Partially Ordered Sets :

A relation R on a set A is called a partial order if R is reflexive, antisymmetric and transitive. The set A together with the partial order R is called a partially ordered set or a poset. It is denoted as (A, R).

Eg: Show that $(Z^+, \leq)$ is a poset.

Soln: Let $a, b \in Z^+$ and aRb iff $a \leq b$.

To prove $(Z^+, \leq)$ is a poset, we have to prove the provided relation R i.e. $\leq$ is a partial order.

$\because$ for any a, $a \leq a$ is true as a is always equal to a thus $(a, a) \in R$, i.e. R is reflexive.

For any a & b that belongs to Z^+, either $a \leq b$ or $b \leq a$ but not both at a time. So either $(a, b) \in R$ or $(b, a) \in R$. Thus, R is antisymmetric.

Now, for transitivity, if $a, b, c \in Z^+$ then if $a \leq b$ i.e. $(a, b) \in R$ and $b \leq c$ i.e. $(b, c) \in R$ then $a \leq c$ ($\because$ a is lesser than b & b is lesser than c then a is lesser than c). Thus, R is transitive.

$\therefore$ the relation R is partial order.

Thus, the set $(Z^+, \leq)$ is a poset.

Dual of a Poset :-

let (A,R) is a poset then, (A,R^{-1}) is a poset and it is called dual of (A,R).

Comparable: If $(A, \leq)$ is a poset, the elements a and b of A are said to be comparable if $a \leq b$ or $b \leq a$.

Linearly Ordered Set:

If every pair of elements in a poset A are comparable we say that A is linearly ordered set.

Theorem 1: If $(A, \leq)$ and $(B, \leq)$ are poset, then $(A \times B, \leq)$ is a poset with partial order defined by $(a,b) \leq (a',b')$ if $a \leq a'$ in A and $b \leq b'$ in B.

→ let $(a,b) \in A \times B$ then $(a,b) \leq (a,b)$ since $(a \leq a,\ b \leq b)$

$\therefore$ the relation is reflexive.

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Let $(a,b) \leq (a',b')$ and $(a',b') \leq (a,b)$ where $a, a' \in A$ $b, b' \in B$. Then, $a \leq a'$ and $a' \leq a$ in $A \Rightarrow a = a'$ [$\because$ A is a poset] and $b \leq b'$ and $b' \leq b$ in $B \Rightarrow$ $b = b'$ [$\because$ B is poset]

$\therefore$ The relation given is antisymmetric.

Suppose $(a,b) \leq (a',b')$ and $(a',b') \leq (a'',b'')$ where $a, a', a'' \in A$ and $b, b', b'' \in B$. Then $a \leq a'$ and $a' \leq a''$ in $A \Rightarrow a \leq a''$ and $b \leq b'$ and $b' \leq b''$ in $B \Rightarrow b \leq b''$

$\therefore (a,b) \leq (a'',b'')$.

Thus, relation is transitive.

Thus, $(A \times B, \leq)$ is poset.

Note: The partial order $\leq$ defined on $A \times B$ is called the product partial order and we denote it by $(a,b) \leq (a',b')$. This ordering is called lexicographic or dictionary order.

Theorem 2:

The diagraph of partial order has no cycle of length greater than 1.

Proof: Assume that a diagraph of partial order has cycle of length greater than or equal to 2. Then there exist distinct elements $a_1, a_2, a_3 \dots a_n$ such that $a_1 \le a_2$, $a_2 \le a_3$, $a_3 \le a_4$, $\dots$ $a_{n-1} \le a_n$, $a_n \le a_1$. And by transitivity property $a_1 \le a_n$ and by assumption $a_n \le a_1$ which implies $a_1 = a_n$ which contradicts our assumptions. Thus by condraticttory method we conclude the diagraph of partial order has no cycle of length greater than 1.

Hasse Diagram:-

A Hasse diagram is a graphical representation of the relation of elements of a partial ordered set (poset) with implied upward orientation.

Steps to Draw Hasse Diagram:

1. Draw the diagraph representing the provided partial ordered set.
   (Keep in mind that all the arrow head except loop are oriented upward only).

