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(25 points) A frog begins at position 0 in the river. (Think of the river as the real line and each position i is the point that corresponds to real number i.) Its goal is to get to position n. There are lilypads at various positions. There is always a lilypad at position 0 and position n. The frog can jump at most r units at a time. Give an algorithm that finds the path the frog should take to minimize the number of jumps, assuming a solution exists. (Yes, the algorithm is indeed very simple. This is a practice for you to prove the correctness of a greedy algorithm using the template we learnt in class.)

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Expert Answer

The goal is to find the path of the frog which is minimal in jumps by assuming that it exists.

Formalizing the Algorithm

- Let J be an empty series of jumps.
- Let our current positon x=0
- While x < n:

1. Find the furthest lilypad L reachable from x that is not after positon n.
2. Add a jump to J from x to L's location.
3. Set x to L's location.

- Return J

This is a simple greedy algorithm for routing the frog as forward as possible at each step.

The algorithm can be proved by using two properties

1.The algorithm doesn't get stuck.

ie .There will always be a legal series of jumps.

2.There isn't a better path available

ie.The algorithm finds an optimal series of jumps.

Lemma 1: The greedy algorithm always finds a path from the start lilypad to the destinaton lilypad.

proof:Lets contradict the lemma by assuming that there isn't a path.Let the positons of lilypads be x1 < x2< .... xm.

Since the algorithm didnt find a path,it must have stopped at some lilypad xk and not been able to jump to a future lilypad. ie It could not jump to lilypad k+1,so xk + r < xk+1.

Since there is a path from lilypad 1 to m,there must be some path that starts before lilypad k+1 and ends at or after lilypad k+1.This jump can't be made from lilypad k,so it must have been made from lilypad s for some s<k.But then we have xs + r < xk + r < xk+1,so this jump is illegal.

Since this is a contradiction ,so our assumption was wrong and our algorithm always finds a path.

Proving Optimality

Let J be the series of jumps produced by our algorithm and let J* be an optimal series of jumps and |J| and |J*| denote the number of jumps in J and J*

Lemma 2: For all $0 \leq i \leq |J^*|$, we have $p(i,J) \geq p(i,J^*)$

Proof

:By induction .As a base case ,if $i=0$, then $p(0,J)=0 \geq 0= p(0,J^*)$ since the frog hasn't moved.

For the inductive step, assume that the claim holds for some $0 \leq i < |J^*|$.We will prove the claim holds for $i+1$ by considering two cases:

Case 1:$p(i,J) \geq p(i + 1,J^*)$. Since each jump moves forward,we hace $p(i + 1, J) \geq p(i,J)$,so we have $p(i + 1,J) \geq p(i + 1, J^*)$

Case 2: $p(i,j) < p(i + 1,J^*)$.Each jump is of size at most r,so $p(i +1,J^*) \leq p(i,J^*) +r$. By our IH,we know $p(i,J) \geq p(i, J^*)$, so $p(i +1,J^*) \leq p(i,J) + r$.

Therefore ,the greedy algorithm can jump to positon at least $p(i +1, J^*)$.Therefore $p(i +1, J) \geq p(i + 1,J^*)$

So $p(i +1, J) \geq p(i + 1,J^*)$,completing the induction.

Theorem: Let J be the series of jumps produced by the greedy algorithm and J^* be any optimal series of jumps .Then $|J| = |J^*|$

Proof:

Since $J^*$is an optimal solution,we know that $|J^*| \leq |J|$

Suppose for contradiction that $|J^*| < |J|$. Let $k = |J^*|$. By Lemma 2,we have $p(k,J^*) \leq p(k,J)$.

Because the frog arrives at position n after k jumps along series J^* , we know $n \leq p(k,J)$ .Because the greedy algorithm never jumps past positon n ,we know $p(k,J) \leq n$, so $n= p(k,J)$. Since $|J^*| <| J|$, the greedy algorithm must have taken another jump after the kth jump,contradicting that the algorithm stops after reaching position n.

We have reached a contradiction,so our assumption was wrong and $|J^*| = |J|$,so the greedy algorithm produces an optimal solution.

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