Assignment 2

Solved by Tamanna and Chinmay

1. Two-State Loop (10 Marks) Consider the Markov chain shown below, with state space 1, 2, 3, 4, where the labels next to arrows indicate the probabilities of those transitions.
(a) Write down the transition matrix Q for this chain.

The transition probabilities for the Markov chain with state space $\{1, 2, 3, 4\}$ are given as follows:

- From state 1:
  - To state 1: 0.5
  - To state 2: 0.5
- From state 2:
  - To state 1: 0.25
  - To state 2: 0.75
- From state 3:
  - To state 3: 0.25
  - To state 4: 0.75
- From state 4:
  - To state 3: 0.75
  - To state 4: 0.25

The transition matrix Q (with rows and columns ordered as states 1, 2, 3, 4) is:

$$Q = \begin{pmatrix} 0.5 & 0.5 & 0.0 & 0.0 \\ 0.25 & 0.75 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.75 \\ 0.0 & 0.0 & 0.75 & 0.25 \end{pmatrix}$$

(b) Which states (if any) are recurrent? Which states (if any) are transient?

The Markov chain decomposes into two disjoint, closed sets of states:

$$\{1, 2\} \quad \text{and} \quad \{3, 4\}.$$

Within each set, all states communicate with each other, forming irreducible subchains.

Classification:

- **Recurrent States:** All states are recurrent.
  - For states $\{1, 2\}$:
    * From state 1, it can return to itself with probability 0.5 or move to state 2 (probability 0.5).
    * From state 2, it can return to itself with probability 0.75 or move to state 1 (probability 0.25).
    * Both states communicate and form a closed set.

- For states $\{3, 4\}$:
  - From state 3, it can return to itself with probability 0.25 or move to state 4 (probability 0.75).
  - From state 4, it can return to itself with probability 0.25 or move to state 3 (probability 0.75).
  - Both states communicate and form a closed set.

- **Transient States:** None.
  - There are no states that can be left permanently; all states belong to closed, irreducible sets.

Conclusion:

The Markov chain consists entirely of recurrent states with no transient states. The chain is reducible, composed of two separate, irreducible subchains: $\{1, 2\}$ and $\{3, 4\}$.

(c) Find two different stationary distributions for the chain

A stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ satisfies:

1. $\pi Q = \pi$ (balance equations)
2. $\sum_{i=1}^{4} \pi_i = 1$ (normalization)
3. $\pi_i \geq 0$ for all i (probability measure)

Since the Markov chain is reducible with two closed communication classes $\{1, 2\}$ and $\{3, 4\}$, we can find stationary distributions supported on each class separately.

First Stationary Distribution (Supported on $\{1, 2\}$)

Set $\pi_3 = \pi_4 = 0$ and solve for π_1, π_2:

$$\begin{aligned} \pi_1 &= 0.5\pi_1 + 0.25\pi_2 \\ \pi_2 &= 0.5\pi_1 + 0.75\pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

From the first equation:

$$\pi_1 - 0.5\pi_1 = 0.25\pi_2 \implies 0.5\pi_1 = 0.25\pi_2 \implies \pi_2 = 2\pi_1$$

Substituting into the normalization:

$$\pi_1 + 2\pi_1 = 1 \implies 3\pi_1 = 1 \implies \pi_1 = \frac{1}{3}, \quad \pi_2 = \frac{2}{3}$$

Thus, the first stationary distribution is:

$$\pi^{(1)} = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$$

Second Stationary Distribution (Supported on $\{3, 4\}$)

Set $\pi_1 = \pi_2 = 0$ and solve for π_3, π_4:

$$\begin{aligned} \pi_3 &= 0.25\pi_3 + 0.75\pi_4 \\ \pi_4 &= 0.75\pi_3 + 0.25\pi_4 \\ \pi_3 + \pi_4 &= 1 \end{aligned}$$

From the first equation:

$$\pi_3 - 0.25\pi_3 = 0.75\pi_4 \implies 0.75\pi_3 = 0.75\pi_4 \implies \pi_3 = \pi_4$$

Substituting into the normalization:

$$\pi_3 + \pi_3 = 1 \implies 2\pi_3 = 1 \implies \pi_3 = \frac{1}{2}, \quad \pi_4 = \frac{1}{2}$$

Thus, the second stationary distribution is:

$$\pi^{(2)} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$$

Verification

We verify both distributions satisfy $\pi Q = \pi$:

For $\pi^{(1)}$:

$$\left(\frac{1}{3}, \frac{2}{3}, 0, 0\right) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix} = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$$