2. Remove all the loops present in diagraph drawn in step 1.

3. Remove all the transitive edges from the resultant structure obtained in step 2.

4. Remove arrow head from all the edges of the resultant graph from step 3.

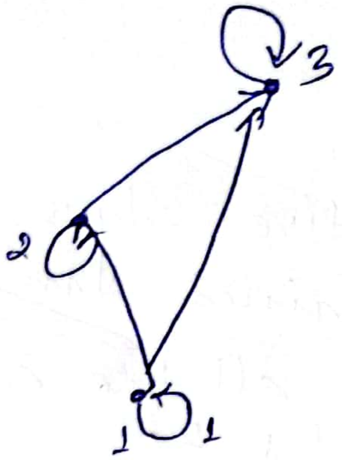
Note :-
A Hasse Diagram can be defined as a structural representation that have enough information to represent a poset.

Eg :- Draw Hass Diagram for the poset (A, R) where $A = \{1, 2, 3\}$ and R is a relation where $(a, b) \in R$ and $a \leq b$.

Solⁿ: The ordered pairs for the provided relation R is given below.

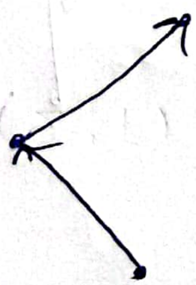
$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

-ep 1: Construction of Diagraph.



Step 2: Removal of loops.

Step 3: Removal of transitive edge.



Step 4: Removal of arrow head.

Extremal Elements of a Poset:

Maximal Element :- An element $a \in A$ is called maximal element of A if there is no element c in A such that $A < C$.

Minimal Element :- An element $b \in A$ is called minimal element of A if there is no element c in A such that $c < b$.

Greatest Element : An element $a \in A$ is called greatest element of A if $x \leq a$ for all $x \in A$.

Least Element :- An element $a \in A$ is called a least element of A if $a \leq x$ for all $x \in A$.

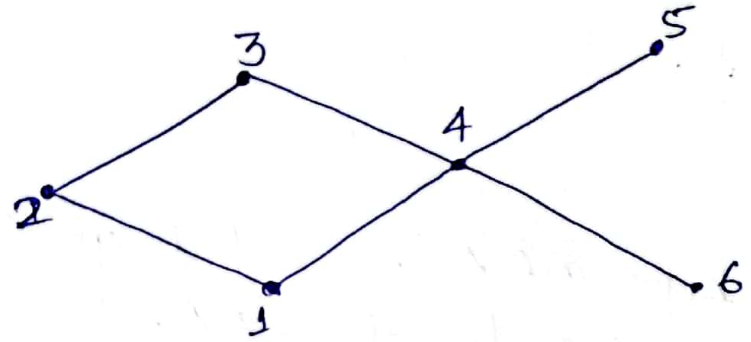
Theorem 1 : Let A be a finite non-empty poset with partial order $\leq$. Then A has at least one maximal element & at least one minimal element.

Unit Element : The greatest element of a poset if it exist is denoted by I and is called the unit element.

Zero Element : The least element of a poset if it exist is denoted by 0 and called zero element.

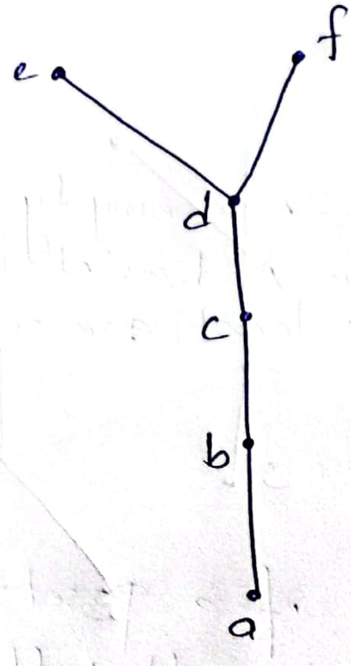
Eg:-

Determine the maximal and minimal element of the provided Hasse Diagram.



Minimal Element : 1,6

Maximal Element : 3,5



Maximal Element = e, f

Minimal Element = a

Defⁿ:-

Minimal Element:- An element x of a set S is called a minimal element if there is no $y \in S$ such that yRx (or $(y,x) \notin R$) and $y \neq x$.

Maximal Element:-

An element x of a set S is called a maximal element if there is no $y \in S$ such that xRy (or $(x,y) \notin R$) and $x \neq y$.

Least element (Minimum Element):-

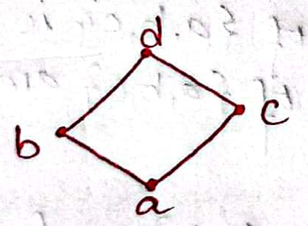
An element $x \in S$ is called the least element of S if $\forall y \in S$, xRy.

Note: Least element is unique if exists.

