Stochastic Modelling of Financial Derivatives

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1. By derangement theorem

Probability that no letter is in the correct envelope $= \dfrac{N\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots\frac{1}{50!}\right)}{N!}$

→ this is tedious to calculate.

We can use an approximation

Let Derangement be $D(N)$ or $D(50)$

then $\dfrac{D(50)}{50!} \approx \dfrac{1}{e}$ (become more accurate as N ↑s)

⇒ Probability of atleast one letter being in the correct envelope $= 1-\dfrac{D(50)}{50!} = 1-\dfrac{1}{e} =$ 0.6321.

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2.) A_i = event that money is in present i

Initially, $P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

H_2 : host open Present 2.

W: Expected winning.

By Law of Total Probability

$P(H_2) = P(H_2 | A_1) . P(A_1) + P(H_2 | A_2) P(A_2) + P(H_2 | A_3) P(A_3)$

$= \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{1}{2}$

Posterior Probability

$P(A_3 | H_2)$

By Bayes theorem

$$P(A_3 | H_2) = \frac{P(H_2 | A_3) \times P(A_3)}{P(H_2)} = \frac{1 \times \frac{1}{3}}{1/2} = \frac{2}{3}$$

Hence expected winning $= \frac{2}{3} \times 1000\$ = 666.66\$$

$\Rightarrow$ Switch to 3 improves probability

we should switch.

This is analogous to the Monty-Hall Problem.

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3) To Prove $P(B \cap C) > 0$; $A, B \& C$ ~~are~~ have common ~~independent~~

Submitted by
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and CHINMAY

(a) $P(A \cap B | C) = P(A | B \cap C) \cdot P(B | C)$

$P(A \cap B | C) = P(A | B \cap C) \cdot P(B | C)$

LHS

$\rightarrow P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

RHS

$$P(A | B \cap C) \cdot P(B | C) = \frac{P(A \cap B \cap C)}{\cancel{P(B \cap C)}} \times \frac{\cancel{P(B \cap C)}}{P(C)}$$

$$= \frac{P(A \cap B \cap C)}{P(C)}$$

Hence, LHS = RHS, Proved.

TRUE

(b) $P(A \cap B | C) = P(A | C) \, P(B | C)$

False, since unconditional independence of A & B doesn't gurrantee conditional independence

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(C)

$P(A \mid D \cap B^c) > P(A \mid D \cap B)$

$P(A \mid D^c \cap B^c) > P(A \mid D^c \cap B)$

then $P(A \mid B)$ must be $>$ than $P(A \mid B^c)$

This statement is false.

This is a case of Simpson's Paradox.

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4) Construct or disprove existence

(a) $E(X)$ finite, $E(X^2)$ not finite

Let $f(x) = \dfrac{1}{x^2(\log x)^2}$

$\int_2^{\infty} f(x)\,dx \Rightarrow \int_{\log 2}^{\infty} \dfrac{1}{e^u u^2}\,du \rightarrow$ converges

valid pdf function.

$xf(x) = \dfrac{1}{x(\log x)^2}$, $E(X) = \sum_{x=2}^{\infty} xf(x)$

we will use integral test $\int_{\log 2}^{\infty} \dfrac{1}{x(\log x)^2}\,dx$

$u = \log x$, $x = e^u$, $\dfrac{dx}{x} = du$

$\int_{\log 2}^{\infty} \dfrac{1}{u^2}\,du = -\dfrac{1}{u}\Big]_{\log 2}^{\infty} = \dfrac{1}{\log 2}$ finite Hence

By integral test $E(X)$ is finite.

