y^2}{b^2}}\,dy$$

$$=4a\int_0^b \sqrt{1-\frac{y^2}{b^2}}\,dy.$$

Setting $y=b\sin\theta$, we obtain $dy=b\cos\theta\,d\theta$. Then,

$$A=4a\int_0^{\pi/2}\cos\theta\cdot b\cos\theta\,d\theta=4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=2ab\times\frac{\pi}{2}=\pi ab.$$

Example 2. Find by double integration, the area which lies inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$.

The area is shown in Fig. 6.15. Because of symmetry, we need consider only the portion of the area above the x-axis. We therefore write

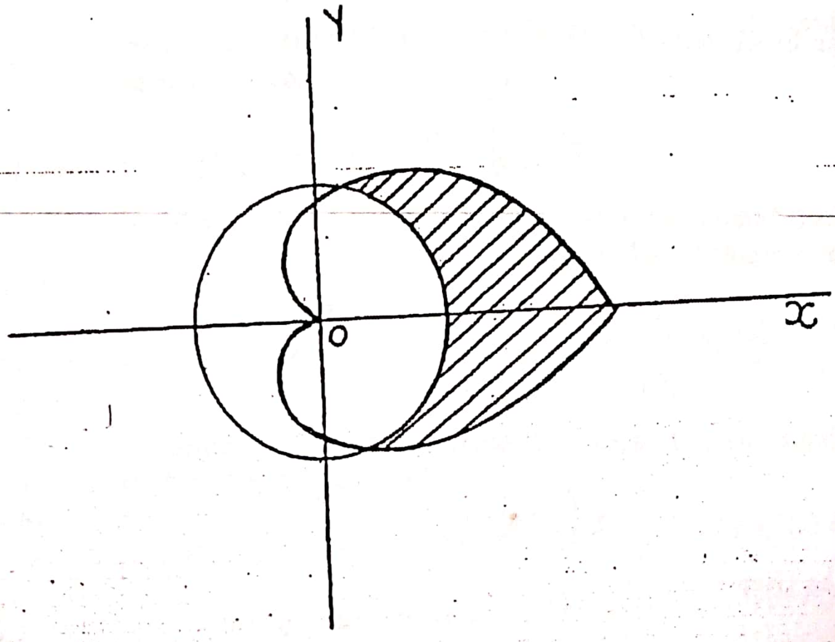


Fig. 6.15.

$$A = 2 \int_0^{\pi/2} \int_{r=a}^{a(1+\cos\theta)} r\, dr\, d\theta$$

$$= 2 \int_0^{\pi/2} \left[\frac{1}{2} r^2\right]_{r=a}^{r=a(1+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2} \left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= a^2 \int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\, d\theta$$

$$= a^2 \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2 \left[2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2}$$