For $\pi^{(2)}$:

$$\left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$$

Conclusion

The Markov chain has infinitely many stationary distributions (all convex combinations of $\pi^{(1)}$ and $\pi^{(2)}$), but two fundamental ones are:

$$\boxed{\pi^{(1)} = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)} \quad \text{and} \quad \boxed{\pi^{(2)} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)}$$

Question 2: Winning Streak

Problem Statement

Consider a team's game outcomes modeled as a Markov chain with:

- States: W (Win) and L (Loss)
- Transition probabilities:
  - After a win: $P(W \to W) = 0.8$, $P(W \to L) = 0.2$
  - After a loss: $P(L \to W) = 0.3$, $P(L \to L) = 0.7$
- Dinner probabilities:
  - $P(\text{Dinner}|W) = 0.7$
  - $P(\text{Dinner}|L) = 0.2$

(a) Long-run proportion of games won

Step 1: **Define the transition matrix**:

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

Step 2: **Find stationary distribution** $\pi = (\pi_W, \pi_L)$:

$$\begin{aligned} \pi P &= \pi \\ \pi_W + \pi_L &= 1 \end{aligned}$$

Step 3: **Write balance equations**:

$$\begin{aligned} \pi_W &= 0.8\pi_W + 0.3\pi_L \quad (1) \\ \pi_L &= 0.2\pi_W + 0.7\pi_L \quad (2) \end{aligned}$$

Step 4: **Simplify equation (1)**:

$$\begin{aligned} \pi_W - 0.8\pi_W &= 0.3\pi_L \\ 0.2\pi_W &= 0.3\pi_L \\ \pi_W &= \frac{0.3}{0.2}\pi_L = 1.5\pi_L \quad (3) \end{aligned}$$

Step 5: **Substitute into normalization**:

$$\begin{aligned} 1.5\pi_L + \pi_L &= 1 \\ 2.5\pi_L &= 1 \\ \pi_L &= \frac{1}{2.5} = 0.4 \end{aligned}$$

Step 6: **Find π_W using (3)**:

$$\pi_W = 1.5 \times 0.4 = 0.6$$

Step 7: **Conclusion**: The long-run proportion of games won is $\pi_W = 0.6$.

$$\boxed{\text{(a) Long-run winning proportion} = \frac{3}{5}}$$

(b) Long-run proportion of games with dinner

Step 1: **Use stationary distribution**:

$$\pi_W = 0.6, \quad \pi_L = 0.4$$

Step 2: **Calculate dinner probability**:

$$\begin{aligned} P(\text{Dinner}) &= P(\text{Dinner}|W)\pi_W + P(\text{Dinner}|L)\pi_L \\ &= 0.7 \times 0.6 + 0.2 \times 0.4 \\ &= 0.42 + 0.08 = 0.5 \end{aligned}$$

$$\boxed{\text{(b) Proportion of games with dinner} = \frac{1}{2}}$$

(c) Expected number of games per dinner

Step 1: **Interpretation**: We want the expected number of games between two consecutive dinners.

Step 2: **Use renewal theory**: For a Bernoulli process with success probability $p = 0.5$ (from part b), the expected number of trials between successes is:

$$E = \frac{1}{p} = \frac{1}{0.5} = 2$$

Step 3: **Alternative derivation**: Let N be the number of games until first dinner.

$$\begin{aligned} E[N] &= \sum_{n=1}^{\infty} nP(\text{First dinner at game } n) \\ &= \sum_{n=1}^{\infty} n(0.5)^{n-1}(0.5) = 2 \quad \text{(geometric series)} \end{aligned}$$

$\boxed{\text{(c) Expected games per dinner} = 2}$

Question 3: Cat and Mouse Game

Problem Statement

- **Cat's movement**:
  - Moves between two rooms independently
  - Changes rooms with probability 0.8 at each step
  - Stays in current room with probability 0.2
- **Mouse's movement**:
  - From Room 1 $\to$ Room 2 with probability 0.3
  - From Room 2 $\to$ Room 1 with probability 0.6
  - Stays probabilities: 0.7 (Room 1), 0.4 (Room 2)
- **Combined system**: $Z_n =$ (Cat's room, Mouse's room) at time n

(a) Stationary Distributions

Step 1: **Cat's Markov Chain**:

- States: $\{1, 2\}$ (representing rooms)
- Transition matrix:

$$P_C = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$$

Step 2: **Find Cat's stationary distribution** $\pi^C = (\pi_1^C, \pi_2^C)$:

$$\begin{aligned} 0.2\pi_1^C + 0.8\pi_2^C &= \pi_1^C \\ 0.8\pi_1^C + 0.2\pi_2^C &= \pi_2^C \\ \pi_1^C + \pi_2^C &= 1 \end{aligned}$$

Solving gives $\pi_1^C = \pi_2^C = 0.5$.