$E(X^2) = \sum_{x=2}^{\infty} x^2 f(x) = \sum_{x=2}^{\infty} \dfrac{1}{(\log x)^2}$

$dx = e^u\,du$

$\int_2^{\infty} \dfrac{1}{(\log x)^2}\,dx = \int_{\log 2}^{\infty} \dfrac{e^u\,du}{u^2}$

~~By Ratio test~~ $\cancel{\dfrac{e^{u+1}}{(u+1)^2} \times \dfrac{u^2}{e^u}}$

$\dfrac{e^u}{u^2} > 0$ for $u \geq \log 2$.

and $\dfrac{e^u}{u^2} \rightarrow \infty$ as $u \rightarrow \infty$ $\Rightarrow \int_{\log 2}^{\infty} \dfrac{e^u}{u^2}\,du$ diverges

(By integral test) $\rightarrow E(X^2)$ diverges.

Both Σ, and $\int$ converge or diverge together.

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(b) the function $f(x) = \frac{1}{x^2(\log x)^2}$ is a continuous function for which $E(X)$ is finite, and $E(X^2)$ is infinite as shown ~~its~~ by integral in part a.

When x takes continuous values, we $\frac{1}{x^2(\log x)^2}$ as a continuous function (as in part b)

when x takes discrete values, $x = 2, 3, 4 \ldots$ it is a discrete random variable as in part a.

(c) $E(X) = 1$, $E(e^{-X}) = \frac{1}{3}$

This is not possible as Jensen's Inequality says

$$E(e^{-X}) \geq e^{-E(X)}$$

In this case

$$E(e^{-X}) \geq e^{-1} = 0.3679 > 0.33 \text{ or } \frac{1}{3}$$

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(5.) Probability distribution of M.

Probability of maximum prize M is at most k is

$$P(M \le k) = \left(\frac{k}{N}\right)^n$$

Since all n draws $\le k$, PMF of M is

$$P(M=k) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

Expectation

$$E(M) = \sum_{k=1}^{N} k \cdot P(M=k) = \sum_{k=1}^{N} k \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

$$E(M) = \frac{1}{N^n} \sum_{k=1}^{N} \left[k^{n+1} - k(k-1)^n\right]$$

$$E(M) = N - \frac{1}{N^n} \sum_{j=0}^{N-1} j^n$$

Eg: For $N=10$ & $n=3$

$$E(M) = 10 - \frac{0^3 + 1^3 + 2^3 \dots 9^3}{10^3}$$

$$= 7.975$$

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(6) $X, Y \rightarrow$ two independent uniform random variables on $[0, d]$

distance $(X, Y) = |X - Y|$.

We want $|X-Y| \leq \frac{1}{3}$

$\Downarrow$

Area between $Y = X + \frac{d}{3}$ & $Y = X - \frac{d}{3}$.

Invalid areas

Two right Δs in the square.

T1 : $Y = X + \frac{d}{3}$, $X = d$, $Y = d$.

T2 : $Y = X - \frac{d}{3}$, $X = 0$, $Y = 0$.

Area of $T_1 = \frac{2d^2}{9}$

Area of $T_2 = \frac{2d^2}{9}$

Invalid area $= \frac{4d^2}{9}$

Valid $= \frac{5d^2}{9}$

Total Area $= d^2$

$\Rightarrow P(|X-Y| < \frac{1}{3}) = \frac{\frac{5d^2}{9}}{d^2} = \frac{5}{9}$

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(7)

Individual Transmission

(a) Rumor starts with originator.

after 1^{st} step

Probability that the person chosen is not the originator is $\frac{n-1}{n}$ at each step.

so for $r-1$ steps after 1st step

$\cancel{\left(\frac{n}{n+1}\right)^{r-1}}$. $\left(\frac{n-1}{n}\right)^{r-1}$.

b) without being repeated to anyone.

$n \times (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$

Total no. of sequences $= \cancel{(n+1)^r}\, n^r$

P (not repeated to anyone) $= \frac{n!}{(n-r)!} \times \frac{1}{\cancel{(n+1)^r}\, n^r}$.