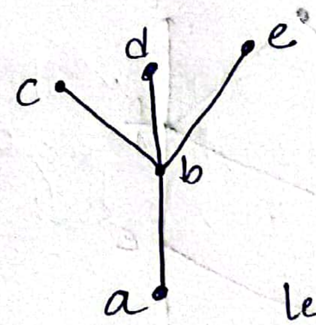
Greatest element (Maximum Element):-

An element $x \in S$ is called the greatest element of S if $\forall y \in S$, yRx.

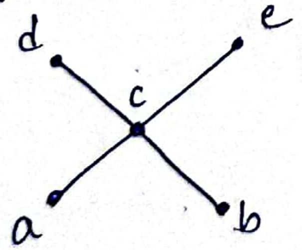
Eg:-



least element is a.
greatest element is d.



Least element is a.
No greatest element.



No least element
No greatest element

Let say we have a poset (S, R) such that S is an arbitary set and R is a partial order defined on set S.
Also, let say $T \subseteq S$.

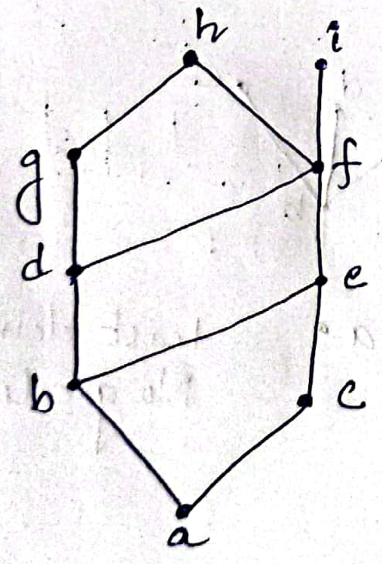
Lower Bound:

An element $x \in S$ is a lower bound of set T if $\forall y \in T \; (x, y) \in R$.

Upper Bound:

An element $x \in S$ is an upper bound of set T if $\forall y \in T$ $(y, x) \in R$.

Eg:- Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$ and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown below.

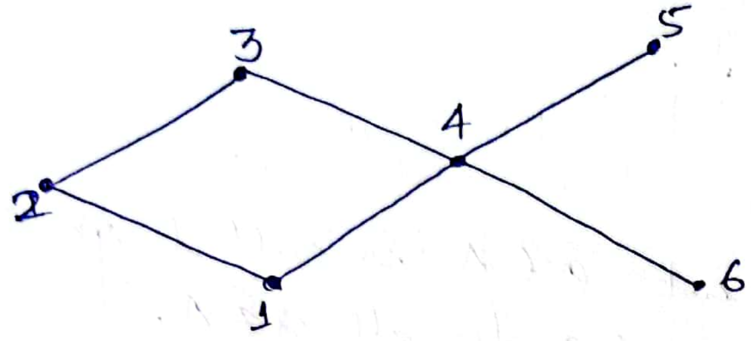


Lower bound of $\{a, b, c\}$ is a.
Upper bounds of $\{a, b, c\}$ are e, f, j and h.

Lower bounds of $\{j, h\}$ are f, d, e, b, c, a
Upper bound of $\{j, h\}$ is ϕ.

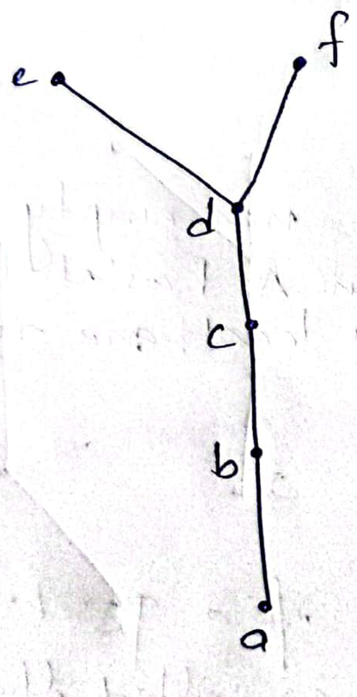
Eg:-

Determine the maximal and minimal element of the provided Hasse Diagram.



Minimal Element : 1,6

Maximal Element : 3,5



Maximal Element = e, f

Minimal Element = a

Greatest Lower Bound :- (GLB or Infimum or Meet) :

let say L is the set of all lower bounds of set T. Then, an element $x \in L$ is called the greatest lower bound if $\forall y \in L, (y, x) \in R$.