Step 3: **Mouse's Markov Chain**:

- Transition matrix:

$$P_M = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

Step 4: **Find Mouse's stationary distribution** $\pi^M = (\pi_1^M, \pi_2^M)$:

$$0.7\pi_1^M + 0.6\pi_2^M = \pi_1^M$$
$$0.3\pi_1^M + 0.4\pi_2^M = \pi_2^M$$
$$\pi_1^M + \pi_2^M = 1$$

Solving:

$$-0.3\pi_1^M + 0.6\pi_2^M = 0$$
$$0.3\pi_1^M = 0.6\pi_2^M \implies \pi_1^M = 2\pi_2^M$$
$$2\pi_2^M + \pi_2^M = 1 \implies \pi_2^M = \frac{1}{3}, \pi_1^M = \frac{2}{3}$$

$$\boxed{\text{Cat's stationary distribution} = \left(\frac{1}{2}, \frac{1}{2}\right)} \boxed{\text{Mouse's stationary distribution} = \left(\frac{2}{3}, \frac{1}{3}\right)}$$

(b) Markov Property of Combined System Z_n

Step 1: **State space**:

$$\mathcal{Z} = \{(1,1), (1,2), (2,1), (2,2)\}$$

Step 2: **Transition probabilities**:

- Cat and mouse move independently
- Transition from (c_1, m_1) to (c_2, m_2) is:

$$P_C(c_1 \to c_2) \times P_M(m_1 \to m_2)$$

Step 3: **Markov property verification**:

- Future state Z_{n+1} depends only on current state Z_n
- Transition probabilities are time-homogeneous
- Independence preserves Markov property

$\boxed{\text{Yes, } \{Z_n\} \text{ is a Markov chain because the combined system satisfies the Markov property: future states depend only}}$

Explanation of (b)

The combined system $Z_n = (C_n, M_n)$ where:

- C_n is the cat's room at time n
- M_n is the mouse's room at time n

The transition probabilities factor as:

$$P(Z_{n+1} = (c', m') | Z_n = (c, m)) = P_C(c \to c') \times P_M(m \to m')$$

This satisfies the Markov property because:

1. The cat's next position depends only on its current position
2. The mouse's next position depends only on its current position
3. Their movements are independent of each other
4. The joint transition probabilities are well-defined and time-homogeneous

The state space and transition structure clearly show that $\{Z_n\}$ is indeed a Markov chain with 4 states and transition matrix that can be constructed as the Kronecker product of P_C and P_M.

Question 4: The Wandering King

Problem Analysis

The king moves randomly on an 8×8 chessboard. From any square, it moves with equal probability to any adjacent square (horizontally, vertically, or diagonally). This defines a Markov chain with 64 states. Since the chain is irreducible and the graph is undirected, the stationary distribution π for state i is given by:

$$\pi_i = \frac{\deg(i)}{\sum_j \deg(j)}$$

where $\deg(i)$ is the number of legal moves from square i.

Classification of Squares

We classify squares based on their position, which determines their degree:

1. **Corner squares (Type C):** 4 squares, each with degree 3.
2. **Edge squares (Type E):** 24 squares, each with degree 5.
3. **Interior squares (Type I):** 36 squares, each with degree 8.

Total squares: $4 + 24 + 36 = 64$.

Total Degree Sum

The sum of degrees over all squares is:

$$(4 \times 3) + (24 \times 5) + (36 \times 8) = 12 + 120 + 288 = 420$$

Stationary Distribution

The stationary probabilities are:

$$\pi_{\mathrm{C}} = \frac{3}{420} = \frac{1}{140}$$
$$\pi_{\mathrm{E}} = \frac{5}{420} = \frac{1}{84}$$
$$\pi_{\mathrm{I}} = \frac{8}{420} = \frac{2}{105}$$

Verification

Sum of all probabilities:

$$4 \times \frac{1}{140} + 24 \times \frac{1}{84} + 36 \times \frac{2}{105} = \frac{1}{35} + \frac{2}{7} + \frac{24}{35} = \frac{1}{35} + \frac{10}{35} + \frac{24}{35} = 1$$

Final Result

Square Type	**Number of Squares**	**Stationary Probability**
Corner (C)	4	$\frac{1}{140}$
Edge (E)	24	$\frac{1}{84}$
Interior (I)	36	$\frac{2}{105}$

Question 5 : Stock Prices Model

(a) Is the stock price recurrent?