$P = \frac{n!}{(n-r)!} \times \frac{1}{\cancel{(n+1)^r}\, n^r}$

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Group Transmission :

(a) without repeating to originator:

$$\frac{\binom{n}{N} \times \binom{n-1}{N}^{r-1}}{\binom{n}{N}^{\cancel{r-1}\, r}}$$

Choose N from n-1 (no originator)

starts with originator

→ favourable originator is excluded

Total ways

→ Originator calls N people.

Continues

(b) without being repeated to any person.

$$\frac{\prod_{i=0}^{r-1} \binom{n-iN}{N}}{\binom{n}{N}^{\cancel{r-1}\, r}}$$

→ everyone excludes the previous listeners and tells to a gathering of N people.

→ Originator call N people and this continues

Total → sequences

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A_i's are independent

(8.)

$$P(\cap A_i^c) \leq e^{-P(A_1)-P(A_2)\cdots-P(A_n)}$$

$$\Rightarrow P(A_1^c) \times P(A_2^c) \times P(A_3^c) \cdots P(A_n^c).$$

Let $P(A_i) = x_i$

$\Rightarrow P(A_i^c) = 1 - x_i$

$$(1-x_1)(1-x_2) \cdots (1-x_n) \leq e^{-\sum_{i=1}^{n} x_i}$$

We need to show $1 - x_i \leq e^{-x_i}$

This can be seen from Taylor Series expansion of e^{-x}.

$$e^{-x} = 1 - \frac{x^1}{1!} + \underbrace{\frac{x^2}{2!}}_{\text{need of add this}} - \frac{x^3}{3!} \cdots$$

need of add this $\frac{x^2}{2!}$ since $1-x$ goes below e^{-x}

Let show by calculus

Let's define

$$f(x) = e^{-x} - (1-x).$$

$$f'(x) = -e^{-x} + 1.$$

For $f'(x) = 0$, $x = 0$.

Also $f(x)$ at $x = 0 = 0$.

$$f''(x) = e^{-x} > 0 \quad \forall x$$

Since $f(x)$ is convex, $x = 0$ is minimum

$\Rightarrow f(x) \geq 0 \quad \forall x \Rightarrow e^{-x} \geq 1 - x$ ✓.

$\Rightarrow$ Multiply to get

$$\boxed{(1-x_1)(1-x_2)\cdots(1-x_n) \leq e^{-x_1-x_2-x_3\cdots-x_n}}$$

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9) $H(x) = (F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y)$,

CDF of $X+Y$, $X \sim F$ & $Y \sim G$ are independent.

- Non-decreasing Property

$$H(x_2) - H(x_1) = \int_{-\infty}^{\infty} \underbrace{[F(x_2-y) - (F(x_1-y))]}_{\geq 0} \, dG(y) \geq 0$$

Probability measure.

since F is non decreasing for all y.

→ Right continuity

Let $x_n \downarrow x$, By right continuity of F

$$\lim_{n \to \infty} F(x_n - y) = F(x-y) \quad \forall y$$

By Dominated Convergence Theorem.

$$F(x_n - y) \leq 1.$$

$$\lim_{n \to \infty} H(x_n) = \int_{-\infty}^{\infty} \lim_{n \to \infty} F(x_n - y)\, dG(y)$$

$$= \int_{-\infty}^{\infty} F(x-y)\, dG(y) = H(x).$$

Thus $H(x)$ is right continuous.

Boundary limits

As $x \to -\infty$

$\forall y$, $F(x-y) \to 0$

By Dominated convergence Theorem

$$\lim_{x \to -\infty} H(x) = \int_{-\infty}^{\infty} 0 \, dG(y) = 0.$$

As $x \to \infty$.