$$= a^2 \left[2 + \frac{1}{2}\left(\frac{\pi}{2} + 0\right)\right]$$

$$= a^2 \left(2 + \frac{\pi}{4}\right).$$

Example 3. Find the area in the xy-plane bounded by the lemniscate

$$r^2 = a^2 \cos 2\theta.$$

The area is shown in Fig. 6.16. By symmetry, the required area is

$$A = 4 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta$$

$$= 4 \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}$$

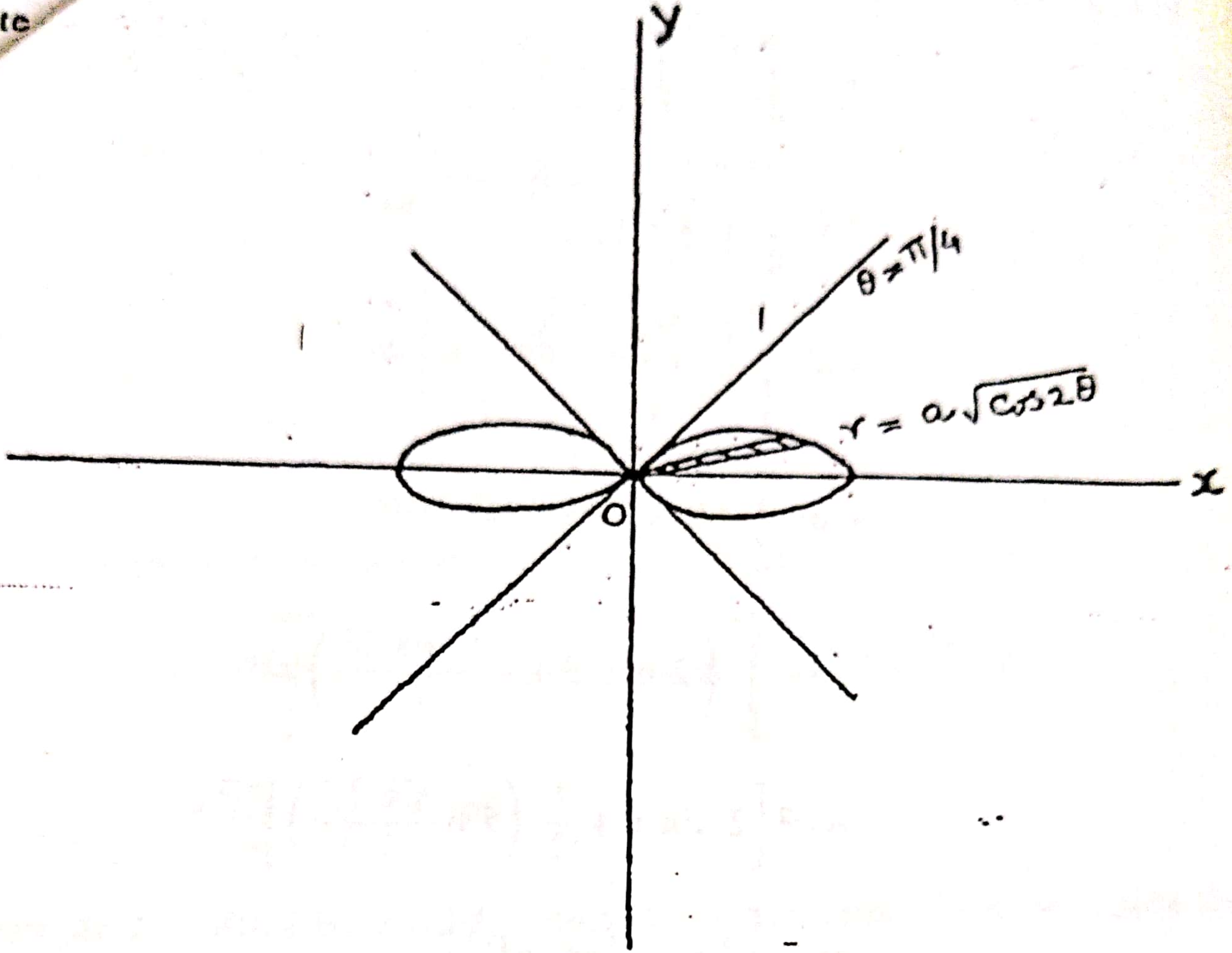
$$= 2 \int_0^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$= 2a^2 \cdot \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2.$$

Beca

write



15. The cardioid $r = a(1 + \cos\theta)$ is rotated about its axis. Find the volume generated.
16. Find the moment of inertia about the x-axis of the area of the triangle with vertices $(1, 1)$, $(2, 1)$ and $(3, 3)$.

6.3 TRIPLE INTEGRALS—VOLUME

The triple integral of a function $f(x, y, z)$ over a closed rectangular parallelopiped is defined in the same manner as a double integral. Proceeding as in the case of the double integral, we obtain sums of the form

$$S = \sum_{i=1}^{N} f(x_i, y_i, z_i)\, \Delta x\, \Delta y\, \Delta z,$$

and the triple integral of $f(x, y, z)$ over the region V is the limiting value to which these sums tend as Δx, Δy and Δz tend to zero. Thus,

$$\iiint_V f(x, y, z)\, dV = \lim \sum_{i=1}^{N} f(x_i, y_i, z_i)\, \Delta x\, \Delta y\, \Delta z \qquad \ldots (6.25)$$