In other words,

$$GLB(T) = \text{maximum}\{LB(T)\}$$

Least Upper Bound :- (LUB or Supremum or Join) :-

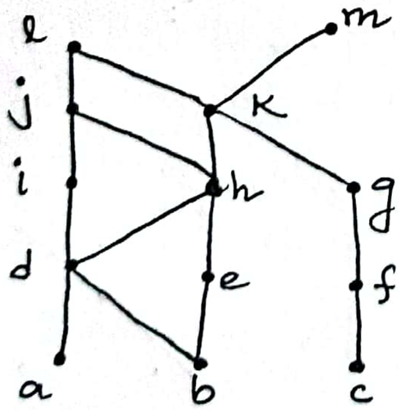
let say U is the set of all upper bounds of set T. Then, an element $x \in U$ is called the least upper bound if $\forall y \in U (x, y) \in R$.

In other words,

$$LUB(T) = \text{minimum}\{UB(T)\}$$

Eg :- Consider the following Hasse Diagram

Find the least upper bound of $\{a, b, c\}$ and the greatest ~~upper~~ lower bound of $\{f, g, h\}$



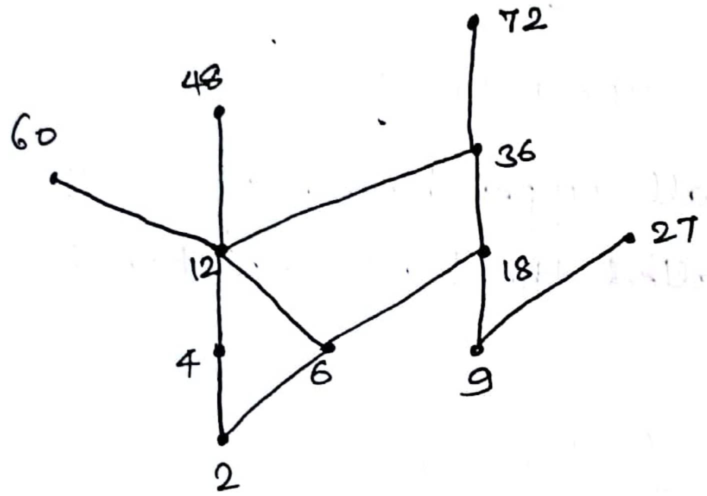
upper bounds of $\{a, b, c\}$ are k, l and m.

Least Upper bound of $\{a, b, c\}$ is k.

Lower bound of $\{f, g, h\}$ is ϕ

Greatest lower bound of $\{f, g, h\}$ is ϕ.

Eg:- Find least upper bound of $\{2,9\}$ and the greatest lower bound of $\{60,72\}$ for the poset $(\{2,4,6,9,12,18,27,36,48,60,72\}, |)$.



upper bounds of $\{2,9\}$ are 18, 36, 72. Least upper bound of $\{2,9\}$ is 18.

Lower bounds of $\{60,72\}$ are 2, 4, 6, 12. Greatest lower bound of $\{60,72\}$ is 12.

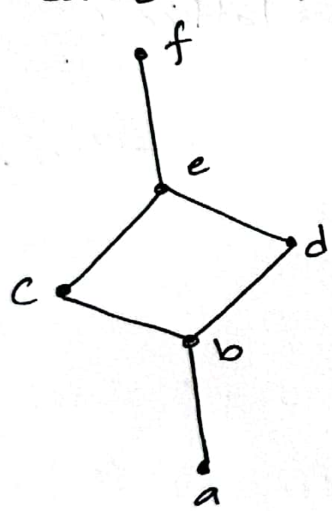
Lattices :

A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of two elements has a least upper bound and a greatest lower bound. We denote $LUB(\{a,b\})$ by $a \vee b$ and call it the join of a & b. Similarly, we denote $GLB(\{a,b\})$ by $a \wedge b$ and called it the meet of a and b.

Meet Semi Lattice :-

The poset $(S, \leq)$ is a meet semilattice if $\forall x, y \in S$, $x \wedge y$ (i.e. $GLB(x,y)$) must not be empty.

Eg: Consider the following Hasse Diagram:



Soln:

In the given Hasse diagram, every pair of element has the greatest lower bound.

For eg:

$GLB(f, e) = e$

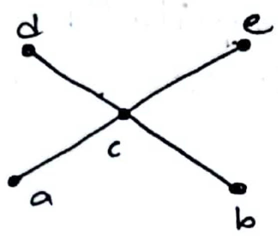
$GLB(c, d) = b$

$GLB(e, d) = d$

$GLB(c, b) = b$

Therefore, the given Hasse diagrams is a meet-semilattice.