$\forall y$ $F(x-y) \to 1$

$$\lim_{x \to \infty} H(x) = \int_{-\infty}^{\infty} 1 \, dG(y) = 1.$$

commutativity

$$H(x) = \int_{-\infty}^{\infty} F(x-y) \, dG(y) = \int_{-\infty}^{\infty} G(x-z) \, dF(z)$$

$$z = x - y.$$

Since $H(x)$ satisfies all properties of a CDF

$\Rightarrow$ $F * G$ is also a CDF

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(10)

For any ω;

$X(\omega) \geq 0$

$$\int_0^\infty \mathbb{I}[0, X(\omega))(x)\,dx = \int_0^{X(\omega)} 1\,dx = X(\omega)$$

$$\int_0^\infty \mathbb{I}_{[0, X(\omega))}(x)\,dx = X(\omega).$$

Taking expectation

$$\int_\Omega \int_0^\infty \mathbb{I}_{[0, X(\omega)]}(x)\,dx\,dP(\omega) = \int_\Omega X(\omega)\,dP(\omega)$$

$$= E(X)$$

Change order of integration (Fubini's Method)

$$\int_\Omega \int_0^\infty I_{[0, X(\omega))}(x)\,dx\,dP(\omega) = \int_0^\infty \int_\Omega I_{[0, X(\omega))}(x)\,dP(\omega)\,dx.$$

$x < X(\omega)$

$$\int_\Omega I_{[0, X(\omega))}(x)\,dP(\omega) = P(X > x) = 1 - F(x)$$

$$\int_0^\infty \int_\Omega I_{[0, X(\omega))}(x)\,dP(\omega)\,dx = \int_0^\infty (1 - F(x))\,dx$$

$$E(x) = \int_\Omega \int_0^\infty I_{[0, X(\omega))}(x)\,dx\,dP(\omega)$$

$$= \int_0^\infty (1 - F(x))\,dx.$$

Hence Proved

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11.)

(i) Verify, $E[e^{\mu X}] = e^{u\mu + \frac{1}{2}\mu^2\sigma^2}$

Set up expectation integral.

$$E[e^{uX}] = \int_{-\infty}^{\infty} e^{\mu x} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$E[e^{\mu X}] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu x - \frac{(x-u)^2}{2\sigma^2}} dx.$$

$$\mu x - \frac{(x-\mu)^2}{2\sigma^2} = \mu x - \frac{x^2 - 2ux + u^2}{2\sigma^2}.$$

$$= \mu x - \frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}.$$

$$= \frac{-x^2}{2\sigma^2} + x\left(\mu + \frac{\mu}{\sigma^2}\right) - \frac{\mu^2}{2\sigma^2}.$$

$$a = \frac{1}{2\sigma^2}, \quad b = \mu + \frac{\mu}{\sigma^2}$$

$$-ax^2 + bx - \frac{\mu^2}{2\sigma^2} = -a\left(x^2 - \frac{bx}{a}\right) - \frac{\mu^2}{2\sigma^2}$$

$$= -a\left(x - \frac{b}{2a}\right)^2 + a\left(\frac{b}{2a}\right)^2 - \frac{\mu^2}{2\sigma^2}$$

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$$\frac{b}{2a} = \frac{u + \frac{\mu}{\sigma^2}}{2 \cdot \frac{1}{2\sigma^2}} = \sigma^2\left(u + \frac{\mu}{\sigma^2}\right) = u\sigma^2 + \mu$$

$$a\left(\frac{b}{2a}\right)^2 = \frac{1}{2\sigma^2}(u\sigma^2+\mu)^2 = \frac{(u\sigma^2+\mu)^2}{2\sigma^2}$$

$$a\left(\frac{b}{2a}\right)^2 - \frac{\mu^2}{2\sigma^2} = \frac{(u\sigma^2+\mu)^2 - \mu^2}{2\sigma^2}$$

$$= \frac{u^2\sigma^4 + 2u\sigma^2\mu}{2\sigma^2} = \frac{u^2\sigma^2}{2} + u\mu.$$