If $f(x, y, z) \equiv 1$, we deduce immediately that the triple integral is just the volume of V. If, on the other hand, $f(x, y, z)$ is the density at (x, y, z), then the integral is the mass in V. As in the case of the double integral, the definition (6.25) is seldom used for the evaluation of the triple integral. Instead, it is usually evaluated as an iterated integral. For example, suppose V is bounded below by a surface $z = z_1(x, y)$ and above by the surface $z = z_2(x, y)$, where $z_1(x, y)$ and $z_2(x, y)$ are functions defined in a region R of the xy-plane, then if $f(x, y, z)$ is continuous in V, we have that

$$\iiint_V f(x, y, z)\, dV = \iint_R \left[\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\, dz\right] dx\, dy \qquad \ldots (6.26)$$

In (6.26), we first integrate with respect to z and then integrate the resulting function of x and y over the region R.

Example 1. Find the volume of the region bounded by the paraboloids

$$z = x^2 + y^2 \quad \text{and} \quad z = 6 - \frac{x^2 + y^2}{2}.$$

The curve of intersection of the two paraboloids is given by

$$x^2 + y^2 = 6 - \frac{x^2 + y^2}{2}$$

or $$3(x^2 + y^2) = 12$$

or, $$x^2 + y^2 = 4.$$

Hence, the volume V is given by

$$V=\iint\limits_R \int_{x^2+y^2}^{6-\frac{1}{2}(x^2+y^2)} dz\,dy\,dx,$$ where R is the region in the (x, y) plane bounded by $x^2+y^2=4$.

Therefore, $$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[6-\frac{1}{2}(x^2+y^2)-(x^2+y^2)\right]dy\,dx$$

$$=6\int_0^2\int_0^{\sqrt{4-x^2}}\left[4-(x^2+y^2)\right]dy\,dx$$

$$=6\int_0^{\pi/2}\int_0^2(4-r^2)\,r\,dr\,d\theta$$

$$=12\pi,$$ on simplification.

Example 2. Find the volume of the region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$.

The volume is given by

$$V=\iint\limits_R\int_{x^2+y^2}^{4} dz\,dy\,dx,$$ where R is the region in the (x, y) plane bounded by the circle $x^2+y^2=4$. We therefore have

$$V=\iint\limits_R\left[4-(x^2+y^2)\right]dy\,dx$$

$$=4\int_0^{\pi/2}\int_0^2(4-r^2)\,r\,dr\,d\theta$$

$$=4\cdot\frac{\pi}{2}\cdot\left[4\frac{r^2}{2}-\frac{r^4}{4}\right]_0^2$$

$$=2\pi\left[2r^2-\frac{r^4}{4}\right]_0^2=2\pi\left[8-4\right]$$

$$=8\pi.$$

Example 3. Find the volume of the solid enclosed by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Because of symmetry, we need compute the volume in the first octant only. Hence, if V is the total volume, then

$$V=8\iint_R \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx,$$

where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant.

Therefore,
$$V=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

We shall first evaluate the inner integral. Let

$$I_1=\int_0^{b\sqrt{p}}\sqrt{p-\frac{y^2}{b^2}}\,dy,\quad \text{where } p=1-\frac{x^2}{a^2}.$$

Setting $\frac{y}{b}=\sqrt{p}\sin\theta$, we obtain $dy=b\sqrt{p}\cos\theta\,d\theta$.

Then,
$$I_1=\int_0^{\pi/2}\sqrt{p}\cdot\cos\theta\cdot b\sqrt{p}\cos\theta\,d\theta$$

$$=bp\int_0^{\pi/2}\cos^2\theta\,d\theta=bp\cdot\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=\frac{bp}{2}\cdot\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$=\frac{bp}{2}\cdot\frac{\pi}{2}=\frac{bp\pi}{4}.$$

We therefore have

$$V=8c\int_0^a\frac{bp\pi}{4}\cdot dx=2cb\pi\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$=2cb\pi\left[x-\frac{1}{a^2}\cdot\frac{x^3}{3}\right]_0^a$$

$$=2cb\pi\left[a-\frac{1}{3}a\right]=\frac{4}{3}cab\pi$$

$$=\frac{4}{3}\pi abc$$

Example 4. Find the volume of the region in space bounded above by the surface $z = 1 - (x^2 + y^2)$, on the sides by the planes $x = 0$, $y = 0$, $x + y = 1$, and below by the plane $z = 0$.