Consider the following Hasse Diagram:



Is the above Hasse diagram a meet semi lattice?

Soln:

Consider the pair (d, e)

LB (d, e) = c, a, b

GLB (d, e) = c

Again,

Consider the pair (a, b)

LB (a, b) = ϕ

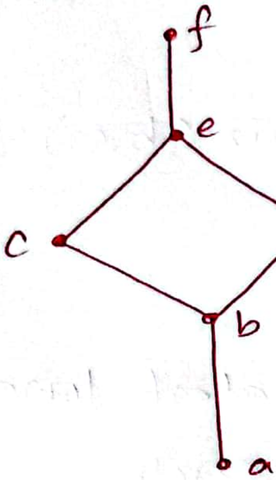
GLB (a, b) = ϕ

$\therefore$ There exist a pair in the poset represented by the given Hasse diagram whose greatest lower bound does not exist, therefore, the given Hasse diagram is NOT a meet semi-lattice.

Join Semilattice:

Consider a poset $(S, \leq)$. The poset $(S, \leq)$ is a join semi lattice if $\forall x, y \in S$, $x \vee y$ (i.e. LUB(x,y)) must not be empty.

Eg: Consider the following Hasse Diagram.

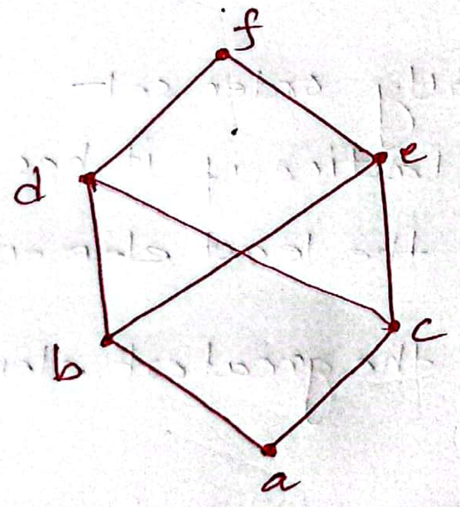


Soln:

LUB$(f, e) = f$
LUB$(c, d) = e$
LUB$(e, d) = e$
LUB$(c, b) = c$
LUB$(a, b) = b$

$\therefore$ all the possible pair have Least upper bound. Thus, the given Hasse Diagram is a join semi lattice.

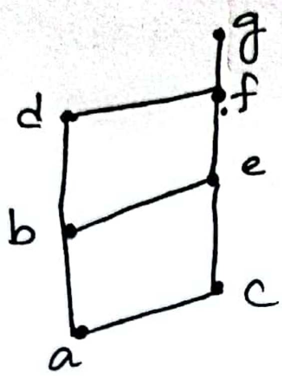
Eg:



Consider the pair (b, c)
UB$(b, c) = \{d, e, f\}$
LUB$(b, c) = \phi$
After tracing the path up from b and c, the first point where they meet are d & e and there is no single meeting point.
Hence, LUB$(b, c) = \phi$

Therefore, the given Hasse diagram is NOT a join semilattice.

Eg: Check if the poset represented by provided Hasse diagram is Lattice or not.



Consider the pair (d, e)

GLB (d, e) = b

LUB (d, e) = f

Again, for the pair (b, c)

GLB (b, c) = a

LUB (b, c) = e

For every pair of elements, the greatest lower bound and the least upper bound exists.

Therefore, the given Hasse diagram is a lattice.

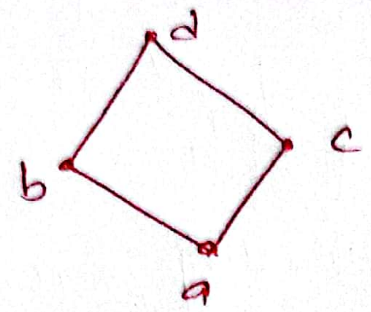
Bounded Lattice:

Consider a poset $(S, \leq)$, A partially order set $(S, \leq)$ is called a bounded lattice if it has the greatest element (1) and the least element (0).

Greatest Element: 1 is called the greatest element if $\forall x \in S$, $x \leq 1$.

Least element: 0 is called the least element if $\forall x \in S$, $0 \leq x$.

Eg: Consider the following Hasse Diagram:



a is ~~the~~ the least element coz $a \le b, a \le c$ and $a \le d$.

d is the greatest element coz $d \ge b, d \ge c$ and $d \ge a$.

Both least and greatest elements exists in the above lattice. Therefore, the given lattice is a bounded lattice.