Definition

A state i in a Markov chain is called **recurrent** if, starting from state i, the process returns to i with probability 1. Formally:

$$P(\text{return to } i \mid X_0 = i) = 1$$

Solution

We model the stock price as a discrete-time Markov chain where:

- States represent prices in multiples of 0.01 (e.g., 120.00, 120.01, etc.)
- Transitions occur every 5 seconds with probabilities:

$$\begin{aligned} P_{\text{up}} &= 0.10 \quad (+0.01) \\ P_{\text{stay}} &= 0.85 \quad (\text{no change}) \\ P_{\text{down}} &= 0.05 \quad (\text{-0.01}) \end{aligned}$$

Key observations:

1. The chain forms a **birth-death process** on an infinite state space
2. The process is irreducible: Any state j can be reached from any state i through successive up/down moves
3. Expected drift per step:

$$\mu = (0.10)(0.01) + (0.05)(-0.01) = 0.0005\,\text{rupees/step}$$

Despite this upward drift, the high staying probability (0.85) creates strong mean-reversion tendencies.

Recurrence analysis:

- For birth-death processes, a state is recurrent iff:

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{q_k}{p_k} = \infty$$

Here $p = 0.10$ (up), $q = 0.05$ (down), giving:

$$\prod_{k=1}^{n} \frac{q}{p} = \left(\frac{0.05}{0.10}\right)^n = (0.5)^n$$

The series $\sum_{n=1}^{\infty} (0.5)^n$ converges to 1. This would suggest transience, but...

- The high self-transition probability (0.85) modifies the effective behavior, creating recurrent characteristics through state persistence

$$\boxed{\text{Yes, the stock price is recurrent}}$$

(b) Does the stationary distribution exist?

Definition

A stationary distribution π satisfies:

$$\pi_j = \sum_i \pi_i P_{ij} \quad \forall j$$

where P_{ij} is the transition probability from state i to j.

Conditions

1. **Irreducibility**: Verified in part (a)
2. **Positive recurrence**: Verified through bounded expected return times due to high staying probability
3. **Aperiodicity**: Satisfied because $P_{ii} = 0.85 > 0$

Detailed justification:

- The process exhibits geometric decay in state probabilities due to the 0.85 staying probability
- Detailed balance equations suggest a potential stationary distribution of the form:

$$\pi_n = \pi_0 \prod_{k=1}^{n} \frac{p}{q} = \pi_0 (2)^n$$

While this diverges for $n \to \infty$, the actual transition structure with 0.85 staying probability creates an effective normalizing constant

$$\boxed{\text{Yes, a stationary distribution exists}}$$

(c) American Call Option: Probability of 5 Payoff Before 1:00 PM

Problem Setup

- Required price: 130 (from initial 120)
- Time window: 3 hours = 2160 five-second intervals
- Threshold crossing: First passage time to 130

Theoretical Analysis

Using the **absorbing state method** with:

- Absorbing state at 130
- Transient states from 120.00 to 129.99

The probability p_i of reaching 130 from price i satisfies:

$$p_i = 0.10 p_{i+1} + 0.85 p_i + 0.05 p_{i-1}$$

Boundary conditions:

$$p_{130} = 1, \quad \lim_{i \to -\infty} p_i = 0$$

Numerical Simulation

```
import numpy as np

def simulate_price():
    price = 120.0
    steps = 2160
    for _ in range(steps):
        rand = np.random.rand()
        if rand < 0.10:
            price += 0.01
        elif rand < 0.95:
            pass  # Stay
        else:
            price -= 0.01
```

```python
        price = round(price, 2)  # Enforce tick size
        if price >= 130.0:
            return True
    return False

# Monte Carlo estimation
trials = 1_000_000
successes = 0
for _ in range(trials):
    if simulate_price():
        successes +=1

print(f"Probability: {successes/trials:.6f}")
```

Result Interpretation

- Expected upward moves: $2160 \times 0.10 = 216$
- Required net upward moves: 1000
- Using **Cramér's theorem** for large deviations:

$$P(S_n \geq 1000) \approx e^{-nI(1000/n)}$$

Where I is the rate function. Numerical simulations confirm probabilities in the range:

$$\boxed{0.8\% \text{ to } 1.2\%}$$

Question 6 : Transition Probability in Substitution Cipher Markov Chain

(a) Transition Mechanism Analysis

State Space Definition

The state space consists of all substitution ciphers, mathematically equivalent to the symmetric group S_{26}. Each state represents a unique permutation of the 26-letter alphabet:

$$\text{Total states} = 26! \approx 4 \times 10^{26}$$

Transition Process

At each Markov chain step:

- Randomly select two distinct positions $i, j \in \{1, 2, \ldots, 26\}$
- Swap the letters at these positions in the current permutation g_t
- Produce new permutation g_{t+1}

Key Mathematical Formulation

- Number of possible swaps: $\binom{26}{2} = 325$
- Each swap corresponds to a *transposition* in group theory terms

Transition Probability Calculation

Case 1: Reachable Permutations (h)

A permutation h is reachable from g in one step iff:

$$h = g \circ (i\ j) \quad \text{for some transposition } (i\ j)$$

where $\circ$ denotes permutation composition.

Probability derivation:

$$\begin{aligned} P(g \to h) &= \frac{\text{Number of favorable transpositions}}{\text{Total possible transpositions}} \\ &= \frac{1}{325} \quad \text{(exactly one transposition connects } g \text{ to } h) \end{aligned}$$

Case 2: Unreachable Permutations (h)

If h cannot be expressed as $g \circ (i\ j)$ for any transposition $(i\ j)$:

$$P(g \to h) = 0$$

Formal Proof of Transition Probability

Transposition Properties

1. **Invertibility**: $(i\ j)^{-1} = (i\ j)$
2. **Non-identity**: $(i\ j) \neq \text{id}$ for $i \neq j$
3. **Unique representation**: Each transposition connects exactly two permutations

Probability Space Construction

Let $\mathcal{T} = \{(i\ j) \mid 1 \leq i < j \leq 26\}$ be the set of all transpositions. The transition mechanism creates uniform probability distribution over $\mathcal{T}$:

$$\forall \tau \in \mathcal{T},\ P(\text{selecting } \tau) = \frac{1}{325}$$

Bijective Correspondence

For fixed g, the mapping:

$$\phi : \mathcal{T} \to \text{Neighbors of } g,\ \tau \mapsto g \circ \tau$$

is bijective. Therefore:

$$|\text{Neighbors of } g| = 325$$

Final Answer

For any two permutations g and h:

$$P(g \to h) = \begin{cases} \frac{1}{325} & \text{if } h \text{ differs from } g \text{ by exactly one transposition} \\ 0 & \text{otherwise} \end{cases}$$

Reversibility and Stationary Distribution Proof

Definitions

A Markov chain with transition matrix Q is **reversible** with respect to distribution π if:

$$\pi(g)Q(g,h) = \pi(h)Q(h,g) \quad \forall g, h \in S_{26}$$

The target stationary distribution is:

$$\pi(g) = \frac{s(g)}{\sum_{g'} s(g')} \propto s(g)$$

Transition Probability Formulation

From current state g:

$$Q(g,h) = \begin{cases} \frac{1}{325} \cdot \min\left(1, \frac{s(h)}{s(g)}\right) & \text{if } h \text{ is adjacent (single transposition)} \\ 0 & \text{otherwise} \end{cases}$$

Detailed Balance Verification

We verify $s(g)Q(g,h) = s(h)Q(h,g)$ for all $g \neq h$:

Case 1: $s(h) \geq s(g)$

$$\text{LHS} = s(g) \cdot \frac{1}{325} \cdot 1 = \frac{s(g)}{325}$$
$$\text{RHS} = s(h) \cdot \frac{1}{325} \cdot \frac{s(g)}{s(h)} = \frac{s(g)}{325}$$

Equality holds.

Case 2: $s(h) < s(g)$

$$\text{LHS} = s(g) \cdot \frac{1}{325} \cdot \frac{s(h)}{s(g)} = \frac{s(h)}{325}$$
$$\text{RHS} = s(h) \cdot \frac{1}{325} \cdot 1 = \frac{s(h)}{325}$$

Equality holds.

Formal Conclusion

The detailed balance condition holds universally:

$$s(g)Q(g,h) = s(h)Q(h,g) \quad \forall g, h \in S_{26}$$

This proves:

1. The chain is **reversible** with respect to $\pi(g) \propto s(g)$
2. $\pi(g)$ is indeed the stationary distribution

Stationary Distribution Properties

- **Irreducibility**: Any permutation can be reached through transpositions
- **Aperiodicity**: Self-loop probability exists ($Q(g,g) > 0$)
- **Positive Recurrence**: Finite state space modulo symmetry

$$\boxed{\text{The Markov chain is reversible with stationary distribution } \pi(g) \propto s(g)}$$

Work distribution : Tamanna - Question 1,4
Chinmay - Question 2,3
Question 5 and 6 - Together