The volume is given by

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(1 - x^2 - y^2\right) dy\, dx$$

$$= \int_0^1 \left[y - x^2 y - \frac{1}{3} y^3\right]_0^{1-x} dx$$

$$= \int_0^1 \left[1 - x - x^2(1-x) - \frac{1}{3}(1-x)^3\right] dx$$

$$= \left[x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{3}\cdot\frac{(1-x)^4}{-4}\right]_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{12}\left[(1-x)^4\right]_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{12}$$

$$= \frac{1}{3}.$$

6.3.1 Cylindrical Coordinates

The use of cylindrical coordinates is particularly suited for problems in which there is an axis of symmetry of the solid. Instead of the volume element $dx\, dy\, dz$, we use, in cylindrical coordinates, the volume element $r\, dr\, d\theta\, dz$, i.e. an element with a cross-sectional area $r\, dr\, d\theta$ and altitude dz.

Example 1. If the radius of the base and altitude of a right circular cone are given by a and h respectively, express its volume as a triple integral and evaluate it using cylindrical coordinates.

Taking r, θ and z as cylindrical coordinates, the equation of the cone may be written as $\frac{z}{r} = \frac{h}{a}$, or $z = r\frac{h}{a}$. Hence, its volume is given by

$$V = \int_0^{2\pi} \int_0^{a} \int_{z=rh/a}^{z=h} r\, dr\, d\theta\, dz$$

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$$= \int_0^{2\pi}\int_0^{a} \left(h - \frac{rh}{a}\right) r\,dr\,d\theta$$

$$= h \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_0^a d\theta = h \cdot 2\pi \cdot \left(\frac{a^2}{2} - \frac{a^2}{3}\right)$$

$$= 2\pi h \cdot \frac{1}{6}\, a^2 = \frac{1}{3}\pi a^2 h.$$

Example 2. Find the volume bounded by the sphere

$$x^2 + y^2 + z^2 = a^2.$$

In cylindrical coordinates, the equation of the sphere becomes

$$r^2 + z^2 = a^2$$

and the volume element is $r\,dr\,d\theta\,dz$. Then the volume is given by

$$V = \int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz$$

$$= 2\int_0^{2\pi}\int_0^{a} \sqrt{a^2 - r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a^2} \sqrt{a^2 - R}\, dR\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{(a^2 - R)^{3/2}}{-3/2}\right]_0^{a^2} d\theta$$

$$= -\frac{2}{3} \cdot 2\pi \cdot (-a^3) = \frac{4}{3}\pi a^3.$$

In the above, the values of ϕ are restricted to the range

$$0 \leqslant \phi \leqslant \pi. \qquad \ldots (6.29)$$

A proof of (6.28) is omitted here.

Example 1. Find the volume of the sphere $x^2+y^2+z^2=a^2$.

The required volume V is eight times that enclosed in the first octant. We integrate the constant function 1 over this region using the volume element $dV=r^2 \sin\phi \, dr\, d\theta\, d\phi$, and limits in spherical polar coordinates. We then obtain

$$V=8\iiint dx\,dy\,dz$$

$$=8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2 \sin\phi \cdot dr\,d\theta\,d\phi$$

$$=8\cdot\frac{a^3}{3}\cdot\frac{\pi}{2}\cdot\Big[-\cos\phi\Big]_0^{\pi/2}$$

$$=\frac{4}{3}\cdot\pi a^3\cdot\left[-\cos\frac{\pi}{2}+1\right]$$

$$=\frac{4}{3}\pi a^3.$$

Example 2. Evaluate $\iiint xyz\,dx\,dy\,dz$

over the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Using (6.27) and (6.28), the given integral transforms into

$$I=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r\cos\theta\sin\phi\cdot r\sin\theta\sin\phi\cdot r\cos\phi\cdot r^2\sin\phi\cdot dr\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^5\cdot\cos\theta\sin\theta\sin^3\phi\cos\phi\,d\phi\,d\theta\,dr$$

$$=\left[\frac{r^6}{6}\right]_0^a\cdot\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\cdot\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2}$$

$$=\frac{1}{6}a^6\cdot\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{48}a^6.$$