Evaluating Integral

$$-\frac{1}{2\sigma^2}(x-(u\sigma^2+\mu))^2 + u\mu + \frac{u^2\sigma^2}{2}$$

$$E[e^{uX}] = \frac{1}{\sigma\sqrt{2\pi}} e^{u\mu + \frac{u^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-(u\sigma^2+\mu)}{2\sigma^2}\right)^2} dx$$

$$\downarrow$$

$$\sigma\sqrt{2\pi}$$

$$\Rightarrow E[e^{uX}] = e^{u\mu + \frac{u^2\sigma^2}{2}}.$$

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(ii) verify Jensen's Inequality

$$E[\varphi(X)] \geq \varphi(EX)$$

$$\varphi(x) = e^{ux} \quad \& \quad X \sim N(\mu, \sigma^2)$$

$$E[\varphi(X)] = E[e^{uX}] = e^{u\mu + \frac{u^2\sigma^2}{2}}$$

$$\varphi(E[X]) = \varphi(\mu) = e^{u\mu}.$$

$$e^{u\mu + \frac{u^2\sigma^2}{2}} \geq e^{u\mu}.$$

$$u\mu + \frac{u^2\sigma^2}{2} \geq u\mu.$$

$$\hookrightarrow \geq 0$$

Hence Proved.

$$E[e^{uX}] = e^{u\mu + \frac{u^2\sigma^2}{2}} \geq e^{u\mu} = e^{uE[X]}.$$

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2. Practical Section.

$\{(x, y) : 0 \le x, y \le n\}$

Start $(0,0)$, End (n, n).

Right $(u, v) \to (u+1, v)$ or Up $(u, v) \to (u, v+1)$

Constraint : Path never crosses above $y = x$.

$\Rightarrow y \le x$ at all times :

No. of valid paths

$$P(i, j) = \begin{cases} 1 & \text{if } i = 0, j = 0. \\ 0 & \text{if } j > i \text{ (above diagonal)} \\ P(i-1, j) + P(i, j-1) & \text{if } j \le i. \end{cases}$$

Closed form solution

$$P_n = C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$$

Code attached (DP- Dynamic Programming approach).

time ~ $O(n^2)$, space $O(n^2)$.

Can also use direct formula $O(n)$ time

Since $1 \le n \le 100$, Both are efficient.

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Result for some n values...

n	P_n
1	1
2	2
3	5
4	15
5	42
6	132
7	429
8	1430

Asymptotic Behaviour

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}} \quad \text{as } n \to \infty.$$

```python
def solve_royal_revenge():
    MOD = 10**9 + 7
    MAX_N = 100

    # Initialize DP table
    dp = [[0] * (MAX_N + 1) for _ in range(MAX_N + 1)]

    # Base case
    dp[0][0] = 1

    # Fill DP table
    for i in range(MAX_N + 1):
        for j in range(min(i + 1, MAX_N + 1)):  # j <= i constraint
            if i == 0 and j == 0:
                continue
            if i > 0:
                dp[i][j] = (dp[i][j] + dp[i-1][j]) % MOD
            if j > 0:
                dp[i][j] = (dp[i][j] + dp[i][j-1]) % MOD

    # Extract results for n = 1 to 100
    results = []
    for n in range(1, 101):
        results.append(dp[n][n])

    return results

# Alternative: Direct Catalan number calculation
def catalan_number(n, MOD):
    # Calculate C(2n, n) - C(2n, n-1) efficiently
    if n == 0:
        return 1

    # Calculate C(2n, n) using multiplicative approach
    result = 1
    for i in range(n):
        result = result * (2 * n - i) // (i + 1)

    return (result // (n + 1)) % MOD

def solve_with_catalan():
    MOD = 10**9 + 7
    results = []

    for n in range(1, 101):
        results.append(catalan_number(n, MOD))

    return results
```

Tamanna → Q 2, Q3a, 3b, Q4a, 4b,
Q5, Q7a, Q10

Chinmay Q1, Q3c, Q4c, Q6, Q7b,
Q8, Q9, Q11.

Q Practical Section → Solved Together