6.3.3 Applications of Triple Integrals

If $\rho=\rho(x, y, z)$ is the density at the point (x, y, z) of a region V of xyz-space, then the mass M is given by

$$M=\iiint \rho\,dV=\iiint \rho\,dx\,dy\,dz \qquad \ldots (6.30)$$

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The coordinates of the centre of gravity are given by

$$\left.\begin{aligned}\bar{x}&=\frac{\iiint x\rho\,dV}{\iiint \rho\,dV}\\ \bar{y}&=\frac{\iiint y\rho\,dV}{\iiint \rho\,dV}\\ \bar{z}&=\frac{\iiint z\rho\,dV}{\iiint \rho\,dV}\end{aligned}\right\} \qquad \ldots(6.31)$$

Similarly, the moment of inertia about the x-axis is given by

$$I_x=\iiint \rho\,(y^2+z^2)\,dV \qquad \ldots(6.32)$$

with similar integrals for I_y and I_z.

The above integrals may be evaluated as triple integrals by taking $dV=dx\,dy\,dz$ or the corresponding volume element in other coordinate systems.

Example 1. Assuming $\rho(x, y, z)=1$, find the centroid of the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

Now, $$V=\frac{1}{8}\cdot\frac{4}{3}\pi a^3=\frac{1}{6}\pi a^3.$$

Here, $$\bar{z}=\frac{\iiint z\,dV}{\iiint dV}$$

i.e., $$\frac{1}{6}\pi a^3\bar{z}=\iiint z\,dV$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r\cos\phi\cdot r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$=\frac{a^4}{4}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}\cdot\frac{\pi}{2}$$

$$=\frac{1}{16}\pi a^4.$$

Therefore,
$$\bar{z} = \frac{\frac{1}{16}\pi a^4}{\frac{1}{6}\pi a^3} = \frac{3}{8} a.$$

By symmetry, we have
$$\bar{x} = \bar{y} = \bar{z} = \frac{3}{8} a.$$

Example 2. Find the mass and moment of inertia of a sphere of radius a with respect to a diameter if the density is proportional to the distance from the centre.

Let $\rho = kr.$

Then,
$$M = \iiint \rho \, dV = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a kr \cdot r^2 \sin\phi \cdot dr \, d\theta \, d\phi$$
$$= 8k \cdot \frac{1}{4} a^4 \cdot \Big[-\cos\phi\Big]_0^{\pi/2} \cdot \frac{\pi}{2}$$
$$= k\pi a^4.$$

We have $I_x + I_y + I_z = 2I_0.$

But, by symmetry, $I_x = I_y = I_z.$

Hence, $I_x = I_y = I_z = \frac{2}{3} I_0.$

Since,
$$I_0 = \iiint \rho \cdot (x^2 + y^2 + z^2) \, dV.$$

We have,
$$I_z = \frac{2}{3} I_0 = \frac{2}{3} \times 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a kr \cdot r^2 \cdot r^2 \sin\phi \cdot dr \, d\theta \, d\phi$$
$$= \frac{16}{3} k \cdot \frac{1}{6} a^6 \cdot \Big[-\cos\phi\Big]_0^{\pi/2} \cdot \frac{\pi}{2}$$
$$= \frac{4}{9} \cdot k\pi a^6.$$
$$= \frac{4}{9} M a^2.$$

6.3.4 Change of Variables

In § 4.2.1, we have seen that the evaluation of an integral can be made easier by a suitable substitution. Similarly, the evaluation of a double integral of the form
$$\iint_R f(x, y) \, dx \, dy$$